Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 29, pp. 46-58, September 2015

ELA

OPEN PROBLEMS IN THE THEORY OF COMPLETELY POSITIVE AND COPOSITIVE MATRICES*

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Abstract. We describe the main open problems which are currently of interest in the theory of copositive and completely positive matrices. We give motivation as to why these questions are relevant and provide a brief description of the state of the art in each open problem.

Key words. completely positive matrices, copositive matrices, doubly nonnegative matrices, extremal matrices, copositive optimization

AMS subject classifications. 15B48, 15A23

1. Introduction. A real symmetric matrix A is called completely positive if it can be written as $A = BB^T$ for some, not necessarily square, nonnegative matrix B. The set of $n \times n$ completely positive matrices forms a proper cone (i.e., closed, convex, pointed, and full dimensional) which we denote by $\mathcal{CP}_n$. A real symmetric $n \times n$ matrix A is called copositive if $x^T Ax \geq 0$ for all $x \in \mathbb{R}^n_+$. The set of $n \times n$ copositive matrices also forms a proper cone which we denote by $\mathcal{COP}_n$. These cones are dual to each other under the trace inner product $\langle A, B\rangle = \text{trace}(AB)$ of the space $\mathcal{S}_n$ of real symmetric $n \times n$ matrices.

It is easy to see that any $n \times n$ completely positive matrix A is also positive semidefinite (i.e., $A \in \mathcal{PSD}_n$) and symmetric entrywise nonnegative (i.e., $A \in \mathcal{N}_n$). Such matrices are called doubly nonnegative, and they also form a proper cone, denoted by $\mathcal{DNN}_n$. Hence we have $\mathcal{CP}_n \subseteq \mathcal{DNN}_n$. On the copositive side, it is easy to see that

$$\mathcal{COP}_n \supseteq \mathcal{PSD}_n + \mathcal{N}_n.$$

*Received on April 27, 2015. Accepted June 2, 2015. Handling Editor: Steve Kirkland. This paper is dedicated to Prof. Ravindra B. Bapat on the occasion of his 60th birthday. This work was supported by grant no. G-18-304.2/2011 by the German-Israeli Foundation for Scientific Research and Development (GIF).

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Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 29, pp. 46-58, September 2015

For $n \le 4$ we have $\mathcal{CP}_n = \mathcal{DNN}_n$ and $\mathcal{COP}_n = \mathcal{PSD}_n + \mathcal{N}_n$, whereas for $n \ge 5$ the inclusions are strict, see [19, 49]. An example of a copositive matrix which is not in $\mathcal{PSD}_n + \mathcal{N}_n$ is the Horn matrix H given by [32]:

$$H = \begin{pmatrix} 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{pmatrix} \in \mathcal{COP}_5 \setminus (\mathcal{PSD}_5 + \mathcal{N}_5) \tag{1.1}$$

Copositive and completely positive matrices have many applications, including block designs, complementarity problems, a model of energy demand, exchangeable probability distributions, a Markovian model of DNA evolution and maximin efficiency-robust tests, see [10, pp. 69–70] and the references therein. More recent applications are in data mining and clustering [25], and in dynamical systems and control [48, 8].

A field where copositive and completely positive matrices have received considerable attention in recent years is mathematical optimization: it has been shown that many combinatorial and nonconvex quadratic optimization problems can be formulated as linear problems over $\mathcal{CP}_n$ or $\mathcal{COP}_n$. In this formulation, the difficulty lies entirely in the cone constraint, as all the other constraints are linear. This has allowed for a completely new angle on combinatorial and nonconvex quadratic optimization problems and has triggered an increased interest in the cones $\mathcal{CP}_n$ and $\mathcal{COP}_n$. For surveys on copositive programming see [13, 29].

In this paper, we describe some of the open problems related to these cones of matrices. The open questions are interesting in their own right, but answering them would also be highly useful for optimization. Our description is divided into four parts: membership, geometry, factorization, optimization.

2. Checking membership in $\mathcal{COP}_n$ and $\mathcal{CP}_n$. It has been proved in [50] that checking whether a given matrix is in $\mathcal{COP}_n$ is a co-NP-complete problem. For the dual cone, the same complexity is expected, and it was shown in [23] that checking membership in $\mathcal{CP}_n$ is NP-hard. *It is open whether checking membership in $\mathcal{CP}_n$ is also NP-complete.* In [9] it was shown that a finite algorithm for deciding whether a matrix $A \in \mathcal{N}_n$ is completely positive does exist, but with a highly nonpolynomial bound on the number of operations required.

For matrices with special structure, the results are obviously better: as shown in [14], copositivity of tridiagonal matrices can be checked in linear time, and the same is true for acyclic matrices [40]. Analogous results have been given for complete positivity in [21].

In view of these complexity results, it is unlikely that an efficient procedure to

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
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verify membership of a matrix in either $\mathcal{COP}_n$ or $\mathcal{CP}_n$ exists. *However, any progress in this question would be useful.* Below we outline some known results. We start with $\mathcal{CP}_n$.

As mentioned in the introduction, an obvious necessary condition for $A \in \mathcal{CP}_n$ is that $A \in \mathcal{DNN}_n$. This necessary condition is not sufficient, for example

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 3 \end{pmatrix} \in \mathcal{DNN}_5 \setminus \mathcal{CP}_5,$$

since $\langle A, H \rangle < 0$, with H as in (1.1).

A number of conditions for complete positivity have been given in terms of the graph of the matrix. Recall that the graph $G(A)$ of an $n \times n$ symmetric matrix A has n vertices and an edge between i and j if and only if $i \neq j$ and $a_{ij} \neq 0$. Conversely, for a graph G, a doubly nonnegative matrix realization of G is a matrix $A \in \mathcal{DNN}_n$ such that $G(A) = G$. With this definition we can state a qualitative condition for the necessary condition to be sufficient: Every doubly nonnegative matrix realization of a graph G is completely positive if and only if G does not contain an odd cycle of length at least 5 (a long odd cycle), see [45, 1, 46]. A new proof was found in [62].

An easily checkable sufficient condition for complete positivity was given in [44]: a diagonally dominant matrix $A \in \mathcal{N}_n$ is completely positive. This result was extended by [28], who showed that if $A \in \mathcal{N}_n$ and its comparison matrix $M(A)$ is positive semidefinite, then A is completely positive, where the comparison matrix $M(A)$ is defined by

$$M(A)_{ij} = \begin{cases} |a_{ii}| & \text{if } i = j \\ -|a_{ij}| & \text{if } i \neq j. \end{cases}$$

The converse result is true for $n \leq 2$, but for $n = 3$ the following matrix provides a counterexample:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

In fact, by [28] and [11], triangle-free graphs are exactly the graphs G that have the following property: A symmetric nonnegative realization A of G is completely positive if and only if $M(A) \in \mathcal{PSD}_n$.

Other graph-based characterizations of complete positivity were given in [3, 2, 12, 58]. More sufficient conditions, which are not graph related, were given in [55, 65].

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A publication of the International Linear Algebra Society
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It would be interesting to obtain additional, graph dependent or not, conditions for complete positivity.

As for the question of membership in $\mathcal{COP}_n$: A matrix A is copositive if and only if no principal submatrix of A has a positive eigenvector corresponding to a negative eigenvalue [43]. Unfortunately, this characterization is obviously not practical for checking copositivity of large matrices. However, spectral information can be useful: it was shown in [41] that for an indefinite symmetric matrix A with exactly one positive eigenvalue we have that $A \in \mathcal{COP}_n$ if and only if $A \in \mathcal{N}_n$.

There are a couple of simple necessary conditions to keep in mind when checking copositivity: for $A \in \mathcal{COP}_n$, we have $a_{ii} \geq 0$ for all i, and if $a_{ii} = 0$, then $a_{ij} \geq 0$ for all j. The matrix A is copositive if and only if its maximal principal submatrix with positive diagonal is copositive.

If A is a symmetric matrix with a positive diagonal, then there exists a positive diagonal matrix D such that $(DAD)_{ii} = 1$ for all i. We have $A \in \mathcal{COP}_n$ if and only if $DAD \in \mathcal{COP}_n$. This scaling invariance of $\mathcal{COP}_n$ was used in [22] to give a full characterization of $\mathcal{COP}_5$:

$$\mathcal{COP}_5 = \{DAD \mid \begin{array}{l} D \text{ is a positive diagonal matrix, and } A \text{ is such that the} \\ \text{polynomial } \left(\sum_{i,j=1}^{5} a_{ij}x_i^2x_j^2\right)\left(\sum_{k=1}^{5} x_k^2\right) \text{ is a sum of squares}\}. \end{array}$$

A necessary condition for copositivity of matrices A with $a_{ii} = 1$ for all i is that $a_{ij} \geq -1$ for all i, j. Moreover, it suffices to consider such matrices whose off-diagonal entries do not exceed 1, since increasing entries of a copositive matrix does not change the copositivity.

The $\{0, \pm 1\}$ copositive matrices were fully characterized in [35] and [39]. The characterization involves the graph of the -1 entries in the matrix.

Another case for which copositivity is fully characterized is that of symmetric matrices whose off-diagonal entries are nonpositive. Such a matrix is copositive if and only if it is positive semidefinite [38]. *For what other classes of matrices can copositivity be fully characterized?*

Matrices can be seen as tensors of order 2, so it seems natural to extend the notions of copositivity and complete positivity to tensors. This was done in [53] and [54], respectively. In [53] it was shown that any symmetric tensor whose off-diagonal entries are nonpositive is copositive if and only if it is positive semidefinite. In [64] a characterization of copositivity in terms of eigenvectors of principal subtensors was proved, similar to the result for matrices. In [54], the diagonal dominance sufficient condition for complete positivity was extended to tensors. *It would be interesting to find a characterization of complete positivity in terms of comparison tensors.*

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3. Geometry of the cones $\mathcal{COP}_n$ and $\mathcal{CP}_n$.

3.1. Extremal rays. We say that a matrix $A \in \mathcal{K}$ generates an extreme ray of a convex cone $\mathcal{K}$, if it cannot be decomposed in a nontrivial manner, i.e., if $A = B+C$ with $B, C \in \mathcal{K}$ implies that B and C are multiples of A. The set of extreme rays spans the cone, so it is of interest to study the extreme rays of $\mathcal{COP}_n$ and $\mathcal{CP}_n$.

It is well known (see, e.g., [10]) that the extreme rays of $\mathcal{CP}_n$ are given by the rank-1 matrices xx^T with $x \in \mathbb{R}^n_+$. By duality, this means that for any $x \in \mathbb{R}^n_+$ the set $\{Y \in \mathcal{S}_n \mid \langle Y, xx^T\rangle = 0\}$ is a supporting hyperplane of $\mathcal{COP}_n$.

The extreme rays of $\mathcal{PSD}_n$ and $\mathcal{N}_n$ are also well known: The extreme rays of $\mathcal{PSD}_n$ are the rank-1 matrices xx^T with $x \in \mathbb{R}^n$, and the extreme rays of $\mathcal{N}_n$ are the matrices E_{ij}, having all entries equal to 0 except entries ij and ji which are 1 (possibly $i = j$).

The extreme rays of the doubly nonnegative cone $\mathcal{DNN}_n$ are not fully understood. Partial results, which include an explicit description of the extreme rays of $\mathcal{DNN}_n$ for $n \leq 6$, are given in [67] and [33].

It is an open question to characterize the extreme rays of $\mathcal{COP}_n$ for $n > 5$. For $n \leq 4$, it is clear that the extreme rays of $\mathcal{COP}_n$ equal the extreme rays of $\mathcal{PSD}_n + \mathcal{N}_n$. It has been shown in [32] that these are given by

(a1) the extreme rays of $\mathcal{N}_n$, i.e., the matrices E_{ij} described above;
(a2) and the rank-1 matrices xx^T where $x \in \mathbb{R}^n$ has both positive and negative entries.

The 5×5 case was solved only a few years ago in [37]. For this, it is important to note that for a given permutation matrix P and a diagonal matrix D with strictly positive diagonal entries, we have that X is extreme for $\mathcal{COP}_n$ if and only if $DPXP^TD$ is. It was proved in [37] that $\mathcal{COP}_5$ has exactly four types of extremal matrices: the matrices given in (a1) and (a2) above, matrices of the form $DPHP^TD$, where H is the Horn matrix from (1.1), and matrices of the form $DPS(\theta)P^TD$, where

$$S(\theta) := \begin{pmatrix} 1 & -\cos\theta_1 & \cos(\theta_1+\theta_2) & \cos(\theta_4+\theta_5) & -\cos\theta_5 \\ -\cos\theta_1 & 1 & -\cos\theta_2 & \cos(\theta_2+\theta_3) & \cos(\theta_5+\theta_1) \\ \cos(\theta_1+\theta_2) & -\cos\theta_2 & 1 & -\cos\theta_3 & \cos(\theta_3+\theta_4) \\ \cos(\theta_4+\theta_5) & \cos(\theta_2+\theta_3) & -\cos\theta_3 & 1 & -\cos\theta_4 \\ -\cos\theta_5 & \cos(\theta_5+\theta_1) & \cos(\theta_3+\theta_4) & -\cos\theta_4 & 1 \end{pmatrix},$$

where $\theta \in \mathbb{R}^5_{++}$ is such that $\sum_{i=1}^5 \theta_i < \pi$. The proof of [37] does not easily carry over to copositive matrices of higher order, as the number of cases to be studied would grow very fast. Attempts in this direction can be found in [24, 36], but it is likely that, to tackle the $n \times n$ case for $n > 5$, different techniques are needed.

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A publication of the International Linear Algebra Society
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We mention that for general n, the extreme matrices of $\mathcal{COP}_n$ which have entries from $\{0, \pm 1\}$ were characterized by [35] and [39].

3.2. Facial structure of $\mathcal{COP}_n$ and $\mathcal{CP}_n$. *A question related to characterizing the extreme rays of $\mathcal{COP}_n$ and $\mathcal{CP}_n$ is the question to characterize the faces of both cones.* Recall that a set $\mathcal{F} \subseteq \mathcal{K}$ is called a face of the closed convex cone $\mathcal{K}$, if any line segment in $\mathcal{K}$ with an interior point in $\mathcal{F}$ has both endpoints in $\mathcal{F}$. Clearly, an extreme ray of $\mathcal{K}$ is a face of dimension 1.

The facial structure of $\mathcal{COP}_n$ and $\mathcal{CP}_n$ is not yet fully understood. Partial results, including a description of the maximal faces, are given in [20].

A question related to this is whether $\mathcal{CP}_n$ is facially exposed. Let $\mathcal{K}$ be a closed convex cone in $\mathcal{S}_n$, and let $\mathcal{F} \neq \emptyset$ be a face of $\mathcal{K}$. Then $\mathcal{F}$ is called an exposed face of $\mathcal{K}$ if it is the intersection of $\mathcal{K}$ and a non-trivial supporting hyperplane, i.e., if there exists $A \in \mathcal{S}_n \setminus \{0\}$ such that $\mathcal{K} \subseteq \{X \in \mathcal{S}_n \mid \langle A, X \rangle \geq 0\}$ and $\mathcal{F} = \{X \in \mathcal{K} \mid \langle A, X \rangle = 0\}$. A cone $\mathcal{K}$ is called facially exposed if all its faces are exposed.

Facial exposedness of a cone and the related concept of niceness of a cone (see [51]) play a role in optimization, since for nice cones it is possible to design so called facial reduction algorithms (cf. [52] and references therein).

It is well known (cf. [51]) that both $\mathcal{PSD}_n$ and $\mathcal{N}_n$ are facially exposed. Since the intersection of facially exposed cones is facially exposed, $\mathcal{DNN}_n$ is facially exposed.

In [20, Theorem 4.4] it is shown that $\mathcal{COP}_n$ is not facially exposed, since the extreme rays E_{ii} of $\mathcal{COP}_n$ are not exposed. This immediately implies that the cone $\mathcal{PSD}_n + \mathcal{N}_n$ is not facially exposed. We see from this that it may happen that a cone is facially exposed whereas its dual is not.

It is unknown whether $\mathcal{CP}_n$ is facially exposed. In [20, Theorem 4.2] it was shown that every extreme ray of $\mathcal{CP}_n$ is also exposed, however the general question remains open.

3.3. Maximal angle between matrices in $\mathcal{COP}_n$. There are different measures for the size of a convex cone $\mathcal{K}$. One such measure, proposed in [38], is the maximal angle

$$\theta_{\max}(\mathcal{K}) := \max\{\arccos\langle X, Y\rangle : X, Y \in \mathcal{K}, \|X\| = \|Y\| = 1\}.$$

It is not difficult to see that for all $n \geq 2$ we have

$$\theta_{\max}(\mathcal{PSD}_n) = \theta_{\max}(\mathcal{N}_n) = \theta_{\max}(\mathcal{DNN}_n) = \theta_{\max}(\mathcal{CP}_n) = \pi/2.$$

In [38, Prop. 6.13] it was proved that $\theta_{\max}(\mathcal{COP}_2) = 3\pi/4$, and it was conjectured that $\theta_{\max}(\mathcal{COP}_n) = 3\pi/4$ for all $n \geq 2$. However, this was disproved in [30], where it

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A publication of the International Linear Algebra Society
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was shown that

$$\lim_{n\to\infty} \theta_{\max}(\mathcal{COP}_n) = \pi. \tag{3.1}$$

The proof is based on constructing sequences of matrices $P_k \in \mathcal{PSD}_{n_k}$ and $N_k \in \mathcal{N}_{n_k}$ of increasing order with $\lim_{k\to\infty} \arccos\langle P_k, N_k\rangle = \pi$. Since $P_k, N_k \in \mathcal{COP}_{n_k}$ and $\mathcal{COP}_n$ is pointed for all n, this implies (3.1). *It remains an open question to compute $\theta_{\max}(\mathcal{COP}_n)$ as well as $\theta_{\max}(\mathcal{PSD}_n + \mathcal{N}_n)$ for finite $n > 2$, and to verify whether these angles are always attained at a pair of matrices $P \in \mathcal{PSD}_n$ and $N \in \mathcal{N}_n$.*

4. Factorization of completely positive matrices.

4.1. Finding a factorization of a matrix in $\mathcal{CP}_n$. Apart from discussing properties of the cone $\mathcal{CP}_n$, it is also interesting to study the factorization of matrices in this cone: a representation $A = BB^T$ with $B \geq 0$ is called a cp-factorzation of A. *The basic open problem is: given $A \in \mathcal{CP}_n$, determine a cp-factorization of A.*

In [66], a factorization algorithm was proposed which is based on projections onto polyhedral inner approximations of $\mathcal{CP}_n$. In theory, this algorithm can factorize any matrix in the interior of $\mathcal{CP}_n$, but it generally fails for a matrix on the boundary. What is more, the computation time is usually quite high, and the resulting factorization is often not very nice in the sense that the matrix B has far more columns than a minimal B would have.

A method to factorize a diagonally dominant matrix in $\mathcal{N}_n$ is proposed in [44]. This method can be extended to matrices whose comparison matrix $M(A)$ is in $\mathcal{PSD}_n$, by applying a scaling that uses the Perron vector associated with $M(A)$, cf. [28].

Only partial results are known as to finding a factorization of A given that $G(A)$ has a certain structure. The case of a matrix with a bipartite graph is treated in [6]. The factorization of circular as well as acyclic matrices is solved in [21], where also some preprocessing strategies are discussed. *Other results would be of interest as well.*

A slightly different open question concerns the existence of rational factorizations: *Given a matrix $A \in \mathcal{CP}_n$ all of whose entries are integral, does A always have a rational cp-factorization?* By a rational cp-factorization of $A \in \mathcal{CP}_n$, we mean a representation of the form

$$A = \sum_{i=1}^{k} \alpha_i b_i b_i^T, \quad \text{where } \alpha_i \in \mathbb{Q}, b_i \in \mathbb{Q}^n \text{ for all } i.$$

This question is related to the open problem of determining whether the membership problem for $\mathcal{CP}_n$ is NP-complete, since such a factorization would be a certificate for $A \in \mathcal{CP}_n$. It would then be necessary to verify that the coding length of the rational cp-factorization is polynomially bounded by the coding length of A.

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A publication of the International Linear Algebra Society
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4.2. Computing the cp-rank. Typically, a completely positive matrix has many cp-factorizations. The minimal number of columns of a nonnegative B such that $A = BB^T$ is called the cp-rank of A, and denoted here by $\mathrm{cpr}(A)$. *Finding the cp-rank of a given completely positive matrix, or estimating it, is a basic open problem.*

A tight upper bound on the cp-rank in terms of the rank is known. For $A \in \mathcal{CP}_n$ with $\mathrm{rank}(A) = r$, we have

$$\mathrm{cpr}(A) \leq \tfrac{1}{2}r(r+1) - 1.$$

This bound was proved in [4], improving on [34]. The bound is attained by a rank r completely positive matrix of unknown order.

A similar problem is that of finding a tight upper bound on the cp-rank in terms of the order, i.e., determining

$$p_n := \max\{\mathrm{cpr}(A) \mid A \in \mathcal{CP}_n\}.$$

For matrices of small order ($n \leq 4$) it is long known that $p_n = n$, see [49]. But for $n > 4$ this problem is still not fully resolved, in spite of significant progress in recent years. In 1994 it was conjectured by [28] that $p_n = \lfloor n^2/4 \rfloor$ for every $n \geq 4$. The proof of this conjecture (the DJL conjecture) for $n = 5$ was completed a couple of years ago in [63], combined with [47]. However, recently the DJL conjecture was refuted for $n \geq 7$ in [17, 16]. By [61] and [16], it is now known that

$$p_n \leq \tfrac{1}{2}n(n+1) - 4 \quad \text{for } n \geq 6,$$

and

$$p_n \geq \tfrac{1}{2}n(n+1) - 4 - \sqrt{2}n^{3/2} + \tfrac{3}{2}n \quad \text{for } n \geq 15.$$

Determining an exact formula for p_n is still an open problem, as is the question whether the DJL conjecture holds for $n = 6$. We conjecture the answer to the latter question is "yes", but so far it has only been shown in [56] that the DJL bound on the cp-rank is valid for certain matrices on the boundary of $\mathcal{CP}_6$. These include all the positive nonsingular matrices on the boundary, and since p_n is attained at a nonsingular matrix on the boundary of $\mathcal{CP}_n$, cf. [63], it remains to show that the cp-rank of nonsingular nonpositive 6×6 completely positive matrices is at most $6^2/4 = 9$.

It is an open problem to characterize those matrices for which the DJL bound holds. Some results of this nature exist, mostly depending on the graph of the matrix. For a graph G on n vertices, let

$$\mathrm{cpr}(G) := \max\{\mathrm{cpr}(A) \mid A \text{ is a completely positive realization of } G\}.$$

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It was proved in [28] that for a triangle-free graph G which is not a tree, $\operatorname{cpr}(G)$ equals the number of edges of G. It is well known that the number of edges in a triangle-free graph on n vertices is at most $\lfloor n^2/4 \rfloor$, thus for any triangle-free graph G on $n \geq 4$ vertices we have $\operatorname{cpr}(G) \leq n^2/4$, which was part of the motivation for the DJL conjecture. It was then shown in [27] that the same bound on $\operatorname{cpr}(G)$ applies also for graphs with no long odd cycle.

Let $\operatorname{tf}(G)$ denote the maximum number of edges in a triangle-free subgraph of G. It was shown in [57] that $\operatorname{tf}(G) \leq \operatorname{cpr}(G)$ for every graph G. If G satisfies $\operatorname{cpr}(G) = \operatorname{tf}(G)$, then the inequality $\operatorname{cpr}(G) \leq n^2/4$ holds. In [57] some graph families were found for which the equality $\operatorname{cpr}(G) = \operatorname{tf}(G)$ holds whenever $\operatorname{tf}(G) \geq n$: the no long odd cycle graphs, graphs that have no triangle-free subgraph with more edges than vertices, and outerplanar graphs (i.e., graphs that can be drawn in the plane so that no two edges cross, and all the vertices lie on the boundary of the outer face). Thus for all such graphs the DJL bound applies. *It is an open problem to find a complete characterization of those graphs for which* $\operatorname{cpr}(G) = \operatorname{tf}(G)$.

Note that in many cases $\operatorname{tf}(G)$ may be much smaller than the DJL bound. For any graph which has no triangle-free subgraph with more edges than vertices, the actual upper bound is the number of vertices of the graph, cf. [60], and for outerplanar graphs the actual bound is smaller than the number of edges in a maximal outerplanar graph, i.e., $2n - 3$.

We mention that the DJL bound on the cp-rank holds for all matrices with a positive semidefinite comparison matrix, even the positive ones [11]. *Are there other graph families for which the DJL bound holds?*

The cp-rank is trivially bounded by the rank of a matrix: for any $A \in \mathcal{CP}_n$, we have $\operatorname{cpr}(A) \geq \operatorname{rank}(A)$. If $n \leq 3$ or $\operatorname{rank}(A) \leq 2$, then $\operatorname{cpr}(A) = \operatorname{rank}(A)$, cf. [31]. But there exists a matrix $A \in \mathcal{CP}_4$ with $\operatorname{cpr}(A) = 4 > 3 = \operatorname{rank}(A)$, see [10, Example 3.1]. *Which conditions guarantee equality between the cp-rank and the rank?*

Graphs having the property that $\operatorname{cpr}(A) = \operatorname{rank}(A)$ for every completely positive realization A of the graph were fully characterized in [58]. These include trees [7], but also graphs obtained from trees by replacing some of the edges by odd cycles, at most one of which has 5 or more vertices. Other cases where this equality holds are discussed in [58] and [65].

A problem related to the cp-rank problem is that of finding a minimal cp-factorization of a given completely positive matrix. A minimal cp-factorization is a cp-factorization $A = BB^T$ where the number of columns of B equals $\operatorname{cpr}(A)$. Note that a matrix in the interior of $\mathcal{CP}_n$ has infinitely many minimal decompositions (e.g., [15]). Some completely positive matrices on the boundary of $\mathcal{CP}_n$ also have infinitely many

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 29, pp. 46-58, September 2015

minimal cp-factorizations, but some have only a finite number of them [21], or even a unique one [58].

Some of the theoretical results suggest a method for finding a minimal cp-factorization in special cases, e.g., [7, 59, 58]. But only very few algorithms to compute a minimal cp-factorizations were developed so far. In [42] the case of a completely positive matrix A which has a diagonal principal submatrix of order $\operatorname{rank}(A)$ is treated, and in [21] linear time algorithms are developed for matrices whose graph is acyclic, and for matrices whose graph is a cycle.

5. Finding cutting planes for completely positive optimization problems. As mentioned in the introduction, it has been shown that several combinatorial and nonconvex quadratic optimization problems can be formulated as linear problems over the cone $\mathcal{CP}_n$. In view of the NP-hardness of the membership problem for $\mathcal{CP}_n$, it is unsurprising that these completely positive optimization problems are numerically very hard to solve. So they are often approximated by semidefinite problems, i.e., instead of optimizing over $\mathcal{CP}_n$, one optimizes over $\mathcal{PSD}_n$ or $\mathcal{DNN}_n$. The latter can be done very efficiently, but one usually obtains a solution which is not in $\mathcal{CP}_n$. One algorithmic way to solve this problem is to generate a cutting plane, i.e., a hyperplane which "cuts off" the inefficient solution. This cutting plane is then added to the semidefinite problem as an additional linear constraint, and the semidefinite problem is re-solved, hopefully with an improved solution. In terms of the cone $\mathcal{CP}_n$, the task of generating a cutting plane can be formulated as:

$$\textit{Given } X \notin \mathcal{CP}_n, \textit{ construct } Y \in \mathcal{COP}_n \textit{ with } \langle X, Y\rangle < 0.$$

Partial answers to this problem for specific structures of X have been given in [18, 26, 66, 5], but in general it is unclear how such a cut Y can be constructed.

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