PAPER

Open Petri nets

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Abstract

The reachability semantics for Petri nets can be studied using open Petri nets. For us, an "open" Petri net is one with certain places designated as inputs and outputs via a cospan of sets. We can compose open Petri nets by gluing the outputs of one to the inputs of another. Open Petri nets can be treated as morphisms of a category Open(Petri), which becomes symmetric monoidal under disjoint union. However, since the composite of open Petri nets is defined only up to isomorphism, it is better to treat them as morphisms of a symmetric monoidal *double* category $\mathbb{O}$**pen**(Petri). We describe two forms of semantics for open Petri nets using symmetric monoidal double functors out of $\mathbb{O}$**pen**(Petri). The first, an operational semantics, gives for each open Petri net a category whose morphisms are the processes that this net can carry out. This is done in a compositional way, so that these categories can be computed on smaller subnets and then glued together. The second, a reachability semantics, simply says which markings of the outputs can be reached from a given marking of the inputs.

1. Introduction

Petri nets are a simple and widely studied model of computation (Girault and Valk 2013; Gorrieri 2017; Peterson 1981), with generalizations applicable to many forms of modeling (Jensen and Kristensen 2009). Recently, more attention has been paid to a compositional treatment in which Petri nets can be assembled from smaller "open" Petri nets (Baez and Pollard 2017; Baldan et al. 2005, 2015; Bruni et al. 2011, 2013, 2001). In particular, the reachability problem for Petri nets, which asks whether one marking of a Petri net can be obtained from another via a sequence of transitions, can be studied compositionally (Rathke et al. 2014; Sassone and Sobociński 2005; Sobociński and Stephens 2013). Here, we seek to give this line of work a firmer footing in category theory. Petri nets are closely tied to symmetric monoidal categories in two ways. First, a Petri net P can be seen as a presentation of a free symmetric monoidal category FP, with the places and transitions of P serving to freely generate the objects and morphisms of FP. We show how to construct this in Section 2, after reviewing a line of previous work going back to Meseguer and Montanari (1990). In these terms, the reachability problem asks whether there is a morphism from one object of FP to another.

Second, there is a symmetric monoidal category where the objects are sets and the morphisms are equivalence classes of open Petri nets. We construct this in Section 3, but the basic idea is very simple. Here is an open Petri net P from a set X to a set Y:
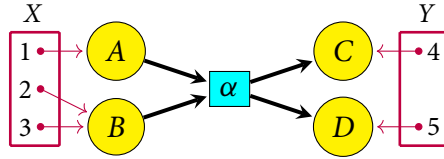
The yellow circles are places and the blue rectangle is a transition. The bold arrows from places to transitions and from transitions to places complete the structure of a Petri net. There are also arbitrary functions from X and Y into the set of places. These indicate points at which tokens could flow in or out, making our Petri net "open." We write this open Petri net as $P\colon X \nrightarrow Y$ for short.

Given another open Petri net $Q\colon Y \nrightarrow Z$:
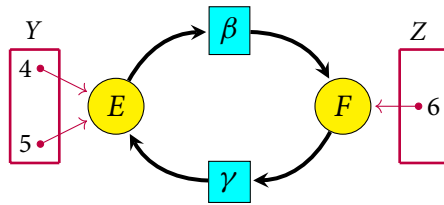
the first step in composing P and Q is to put the pictures together:
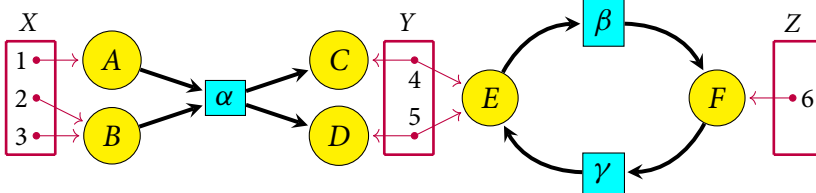
At this point, if we ignore the sets X, Y, Z, we have a new Petri net whose set of places is the disjoint union of those for P and Q. The second step is to identify a place of P with a place of Q whenever both are images of the same point in Y. We can then stop drawing everything involving Y and get an open Petri net $Q \odot P\colon X \nrightarrow Z$:
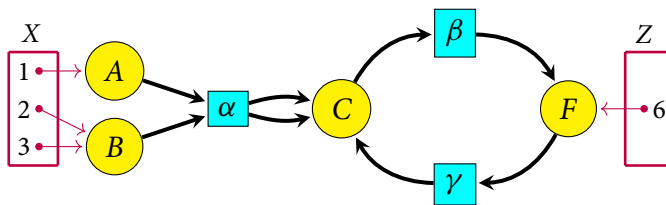
Formalizing this simple construction leads us into a bit of higher category theory. The process of taking the disjoint union of two sets of places and then quotienting by an equivalence relation is a pushout. Pushouts are defined only up to canonical isomorphism: for example, the place labeled C in the last diagram above could equally well have been labeled D or E. This is why to get a category, with composition strictly associative, we need to use *isomorphism classes* of open Petri nets as morphisms. But there are advantages to working with open Petri nets rather than isomorphism classes. For example, we cannot point to a specific place or transition in an isomorphism class of Petri nets. If we work with actual open Petri nets, we obtain not a category but a bicategory (Sassone and Sobociński 2005).

However, this bicategory is equipped with more structure. Besides composing open Petri nets, we can also "tensor" them via disjoint union: this describes Petri nets being run in parallel rather than in series. The result is a symmetric monoidal bicategory. Unfortunately, the axioms for a symmetric monoidal bicategory are cumbersome to check directly (Stay 2016). Double categories turn

out to be much more convenient. Double categories were introduced in the 1960s by Ehresmann (1963, 1965). More recently, they have been used to study open dynamical systems (Lerman 2018; Lerman and Spivak 2016; Ngotiaoco 2017), open electrical circuits and chemical reaction networks (Courser 2017), open discrete-time Markov chains (Clerc et al. 2017), coarse-graining for open continuous-time Markov chains (Baez and Courser 2017), and "tile logic" for concurrency in computer science (Bruni et al. 2002).

A 2-morphism in a double category can be drawn as a square:

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\ M\ } & Y_1 \\
{\scriptstyle f}\Big\downarrow & \Downarrow \alpha & \Big\downarrow{\scriptstyle g} \\
X_2 & \xrightarrow{\ N\ } & Y_2.
\end{array}
$$

We call X_1, X_2, Y_1, and Y_2 "objects," f and g "vertical 1-morphisms," M and N "horizontal 1-cells," and α a "2-morphism." We can compose vertical 1-morphisms to get new vertical 1-morphism and compose horizontal 1-cells to get new horizontal 1-cells. We can compose the 2-morphisms in two ways: horizontally and vertically. This is just a quick sketch of the ideas; for full definitions see Appendix A.

In Theorem 13, we construct a symmetric monoidal double category $\mathbb{O}\mathbf{pen}(\text{Petri})$ with:

- sets $X, Y, Z, \ldots$ as objects,
- functions $f \colon X \to Y$ as vertical 1-morphisms,
- open Petri nets $P \colon X \nrightarrow Y$ as horizontal 1-cells,
- morphisms between open Petri nets as 2-morphisms.

To get a feeling for morphisms between open Petri nets, some examples may be helpful. There is a morphism from this open Petri net:
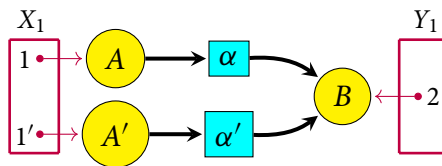
to this one:
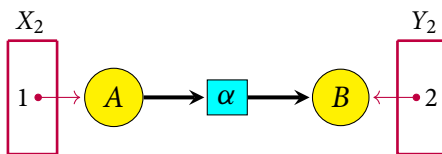
mapping both primed and unprimed symbols to unprimed ones. This describes a process of "simplifying" an open Petri net. There are also morphisms that include simple open Petri nets more complicated ones. For example, the above morphism of open Petri nets has a right inverse.

The main goal of this paper is to describe two forms of semantics for open Petri nets. The first is an "operational" semantics. In Theorem 17, we show that this semantics gives a map from $\mathbb{O}\mathbf{pen}(\text{Petri})$ to a double category $\mathbb{O}\mathbf{pen}(\text{CMC})$. This map sends any Petri net P to the symmetric monoidal category FP, but it also acts on open Petri nets in a compositional way. The second is a "reachability" semantics. This gives a map from $\mathbb{O}\mathbf{pen}(\text{Petri})$ to the double category of relations, $\mathbb{R}\mathbf{el}$, which has

- sets $X, Y, Z, \ldots$ as objects,
- functions $f\colon X \to Y$ as vertical 1-morphisms,
- relations $R \subseteq X \times Y$ as horizontal 1-cells,
- squares

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\;\; R \subseteq X_1 \times Y_1 \;\;} & Y_1 \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\
X_2 & \xrightarrow{\;\; S \subseteq X_2 \times Y_2 \;\;} & Y_2
\end{array}
$$
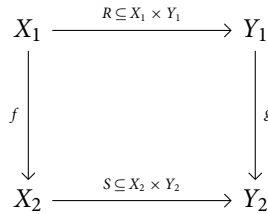
obeying $(f \times g)R \subseteq S$ as 2-morphisms.

In Petri net theory, a "marking" of a set X is a finite multisubset of X: we can think of this as a way of placing finitely many tokens on the points of X. Let $\mathbb{N}[X]$ denote the set of markings of X. Given an open Petri net $P\colon X \nrightarrow Y$, there is a "reachability relation" saying when a given marking of X can be carried by a sequence of transitions in P to a given marking of Y, leaving no tokens behind. We write the reachability relation of P as
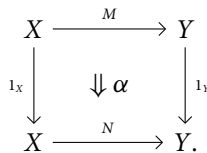
$$
\blacksquare P \subseteq \mathbb{N}[X] \times \mathbb{N}[Y].
$$

In Theorem 23, we show that the map sending P to $\blacksquare P$ extends to a lax double functor

$$
\blacksquare\colon \mathbb{O}\mathbf{pen}(\mathsf{Petri}) \to \mathbb{R}\mathbf{el}.
$$

In Theorem 24, we go further and show that this double functor is symmetric monoidal.

If the reader prefers bicategories to double categories, they may be relieved to learn that any double category $\mathbb{D}$ gives rise to a bicategory $H(\mathbb{D})$ whose 2-morphisms are those 2-morphisms of $\mathbb{D}$ of the form

$$
\begin{array}{ccc}
X & \xrightarrow{\;\; M \;\;} & Y \\
{\scriptstyle 1_X}\downarrow & \Downarrow \alpha & \downarrow{\scriptstyle 1_Y} \\
X & \xrightarrow{\;\; N \;\;} & Y.
\end{array}
$$

Shulman has described conditions under which symmetric monoidal double categories give rise to symmetric monoidal bicategories (Shulman 2010), and using his work, one can show that the operational and reachability semantics for open Petri nets give maps between symmetric monoidal bicategories (Baez and Courser 2019). However, only the double category framework presents the operational and reachability semantics in their full glory. Namely, using double categories, we can describe how these semantics behave on composite open Petri nets, tensor products of open Petri nets, and also morphisms between open Petri nets.

2. From Petri Nets to Commutative Monoidal Categories

In this section, we treat Petri nets as presentations of symmetric monoidal categories. As we shall explain, this has already been done by various authors. Unfortunately, there are different notions of symmetric monoidal category, and also different notions of morphism between Petri nets, which combine to yield a confusing variety of possible approaches.

Here, we take the maximally strict approach and work with "commutative" monoidal categories. This means that we are treating tokens in Petri nets as indistinguishable rather than merely swappable – an approach known as the "collective token philosophy" (van Glabbeek and Plotkin 2009). A commutative monoidal category is a commutative monoid object in Cat, so its associator:

$$\alpha_{a,b,c}\colon (a \otimes b) \otimes c \xrightarrow{\sim} a \otimes (b \otimes c),$$

its left and right unitor:

$$\lambda_a\colon I \otimes a \xrightarrow{\sim} a, \qquad \rho_a\colon a \otimes I \xrightarrow{\sim} a,$$

and even – disturbingly – its symmetry:

$$\sigma_{a,b}\colon a \otimes b \xrightarrow{\sim} b \otimes a$$

are all identity morphisms. The last would ordinarily be seen as "going too far," since while every symmetric monoidal category is equivalent to one with trivial associator and unitors, this ceases to be true if we also require the symmetry to be trivial. However, it seems that Petri nets most naturally serve to present symmetric monoidal categories of this very strict sort. Thus, we construct a functor from the category of Petri nets to the category of commutative monoidal categories, which we call CMC:

$$F\colon \mathsf{Petri} \to \mathsf{CMC}.$$

This functor sends any Petri net P to the free commutative monoidal category on P, and indeed, it is a left adjoint.

It seems that Montanari and Meseguer were the first to treat Petri nets as presentations of commutative monoidal categories (Meseguer and Montanari 1990). They constructed a closely related but different left adjoint functor from a category of Petri nets to a category of "Petri categories," which they call CatPetri. Our category Petri is a subcategory of their category of Petri nets: our morphisms of Petri nets send places to places, while they allow more general maps that send a place to a formal linear combination of places. On the other hand, their CatPetri is the full subcategory of CMC containing only commutative monoidal categories whose objects form a *free* commutative monoid.

In short, the situation is surprisingly subtle given the elementary nature of the concepts involved. The paper by Montanari and Meseguer actually discusses over half a dozen categories of Petri nets and commutative monoidal categories. Further work by Degano et al. (1989) and Sassone (1994, 1995, 1996) explores other variations on the theme of generating symmetric monoidal categories from Petri nets. Resisting the temptation to dwell on the subtleties of this topic, we present our approach with no further ado.

Definition 1. *Let* CommMon *be the category of commutative monoids and monoid homomorphisms.*

Definition 2. *Let* $J\colon \mathsf{Set} \to \mathsf{CommMon}$ *be the free commutative monoid functor, that is, the left adjoint of the functor* $K\colon \mathsf{CommMon} \to \mathsf{Set}$ *that sends commutative monoids to their underlying sets and monoid homomorphisms to their underlying functions. Let*

$$\mathbb{N}\colon \mathsf{Set} \to \mathsf{Set}$$

be the free commutative monoid monad given by the composite KJ.

For any set X, $\mathbb{N}[X]$ is the set of formal finite linear combinations of elements of X with natural number coefficients. The set X naturally includes in $\mathbb{N}[X]$, and for any function $f\colon X \to Y$, $\mathbb{N}[f]\colon \mathbb{N}[X] \to \mathbb{N}[Y]$ is given by the unique monoid homomorphism that extends f.

Definition 3. *We define a **Petri net** to be a pair of functions of the following form:*

$$T \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{N}[S].$$

*We call T the set of **transitions**, S the set of **places**, s the **source** function, and t the **target** function.*

Definition 4. *A **Petri net morphism** from the Petri net $s, t \colon T \to \mathbb{N}[S]$ to the Petri net $s', t' \colon T \to \mathbb{N}[S']$ is a pair of functions $(f \colon T \to T', g \colon S \to S')$ such that the following diagrams commute:*

$$
\begin{array}{ccc}
T & \xrightarrow{\ s\ } & \mathbb{N}[S] \\
f \downarrow & & \downarrow \mathbb{N}[g] \\
T' & \xrightarrow{\ s'\ } & \mathbb{N}[S']
\end{array}
\qquad
\begin{array}{ccc}
T & \xrightarrow{\ t\ } & \mathbb{N}[S] \\
f \downarrow & & \downarrow \mathbb{N}[g] \\
T' & \xrightarrow{\ t'\ } & \mathbb{N}[S'].
\end{array}
$$

Definition 5. *Let* Petri *be the category of Petri nets and Petri net morphisms, with composition defined by*

$$(f, g) \circ (f', g') = (f \circ f', g \circ g').$$

As mentioned above, Meseguer and Montanari (1990) use a more general definition of Petri net morphism: they allow an arbitrary commutative monoid homomorphism from $\mathbb{N}[S]$ to $\mathbb{N}[S']$, not necessarily of the form $\mathbb{N}[g]$ for some function $g \colon S \to S'$. Sassone (1994, 1995, 1996) and Degano et al. (1989) also use this more general definition, but Baldan et al. (2005, 2015) use the definition we are using here.

Definition 6. *A **commutative monoidal category** is a commutative monoid object internal to* Cat. *Explicitly, a commutative monoidal category is a strict monoidal category $(C, \otimes, I)$ such that for all objects a and b and morphisms f and g in C*

$$a \otimes b = b \otimes a \text{ and } f \otimes g = g \otimes f.$$

Note that a commutative monoidal category is the same as a strict symmetric monoidal category where the symmetry isomorphisms $\sigma_{a,b} \colon a \otimes b \xrightarrow{\sim} b \otimes a$ are all identity morphisms. Every strict monoidal functor between commutative monoidal categories is automatically a strict symmetric monoidal functor. This motivates the following definition:

Definition 7. *Let* CMC *be the category whose objects are commutative monoidal categories and whose morphisms are strict monoidal functors.*

We can turn a Petri net $P = (s, t \colon T \to \mathbb{N}[S])$ into a commutative monoidal category FP as follows. We take the commutative monoid of objects $\mathrm{Ob}(FP)$ to be the free commutative monoid on S. We construct the commutative monoid of morphisms $\mathrm{Mor}(FP)$ as follows. First, we generate morphisms recursively:

- for every transition $\tau \in T$, we include a morphism $\tau \colon s(\tau) \to t(\tau)$;
- for any object a, we include a morphism $1_a \colon a \to a$;
- for any morphisms $f \colon a \to b$ and $g \colon a' \to b'$, we include a morphism denoted $f + g \colon a + a' \to b + b'$ to serve as their tensor product;
- for any morphisms $f \colon a \to b$ and $g \colon b \to c$, we include a morphism $g \circ f \colon a \to c$ to serve as their composite.

Then, we mod out by an equivalence relation on morphisms that imposes the laws of a commutative monoidal category, obtaining the commutative monoid $\mathrm{Mor}(FP)$.

Definition 8. *Let $F \colon$* Petri $\to$ CMC *be the functor that makes the following assignments on Petri nets and morphisms:*

$$T \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{N}[S] \qquad FP$$

$$f \downarrow \qquad \qquad \downarrow \mathbb{N}[g] \quad \mapsto \quad \downarrow F(f,g)$$

$$T' \underset{t'}{\overset{s'}{\rightrightarrows}} \mathbb{N}[S'] \qquad FP'.$$

Here, $F(f,g)\colon FP \to FP'$ is defined on objects by $\mathbb{N}[g]$. On morphisms, $F(f,g)$ is the unique map extending f that preserves identities, composition, and the tensor product.

Lemma 9. *The functor*

$$F\colon \mathsf{Petri} \to \mathsf{CMC}$$

is a left adjoint.

Proof. This is a special case of a result of Master (2019, Theorem 5.1) which shows that there is similar adjunction for any Lawvere theory Q. When Q is set equal to the Lawvere theory for commutative monoids, this theorem gives the desired adjunction.   □

3. Open Petri Nets

Our goal in this paper is to use the language of double categories to develop a theory of Petri nets with inputs and outputs that can be glued together. The first step is to construct a double category $\mathbb{O}\mathbf{pen}(\mathsf{Petri})$ whose horizontal 1-morphisms are open Petri nets. For this, we need a functor $L\colon \mathsf{Set} \to \mathsf{Petri}$ that maps any set S to a Petri net with S as its set of places, and we need L to be a left adjoint.

Definition 10. *Let $L\colon \mathsf{Set} \to \mathsf{Petri}$ be the functor defined on sets and functions as follows:*

$$X \qquad \qquad \varnothing \rightrightarrows \mathbb{N}[X]$$

$$f \downarrow \quad \mapsto \quad \downarrow \qquad \qquad \downarrow \mathbb{N}[f]$$

$$Y \qquad \qquad \varnothing \rightrightarrows \mathbb{N}[Y]$$

where the unlabeled maps are the unique maps of that type.

Lemma 11. *The functor L has a right adjoint $R\colon \mathsf{Petri} \to \mathsf{Set}$ that acts as follows on Petri nets and Petri net morphisms:*

$$T \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{N}[S] \qquad S$$

$$f \downarrow \qquad \qquad \downarrow \mathbb{N}[g] \quad \mapsto \quad \downarrow g$$

$$T' \underset{t'}{\overset{s'}{\rightrightarrows}} \mathbb{N}[S] \qquad S'.$$

Proof. For any set X and Petri net $P = (s, t\colon T \to \mathbb{N}[S])$, we have natural isomorphisms

$$\hom_{\mathsf{Petri}}\big(L(X),\ T \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{N}[S]\,\big) \cong \hom_{\mathsf{Petri}}\big(\ \varnothing \rightrightarrows \mathbb{N}[X]\ ,\ T \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{N}[S]\ \big)$$

$$\cong \hom_{\mathsf{Set}}(X, S)$$

$$\cong \hom_{\mathsf{Set}}\big(X, R(\,T \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{N}[S]\,)\big).$$

□

An "open" Petri net is a Petri net P equipped with maps from two sets X and Y into its set of places, RP. We can write this as a cospan in Set of the form

$$
\begin{array}{ccc}
 & RP & \\
\nearrow & & \nwarrow \\
X & & Y.
\end{array}
$$
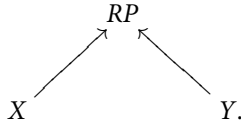
Using the left adjoint L, we can reexpress this as a cospan in Petri, and this gives our official definition:

Definition 12. *An **open Petri net** is a diagram in Petri of the form*

$$
\begin{array}{ccc}
 & P & \\
{\scriptstyle i}\nearrow & & \nwarrow{\scriptstyle o} \\
LX & & LY
\end{array}
$$
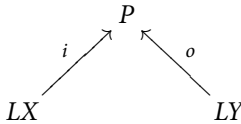
for some sets X and Y. We sometimes write this as $P\colon X \nrightarrow Y$ for short.

We now introduce the main object of study: the double category $\mathbb{O}$**pen**(Petri), which has open Petri nets as its horizontal 1-cells. Since this is a symmetric monoidal double category, it involves quite a lot of structure. The definition of symmetric monoidal double category can be found in Appendix A.

Theorem 13. *There is a symmetric monoidal double category $\mathbb{O}$**pen**(Petri) for which:*

- *objects are sets,*
- *vertical 1-morphisms are functions,*
- *horizontal 1-cells from a set X to a set Y are open Petri nets,*

$$
\begin{array}{ccc}
 & P & \\
{\scriptstyle i}\nearrow & & \nwarrow{\scriptstyle o} \\
LX & & LY
\end{array}
$$
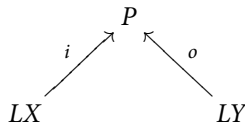
- *2-morphisms $\alpha\colon P \Rightarrow P'$ are commutative diagrams*

$$
\begin{array}{ccccc}
LX & \xrightarrow{\ i\ } & P & \xleftarrow{\ o\ } & LY \\
{\scriptstyle Lf}\downarrow & & {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle Lg} \\
LX' & \xrightarrow{\ i'\ } & P' & \xleftarrow{\ o'\ } & LY'
\end{array}
$$
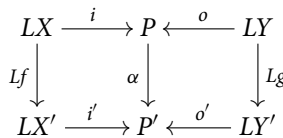
   in Petri

Composition of vertical 1-morphisms is the usual composition of functions. Composition of horizontal 1-cells is composition of cospans via pushout: given two horizontal 1-cells

$$
\begin{array}{ccccccc}
 & P & & & & Q & \\
{\scriptstyle i_1}\nearrow & & \nwarrow{\scriptstyle o_1} & & {\scriptstyle i_2}\nearrow & & \nwarrow{\scriptstyle o_2} \\
LX & & LY & & LY & & LZ
\end{array}
$$

their composite is given by this cospan from LX to LZ:
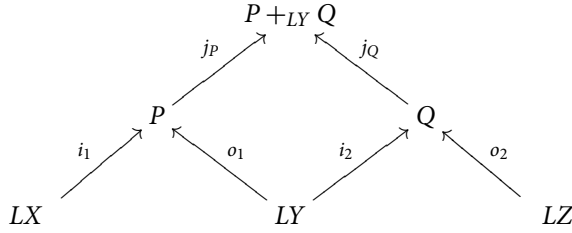
$$P +_{LY} Q$$

where the diamond is a pushout square. The horizontal composite of 2-morphisms

$$
\begin{array}{ccccc}
LX & \xrightarrow{i_1} & P & \xleftarrow{o_1} & LY \\
{\scriptstyle Lf}\downarrow & & {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle Lg} \\
LX' & \xrightarrow{i_1'} & P' & \xleftarrow{o_1'} & LY'
\end{array}
\qquad
\begin{array}{ccccc}
LY & \xrightarrow{i_2} & Q & \xleftarrow{o_2} & LZ \\
{\scriptstyle Lg}\downarrow & & {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle Lh} \\
LY' & \xrightarrow{i_2'} & Q' & \xleftarrow{o_2'} & LZ'
\end{array}
$$

is given by

$$
\begin{array}{ccccc}
LX & \xrightarrow{j_P i_1} & P +_{LY} Q & \xleftarrow{j_Q o_1} & LZ \\
{\scriptstyle Lf}\downarrow & & {\scriptstyle \alpha +_{Lg}\beta}\downarrow & & \downarrow{\scriptstyle Lh} \\
LX' & \xrightarrow{j_{P'} i_1'} & P' +_{LY'} Q' & \xleftarrow{j_{Q'} o_2'} & LZ'.
\end{array}
$$

Vertical composition of 2-morphisms is done using composition of functions. The symmetric monoidal structure comes from coproducts in Set *and* Petri.

Proof. We construct this symmetric monoidal double category using the machinery of "structured cospans" (Baez and Courser 2019). The main tool is the following lemma, which explains the symmetric monoidal structure in more detail:

Lemma 14. *Let* A *be a category with finite coproducts and* X *be a category with finite colimits. Given a left adjoint $L\colon$ A $\to$ X, there exists a unique symmetric monoidal double category $_L\mathbb{C}\mathbf{sp}(\mathsf{X})$, such that:*

- *objects are objects of* A,
- *vertical 1-morphisms are morphisms of* A,
- *a horizontal 1-cell from $a \in$ A to $b \in$ A is a cospan in* X *of this form:*

$$La \longrightarrow x \longleftarrow Lb$$

- *a 2-morphism is a commutative diagram in* X *of this form:*

$$
\begin{array}{ccc}
La & \longrightarrow x \longleftarrow & Lb \\
{\scriptstyle Lf}\downarrow & \quad {\scriptstyle h}\downarrow \quad & \downarrow{\scriptstyle Lg} \\
Lc & \longrightarrow y \longleftarrow & Ld.
\end{array}
$$

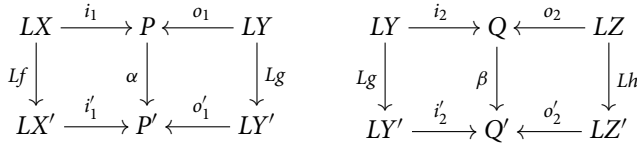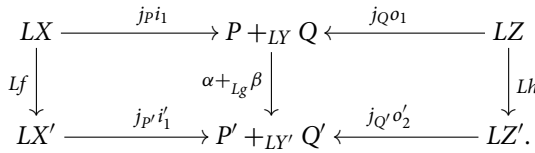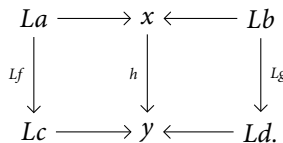*Composition of vertical 1-morphisms is composition in* A. *Composition of horizontal 1-cells is composition of cospans in* X *via pushout: given horizontal 1-cells*
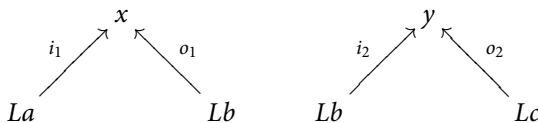
their composite is this cospan from La to Lc:
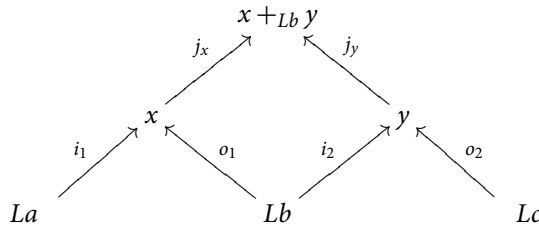
$$x +_{Lb} y$$

$$j_x \nearrow \quad \nwarrow j_y$$

$$x \quad\quad y$$

$$i_1 \nearrow \quad \nwarrow o_1 \quad i_2 \nearrow \quad \nwarrow o_2$$

$$La \quad\quad Lb \quad\quad Lc$$

where the diamond is a pushout square. The horizontal composite of 2-morphisms

$$
\begin{array}{ccccc}
La & \xrightarrow{i_1} & x & \xleftarrow{o_1} & Lb \\
Lf \downarrow & & \alpha \downarrow & & \downarrow Lg \\
La' & \xrightarrow{i_1'} & x' & \xleftarrow{o_1'} & Lb'
\end{array}
\qquad
\begin{array}{ccccc}
Lb & \xrightarrow{i_2} & y & \xleftarrow{o_2} & Lc \\
Lg \downarrow & & \beta \downarrow & & \downarrow Lh \\
Lb' & \xrightarrow{i_2'} & y' & \xleftarrow{o_2'} & Lc'
\end{array}
$$

is given by

$$
\begin{array}{ccccc}
La & \xrightarrow{j_x i_1} & x +_{Lb} y & \xleftarrow{j_y o_2} & Lc \\
Lf \downarrow & & \alpha +_{Lg} \beta \downarrow & & \downarrow Lh \\
La' & \xrightarrow{j_{x'} i_1'} & x' +_{Lb'} y' & \xleftarrow{j_{y'} o_2'} & Lc'.
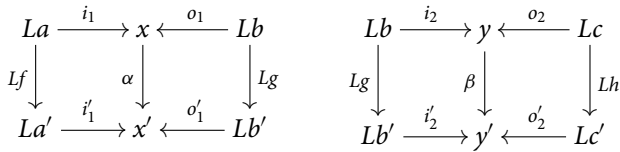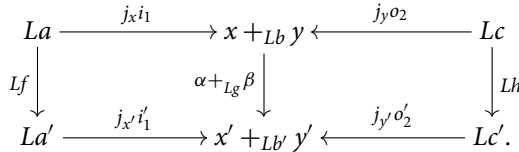\end{array}
$$

The vertical composite of 2-morphisms

$$
\begin{array}{ccccc}
La & \xrightarrow{i_1} & x & \xleftarrow{o_1} & Lb \\
Lf \downarrow & & \alpha \downarrow & & \downarrow Lg \\
La' & \xrightarrow{i_1'} & x' & \xleftarrow{o_1'} & Lb'
\end{array}
$$

$$
\begin{array}{ccccc}
La' & \xrightarrow{i_1'} & x' & \xleftarrow{o_1'} & Lb' \\
Lf' \downarrow & & \alpha' \downarrow & & \downarrow Lg' \\
La'' & \xrightarrow{i_1''} & x'' & \xleftarrow{o_1''} & Lb''
\end{array}
$$

is given by
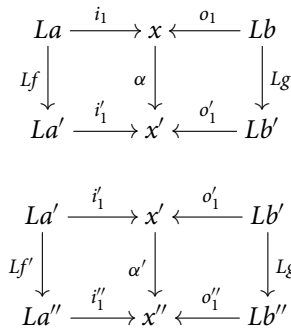
$$
\begin{array}{ccccc}
La & \xrightarrow{i_1} & x & \xleftarrow{o_1} & Lb \\
L(f'f) \downarrow & & \alpha' \alpha \downarrow & & \downarrow L(g'g) \\
La'' & \xrightarrow{i_1''} & x'' & \xleftarrow{o_1''} & Lb''.
\end{array}
$$
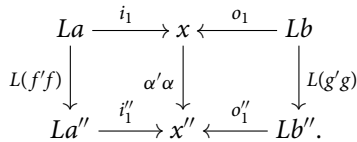
The tensor product is defined using chosen coproducts in A *and* X. *Thus, the tensor product of two objects a_1 and a_2 is $a_1 + a_2$, the tensor product of two vertical 1-morphisms*
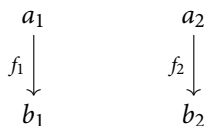
$$
\begin{array}{cc}
a_1 & a_2 \\
f_1 \downarrow & f_2 \downarrow \\
b_1 & b_2
\end{array}
$$

is

$$a_1 + a_2$$
$$\downarrow {\scriptstyle f_1+f_2}$$
$$b_1 + b_2,$$

the tensor product of two horizontal 1-cells

$$La_1 \xrightarrow{\ i_1\ } x_1 \xleftarrow{\ o_1\ } Lb_1 \qquad\qquad La_2 \xrightarrow{\ i_2\ } x_2 \xleftarrow{\ o_2\ } Lb_2$$

is

$$L(a_1 + a_2) \xrightarrow{\ i_1+i_2\ } x_1 + x_2 \xleftarrow{\ o_1+o_2\ } L(b_1 + b_2),$$

and the tensor product of two 2-morphisms

$$La_1 \xrightarrow{\ i_1\ } x_1 \xleftarrow{\ o_1\ } Lb_1 \qquad La_2 \xrightarrow{\ i_2\ } x_2 \xleftarrow{\ o_2\ } Lb_2$$

with vertical maps Lf_1, α_1, Lg_1 and Lf_2, α_2, Lg_2

$$La'_1 \xrightarrow{\ i'_1\ } x'_1 \xleftarrow{\ o'_1\ } Lb'_1 \qquad La'_2 \xrightarrow{\ i'_2\ } x'_2 \xleftarrow{\ o'_2\ } Lb'_2$$

is

$$L(a_1 + a_2) \xrightarrow{\ i_1+i_2\ } x_1 + x_2 \xleftarrow{\ o_1+o_2\ } L(b_1 + b_2)$$

with vertical maps $L(f_1+f_2)$, $\alpha_1+\alpha_2$, $L(g_1+g_2)$

$$L(a'_1 + a'_2) \xrightarrow{\ i'_1+i'_2\ } x'_1 + x'_2 \xleftarrow{\ o'_1+o'_2\ } L(b'_1 + b'_2).$$

The units for these tensor products are taken to be initial objects, and the symmetry is defined using the canonical isomorphisms $a + b \cong b + a$.

Proof. This is a result of Baez and Courser (2019, Theorem 3.9). Note that we are abusing language slightly above. We must choose a specific coproduct for each pair of objects in X and A to give $_L\mathbb{C}\mathbf{sp}(X)$ its tensor product. Given morphisms $i_1 \colon La_1 \to x_1$ and $i_2 \colon La_2 \to x_2$, their coproduct is really a morphism $i_1 + i_2 \colon La_1 + La_2 \to x_1 + x_2$ between these chosen coproducts. But since L preserves coproducts, we can compose this morphism with the canonical isomorphism $L(a_1 + a_2) \cong La_1 + La_2$ to obtain the morphism that we call $i_1 + i_2 \colon L(a_1 + a_2) \to x_1 + x_2$ above. □

To apply this lemma to the situation at hand, we need the following result.

Lemma 15. Petri *has small colimits.*

Proof. Note that Petri is equivalent to the comma category f/g where $f\colon \mathsf{Set} \to \mathsf{Set}$ is the identity and $g\colon \mathsf{Set} \to \mathsf{Set}$ is the functor $\mathbb{N}[-]^2$. Whenever categories A and B have small colimits, $f\colon \mathsf{A} \to \mathsf{C}$ is a functor preserving such colimits, and $g\colon \mathsf{B} \to \mathsf{C}$ is any functor, then f/g has small colimits (Burstall and Rydeheard 1988, Theorem 3, Section 5.2). Thus, Petri has small colimits.

For completeness, we recall how these colimits are constructed. The notation is simpler in the general case. A diagram $D\colon \mathsf{J} \to f/g$ consists of diagrams $D_A\colon \mathsf{J} \to \mathsf{A}$ and $D_B\colon \mathsf{J} \to \mathsf{B}$ together with a natural transformation

$$\gamma \colon f \circ D_A \to g \circ D_B.$$

To construct the colimit of D, we use the canonical morphisms

$$\alpha \colon \operatorname{colim} f \circ D_A \to f(\operatorname{colim} D_A),$$
$$\beta \colon \operatorname{colim} g \circ D_B \to g(\operatorname{colim} D_B)$$

defined using the universal property of the colimits at left. Since f preserves colimits, α is an isomorphism. We also use the fact that colimits are functorial, so that γ gives a natural transformation that we may call

$$\operatorname{colim} \gamma : \operatorname{colim} f \circ D_A \to \operatorname{colim} g \circ D_B.$$

The desired colimiting object $\operatorname{colim} D$ in f/g consists of the objects $\operatorname{colim} D_A \in \mathsf{A}$, $\operatorname{colim} D_B \in \mathsf{B}$ and the morphism

$$f(\operatorname{colim} D_A) \xrightarrow{\alpha^{-1}} \operatorname{colim} f \circ D_A \xrightarrow{\operatorname{colim} \gamma} \operatorname{colim} g \circ D_B \xrightarrow{\beta} g(\operatorname{colim} D_B).$$

In particular, a diagram of Petri nets $D\colon \mathsf{J} \to \mathsf{Petri}$ gives rise to functors $D_A, D_B \colon \mathsf{J} \to \mathsf{Set}$, a Petri net

$$D_A(j) \xrightarrow[t_j]{s_j} \mathbb{N}[D_B(j)]$$

for each object j of J, and a morphism between these Petri nets for each morphism of J. The colimit of D takes the form

$$\operatorname{colim} D_A \xrightarrow[t]{s} \mathbb{N}[\operatorname{colim} D_B].$$

where s and t are constructed using the general prescription just described. $\qquad \square$

We now have all of the ingredients to apply Lemma 14 to the functor $L\colon \mathsf{Set} \to \mathsf{Petri}$. Theorem 13 follows from realizing that $\mathbb{O}\mathbf{pen}(\mathsf{Petri})$ as described in the theorem is the symmetric monoidal double category $_L\mathbb{C}\mathbf{sp}(\mathsf{Petri})$. $\qquad \square$

4. The Operational Semantics

In Section 2, we saw how a Petri net P gives a commutative monoidal category FP, and in Section 3, we constructed a double category $\mathbb{O}\mathbf{pen}(\mathsf{Petri})$ of open Petri nets. Now we construct a double category $\mathbb{O}\mathbf{pen}(\mathsf{CMC})$ of "open commutative monoidal categories" and a map

$$\mathbb{C}\mathbf{sp}(F)\colon \mathbb{O}\mathbf{pen}(\mathsf{Petri}) \to \mathbb{O}\mathbf{pen}(\mathsf{CMC}).$$

This can be seen as providing an operational semantics for open Petri nets in which any open Petri net is mapped to the commutative monoidal category it presents. The reachability semantics for open Petri nets is based on this more fundamental form of semantics.

The key is this commutative diagram of left adjoint functors:

$$
\begin{array}{ccc}
\mathsf{Set} & \xrightarrow{\;\;L\;\;} & \mathsf{Petri} \\
& \searrow{\scriptstyle L'} & \downarrow{\scriptstyle F} \\
& & \mathsf{CMC}
\end{array}
$$

where $L' = FL$ sends any set to the free commutative monoidal category on this set: $L'X$ has $\mathbb{N}[X]$ as its set of objects, and only identity morphisms. Using Lemma 14, we can produce two symmetric monoidal double categories from this diagram. We have already seen one: $\mathbb{O}\mathbf{pen}(\mathsf{Petri}) = {}_L\mathbb{C}\mathbf{sp}(\mathsf{Petri})$. We now introduce the other: $\mathbb{O}\mathbf{pen}(\mathsf{CMC}) = {}_{L'}\mathbb{C}\mathbf{sp}(\mathsf{CMC})$.

Theorem 16. *There is a symmetric monoidal double category* $\mathbb{O}\mathbf{pen}(\mathsf{CMC})$ *for which:*

- *objects are sets*
- *vertical 1-morphisms are functions*

- *horizontal 1-cells from a set X to a set Y are **open commutative monoidal categories** $C \colon X \nrightarrow Y$, that is, cospans in* CMC *of the form*

$$
\begin{array}{ccc}
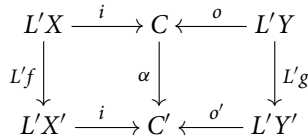 & C & \\
{\scriptstyle i}\nearrow & & \nwarrow{\scriptstyle o} \\
L'X & & L'Y
\end{array}
$$
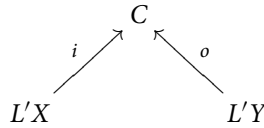
  where C is a commutative monoidal category and i and o are strict monoidal functors,
- *2-morphisms $\alpha \colon C \Rightarrow C'$ are commutative diagrams in* CMC *of the form*

$$
\begin{array}{ccccc}
L'X & \xrightarrow{\ i\ } & C & \xleftarrow{\ o\ } & L'Y \\
{\scriptstyle L'f}\downarrow & & {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle L'g} \\
L'X' & \xrightarrow{\ i\ } & C' & \xleftarrow{\ o'\ } & L'Y'
\end{array}
$$

and the rest of the structure is given as in Lemma 14.

Proof. To apply Lemma 14 to the functor $L' \colon$ Set $\to$ CMC, we just need to check that CMC has finite colimits. First note that

$$
\mathsf{CMC} \simeq \mathsf{Mod}(\mathsf{CMON}, \mathsf{Cat})
$$

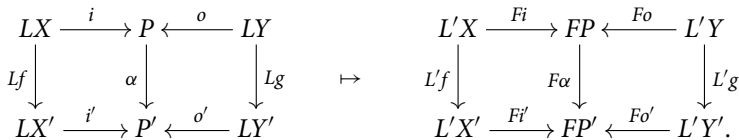where $\mathsf{Mod}(\mathsf{CMON}, \mathsf{Cat})$ is the category of finite-product-preserving functors from the Lawvere theory for commutative monoids to Cat. The cocompleteness of this category then follows from various classical results, some listed in the introduction of Freyd and Kelly (1972). More recently, Trimble (2019, Proposition 3.1) showed that for any Lawvere theory Q and any cocomplete cartesian category X with finite products distributing over colimits, the category of finite-product-preserving functors $\mathsf{Mod}(\mathsf{Q}, \mathsf{X})$ is cocomplete. $\square$

The functor $F \colon$ Petri $\to$ CMC induces a map sending open Petri nets to open commutative monoidal categories. This map is actually part of a "symmetric monoidal double functor," a concept recalled in Appendix A.
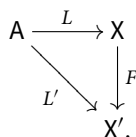
Theorem 17. *There is a symmetric monoidal double functor*

$$
\mathbb{O}\mathbf{pen}(F) \colon \mathbb{O}\mathbf{pen}(\mathsf{Petri}) \to \mathbb{O}\mathbf{pen}(\mathsf{CMC})
$$

that is the identity on objects and vertical 1-morphisms, and makes the following assignments on horizontal 1-cells and 2-morphisms:

$$
\begin{array}{ccccc}
LX & \xrightarrow{\ i\ } & P & \xleftarrow{\ o\ } & LY \\
{\scriptstyle Lf}\downarrow & & {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle Lg} \\
LX' & \xrightarrow{\ i'\ } & P' & \xleftarrow{\ o'\ } & LY'
\end{array}
\quad \mapsto \quad
\begin{array}{ccccc}
L'X & \xrightarrow{\ Fi\ } & FP & \xleftarrow{\ Fo\ } & L'Y \\
{\scriptstyle L'f}\downarrow & & {\scriptstyle F\alpha}\downarrow & & \downarrow{\scriptstyle L'g} \\
L'X' & \xrightarrow{\ Fi'\ } & FP' & \xleftarrow{\ Fo'\ } & L'Y'.
\end{array}
$$

Proof. This follows from the theory of structured cospans. More generally, suppose A is a category with finite coproducts and X, X' are categories with finite colimits. Suppose there is a commuting triangle of left adjoints

$$
\begin{array}{ccc}
\mathsf{A} & \xrightarrow{\ L\ } & \mathsf{X} \\
{\scriptstyle L'}\searrow & & \downarrow{\scriptstyle F} \\
 & & \mathsf{X'}.
\end{array}
$$

Then, Lemma 14 gives us symmetric monoidal double categories $_L\mathbb{C}\mathbf{sp}(\mathsf{X})$ and $_{L'}\mathbb{C}\mathbf{sp}(\mathsf{X}')$, and Baez and Courser (2019, Theorem 4.3) give a symmetric monoidal double functor

$$\mathbb{C}\mathbf{sp}(F)\colon {}_L\mathbb{C}\mathbf{sp}(\mathsf{X}) \to {}_{L'}\mathbb{C}\mathbf{sp}(\mathsf{X}')$$

that is the identity on objects and vertical morphisms and acts as follows on horizontal 1-cells and 2-morphisms:

$$
\begin{array}{ccccc}
La & \xrightarrow{\;i\;} & x & \xleftarrow{\;o\;} & Lb \\
\scriptstyle{Lf}\downarrow & & \scriptstyle{\alpha}\downarrow & & \downarrow\scriptstyle{Lg} \\
La' & \xrightarrow{\;i'\;} & x' & \xleftarrow{\;o'\;} & Lb'
\end{array}
\quad\mapsto\quad
\begin{array}{ccccc}
L'a & \xrightarrow{\;Fi\;} & Fx & \xleftarrow{\;Fo\;} & L'b \\
\scriptstyle{L'f}\downarrow & & \scriptstyle{F\alpha}\downarrow & & \downarrow\scriptstyle{L'g} \\
L'a' & \xrightarrow{\;Fi'\;} & Fx' & \xleftarrow{\;Fo'\;} & L'b'.
\end{array}
$$

In the case at hand, where the commutative triangle is

$$
\begin{array}{ccc}
\mathsf{Set} & \xrightarrow{\;L\;} & \mathsf{Petri} \\
& \scriptstyle{L'}\searrow & \downarrow\scriptstyle{F} \\
& & \mathsf{CMC},
\end{array}
$$

this double functor $\mathsf{Csp}(F)$ is what we are calling $\mathbb{O}\mathbf{pen}(F)$. □

We can think of the commutative monoidal category FP as providing an operational semantics for the Petri net P: morphisms in this category are processes allowed by the Petri net. The above theorem says that this semantics is compositional. That is, if we write P as a composite (or tensor product) of smaller open Petri nets, FP will be the composite (or tensor product) of the corresponding open commutative monoidal categories.

It is worthwhile comparing the work of some other authors. Baldan et al. (2005) consider a category of Petri nets that is the same as our Petri. They define an "open net" to a Petri net P equipped two subsets X and Y of its set of places. If one weakened this requirement slightly to demand merely that X and Y are equipped with injections into the set of places, the corresponding class of open Petri nets

$$
\begin{array}{ccc}
& P & \\
\scriptstyle{i}\nearrow & & \nwarrow\scriptstyle{o} \\
LX & & LY
\end{array}
$$

would be precisely those for which i and o are monic. This class of open Petri nets is closed under our form of horizontal composition. However, the authors take a different approach to composing open nets. They consider a compositional semantics for open nets, but only for those of a special kind, called "deterministic occurrence nets" because there is never any choice about what a token can do. They do not describe this semantics as a functor.

Bruni et al. (2011, 2013) also consider a category of Petri nets that matches our Petri. Given $m, n \in \mathbb{N}$, they define a "P/T-net with boundary" $P\colon m \to n$ to be a Petri net $P = (s, t\colon T \to \mathbb{N}[S])$ equipped with maps $i\colon T \to \mathbb{N}^m$, $o\colon T \to \mathbb{N}^n$. Thus, we may think of each transition as having, besides its usual source and target, an input which is a multisubset of $\{1, \ldots, m\}$ and an output which is a multisubset of $\{1, \ldots, n\}$. They define a way to compose P/T-nets with boundary using "synchronization" and show that this makes isomorphism classes of P/T-nets into the morphisms of a category. They also describe an operational semantics for P/T nets with boundary using a "tile calculus," which is essentially a double category (Bruni et al. 2002). However, the vertical direction in this double category has a fundamentally different meaning that in $\mathbb{O}\mathbf{pen}(\mathsf{Petri})$, it is used to describe the process of firing transitions.

As already mentioned, the operational semantics used here implements the "collective token philosophy," meaning that tokens are treated as indistinguishable. By contrast, in the "individual token philosophy," swapping two tokens is treated as a nontrivial process. Van Glabbeek and Plotkin have argued that these philosophies give different interpretations of causality in Petri nets (van Glabbeek and Plotkin 2009). The key mathematical difference is that the individual token philosophy uses symmetric monoidal categories that are not commutative, so their symmetries are not identity morphisms. Bruni et al. showed that for a Petri net P, a category whose morphisms represent processes of P under the individual token philosophy can be freely generated by equipping the inputs and outputs of each transition with an ordering (Bruni et al. 2001). Petri nets equipped with these orders are called "pre-nets." In Master (2019, Section 6.1), an operational semantics for pre-nets is described as a left adjoint

$$Z \colon \mathsf{PreNet} \to \mathsf{SSMC}$$

where PreNet is an appropriate category of pre-nets and SSMC is the category of strict symmetric monoidal categories. In a similar way to Theorem 17, this left adjoint can be extended to a symmetric monoidal double functor

$$\mathbb{O}\mathbf{pen}(Z) \colon \mathbb{O}\mathbf{pen}(\mathsf{PreNet}) \to \mathbb{O}\mathbf{pen}(\mathsf{SSMC})$$
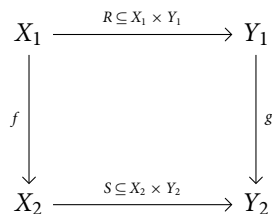
This double functor explicates the way in which the more nuanced semantics of the individual token philosophy can be built in a compositional way. A proof of existence and a detailed explanation of this double functor will be left to future work.

5. The Double Category of Relations

Using the language of functorial semantics, $\mathbb{O}\mathbf{pen}(\mathsf{Petri})$ can be thought of as a syntax for describing open systems, and reachability as a choice of semantics. To implement this, we show that the reachability relation of a Petri net can be defined for open Petri nets in a way that gives a lax double functor from $\mathbb{O}\mathbf{pen}(\mathsf{Petri})$ to the double category of relations constructed in Grandis and Paré (1999, Section 3.4). Here, we recall this double category and give it a symmetric monoidal structure.

This double category, which we call $\mathbb{R}\mathbf{el}$, has

- sets as objects,
- functions $f \colon X \to Y$ as vertical 1-morphisms from X to Y,
- relations $R \subseteq X \times Y$ as horizontal 1-cells from X to Y,
- squares

$$
\begin{array}{ccc}
X_1 & \xrightarrow{R \subseteq X_1 \times Y_1} & Y_1 \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\
X_2 & \xrightarrow{S \subseteq X_2 \times Y_2} & Y_2
\end{array}
$$

obeying $(f \times g)R \subseteq S$ as 2-morphisms.

The last item deserves some explanation. A preorder is a category such that for any pair of objects a, b there exists at most one morphism $\alpha \colon x \to y$. When such a morphism exists, we usually write $x \leq y$. Similarly, there is a kind of double category for which given any **frame**—that is, any collection of objects, vertical 1-morphisms and horizontal 1-cells as follows:

$$X_1 \xrightarrow{\ M\ } Y_1$$
$$\downarrow{\scriptstyle f} \qquad \downarrow{\scriptstyle g}$$
$$X_2 \xrightarrow{\ N\ } Y_2$$

there exists at most one 2-morphism

$$X_1 \xrightarrow{\ M\ } Y_1$$
$$\downarrow{\scriptstyle f} \quad \Downarrow \alpha \quad \downarrow{\scriptstyle g}$$
$$X_2 \xrightarrow{\ N\ } Y_2$$

filling this frame. Following Baez and Courser (2017), we call this a **degenerate** double category. Our definition of the 2-morphisms in $\mathbb{R}\mathbf{el}$ will imply that this double category is degenerate.

Composition of vertical 1-morphisms in $\mathbb{R}\mathbf{el}$ is the usual composition of functions, while composition of horizontal 1-cells is the usual composition of relations. Since composition of relations obeys the associative and unit laws strictly, $\mathbb{R}\mathbf{el}$ will be a *strict* double category. Since $\mathbb{R}\mathbf{el}$ is degenerate, there is at most one way to define the vertical composite of 2-morphisms

$$X_1 \xrightarrow{R \subseteq X_1 \times Y_1} Y_1$$
$$\downarrow{\scriptstyle f} \quad \Downarrow \alpha \quad \downarrow{\scriptstyle g}$$
$$X_2 \xrightarrow{S \subseteq X_2 \times Y_2} Y_2$$
$$\downarrow{\scriptstyle f'} \quad \Downarrow \beta \quad \downarrow{\scriptstyle g'}$$
$$X_3 \xrightarrow{T \subseteq X_3 \times Y_3} Y_3$$
$$=$$
$$X_1 \xrightarrow{R \subseteq X_1 \times Y_1} Y_1$$
$$\downarrow{\scriptstyle f'f} \quad \Downarrow \beta\alpha \quad \downarrow{\scriptstyle g'g}$$
$$X_3 \xrightarrow{T \subseteq X_3 \times Y_3} Y_3$$

so we need merely check that a 2-morphism $\beta\alpha$ filling the frame at right exists. This amounts to noting that

$$(f \times g)R \subseteq S, \ (f' \times g')S \subseteq T \implies (f' \times g')(f \times g)R \subseteq T.$$

Similarly, there is at most one way to define the horizontal composite of 2-morphisms

$$X_1 \xrightarrow{R \subseteq X_1 \times Y_1} Y_1 \xrightarrow{R' \subseteq Y_1 \times Z_1} Z_1$$
$$\downarrow{\scriptstyle f} \quad \Downarrow \alpha \quad \downarrow{\scriptstyle g} \quad \Downarrow \alpha' \quad \downarrow{\scriptstyle h}$$
$$X_2 \xrightarrow{S \subseteq X_2 \times Y_2} Y_2 \xrightarrow{S' \subseteq Y_2 \times Z_2} Z_2$$
$$=$$
$$X_1 \xrightarrow{R'R \subseteq X_1 \times Z_1} Z_1$$
$$\downarrow{\scriptstyle f} \quad \Downarrow \alpha'\circ\alpha \quad \downarrow{\scriptstyle h}$$
$$X_2 \xrightarrow{S'S \subseteq X_2 \times Z_2} Z_2$$

so we need merely check that a filler $\alpha' \circ \alpha$ exists, which amounts to noting that

$$(f \times g)R \subseteq S, \ (g \times h)R' \subseteq S' \implies (f \times h)(R'R) \subseteq S'S.$$

Theorem 18. *There exists a strict double category $\mathbb{R}\mathbf{el}$ with the above properties.*

Proof. We use the definition of double category in Appendix A (Definition A.1), which introduces two concepts not mentioned so far: the category of objects and the category of arrows. We define the category of objects $\mathbb{R}\mathbf{el}_0$ to have sets as objects and functions as morphisms. We define the category of arrows $\mathbb{R}\mathbf{el}_1$ to have relations as objects and squares

$$
\begin{array}{ccc}
X_1 & \xrightarrow{R \subseteq X_1 \times X_2} & X_2 \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\
Y_1 & \xrightarrow{S \subseteq Y_1 \times Y_2} & Y_2
\end{array}
$$

with $(f \times g)R \subseteq S$ as morphisms. The source and target functors $S, T \colon \mathbb{R}\mathbf{el}_1 \to \mathbb{R}\mathbf{el}_0$ are clear. The identity-assigning functor $u \colon \mathbb{R}\mathbf{el}_0 \to \mathbb{R}\mathbf{el}_1$ sends a set X to the identity function 1_X and a function $f \colon X \to Y$ to the unique 2-morphism

$$
\begin{array}{ccc}
X & \xrightarrow{1_X} & X \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\
Y & \xrightarrow{1_Y} & Y
\end{array}
$$

The composition functor $\odot \colon \mathbb{R}\mathbf{el}_1 \times_{\mathbb{R}\mathbf{el}_0} \mathbb{R}\mathbf{el}_1 \to \mathbb{R}\mathbf{el}_1$ acts on objects by the usual composition of relations, and it acts on 2-morphisms by horizontal composition as described above. These functors can be shown to obey all the axioms of a double category. In particular, because $\mathbb{R}\mathbf{el}$ is degenerate, all the required equations between 2-morphisms, such as the interchange law, hold automatically. $\qquad\square$

Next we make $\mathbb{R}\mathbf{el}$ into a symmetric monoidal double category. To do this, we first give $\mathbb{R}\mathbf{el}_0 = \mathbf{Set}$ the symmetric monoidal structure induced by the cartesian product. Then, we give $\mathbb{R}\mathbf{el}_1$ a symmetric monoidal structure as follows. Given relations $R_1 \subseteq X_1 \times Y_1$ and $R_2 \subseteq X_2 \times Y_2$, we define

$$
R_1 \times R_2 = \{(x_1, x_2, y_1, y_2) \colon (x_1, y_1) \in R_1, (x_2, y_2) \in R_2\} \subseteq X_1 \times X_2 \times Y_1 \times Y_2.
$$

Given two 2-morphisms in $\mathbb{R}\mathbf{el}_1$:

$$
\begin{array}{ccc}
X_1 & \xrightarrow{R \subseteq X_1 \times Y_1} & Y_1 \\
\downarrow{\scriptstyle f} & \Downarrow \alpha & \downarrow{\scriptstyle g} \\
X_2 & \xrightarrow{S \subseteq X_2 \times Y_2} & Y_2
\end{array}
\qquad
\begin{array}{ccc}
X_1' & \xrightarrow{R' \subseteq X_1' \times Y_1'} & Y_1' \\
\downarrow{\scriptstyle f'} & \Downarrow \alpha' & \downarrow{\scriptstyle g'} \\
X_2' & \xrightarrow{S' \subseteq X_2' \times Y_2'} & Y_2',
\end{array}
$$

there is at most one way to define their product

$$
\begin{array}{ccc}
X_1 \times X_1' & \xrightarrow{R \times R' \subseteq (X_1 \times X_1') \times (Y_1 \times Y_1')} & Y_1 \times Y_1' \\
\downarrow{\scriptstyle f \times f'} & \Downarrow \alpha \times \alpha' & \downarrow{\scriptstyle g \times g'} \\
X_2 \times X_2' & \xrightarrow{S \times S' \subseteq (X_2 \times X_2') \times (Y_2 \times Y_2')} & Y_2 \times Y_2'
\end{array}
$$

because $\mathbb{R}\mathbf{el}$ is degenerate. To show that $\alpha \times \alpha'$ exists, we need merely note that

$$(f \times g)R \subseteq S, \ (f' \times g')R' \subseteq S' \implies (f \times f' \times g \times g')(R \times R') \subseteq S \times S'.$$

Theorem 19. *The double category $\mathbb{R}\mathbf{el}$ can be given the structure of a symmetric monoidal double category with the above properties.*

Proof. We have described $\mathbb{R}\mathbf{el}_0$ and $\mathbb{R}\mathbf{el}_1$ as symmetric monoidal categories. The source and target functors $S, T \colon \mathbb{R}\mathbf{el}_1 \to \mathbb{R}\mathbf{el}_0$ are strict symmetric monoidal functors. We must also equip $\mathbb{R}\mathbf{el}$ with two other pieces of structure. One, called χ, says how the composition of horizontal 1-cells interacts with the tensor product in the category of arrows. The other, called μ, says how the identity-assigning functor u relates the tensor product in the category of objects to the tensor product in the category of arrows. These are defined as follows. Given four horizontal 1-cells

$$R_1 \subseteq X_1 \times Y_1, \quad R_2 \subseteq Y_1 \times Z_1,$$

$$S_1 \subseteq X_2 \times Y_2, \quad S_2 \subseteq Y_2 \times Z_2,$$

the globular 2-isomorphism $\chi \colon (R_2 \times S_2)(R_1 \times S_1) \Rightarrow (R_2 R_1) \times (S_2 S_1)$ is the identity 2-morphism

$$
\begin{array}{ccc}
X_1 \times X_2 & \xrightarrow{(R_2 \times S_2)(R_1 \times S_1)} & Z_1 \times Z_2 \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\
X_1 \times X_2 & \xrightarrow{(R_2 R_1) \times (S_2 S_1)} & Z_1 \times Z_2.
\end{array}
$$

The globular 2-isomorphism $\mu \colon u(X \times Y) \Rightarrow u(X) \times u(Y)$ is the identity 2-morphism

$$
\begin{array}{ccc}
X \times Y & \xrightarrow{1_{X \times Y}} & X \times Y \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\
X \times Y & \xrightarrow{1_X \times 1_Y} & X \times Y.
\end{array}
$$

All the commutative diagrams in the definition of symmetric monoidal double category (Definitions A.5 and A.6) can be checked straightforwardly. In particular, all diagrams of 2-morphisms commute automatically because $\mathbb{R}\mathbf{el}$ is degenerate. $\qquad\square$

6. The Reachability Semantics

Now, we explain how $\mathbb{O}\mathbf{pen}(\mathsf{Petri})$ provides a compositional approach to the reachability problem. In particular, we prove that the reachability semantics defines a lax double functor

$$\blacksquare \colon \mathbb{O}\mathbf{pen}(\mathsf{Petri}) \to \mathbb{R}\mathbf{el}$$

which is symmetric monoidal.

Definition 20. *Let P be a Petri net $(s, t \colon T \to \mathbb{N}[S])$. A **marking** of P is an element $m \in \mathbb{N}[S]$. Given a transition $\tau \in T$, a **firing** of τ is a tuple (τ, m, n) such that $m \geq s(\tau)$ and $n + s(\tau) = m + t(\tau)$. We say that a marking n is reachable from a marking m if for some $k \geq 1$ there is a sequence of markings $m = m_1, \ldots, m_k = n$ and firings $\{(\tau_i, m_i, m_{i+1})\}_{i=1}^{k-1}$. In particular, taking $k = 1$, any marking is **reachable** from itself with no firings.*

Given two markings of a Petri net, the problem of deciding whether one is reachable from the other is called the "reachability problem." In 1984, Mayr showed that the reachability problem is decidable (Mayr 1984). However, it is a very hard problem: in 1976, Lipton had showed that it requires at least exponential space, and in fact, any EXPSPACE algorithm can be reduced in polynomial time to a Petri net reachability problem (Lipton 1976). More recently, lower and upper bounds on the time to solve the reachability problem have been found (Czerwinski et al. 2018; Leroux and Schmitz 2015). The lower bound grows much faster than the Ackermann function.

There is a close connection between reachability and the free commutative monoidal category on a Petri net constructed in Lemma 9.

Proposition 21. *If m and n are markings of a Petri net P, then n is reachable from m if and only if there is a morphism $f: m \to n$ in FP.*

Proof. If n is reachable from m, there is a sequence of markings $m = m_1, \ldots, m_k = n$ and firings $\{(\tau_i, m_i, m_{i+1})\}_{i=1}^{k-1}$. For each firing (τ_i, m_i, m_{i+1}), there is a morphism in FP given by

$$\tau_i + 1_{m_i - s(\tau_i)}: m_i \to m_{i+1}.$$

Taking the composite of these morphisms gives a morphism $f: m \to n$ in FP.

Conversely, if $f: m \to n$ is a morphism in FP, it can be obtained by composition and addition (i.e., the tensor product) from morphisms arising from the basic transitions and symmetry morphisms. Because $+$ is a functor, we have the interchange law

$$(f_1 \circ g_1) + (f_2 \circ g_2) = (f_1 + f_2) \circ (g_1 + g_2)$$

whenever f_1, g_1 and f_2, g_2 are pairs of composable morphisms in FP. We can use this inductively to simplify f into a composite of sums. If $f_1: a_1 \to b_1$ and $f_2: a_2 \to b_2$ are morphisms in FP, the interchange law also tells us that

$$f_1 + f_2 = (f_1 \circ 1_{a_1}) + (1_{b_2} \circ f_2) = (f_1 + 1_{b_2}) \circ (1_{a_1} + f_2).$$

This fact allows us to inductively simplify f to a composite of sums each containing one transition. The factors in this composite correspond to firings that make n reachable from m. (Here, we allow the possibility of an empty composite, which corresponds to an identity morphism.) $\qquad\square$

Definition 22. *We define the **reachability relation** of an open Petri net*

$$LX \xrightarrow{\ i\ } P \xleftarrow{\ o\ } LY$$

to be the relation

$$\blacksquare P = \{(x, y) \in \mathbb{N}[X] \times \mathbb{N}[Y] \mid o(y) \text{ is reachable from } i(x)\} \subseteq \mathbb{N}[X] \times \mathbb{N}[Y].$$

Note that $\blacksquare P$ depends on the whole open Petri net $P: X \nrightarrow Y$, not just its underlying Petri net P. By Proposition 21,

$$\blacksquare P = \{(x, y) \in \mathbb{N}[X] \times \mathbb{N}[Y] \mid \exists h: F(i)(x) \to F(o)(y)\}.$$

Here, $F(i)(x)$ and $F(o)(y)$ are objects of the category FP, and the reachability relation holds iff there is a morphism in FP from the first of these to the second.

Theorem 23. *There is a lax double functor $\blacksquare: \mathbb{O}\mathbf{pen}(\mathsf{Petri}) \to \mathbb{R}\mathbf{el}$, called the **reachability semantics**, that sends*

- *any object X to the underlying set of the free commutative monoid $\mathbb{N}[X]$, which we denote simply as $\mathbb{N}[X]$,*
- *any vertical 1-morphism $f: X \to Y$ to the underlying function of $\mathbb{N}[f]$,*

- *any horizontal 1-cell, that is, any open Petri net*

$$LX \xrightarrow{\;\;i\;\;} P \xleftarrow{\;\;o\;\;} LY,$$

  to the reachability relation $\blacksquare P$.
- *any 2-morphism* $\alpha \colon P \Rightarrow P'$, *that is any commuting diagram*

$$
\begin{array}{ccccc}
LX & \xrightarrow{\;i\;} & P & \xleftarrow{\;o\;} & LY \\
{\scriptstyle Lf}\downarrow & & {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle Lg} \\
LX' & \xrightarrow{\;i'\;} & P' & \xleftarrow{\;o'\;} & LY',
\end{array}
$$

  to the square

$$
\begin{array}{ccc}
\mathbb{N}[X] & \xrightarrow{\;\blacksquare P \subseteq X \times Y\;} & \mathbb{N}[Y] \\
{\scriptstyle \mathbb{N}[f]}\downarrow & & \downarrow{\scriptstyle \mathbb{N}[g]} \\
\mathbb{N}[X'] & \xrightarrow{\;\blacksquare P' \subseteq X' \times Y'\;} & \mathbb{N}[Y'].
\end{array}
$$

Proof. We construct $\blacksquare$ as the composite $G \circ \mathbb{C}\mathbf{sp}(F)$ where

$$\mathbb{C}\mathbf{sp}(F) \colon \mathbb{O}\mathbf{pen}(\mathrm{Petri}) \to \mathbb{O}\mathbf{pen}(\mathrm{CMC})$$

is the double functor constructed in Theorem 17 and

$$G \colon \mathbb{O}\mathbf{pen}(\mathrm{CMC}) \to \mathbb{R}\mathbf{el}$$

is defined as follows. Recall that we have categories of objects

$$\mathbb{O}\mathbf{pen}(\mathrm{CMC})_0 = \mathbb{R}\mathbf{el}_0 = \mathsf{Set}.$$

We define $G_0 \colon \mathbb{O}\mathbf{pen}(\mathrm{PetriCat})_0 \to \mathbb{R}\mathbf{el}_0$ to be the functor $\mathbb{N} \colon \mathsf{Set} \to \mathsf{Set}$. We define $G_1 \colon \mathbb{O}\mathbf{pen}(\mathrm{CMC})_1 \to \mathbb{R}\mathbf{el}_1$ as follows:

$$
\begin{array}{ccccc}
L'X & \xrightarrow{\;i\;} & C & \xleftarrow{\;o\;} & L'Y \\
{\scriptstyle L'f}\downarrow & & {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle L'g} \\
L'X' & \xrightarrow{\;i'\;} & C' & \xleftarrow{\;o'\;} & L'Y'
\end{array}
\qquad \mapsto \qquad
\begin{array}{ccc}
\mathbb{N}[X] & \xrightarrow{\;G_1 C \subseteq \mathbb{N}[X] \times \mathbb{N}[Y]\;} & \mathbb{N}[Y] \\
{\scriptstyle \mathbb{N}[f]}\downarrow & & \downarrow{\scriptstyle \mathbb{N}[g]} \\
\mathbb{N}[X'] & \xrightarrow{\;G_1 C' \subseteq \mathbb{N}[X'] \times \mathbb{N}[Y']\;} & \mathbb{N}[Y'].
\end{array}
$$

Recall that the set of objects of $L'X$ is $\mathbb{N}[X]$ and the set of objects of $L'Y$ is $\mathbb{N}[Y]$. We define $G_1 C$ to be the relation

$$\{(x, y) \in L'X \times L'Y \mid h \colon i(x) \to o(y) \text{ for some } h \text{ in } C\} \subseteq \mathbb{N}[X] \times \mathbb{N}[Y]$$
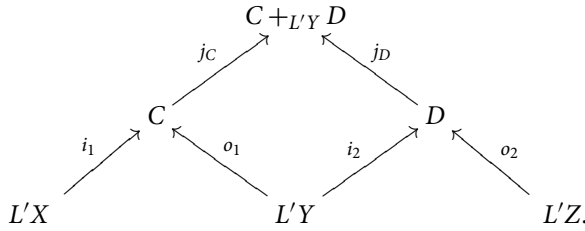
and $G_1 \alpha$ to be the inclusion

$$(\mathbb{N}[f] \times \mathbb{N}[g]) G_1 C \subseteq G_1 C'.$$

To see that this inclusion is well defined, suppose $(x, y) \in G_1 C$. Then, there exists a morphism $h \colon i(x) \to o(y)$ in C. We thus have a morphism $\alpha(h) \colon \alpha(i(x)) \to \alpha(o(y))$ in C'. However, on objects, we have $\alpha \circ i = i' \circ L'f = i' \circ \mathbb{N}[f]$ and similarly $\alpha \circ o = o' \circ \mathbb{N}[g]$, so $\alpha(h) \colon i'(\mathbb{N}[f](x)) \to o'(\mathbb{N}[g](y))$. It follows that $(\mathbb{N}[f] \times \mathbb{N}[g])(x, y) \in G_1 C'$.

Next we prove that G is a lax double functor. First note that by construction we have the following equalities:

$$S \circ G_1 = G_0 \circ S, \qquad T \circ G_1 = G_0 \circ T.$$

Next we need the composition comparison required by Definition A.3. Suppose we compose $C\colon X \nrightarrow Y$ and $D\colon Y \nrightarrow Z$ in $\mathbb{O}\mathbf{pen}(\mathsf{CMC})$:

$$
\begin{array}{c}
C +_{L'Y} D \\
\overset{j_C}{\nearrow} \quad \overset{j_D}{\nwarrow} \\
C \qquad\qquad D \\
\overset{i_1}{\nearrow} \quad \overset{o_1}{\nwarrow} \quad \overset{i_2}{\nearrow} \quad \overset{o_2}{\nwarrow} \\
L'X \qquad\qquad L'Y \qquad\qquad L'Z.
\end{array}
$$

We need to prove that

$$G_1(D) \odot G_1(C) \subseteq G_1(D \odot C).$$

We have

$$G_1(D \odot C) = \{(x, z) \in L'X \times L'Z \mid \exists h\colon j_C i_1(x) \to j_D o_2(z)\}.$$

On the other hand,

$$G_1 C = \{(x, y) \in L'X \times L'Y \mid \exists m\colon i_1(x) \to o_1(y)\}$$

and

$$G_1 D = \{(y, z) \in L'Y \times L'Z \mid \exists n\colon i_2(y) \to o_2(z)\}$$

which compose to give the relation

$$G_1 D \odot G_1 C = \{(x, z) \in L'X \times L'Z \mid \exists y\ (x, y) \in G_1 C \text{ and } (y, z) \in G_1 D\}.$$

Suppose $(x, z) \in G_1 D \odot G_1 C$. Then, there exist morphisms $m\colon i_1(x) \to o_1(y)$ in C and $n\colon i_2(y) \to o_2(z)$ in D. By commutativity of the pushout square, $j_C o_1 = j_D i_2$. Therefore, the codomain of $j_C(m)$ is $j_C o_1(y) = j_D i_2(y)$, which is also the domain of $j_D(n)$. This allows us to form the composite

$$j_D(n) \circ j_C(m)\colon j_C i_1(x) \to j_D o_2(z).$$

Thus, $(x, z) \in G_1(D \odot C)$ as desired.

We also need the identity comparison required by Definition A.3. Thus, we need

$$U_{G_0(X)} \subseteq G_1(U_X)$$

for any set X. By definition, $U_X \in \mathbb{O}\mathbf{pen}(\mathsf{CMC})_1$ is the cospan

$$L'X \xrightarrow{1} L'X \xleftarrow{1} L'X.$$

Because $L'X$ has no nonidentity morphisms, G_1 maps this to the identity relation on the set $\mathbb{N}[X]$. On the other hand, $G_0(X) = \mathbb{N}[X]$ and $U_{G_0(X)}$ is the identity relation on this set. So, the desired inclusion is actually an equality.

Finally, because $\mathbb{R}\mathbf{el}$ is a degenerate double category, the composition and identity comparisons for G are trivially natural transformations. For the same reason, the diagrams in Definition A.3 expressing compatibility with the associator, left unitor, and right unitor also commute trivially. It follows that G is a lax double functor.
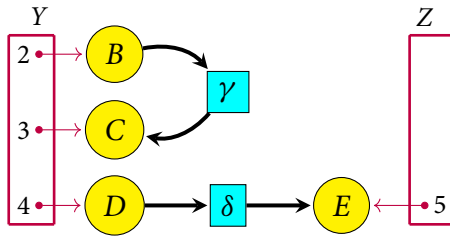
To complete the proof, one simply computes the composite $\blacksquare = G \circ \mathbb{C}\mathbf{sp}(F)$ and checks that it matches the description in the theorem statement. $\qquad\square$

The reachability semantics is only lax: given two open Petri nets $P\colon X \nrightarrow Y$ and $Q\colon Y \nrightarrow Z$, the composite of $\blacksquare Q$ and $\blacksquare P$ is in general a proper subset of $\blacksquare(Q \odot P)$. To see this, take P to be this
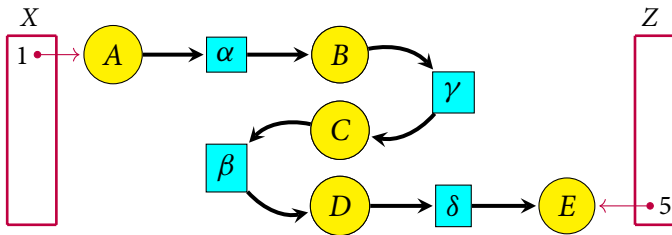
open Petri net:
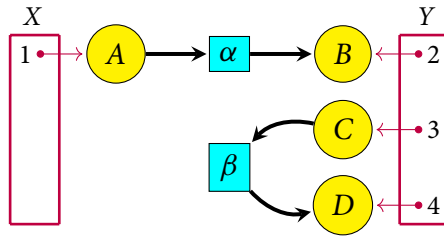
and take Q to be this:

Then, their composite, $Q \odot P \colon X \nrightarrow Z$, looks like this:

We have

$$\blacksquare P = \{(n, n, 0, 0) \mid n \in \mathbb{N}\} \subseteq \mathbb{N} \times \mathbb{N}^3$$

since tokens starting at A can only move to B, and similarly

$$\blacksquare Q = \{(0, 0, n, n) \mid n \in \mathbb{N}\} \subseteq \mathbb{N}^3 \times \mathbb{N}.$$

It follows that

$$\blacksquare Q \odot \blacksquare P = \{(0, 0)\} \subseteq \mathbb{N} \times \mathbb{N}.$$

On the other hand,

$$\blacksquare(Q \odot P) = \{(n, n) \mid n \in \mathbb{N}\} \subseteq \mathbb{N} \times \mathbb{N}$$

since in the composite open Petri net QP tokens can move from A to E. The point is that tokens can only accomplish this by leaving the open Petri net P, going to Q, then returning to P, then going to Q. The composite relation $\blacksquare Q \odot \blacksquare P$ only keeps track of processes where tokens leave P, move to Q, and never reenter P.

This makes it all the more impressive that the operational semantics

$$\mathbb{O}\mathbf{pen}(F) \colon \mathbb{O}\mathbf{pen}(\mathsf{Petri}) \to \mathbb{O}\mathbf{pen}(\mathsf{CMC})$$

is not lax:

$$\mathbb{O}\mathbf{pen}(Q \odot P) \cong \mathbb{O}\mathbf{pen}(Q) \odot \mathbb{O}\mathbf{pen}(P).$$

We can see the difference in the example above: $\mathbb{O}\mathbf{pen}(Q) \odot \mathbb{O}\mathbf{pen}(P)$ contains a morphism $\delta\beta\gamma\alpha\colon A \to E$ which describes a process where tokens start in P, go to Q, then reenter P, and finally end in Q.

On the other hand, the reachability semantics is maximally compatible with running Petri nets in parallel:

Theorem 24. *The reachability semantics* $\blacksquare\colon \mathbb{O}\mathbf{pen}(\mathsf{Petri}) \to \mathbb{R}\mathbf{el}$ *is symmetric monoidal.*

Proof. Because $\mathbb{C}\mathbf{sp}(F)$ is symmetric monoidal, it suffices to show that

$$G\colon \mathbb{O}\mathbf{pen}(\mathsf{CMC}) \to \mathbb{R}\mathbf{el}$$

is symmetric monoidal. This is simplified by that fact that $\mathbb{R}\mathbf{el}$ is a degenerate double category. Following Definition A.7, it suffices to show that

- $G_0\colon (\mathsf{Set}, +) \to (\mathsf{Set}, \times)$ is symmetric monoidal,
- $G_1\colon \mathbb{O}\mathbf{pen}(\mathsf{CMC})_1 \to \mathbb{R}\mathbf{el}_1$ is symmetric monoidal,
- we have equations of monoidal functors

$$S \circ G_1 = G_0 \circ S, \qquad T \circ G_1 = G_0 \circ T,$$

- the composition and unit comparisons are monoidal natural transformations.

To show these things, first recall that $G_0 = \mathbb{N} = K \circ J$ where $K\colon \mathsf{CommMon} \to \mathsf{Set}$ is the forgetful functor and $J\colon \mathsf{Set} \to \mathsf{CommMon}$ is its left adjoint. Since J is a left adjoint, it preserves finite coproducts. Since $K\colon \mathsf{CommMon} \to \mathsf{Set}$ is a right adjoint, it preserves finite products. However, finite products in $\mathsf{CommMon}$ are also finite coproducts. Thus, G_0 maps finite coproducts to finite products and is thus a symmetric monoidal functor from $(\mathsf{Set}, +)$ to $(\mathsf{Set}, \times)$.

Next, suppose we are given two open commutative monoidal categories

$$L'X \xrightarrow{i} C \xleftarrow{o} L'Y, \qquad L'X' \xrightarrow{i'} C' \xleftarrow{o'} L'Y'.$$

Their tensor product is

$$L'(X + X') \xrightarrow{i+i'} C + C' \xleftarrow{o+o'} L'(Y + Y') \,.$$

The set of objects of $L'(X + X')$ is naturally isomorphic to $\mathbb{N}[X] \times \mathbb{N}[X']$, and similarly for $L'(Y + Y')$, so we have natural isomorphisms

$$G_1(C + C') \cong$$

$$\{((x, x', y, y') \in \mathbb{N}[X] \times \mathbb{N}[X'] \times \mathbb{N}[Y] \times \mathbb{N}[Y'] \mid \exists h\colon i(x) \to o(y) \text{ and } \exists h'\colon i'(x') \to o'(y')\}$$

$$\cong G_1(C) \times G_1(C').$$

Using this fact, one can check that G_1 is symmetric monoidal.

One can check that the equations $S \circ G_1 = G_0 \circ S$ and $T \circ G_1 = G_0 \circ T$ are equations of monoidal functors, and the composition and unit comparisons of G are trivially monoidal natural transformations because $\mathbb{R}\mathbf{el}$ is degenerate. $\square$

7. Conclusions

The ideas presented here can be adapted to handle timed Petri nets, colored Petri nets with guards, and other kinds of Petri nets. One can also develop a reachability semantics for open Petri nets that are glued together along transitions as well as places. We hope to treat some of these generalizations in future work.

It would be valuable to have $\blacksquare(QP) = \blacksquare Q \odot \blacksquare P$, since then the reachability relation for an open Petri net could be computed compositionally, not merely "approximated from below" using $\blacksquare Q \odot \blacksquare P \subseteq \blacksquare(Q \odot P)$. We conjecture that $\blacksquare(Q \odot P) = \blacksquare Q \odot \blacksquare P$ if P and Q are "one-way" open Petri nets. Here, an open Petri net

$$LX \xrightarrow{\ i\ } P \xleftarrow{\ o\ } LY$$

is one way if no place in the image of i appears in the target $t(\tau)$ of any transition τ of P, and no place in the image of o appears in the source $s(\tau)$ of any transition τ of P. One-way open Petri nets should be the horizontal 1-cells in a full subdouble category $\mathbb{O}\mathbf{neWay}(\mathsf{Petri})$ of $\mathbb{O}\mathbf{pen}(\mathsf{Petri})$, and we conjecture that the reachability semantics restricts to an actual (not merely lax) double functor

$$\blacksquare \colon \mathbb{O}\mathbf{neWay}(\mathsf{Petri}) \to \mathbb{R}\mathbf{el}.$$

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Appendix A. Double Categories

What follows is a brief introduction to double categories. A more detailed exposition can be found in Grandis and Paré (1999, 2004), and for monoidal double categories in Shulman (2010). We use "double category" to mean what earlier authors called a "pseudo double category."

Definition A.1. *A **double category** is a category weakly internal to* Cat. *More explicitly, a double category* $\mathbb{D}$ *consists of*

- *a **category of objects** $\mathbb{D}_0$ and a **category of arrows** $\mathbb{D}_1$,*
- *source and target functors*

$$S, T\colon \mathbb{D}_1 \to \mathbb{D}_0,$$

  *an **identity-assigning** functor*

$$U\colon \mathbb{D}_0 \to \mathbb{D}_1,$$

  *and a **composition** functor*

$$\odot\colon \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \to \mathbb{D}_1,$$

  where the pullback is taken over $\mathbb{D}_1 \xrightarrow{T} \mathbb{D}_0 \xleftarrow{S} \mathbb{D}_1$, *such that*

$$S(U_A) = A = T(U_A), \quad S(M \odot N) = SN, \quad T(M \odot N) = TM,$$

- *natural isomorphisms called the **associator***

$$\alpha_{N,N',N''}\colon (N \odot N') \odot N'' \xrightarrow{\sim} N \odot (N' \odot N''),$$

*the **left unitor***

$$\lambda_N \colon U_{T(N)} \odot N \xrightarrow{\sim} N,$$

*and **the right unitor***

$$\rho_N \colon N \odot U_{S(N)} \xrightarrow{\sim} N$$

such that $S(\alpha), S(\lambda), S(\rho), T(\alpha), T(\lambda),$ and $T(\rho)$ are all identities and such that the standard coherence axioms hold: the pentagon identity for the associator and the triangle identity for the left and right unitor (Mac Lane, 1998, Section VII.1).

If α, λ, and ρ are identities, we call $\mathbb{D}$ a **strict** double category.

Objects of $\mathbb{D}_0$ are called **objects**, and morphisms in $\mathbb{D}_0$ are called **vertical 1-morphisms**. Objects of $\mathbb{D}_1$ are called **horizontal 1-cells** of $\mathbb{D}$ and morphisms in $\mathbb{D}_1$ are called **2-morphisms**. A morphism $\alpha \colon M \to N$ in $\mathbb{D}_1$ can be drawn as a square:

$$
\begin{array}{ccc}
A & \xrightarrow{\ M\ } & B \\
f \downarrow & \quad \Downarrow \alpha & \downarrow g \\
C & \xrightarrow{\ N\ } & D
\end{array}
$$

where $f = S\alpha$ and $g = T\alpha$. If f and g are identities, we call α a **globular 2-morphism**. These give rise to a bicategory:

Definition A.2. *Let $\mathbb{D}$ be a double category. Then, the **horizontal bicategory** of $\mathbb{D}$, denoted $H(\mathbb{D})$, is the bicategory consisting of objects, horizontal 1-cells, and globular 2-morphisms of $\mathbb{D}$.*

We have maps between double categories and also transformations between maps:

Definition A.3. *Let $\mathbb{A}$ and $\mathbb{B}$ be double categories. A **double functor** $F \colon \mathbb{A} \to \mathbb{B}$ consists of*

- *functors $F_0 \colon \mathbb{A}_0 \to \mathbb{B}_0$ and $F_1 \colon \mathbb{A}_1 \to \mathbb{B}_1$ obeying the following equations:*

$$S \circ F_1 = F_0 \circ S, \qquad T \circ F_1 = F_0 \circ T,$$

- *natural isomorphisms called the **composition comparison**:*

$$\phi(N, N') \colon F_1(N) \odot F_1(N') \xrightarrow{\sim} F_1(N \odot N')$$

  *and the **identity comparison**:*

$$\phi_A \colon U_{F_0(A)} \xrightarrow{\sim} F_1(U_A)$$

  whose components are globular 2-morphisms,

such that the following diagram commmute:

- *a diagram expressing compatibility with the associator:*

$$
\begin{array}{ccc}
(F_1(N) \odot F_1(N')) \odot F_1(N'') & \xrightarrow{\ \alpha\ } & F_1(N) \odot (F_1(N') \odot F_1(N'')) \\
{\scriptstyle \phi(N,N')\odot 1}\downarrow & & \downarrow{\scriptstyle 1\odot\phi(N',N'')} \\
F_1(N \odot N') \odot F_1(N'') & & F_1(N) \odot F_1(N' \odot N'') \\
{\scriptstyle \phi(N\odot N',N'')}\downarrow & & \downarrow{\scriptstyle \phi(N,N'\odot N'')} \\
F_1((N \odot N') \odot N'') & \xrightarrow{\ F_1(\alpha)\ } & F_1(N \odot (N' \odot N''))
\end{array}
$$

- *two diagrams expressing compatibility with the left and right unitors:*

$$
\begin{array}{ccc}
F_1(N) \odot U_{F_0(A)} & \xrightarrow{\ \rho_{F_1(N)}\ } & F_1(N) \\
{\scriptstyle 1 \odot \phi_A} \downarrow & & \uparrow {\scriptstyle F_1(\rho_N)} \\
F_1(N) \odot F_1(U_A) & \xrightarrow{\ \phi(N,\,U_A)\ } & F_1(N \odot U_A)
\end{array}
$$

$$
\begin{array}{ccc}
U_{F_0(B)} \odot F_1(N) & \xrightarrow{\ \lambda_{F_1(N)}\ } & F_1(N) \\
{\scriptstyle \phi_B \odot 1} \downarrow & & \uparrow {\scriptstyle F_1(\lambda_N)} \\
F_1(U_B) \odot F_1(N) & \xrightarrow{\ \phi(U_B,\,N)\ } & F_1(U_B \odot N).
\end{array}
$$

*If the 2-morphisms $\phi(N, N')$ and ϕ_A are identities for all $N, N' \in \mathbb{A}_1$ and $A \in \mathbb{A}_0$, we say $F \colon \mathbb{A} \to \mathbb{B}$ is a **strict** double functor. If on the other hand we drop the requirement that these 2-morphisms be invertible, we call F a **lax** double functor.*

Definition A.4. *Let $F \colon \mathbb{A} \to \mathbb{B}$ and $G \colon \mathbb{A} \to \mathbb{B}$ be lax double functors. A **transformation** $\beta \colon F \Rightarrow G$ consists of natural transformations $\beta_0 \colon F_0 \Rightarrow G_0$ and $\beta_1 \colon F_1 \Rightarrow G_1$ (both usually written as β) such that*

- *$S(\beta_M) = \beta_{SM}$ and $T(\beta_M) = \beta_{TM}$ for any object $M \in \mathsf{A}_1$,*
- *β commutes with the composition comparison, and*
- *β commutes with the identity comparison.*

Shulman defines a 2-category **Dbl** of double categories, double functors, and transformations (Shulman 2010). This has finite products. In any 2-category with finite products, we can define a pseudomonoid (Day and Street 1997), which is a categorification of the concept of monoid. For example, a pseudomonoid in Cat is a monoidal category.

Definition A.5. *A **monoidal double category** is a pseudomonoid in **Dbl**. Explicitly, a monoidal double category is a double category equipped with double functors $\otimes \colon \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ and $I \colon \ast \to \mathbb{D}$, where $\ast$ is the terminal double category, along with invertible transformations called the **associator**:*

$$
A \colon\ \otimes \circ (1_{\mathbb{D}} \times \otimes) \Rightarrow \otimes \circ (\otimes \times 1_{\mathbb{D}}),
$$

left unitor:

$$
L \colon\ \otimes \circ (1_{\mathbb{D}} \times I) \Rightarrow 1_{\mathbb{D}},
$$

and right unitor:

$$
R \colon\ \otimes \circ (I \times 1_{\mathbb{D}}) \Rightarrow 1_{\mathbb{D}}
$$

satisfying the pentagon axiom and triangle axioms.

This definition neatly packages a large quantity of information. Namely,

- $\mathbb{D}_0$ and $\mathbb{D}_1$ are both monoidal categories.
- If I is the monoidal unit of $\mathbb{D}_0$, then U_I is the monoidal unit of $\mathbb{D}_1$.
- The functors S and T are strict monoidal.

- $\otimes$ is equipped with composition and identity comparisons

$$\chi : (M_1 \otimes N_1) \odot (M_2 \otimes N_2) \xrightarrow{\sim} (M_1 \odot M_2) \otimes (N_1 \odot N_2)$$

$$\mu : U_{A \otimes B} \xrightarrow{\sim} (U_A \otimes U_B)$$

making three diagrams commute as in Definition A.3.
- The associativity isomorphism for $\otimes$ is a transformation between double functors.
- The unit isomorphisms are transformations between double functors.

Definition A.6. *A **braided monoidal double category** is a monoidal double category equipped with an invertible transformation*

$$\beta : \otimes \Rightarrow \otimes \circ \tau$$

*called the **braiding**, where $\tau : \mathbb{D} \times \mathbb{D} \to \mathbb{D} \times \mathbb{D}$ is the twist double functor sending pairs in the object and arrow categories to the same pairs in the opposite order. The braiding is required to satisfy the usual two hexagon identities (Mac Lane, 1998, Section XI.1). If the braiding is self-inverse, we say that $\mathbb{D}$ is a **symmetric monoidal double category**.*

In other words:

- $\mathbb{D}_0$ and $\mathbb{D}_1$ are braided (respectively, symmetric) monoidal categories,
- the functors S and T are strict braided monoidal functors, and
- the braiding is a transformation between double functors.

Definition A.7. *A **monoidal lax double functor** $F \colon \mathbb{D} \to \mathbb{D}$ between monoidal double categories $\mathbb{D}$ and $\mathbb{D}$ is a lax double functor $F \colon \mathbb{D} \to \mathbb{D}$ such that*

- F_0 *and* F_1 *are monoidal functors,*
- $SF_1 = F_0 S$ *and* $TF_1 = F_0 T$ *are equations between monoidal functors, and*
- *the composition and unit comparisons* $\phi(N_1, N_2) \colon F_1(N_1) \odot F_1(N_2) \to F_1(N_1 \odot N_2)$ *and* $\phi_A \colon U_{F_0(A)} \to F_1(U_A)$ *are monoidal natural transformations.*

*The monoidal lax double functor is **braided** if F_0 and F_1 are braided monoidal functors and symmetric if they are **symmetric** monoidal functors.*