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Open String Field Theory on Noncommutative Space

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We study Witten's open string field theory in the presence of a constant B field. We construct the string field theory in the operator formalism and find that, compared to the ordinary theory with no B field, the vertices in the resulting theory has an additional factor. This factor makes the zero modes of strings noncommutative. This is in agreement with the results in the first-quantized formulation. We also discuss the background independence of the purely cubic action derived from the above mentioned string field theory and then make a redefinition of string fields to remove the additional factor from the vertex. Furthermore, we briefly discuss the supersymmetric extension of our string field theory.

§1. Introduction

Since the appearance of the seminal paper,[1)] noncommutative geometry has received much attention in Matrix theory and string theory.[2)-5)] (See Ref. 5) for further references.) In string theory, we have the familiar antisymmetric tensor field B_{ij}, which directly couples to fundamental strings. If we turn on the background B field, spacetime becomes noncommutative on D-branes with the nonvanishing B field. D-branes can be described by open strings whose ends are on the D-branes.[6)] By the quantization of the open strings, we have gauge field on the D-branes, and the low-energy effective theory of the gauge field is described by the Dirac-Born-Infeld (DBI) action.[7)] Therefore, turning on the B field, we can find the DBI action on the noncommutative space.

Recently, Seiberg and Witten have shown that the noncommutative DBI action is equivalent to the ordinary one.[5)] To prove this equivalence, they have given a relation between the gauge fields in the noncommutative DBI action and the ordinary one.[5)] Some closely related topics are discussed in Refs. 8) – 11). However, at present it seems unclear how we can embed the relation into a whole tower of the excitation modes of strings. To uncover such a relation, string field theories seem a natural framework, where we can deal with string fields which include all the excitations as well as the gauge field.

In Ref. 12), Witten constructed, on a commutative flat Minkowski spacetime, a covariant open string field theory based on noncommutative geometry. This noncommutativity comes from the nature of the manner in which open strings join together to become a new string. Therefore, we may expect that Witten's string field theory in the background B field has an additional noncommutativity. In this paper,

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we derive Witten's open string field theory in the above-mentioned background in the operator formalism.[13)-15)] By solving the overlap conditions, we show that the string field theory has an additional factor in its vertex. This factor accounts for the noncommutativity of spacetime and is in agreement with the result of Refs. 16) and 5) in the first-quantized formulation. In Ref. 17), this factor has also been found in Witten's open string field theory with a constant background magnetic field F_{ij}. By the gauge invariance $B_{ij} \to B_{ij} + \partial_i \Lambda_j - \partial_j \Lambda_i$, $A_i \to A_i + \Lambda_i$, we can see that this background is the same as ours. However, the physical significance of this factor has not been fully realized. Also, open string field theories in general backgrounds have been discussed in terms of the conformal field theory.[18)] The string field theory presented in this paper could be studied in the same way.

Pregeometrical string field theories have been proposed to give a background-independent formulation of string theory.[19),20)] In particular, the pregeometrical theory given in Ref. 19) is Witten's open string field theory on a flat Minkowski spacetime without the kinetic term, and so it is sometimes referred to as a purely cubic action. Therefore, if we drop the kinetic term from our string theory in the background B field, it is tempting to ask whether the resulting theory can be background independent. Since the additional noncommutative factor explicitly depends on the background B field, we may at first think that it cannot be background independent. If we were considering a particle field theory, this would be true. However, as we will show in this paper, we can remove the noncommutative factor from the three-string vertex through a redefinition of string fields. In addition, we explicitly demonstrate that the three-string vertex is independent of the background metric which we use to express the vertex in terms of the oscillators of strings. To this end, we apply the method given in Ref. 21) to open string field theory.

This paper is organized as follows: In §2, we construct Witten's open string field theory in the background B field by using the operator formalism and solving the overlap conditions for the vertices. In §3, we explicitly show the background independence of our pregeometrical theory in great detail. Section 4 is devoted to discussion. In Appendix A, we briefly summarize the operator formalism of the first-quantized string theory.[22)-24)] In Appendix B, we give a derivation of Yoneya's identities[15),25)] of the Neumann coefficients for Witten's string field theory, which we need in §3.

When we had almost finished writing this paper, we found a paper (Ref. 26)) written by Sugino that has considerable overlap with ours. The main difference between that paper and ours is the following two points. First, he argues that the dependence on the B field can be eliminated from the string field theory by a redefinition of string fields. This suggests that we can 'gauge away' the background field. We discuss this point in further detail in §4. Second, we explicitly show background independence of our pregeometrical theory. In §4, we also mention our main results regarding an open-closed string field theory with light-cone type interactions[27),28)] in the background we are considering. Furthermore, we discuss the supersymmetric extension[29)] of our string field theory.

§2. String Field Theory in the Background B Field

We study the bosonic open string field theory proposed by Witten [12] with a constant metric g_{ij} and a constant antisymmetric field B_{ij}. The open string field theory in the presence of background fields is discussed in Refs. 18) and 30). We show that we can construct the field theory in our background explicitly by using the operator formalism. [13)-15)] To this end, it is appropriate to begin with a review of the operator formalism of the first-quantized string theory with the B field. [22)-24)] In Appendix A, we give a simple derivation of the result in Refs. 22) – 24) to make this paper self-contained.

In the first-quantized string theory, the worldsheet action is given by

$$S = -\frac{1}{4\pi\alpha'}\int d\sigma\, d\tau \left(g_{ij}\eta^{ab}\partial_a X^i \partial_b X^j - 2\pi\alpha' B_{ij}\epsilon^{ab}\partial_a X^i \partial_b X^j\right). \qquad (2\cdot 1)$$

From this action, if the Dirichlet boundary condition is not chosen for all the directions of the string coordinates, the boundary condition can be seen to be $g_{ij}X^{j\prime} + (2\pi\alpha')B_{ij}\dot{X}^j = 0$ at $\sigma = 0, \pi$, where we denote differentiation with respect to τ and σ by a dot and a prime, respectively. For simplicity, in this paper, we impose this boundary condition on all the string coordinates $X^i(\tau,\sigma)$. The conjugate momenta of the string coordinates $X^i(\sigma)$ turn out to be $P_i(\sigma) = \frac{1}{2\pi\alpha'}g_{ij}\dot{X}^j(\sigma) + B_{ij}X^{j\prime}(\sigma)$.

The authors of Refs. 22) – 24) have shown that we can quantize our system by the Dirac quantization procedure if we treat the boundary condition as a constraint. (See also Appendix A for further details.) The resulting commutation relations can be seen to be

$$\begin{aligned}
&\left[X^i(\sigma),\ P_j(\sigma')\right] = i\,\delta^i_j\,\delta(\sigma-\sigma'),\\
&\left[P_i(\sigma),\ P_j(\sigma')\right] = 0,\\
&\left[X^i(\sigma),\ X^j(\sigma')\right] = \begin{cases} \ \ i\theta^{ij}, & (\sigma=\sigma'=0)\\ -i\theta^{ij}, & (\sigma=\sigma'=\pi)\\ 0, & (\text{otherwise})\end{cases}
\end{aligned} \qquad (2\cdot 2)$$

where we use the same definitions of the open string metric G_{ij} and the theta parameter θ^{ij} as those in Ref. 5):

$$\begin{aligned}
G^{ij} &= \left(\frac{1}{g+2\pi\alpha' B}\, g\, \frac{1}{g-2\pi\alpha' B}\right)^{ij},\\
\theta^{ij} &= -(2\pi\alpha)^2\left(\frac{1}{g+2\pi\alpha' B}\, B\, \frac{1}{g-2\pi\alpha' B}\right)^{ij}.
\end{aligned} \qquad (2\cdot 3)$$

As we can see in Appendix A, the mode expansion of the string coordinates $X^i(\sigma)$ turns out to be

$$X^i(\sigma) = \tilde{X}^i(\sigma) + (\theta G)^i{}_j Q^j(\sigma), \qquad (2\cdot 4)$$

where $\tilde{X}^i(\sigma)$ and $Q^i(\sigma)$ are defined with $l_s = \sqrt{2\alpha'}$ as follows:

$$\tilde{X}^i(\sigma) = \tilde{x}^i + l_s \sum_{n\neq 0} \frac{i}{n}\alpha_n^i \cos(n\sigma),$$
$$Q^i(\sigma) = \frac{1}{\pi} G^{ij} p_j \left(\sigma - \frac{\pi}{2}\right) + \frac{1}{\pi l_s} \sum_{n\neq 0} \frac{1}{n}\alpha_n^i \sin(n\sigma). \tag{2·5}$$

We can see here that the variables $\tilde{X}^i(\sigma)$ satisfy the Neumann boundary condition. Similarly, the expansion of the momenta is given by

$$P_i(\sigma) = \frac{1}{\pi l_s} \sum_n G_{ij}\alpha_n^j \cos(n\sigma). \tag{2·6}$$

Note that $Q^i(\sigma)$ and $P_i(\sigma)$ have the relation

$$Q^i(\sigma) = \int_{\frac{\pi}{2}}^{\sigma} d\sigma' G^{ij} P_j(\sigma') + \frac{1}{2} G^{ij}\left(p_{L\,j} - p_{R\,j}\right), \tag{2·7}$$

where we introduce the momentum operators integrated over half a string, [31]

$$p_{L\,i} = \int_0^{\frac{\pi}{2}} d\sigma\; P_i(\sigma), \qquad p_{R\,i} = \int_{\frac{\pi}{2}}^{\pi} d\sigma\; P_i(\sigma). \tag{2·8}$$

The commutation relations of the mode variables $\tilde{x}^i$, p_i and α_n^i can be shown [22)-24)] to be

$$[\tilde{x}^i, p_j] = i\delta^i{}_j, \qquad \left[\alpha_m^i, \alpha_n^j\right] = m\delta_{m+n} G^{ij}, \tag{2·9}$$

and the others vanish, as is seen in Appendix A.

The BRS charge in string field theories is necessary to construct their kinetic terms. In order to obtain the BRS charge, we need to know the energy-momentum tensor in the worldsheet theory (2·1). While the contribution from the reparametrization ghosts to the energy-momentum tensor is the same as usual, the energy-momentum tensor from the matter sector, i.e. the string coordinates, can be found to be

$$T(z) = -\frac{1}{\alpha'} g_{ij}\partial X^i \partial X^j(z) = -\frac{1}{\alpha'} G_{ij}\partial \tilde{X}^i \partial \tilde{X}^j(z),$$
$$\tilde{T}(\bar{z}) = -\frac{1}{\alpha'} g_{ij}\bar{\partial} X^i \bar{\partial} X^j(\bar{z}) = -\frac{1}{\alpha'} G_{ij}\bar{\partial} \tilde{X}^i \bar{\partial} \tilde{X}^j(\bar{z}), \tag{2·10}$$

with $z = \exp(\tau + i\sigma)$. Note from the boundary condition that $T(z) = \tilde{T}(\bar{z})$ for $z = \bar{z}$. Therefore, we can make use of the doubling technique for open strings to extend the worldsheet of the upper half-plane to a whole complex plane when we define the BRS charge by using the energy momentum tensor in the usual way.

In the remainder of this section, we construct a string field theory with the midpoint interaction in our background. To this end, the reflector and the three-string vertex will be constructed by using the overlap conditions, as usual. From the result of Refs. 16) and 5), we expect that the noncommutativity of spacetime will also appear in our string field theory, in addition to the usual noncommutativity of an open string field theory. This is actually the case, as we will see below. Although the mode expansion of the string coordinates is different from that with the Neumann boundary condition, due to the presence of the background B field, the resulting vertices will be shown to be the same as usual vertices, except for one factor. It is this factor that accounts for the noncommutativity of spacetime. By the 'noncommutative spacetime', we mean a spacetime such that, given two arbitrary functions $f(x)$ and $g(x)$ on the space, the product of these functions is given by the Moyal product

$$f * g(x) = f(x) \exp\left[\frac{i}{2}\theta^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right] g(x). \tag{2·11}$$

If we identify the 'zero mode' $\tilde{x}$ of string coordinates with coordinates of our spacetime, the above-mentioned factor turns out to be the exponential factor in (2·11), as we show below.

In addition to the BRS charge, in order to obtain a kinetic term in string field theory, we need the reflector $\langle R|$, which is used to give the inner product of string fields. The reflector $\langle R|$ is defined up to an overall normalization by the overlap conditions

$$\begin{aligned}\langle R| \left(X^{i(1)}(\sigma) - X^{i(2)}(\pi-\sigma)\right) &= 0,\\ \langle R| \left(P_j^{(1)}(\sigma) + P_j^{(2)}(\pi-\sigma)\right) &= 0.\end{aligned} \tag{2·12}$$

Since the ghost part of the reflector remains unchanged even in our case, we focus on only its matter part. This will also be the case later for the three-string vertex. In the case that $\theta = 0$, the matter part of the reflector is thus given by

$$\langle R^x| = (2\pi)^{26}\delta^{26}(p_1+p_2)_{21}\langle 0| \exp\left(-\sum_{n\geq 1}\frac{(-)^n}{n}G_{ij}\alpha_n^{i\,(1)}\alpha_n^{j\,(2)}\right), \tag{2·13}$$

where ${}_{21}\langle 0|$ denotes ${}_2\langle 0|_1\langle 0|$. We can see that this reflector still satisfies the connection conditions (2·13) even in our case of $\theta \neq 0$. Therefore, using the BRS charge Q_{B} and the reflector $\langle R|$, we can write the kinetic term of our string field theory. Obviously, this kinetic term does not have any dependence on the theta parameter θ^{ij}.

Now, let us move on to the three-string vertex. The three-string vertex can also be specified up to an overall normalization by the connection equations

$$\begin{aligned}\langle V_3| \left(X^{i(r)}(\sigma) - X^{i(r+1)}(\pi-\sigma)\right) &= 0, \quad \left(\frac{\pi}{2} < \sigma \leq \pi\right)\\ \langle V_3| \left(P_i^{(r)}(\sigma) + P_i^{(r+1)}(\pi-\sigma)\right) &= 0, \quad \left(\frac{\pi}{2} < \sigma \leq \pi\right)\end{aligned} \tag{2·14}$$

where $r = 1, 2, 3$ denotes the r-th string and $r+3$ equals r. Before proceeding to solve these equations, let us consider the three-string vertex with $\theta = 0$.

$$\left\langle \tilde{V}_3\right| \left(\tilde{X}^{i\,(r)}(\sigma) - \tilde{X}^{i\,(r+1)}(\pi-\sigma)\right) = 0, \quad \left(\frac{\pi}{2} < \sigma \leq \pi\right)$$

$$\left\langle \tilde{V}_3 \right| \left(P_i{}^{(r)}(\sigma) + P_i{}^{(r+1)}(\pi - \sigma) \right) = 0. \quad \left(\frac{\pi}{2} < \sigma \leq \pi \right) \tag{2·15}$$

The variables $\tilde{X}^i(\sigma)$ and $P_i(\sigma)$ are expressed by the mode expansions of (2·4) and (2·6), and these correspond to strings with the Neumann boundary condition, as mentioned above. Therefore, we find that this vertex $\left\langle \tilde{V}_3 \right|$ agrees with the usual three-string vertex given by

$$\left\langle \tilde{V}_3^x \right| = (2\pi)^{26} \delta^{26} \left(\sum_{r=1}^{3} p^{(r)} \right) {}_{321} \langle 0 | \, e^{E_{123}},$$
$$E_{123} = \sum_{\substack{m,n \geq 0 \\ r,s=1,2,3}} \frac{1}{2} \bar{N}^{rs}_{mn} G_{ij} \alpha^{i}_{m}{}^{(r)} \alpha^{j}_{n}{}^{(s)}, \tag{2·16}$$

where the Neumann coefficients have the same forms as those in the Minkowski spacetime,[13)-15)] and ${}_{321}\langle 0|$ denotes ${}_3\langle 0| \, {}_2\langle 0| \, {}_1\langle 0|$.

Using (2·7), (2·4), and (2·15), we can evaluate how the string coordinates $X^{i(r)}(\sigma)$ connect with each other on the vertex $\left\langle \tilde{V}_3 \right|$. We find

$$\left\langle \tilde{V}_3 \right| \left(X^{i(r)}(\sigma) - X^{i(r+1)}(\pi - \sigma) \right) = -\frac{1}{2} \left\langle \tilde{V}_3 \right| \theta^{ij} p_j^{(r+2)}, \quad \left(\frac{\pi}{2} < \sigma \leq \pi \right) \tag{2·17}$$

where, due to the momentum conservation on the worldsheet,[31)]

$$p_{L\,i}^{(r)} + p_{R\,i}^{(r)} = p_i^{(r)}, \qquad p_{L\,i}^{(r+1)} + p_{R\,i}^{(r)} = 0 \tag{2·18}$$

are used. From the relation

$$\left[\sum_{r<s} \theta^{ij} p_i^{(r)} p_j^{(s)}, \ X^{i(t)}(\sigma) - X^{i(t+1)}(\pi - \sigma) \right] = i\theta^{ij} p_j^{(t+2)}, \tag{2·19}$$

(2·17) leads to the three-string vertex with non-zero B field [*)]

$$\langle V_3 | = \left\langle \tilde{V}_3 \right| \exp\left(-\frac{i}{2} \sum_{r<s} \theta^{ij} p_i^{(r)} p_j^{(s)} \right). \tag{2·20}$$

Thus, the three-string vertex in the background B field can be obtained by multiplying the usual vertex $\left\langle \tilde{V}_3 \right|$ by the factor $e^{-\frac{i}{2}\sum \theta^{ij} p_i^{(r)} p_j^{(s)}}$, which is characteristic of a noncommutative space. Since the BRS charge $Q_{\rm B}$ can be expressed by the variables $\tilde{X}^i(\sigma)$ and $P_j(\sigma)$ and commutes with the zero mode p_j of the momenta, we can see that the three-string vertex satisfies the BRS invariance:

$$\langle V_3 | \sum_{r=1}^{3} Q_{\rm B}^{(r)} = 0. \tag{2·21}$$

*) A similar expression for the three-string vertex is discussed in Ref. 17).

Finally, we find that our string field theory has the following action:

$$S[\Psi] = \int \left(\frac{1}{2} \Psi \star Q_{\mathrm{B}} \Psi + \frac{1}{3} \Psi \star \Psi \star \Psi \right)$$
$$= \frac{1}{2} {}_{21} \langle R| \, |\Psi\rangle_1 \, Q_{\mathrm{B}}^{(2)} \, |\Psi\rangle_2 + \frac{1}{3} {}_{321} \langle V_3| \, |\Psi\rangle_1 \, |\Psi\rangle_2 \, |\Psi\rangle_3 \, . \qquad (2{\cdot}22)$$

This $\star$ product differs from the ordinary product by the factor which represents the noncommutativity of space-time, as we have mentioned above. However, except for this factor, the action (2·22) is the same as that of the theory without B field. Note that in the kinetic term, the ordinary product can be replaced by the $\star$ product, due to momentum conservation. This action can be verified to be invariant under the gauge transformation

$$\delta\Psi = Q_{\mathrm{B}}\Lambda + \Psi \star \Lambda - \Lambda \star \Psi. \qquad (2{\cdot}23)$$

In the perturbative expansion of this string field theory, if we expand the string field Ψ in its component fields, for example, a tachyon field and a vector field, the product of these component fields in the resulting effective action turns out to be the product of functions on a noncommutative space. Therefore, the low-energy effective theory becomes noncommutative Yang-Mills theory. Also, in this theory, we have the open string metric G_{ij}, but not the closed one g_{ij}. This is in agreement with the result in Ref. 5).

§3. Background Independence of String Field Theory

As a background independent formulation of string theory, pregeometrical string field theories have been proposed in Ref. 19) and 20), where the kinetic terms have been dropped from the actions of the ordinary string field theories on a flat Minkowski space and only a cubic term is retained.

If we drop the kinetic term of our string field theory, we may expect the resulting theory to be a pregeometrical theory on the same footing as the theory proposed in Ref. 19). In this section, we show that this is the case. Although this seems to depend on the theta parameter θ^{ij}, we find that its background dependence can be absorbed into a redefinition of a string field and that the resulting theory turns out to be the theory in Ref. 19). In addition, we explicitly show in the oscillator representation that the three-string vertex is also independent of G_{ij}. This is an application of the method given by Kugo and Zwiebach [21)] to Witten's open string field theory.

In Ref. 21), background independence is discussed in $\alpha = p^+$ closed HIKKO theory compactified on a torus. Kugo and Zwiebach proposed that $X^i(\sigma)$ and $P_i(\sigma)$ are independent of background fields. We can therefore read the dependence of the oscillators on the background fields, which allows us to explicitly verify in terms of the oscillators that the three-string vertex is background independent.

However, in our open string field theory, the coordinates $X^i(\sigma)$ are no longer universal objects, because the commutation relation of the string coordinates itself depends on θ, as in (2·2). What objects should we regard as universal? From Eq. (2·4) and (2·7), $X^i(\sigma)$ can be rewritten as

$$X^i(\sigma) = \tilde{X}^i(\sigma) + \int_{\frac{\pi}{2}}^{\sigma} d\sigma' \theta^{ij} P_j(\sigma') + \frac{1}{2}\theta^{ij}\left(p_{Lj} - p_{Rj}\right).$$

Thus, $X^i(\sigma)$ and $P_i(\sigma)$ can be expressed by $\tilde{X}^i(\sigma)$ and $P_i(\sigma)$. Furthermore, their commutation relations,

$$\begin{aligned} &\left[\tilde{X}^i(\sigma),\ P_j(\sigma')\right] = i\,\delta^i_j\,\delta(\sigma-\sigma'), \qquad \left[P_i(\sigma),\ P_j(\sigma')\right] = 0, \\ &\left[\tilde{X}^i(\sigma),\ \tilde{X}^j(\sigma')\right] = 0, \end{aligned} \tag{3·1}$$

have no apparent dependence on background fields. Thus, we propose that $\tilde{X}^i(\sigma)$ and $P_i(\sigma)$ are background-independent objects. Namely, under an infinitesimal variation of G^{ij} and θ^{ij}, $\delta\tilde{X}^i(\sigma) = 0$ and $\delta P_i(\sigma) = 0$. Therefore, under this variation, we can obtain the change of the oscillators

$$\delta\alpha^i_n = -\frac{1}{2}G^{ij}\delta G_{jk}\left(\alpha^k_n + \alpha^k_{-n}\right). \tag{3·2}$$

The oscillators α^i_n can be seen to depend only upon the open string metric G^{ij}. This implies that the theta parameter θ^{ij} in our theory is included only in the above-mentioned factor of the three-string vertex.

Now, let us consider a purely cubic action with our three-string vertex:

$$S = \int \Psi \star \Psi \star \Psi = {}_{321}\langle V_3|\,|\Psi\rangle_1\,|\Psi\rangle_2\,|\Psi\rangle_3\,. \tag{3·3}$$

If we expand the string field around a classical solution as $\Psi = Q_{\mathrm{L}} I + \tilde{\Psi}$, we can recover the action Eq. (2·22), as discussed in Ref. 19). Here, Q_{L} is the BRS charge density integrated over the left half of a string, and, in terms of the oscillators, I can be given [14), 15)] by

$$|I\rangle = \exp\left(-\sum_{n\geq 1}\frac{(-1)^n}{2n}G_{ij}\alpha^i_n\alpha^j_n\right)|0\rangle\,(2\pi)^{26}\delta^{26}(p). \tag{3·4}$$

In the following two subsections, we in turn discuss the dependence of our theory (3·3) on the theta parameter θ^{ij} and on the open string metric G_{ij}.

3.1. *Similarity transformation of string fields and the theta parameter*

The three-string vertex in our theory differs from that in Ref. 19) by the noncommutative factor $\exp[-(i/2)\sum_{r<s}\theta^{ij}p^{(r)}_i p^{(s)}_j]$. Nothing but this factor depends on the theta parameter, as we have seen previously. Therefore, our theory at first seems dependent on the background field θ^{ij}. If we are dealing with a particle field theory, say ϕ^3 theory, on a noncommutative space, this is true. However, if, in our string theory, we can express the noncommutative factor by a product of operators from each of the three strings, we can eliminate the factor by a redefinition of

string fields. Interestingly, this is indeed the case, as we show below. Therefore, our pregeometrical theory is independent of the theta parameter.

For the above-stated purpose, let us consider the operator $\sum_{r<s} -(i/2)\theta^{ij}p_i^{(r)}p_j^{(s)}$ on the ordinary three-string vertex $\left\langle \tilde{V}_3 \right|$. This operator can be rewritten as $\sum_{r=1}^{3}(i/2)\theta^{ij}{p_L}_i^{(r)}{p_R}_j^{(r)}$ by using the momentum conservation (2·18) on the vertex $\left\langle \tilde{V}_3 \right|$. Therefore, our three-string vertex $\langle V_3|$ can be rewritten as

$$\left\langle \tilde{V}_3 \right| \prod_{r=1}^{3} e^{M^{(r)}} = \langle V_3| \,, \tag{3·5}$$

where $M^{(r)} = (i/2)\theta^{ij}{p_L}_i^{(r)}{p_R}_j^{(r)}$.

Since the noncommutative factor can be given by operating with the operators $e^{M^{(r)}}$ on the three-string vertex, we can eliminate it from the vertex by the redefinition of string fields $\Psi \to e^{-M}\Psi$, and we find that our theory turns into the ordinary theory proposed in Ref. 19), as we have mentioned above.

Before examining the dependence of our theory on the open string metric, we would like to make some comments about this similarity transformation. When we apply this field redefinition to the theory we discussed in the last section, we can eliminate the noncommutative factor from the three-string vertex. But this redefinition also affects the kinetic term, and then the BRS charge is transformed into $e^{M}Q_{\mathrm{B}}e^{-M}$. This transformed BRS charge can be found to possess a divergent term. Very recently, using an interesting technique, Sugino has argued that the transformed operator indeed remains the original BRS operator Q_{B} in the kinetic term.[26)] This seems to imply that the background B field is physically meaningless. We discuss this puzzle in some detail in §4.

3.2. *Independence of the three-string vertex on the background metric*

In string field theory, the reflector and the three-string vertex are defined by the overlap conditions up to an overall normalization. Since the overlap conditions do not include any background fields, we can expect that these vertices are independent of background fields. However, we need at least a background metric to concretely construct these vertices in terms of the oscillators. Therefore, it is interesting to examine the background independence of the vertices. In this subsection, we consider the independence of the ordinary three-string vertex $\left\langle \tilde{V}_3 \right|$ on the background metric by using the method proposed by Kugo and Zwiebach, who applied it to the $\alpha' = p^+$ HIKKO closed string theory. We could also study the background independence by using the general method of Sen.[18)]

Before considering the three-string vertex, we demonstrate the independence of the reflector on the open string metric, as an illustration of the method in Ref. 21). In this subsection, we focus only on the matter sector.

The Fock vacuum of string fields is defined by ${}_G\langle 0|\,\alpha_{-n} = 0$ for $n \geq 1$, where the oscillators α_n depend on the open string metric G_{ij}. Thus, the vacuum ${}_G\langle 0|$ also depends on the metric G_{ij}. As we have seen in (3·2), the oscillators change under an infinitesimal variation of G_{ij} by $\delta\alpha_n^i = -\frac{1}{2}G^{ij}\delta G_{jk}\left(\alpha_n^k + \alpha_{-n}^k\right)$. It is useful to introduce the operator

$$\mathcal{B} = -\sum_{n\geq 1}\frac{1}{4n}\delta G_{ij}\left(\alpha_n^i\alpha_n^j - \alpha_{-n}^i\alpha_{-n}^j\right), \tag{3·6}$$

which satisfies $[\mathcal{B},\, \alpha_n^i] = -\frac{1}{2}G^{ij}\delta G_{jk}\alpha_{-n}^k$. According to the above definition of the Fock vacuum, it is changed under the variation δG_{ij} into

$$_{G+\delta G}\langle 0| = {}_G\langle 0| - {}_G\langle 0|\mathcal{B}. \tag{3·7}$$

The part of the reflector relevant to this paper is $\langle R^x| \sim {}_{21}\langle 0| e^{E_{12}}$, where E_{12} is given[15] by $E_{12} = -\sum_{n\geq 1}\frac{(-)^n}{n}G_{ij}\alpha_n^{i\,(1)}\alpha_n^{j\,(2)}$. (We will omit the delta function of the zero mode p_j, which, as we mentioned above, is background independent.) By making use of (3·7) and the formula $\delta\left(e^{E_{12}}\right) = \left[\mathcal{B}^{(1)} + \mathcal{B}^{(2)},\, e^{E_{12}}\right]$ under the variation, we obtain

$$\delta\langle R^x| = -\langle R^x|\left(\mathcal{B}^{(1)} + \mathcal{B}^{(2)}\right). \tag{3·8}$$

The right-hand side of (3·8) vanishes because, on the reflector, the oscillators satisfy

$$\langle R^x|\left(\alpha_n^{i\,(1)} + (-)^n\alpha_n^{i\,(2)}\right) = 0.$$

Thus, the reflector is independent of the background G_{ij}.

Similarly, under the variation of the metric, we find the variation of the three-string vertex $\left\langle\tilde{V}_3\right|$ to be

$$\delta\left\langle\tilde{V}_3^x\right| = -{}_{321}\langle 0| e^{E_{123}}\sum_{r=1}^{3}\mathcal{B}^{(r)} + {}_{321}\langle 0| e^{E_{123}}\delta_0 E_{123}, \tag{3·9}$$

where $\delta_0 E_{123}$ corresponds to the change in the zero-mode parts and is given by

$$\delta_0 E_{123} = -\frac{1}{2}\sum_{rs}\bar{N}_{00}^{rs}\delta G_{ij}\alpha_0^{i\,(r)}\alpha_0^{j\,(s)} - \frac{1}{2}\sum_{rs}\sum_{m\geq 1}\bar{N}_{0m}^{rs}\delta G_{ij}\alpha_0^{i\,(r)}\alpha_m^{j\,(s)}.$$

The first term of (3·9) can be evaluated to be ${}_{321}\langle 0| e^{E_{123}}$ multiplied by

$$\begin{aligned}
&-\frac{1}{4}\sum_{n\geq 1}\sum_{r=1}^{3}\frac{1}{n}\delta G_{ij}\alpha_n^{i\,(r)}\alpha_n^{j\,(r)} + \frac{1}{4}\sum_{\substack{m,l\geq 1\\ r,s}}\left(\sum_{n\geq 1}\sum_{t=1}^{3}\bar{N}_{mn}^{rt}n\bar{N}_{nl}^{ts}\right)\delta G_{ij}\alpha_m^{i\,(r)}\alpha_l^{j\,(s)}\\
&-\frac{1}{2}\sum_{\substack{l\geq 1\\ r,s}}\left(\sum_{n\geq 1}\sum_{t=1}^{3}\bar{N}_{0n}^{rt}n\bar{N}_{nl}^{ts}\right)\delta G_{ij}\alpha_0^{i\,(r)}\alpha_l^{j\,(s)}\\
&-\frac{1}{4}\sum_{r,s}\left(\sum_{n\geq 1}\sum_{t=1}^{3}\bar{N}_{0n}^{rt}n\bar{N}_{0l}^{ts}\right)\delta G_{ij}\alpha_0^{i\,(r)}\alpha_0^{j\,(s)}\\
&-\frac{1}{4}\sum_{n\geq 1}\sum_{r=1}^{N}n\bar{N}_{nn}^{rr}\delta G_{ij}G^{ij}.
\end{aligned} \tag{3·10}$$

Note that, although our argument is parallel to that in Ref. 21), the last term of (3·10) is a new term, whose counterpart in the closed string case dose not exist.

To prove the background independence, we can use the identities of the Neumann coefficients,[15),25)]

$$\sum_{n\geq 1}\sum_{t=1}^{3}\bar{N}^{rt}_{mn}n\bar{N}^{ts}_{nl}=\frac{1}{m}\delta_{m,l}\delta^{rs},\qquad \sum_{n\geq 1}\sum_{t=1}^{3}\bar{N}^{rt}_{mn}n\bar{N}^{ts}_{n0}=-\bar{N}^{rs}_{m0},$$
$$\sum_{n\geq 1}\sum_{t=1}^{3}\bar{N}^{rt}_{0n}n\bar{N}^{ts}_{n0}=-2\bar{N}^{rs}_{00}. \tag{3·11}$$

Note that the second and third equalities need the momentum conservation for the zero-modes, which is guaranteed by the vertex. These identities are proven in Appendix B. Therefore, we can see that the second term of (3·9) cancels the first three terms of (3·10). For the last term of (3·10), we need another identity,

$$\sum_{n\geq 1}\sum_{r=1}^{N}n\bar{N}^{rr}_{nn}=0, \tag{3·12}$$

which is also proved in Appendix B. Thus, we can see that the three-string vertex $\left\langle\tilde{V}_3\right|$ is background independent.

§4. Discussion

In this paper, we derived Witten's open string field theory in a background B field by using the standard overlap conditions in the operator formalism. The resulting three-string vertex naturally contains an additional factor which gives the Moyal product to the zero modes, compared to the ordinary vertex with no B field. Thus, the zero modes $\tilde{x}^i$ can be found to be noncommutative. In addition to this noncommutative factor, the three-string vertex can be written by using the open string metric G_{ij}. Therefore, the low-energy effective theory of the gauge field should be described by noncommutative Yang-Mills theory. This result is in agreement with the result in the first-quantization formulation in Ref. 5) and 16).

Following the idea of the pregeometrical formulation,[19),20)] we dropped the kinetic term from our string field theory and explicitly demonstrated the background independence of the resulting theory by using the method of Ref. 21).

In order to prove that the three-string vertex is independent of the theta parameter θ^{ij}, we have shown that the noncommutative factor can be eliminated by field redefinition. If we also apply the redefinition to the theory with the kinetic term, we can easily see that, in addition to elimination of the noncommutative factor of the three-string vertex, it also affects the kinetic term and transforms the BRS charge $Q_{\rm B}$ into $e^{M}Q_{\rm B}e^{-M}$. Here, the operator M is $(i/2)\theta^{ij}p_{Li}p_{Rj}$. This transformed BRS charge can be found to possess a divergent term. This divergent term seems to come from the mid-point $\sigma=\pi/2$ of strings. Very recently, using an interesting technique, Sugino has argued[26)] that the transformed operator indeed remains the original BRS operator Q_B in the kinetic term and that the kinetic term remains intact under the field redefinition. This seems to imply that the background B field is physically meaningless.

In Ref. 5), Seiberg and Witten showed that the noncommutative Dirac-Born-Infeld (DBI) action is equivalent in the slowly varying field approximation to the commutative DBI action with the background B field. In addition, they emphasized that the B field parallel to D-branes cannot be gauged away, due to the gauge invariance $B_{ij} \to B_{ij} + \partial_i \Lambda_j - \partial_j \Lambda_i$, $A_i \to A_i + \Lambda_i$. Here A_i is the gauge field on the D-branes. Therefore, to be consistent with the result in Ref. 5), we may expect that, even after the field redefinition, the dependence of the B field remains in the kinetic term of the string field theory and that the B field appears in the low-energy effective action only through the gauge-invariant combination $\mathcal{F}_{ij} = B_{ij} + F_{ij}$. However, this apparently conflicts with Sugino's recent result. [26] Thus, we think that we have an interesting puzzle to solve.

To our knowledge, there is no study which shows that Witten's open string field theory has the above-mentioned gauge invariance. In addition, it seems difficult to prove it, because the theory has no explicit field from the closed string sector in its action. For this purpose, it may be more suitable to study the gauge invariance and the dependence of B field in an open-closed string field theory with the mid-point interaction in Ref. 32).

Apart from the gauge invariance, we would like to discuss the operator M which was used for the field redefinition. If we do not use Sugino's technique to show that the kinetic term remains intact by the field redefinition, we must deal with the divergent term in $e^M Q_\mathrm{B} e^{-M}$. This divergent term seems to come from the mid-point of strings, as mentioned above. Since the operator M consists of the half-integrated momenta p_L and p_R, we are led to wonder if the singularity may be related to the mid-point interaction. In addition, the operator M seems to be suitable only for the mid-point interaction, because we cannot apply it to the light-cone type interaction. Since the kinetic term of string field theories does not depend on the types of interactions of strings, it is desirable to have a field redefinition which is independent of types of string interactions. Therefore, it may be useful to introduce another candidate,

$$\tilde{M} = -\frac{i}{4}\int_0^\pi d\sigma \int_0^\pi d\sigma' \epsilon(\sigma - \sigma')\theta^{ij} P_i(\sigma) P_j(\sigma'), \tag{4·1}$$

where $\epsilon(\sigma)$ is the step function which is 1 for $\sigma > 0$ and -1 for $\sigma < 0$. Indeed, rewriting this operator as

$$\frac{i}{2}\theta^{ij} p_{Li} p_{Rj} - \frac{i}{4}\int_{\frac{\pi}{2}}^\pi d\sigma \int_{\frac{\pi}{2}}^\pi d\sigma' \epsilon(\sigma - \sigma')\theta^{ij}\{P_i(\sigma)P_j(\sigma') - P_i(\pi-\sigma)P_j(\pi-\sigma')\} \tag{4·2}$$

and putting the sum $\sum_{r=1}^3 \tilde{M}^{(r)}$ on the usual three-string vertex $\left\langle \tilde{V}_3 \right|$, we can see that the second terms in the operators $\tilde{M}^{(r)}$ cancel each other, due to the overlap condition (2·15). Furthermore, as we discuss below, this operator $\tilde{M}$ can be used to give the field redefinition to remove the noncommutative factor from the light-cone type interactions.

To this point we have been discussing the problem with the mid-point interaction concerning the relation between our string field theory and the ordinary one. However, it is plausible that, if there is such a relation, we can find the same relation in other string field theories. In particular, since we cannot apply the operator M to string field theories with light-cone type interactions like that given by Refs. 27) and 28), we can expect that the dependence on the B field cannot be completely removed. Therefore, in order to obtain some information regarding this problem, it may be helpful to study the string field theory with the light-cone type interaction[27),28)] by using the operator $\tilde{M}$. Since this theory explicitly has closed string fields in its Lagrangian, as well as open strings, it may also help to find some relation between the condensation of the antisymmetric tensor B_{ij} from the closed string field and the above redefinition of the open string field.

Now, let us just sketch the main points of our results about the string field theory with the light-cone type interaction[27),28)] in the background B field. These results will be explained in more detail in another paper.[33)] As we can verify by the method explained in §2, the light-cone type vertices are also modified to include the noncommutative factor, and there is no other modification due to the background B field. This noncommutative factor can be expressed by a product of the operators $\tilde{M}$ from each string. Since the operator $\tilde{M}$ thus plays an important role, it is useful to give some discussion on it.

The commutation relation between the operator $\tilde{M}$ and the string coordinates can be shown to be

$$[\tilde{M},\, X^i(\sigma)] = -\frac{1}{\pi^2 l_s}\sum_{n\neq 0}\frac{1-(-)^n}{n^2}(\theta G)^i_j\alpha^j_n + \frac{2}{\pi^2}\theta^{ij}\,p_j\sum_{m\geq 1}\frac{1-(-)^m}{m^2}\cos(m\sigma)$$
$$+\frac{2}{\pi^2 l_s}\sum_{m\geq 1}\left(\sum_{n\neq\pm m}\frac{1-(-)^{m+n}}{m^2-n^2}(\theta G)^i_j\alpha^j_n\right)\cos(m\sigma). \qquad (4\cdot 3)$$

If we naively exchange the order of the summations on the right-hand side of (4·3), we obtain $[\tilde{M},\, X^i(\sigma)] = -(\theta G)^i_j\, Q^j(\sigma)$. Therefore, we would find that $e^{\tilde{M}}\, X^i(\sigma)\, e^{-\tilde{M}} = \tilde{X}^i(\sigma)$. But this result must be false, because it is inconsistent with the commutation relation $[X^i(\sigma),\ X^j(\sigma')] \sim \theta^{ij}$, which cannot be changed to $[\tilde{X}^i(\sigma),\ \tilde{X}^j(\sigma')] = 0$ by the similarity transformation $e^{\tilde{M}}\, X^i(\sigma)\, e^{-\tilde{M}}$. A closer examination shows that the relation $[\tilde{M},\, X^i(\sigma)] = -(\theta G)^i_j\, Q^j(\sigma)$ holds only for $0 < \sigma < \pi$. Therefore, the transformed BRS operator $e^{\tilde{M}}\, Q_{\mathrm{B}}\, e^{-\tilde{M}}$ naively becomes the ordinary BRS operator with the closed string metric g_{ij}, but, due to subtleties caused by the ends of strings, it has additional terms for which, at least at present, we do not have any interpretation. In spite of the discrepancy between $e^{\tilde{M}}\, X^i(\sigma)\, e^{-\tilde{M}}$ and $\tilde{X}^i(\sigma)$, since the operator $\tilde{M}$ allows us to relate our three-string vertex to the ordinary vertex, regardless of the types of string interactions, we are tempted to speculate that the operator $\tilde{M}$ would give us some clue about a relation between the noncommutative string field theory and the ordinary one, like the relation found by Seiberg and Witten[5)] between noncommutative and commutative DBI theories.

Finally, we would like to touch on superstring field theory. We can easily extend our theory to Witten's superstring field theory [29] by constructing other necessary vertices in a manner similar to that in the bosonic case. In the worldsheet picture of superstring theory, we add to the bosonic sector

$$S_\psi = -\frac{1}{4\pi\alpha'}\int d^2z\left(g_{ij}\psi^i\bar{\partial}\psi^j + g_{ij}\tilde{\psi}^i\partial\tilde{\psi}^j\right), \tag{4·4}$$

and the boundary conditions are given by

$$(g+2\pi\alpha' B)_{ij}\psi(z) = (g-2\pi\alpha' B)_{ij}\tilde{\psi}(\bar{z}), \quad \text{at } z=\bar{z}. \tag{4·5}$$

It is convenient to introduce the new fields $\varphi^i(z)$ and $\tilde{\varphi}^i(z)$ defined by

$$\begin{aligned}\varphi^i(z) &= G^{ij}(g+2\pi\alpha' B)_{jk}\psi^k(z),\\ \tilde{\varphi}^i(\bar{z}) &= G^{ij}(g-2\pi\alpha' B)_{jk}\tilde{\psi}^k(\bar{z}).\end{aligned} \tag{4·6}$$

These fields play a role similar to the string coordinates $\tilde{X}^i$ in the bosonic case. Since the fields $\varphi^i(z)$ and $\tilde{\varphi}^i(z)$ satisfy the same boundary condition as that with no B field, we have the ordinary mode expansion of these fields $\varphi^i(z)$ and $\tilde{\varphi}^i(z)$. Therefore, solving the overlap conditions for these fields, we obtain the ordinary three-string vertex as well as the ordinary reflector. The oscillator expression of the vertices can be found in Refs. 34) and 15). From (4·6), we can immediately verify that these vertices also satisfy the overlap conditions for $\psi(z)$ and $\tilde{\psi}(\bar{z})$. Moreover, the picture changing operators can be expressed by using only $\varphi^i(z)$, $\tilde{\varphi}^i(z)$ and $\tilde{X}^i$, and can be found to have the ordinary expression. Thus, we can extend our string field theory to superstring cases, though we still face the mid-point singularity problem, as in Ref. 35).[*]

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[*] Recently, a new approach has been proposed to solve this difficulty. [36]

Appendix A

—— *Operator Formalism of Strings in a Background B field* ——

We consider the operator formalism of first-quantized string theory in a constant background B field by following Refs. 22) – 24). Although Chu and Ho have discussed different methods of quantization in their first and second papers of Ref. 22), our strategy is slightly different from both of theirs; namely, we simplify their methods by combining them.

The worldsheet action is given by

$$S = -\frac{1}{4\pi\alpha'}\int d\sigma\, d\tau \left(g_{ij}\eta^{ab}\partial_a X^i \partial_b X^j - 2\pi\alpha' B_{ij}\epsilon^{ab}\partial_a X^i \partial_b X^j\right), \qquad \text{(A·1)}$$

where g_{ij} is a constant background metric and B_{ij} is a constant background antisymmetric tensor. We also denote $(2\pi\alpha')B_{ij}$ by b_{ij}. The signature of the worldsheet metric η^{ab} is $(-,+)$ and the invariant antisymmetric tensor ϵ^{ab} is defined by $\epsilon^{01}=1$. The equation of motion of the string coordinates is $\partial_a\eta^{ab}\partial_b X^i = 0$. The boundary condition turns out to be

$$g_{ij}X'^j + 2\pi\alpha' B_{ij}\dot{X}^j = 0 \qquad \text{(A·2)}$$

at $\sigma = 0, \pi$. The conjugate momenta are given by $P_i = (1/2\pi\alpha')g_{ij}\dot{X}^j + B_{ij}X'^j$, where the dot and the prime denote differentiation with respect to τ and σ, respectively. As in the usual way, we can obtain the Hamiltonian density

$$\mathcal{H} = \frac{1}{4\pi\alpha'}\left[(2\pi\alpha')^2 g^{ij}P_iP_j + (4\pi\alpha')b_{ik}g^{kj}x'^i P_j + G_{ij}X'^i X'^j\right]. \qquad \text{(A·3)}$$

Here, the open string metric G_{ij} is given by $G_{ij} = g_{ij} - (bg^{-1}b)_{ij}$. The Poisson brackets of the string coordinates and the momenta are all vanishing except $\{X^i(\sigma),\, P_j(\sigma')\}_P = \delta^i_j\delta(\sigma-\sigma')$.

The idea in the second paper of Ref. 22) and in Refs. 23) and 24) is to deal with the boundary condition (A·2) as a constraint $\phi_i = G_{ij}X'^j + (2\pi\alpha')b_{ik}g^{kj}P_j$ in the operator formalism and to quantize the system by following the Dirac quantization method. By using the consistency condition $\dot{\phi}_i = \{\phi_i, H\}_P \approx 0$ with the Hamiltonian $H = \int d\sigma\mathcal{H}$, all the second-class constraints are found [22)-24)] to be

$$\frac{\partial^{2n}\phi_i}{\partial\sigma^{2n}}(\sigma) = 0, \qquad \frac{\partial^{2n+1}P_i}{\partial\sigma^{2n+1}}(\sigma) = 0, \qquad \text{(A·4)}$$

with $n \geq 0$ at $\sigma = 0, \pi$, and there is no first-class constraint. Here, we first solve these constraints (A·4) and find that

$$\phi_i(\sigma) = -\sum_{n=1}^{\infty} nG_{ij}x_n^j\sin(n\sigma), \qquad P_i(\sigma) = \sum_{n=0}^{\infty} p_{ni}\cos(n\sigma). \qquad \text{(A·5)}$$

Therefore, using $\theta^{ij} = -(2\pi\alpha')(G^{-1}bg^{-1})^{ij}$, we obtain

$$X^i(\sigma) = \sum_{n=0}^{\infty} x_n^i\cos(n\sigma) + \theta^{ij}\left[p_{0j}\sigma + \sum_{n=1}^{\infty}\frac{1}{n}p_{nj}\sin(n\sigma)\right]. \qquad \text{(A·6)}$$

Chu and Ho in the first paper of Ref. 22) expressed the string coordinates as a mode expansion in terms of the solutions of the equation of motion and used the invariant symplectic form

$$\omega = \int d\sigma \left[-dX^i(\sigma) \wedge dP_i(\sigma) + dP_i(\sigma) \wedge dX^i(\sigma) \right] \tag{A·7}$$

to find the commutation relations of the modes. Now we apply their idea to our system by using the above mode expansion of string coordinates and the momenta, instead of the mode expansion in terms of the solutions of the equation of motion, to find the commutation relations of x_n^i and p_{nj}. After applying this procedure, we find the equation of motion of these variables x_n^i and p_{nj}. Substituting the expansions (A·5) and (A·6) into the symplectic form (A·7), we find the Poisson brackets of the modes to be

$$\left\{ x_0^i, p_{0j} \right\}_P = \frac{1}{\pi}\delta_j^i, \qquad \left\{ x_n^i, p_{mj} \right\}_P = \frac{2}{\pi}\delta_j^i \delta_{n,m}, \qquad \left\{ x_0^i, x_0^j \right\}_P = \theta^{ij}, \tag{A·8}$$

and the others vanish. Since these variables x_n^i and p_{nj} are the solutions of the constraints (A·4), they are all physical variables. Thus, we can obtain their commutation relations by the usual prescription $[A, B] = i\{A, B\}_P$ to quantize our system.

Since the time derivative of a physical variable $\mathcal{O}$ can be obtained by $\dot{\mathcal{O}} = \{\mathcal{O}, H\}_P$, we can see that $\dot{\phi}_i(\sigma) = (2\pi\alpha')P'_i(\sigma)$ and $\dot{P}_i(\sigma) = (1/2\pi\alpha')\phi'_i(\sigma)$. Substituting the mode expansion (A·5) into these equations to get the equations of motion of the variables x_n^i and p_{nj} and solving the resulting equations, we find that

$$x_n^i = i\frac{l_s}{n}\left(\alpha_n^i e^{-in\tau} - \alpha_{-n}^i e^{in\tau} \right), \quad p_{nj} = \frac{1}{\pi l_s} G_{jk} \left(\alpha_n^k e^{-in\tau} + \alpha_{-n}^k e^{in\tau} \right), \tag{A·9}$$

with $l_s = \sqrt{2\alpha'}$ for $n \neq 0$. We also have $x_0^i = x^i + {l_s}^2 G^{ij} p_j \tau$ and $p_{0j} = (1/\pi)p_j$. Putting these new mode variables, α_n^i, x^i and p_i, into (A·8), we obtain

$$\left[x^i, p_j \right] = i\delta_j^i, \qquad \left[x^i, x^j \right] = i\theta^{ij}, \qquad \left[\alpha_m^i, \alpha_n^j \right] = mG^{ij}\delta_{m+n}. \tag{A·10}$$

These commutation relations are in agreement with the results in Ref. 22). Note here that, if we define the new 'center of mass' coordinates $\tilde{x}^i$ by $\tilde{x}^i = x^i + (1/2)\theta^{ij}p_j$, the coordinates $\tilde{x}^i$ turn out to be commutative variables: $[\tilde{x}^i, \tilde{x}^j] = 0$.

Finally, with the mode variables α_n^i, $\tilde{x}^i$ and p_i, we can express the string coordinates $X^i(\sigma)$ and the conjugate momenta $P_i(\sigma)$ as

$$\begin{aligned} X^i(\sigma) &= \tilde{X}^i(\sigma) + \frac{1}{\pi l_s}\theta^{ij} \left[l_s p_j \left(\sigma - \frac{\pi}{2} \right) + G_{jk} \sum_{n \neq 0} \frac{1}{n} \alpha_n^k e^{-in\tau} \sin(n\sigma) \right], \\ P_j(\sigma) &= \frac{1}{\pi l_s} G_{jk} \sum_{n=-\infty}^{\infty} \alpha_n^k e^{-in\tau} \cos(n\sigma), \end{aligned} \tag{A·11}$$

where $\tilde{X}^i(\sigma) = \tilde{x}^i + {l_s}^2 G^{ij} p_j \tau + l_s \sum_{n \neq 0} (i/n) \alpha_n^i e^{-in\tau} \cos(n\sigma)$ and $p_j = (1/l_s) G_{jk} \alpha_0^k$.

Appendix B
—— *Identities for the Neumann coefficients* ——

Consider an N-string vertex with a midpoint interaction [15), 13), 14)]. The Neumann function on the strip is given by

$$N(\rho_r, \rho'_s) = -\delta_{rs}\left[2\sum_{n\geq 1}\frac{1}{n}e^{-n|\tau_r-\tau'_s|}\cos(n\sigma_r)\cos(n\sigma'_s) - 2\max(\tau_r,\tau'_s)\right] +2\sum_{m,n\geq 0}\bar{N}^{rs}_{mn}e^{m\tau_r+n\tau'_s}\cos(m\sigma_r)\cos(n\sigma'_s). \qquad (\text{B}\cdot 1)$$

In the case $\tau_r > \tau'_s$, we find that

$$\frac{\partial}{\partial\tau_r}N(\rho_r, \rho'_s) = 2\delta_{rs}\left[\sum_{n\geq 1}e^{-n|\tau_r-\tau'_s|}\cos(n\sigma_r)\cos(n\sigma'_s)+1\right] +2\sum_{m\geq 1}\sum_{n\geq 0}m\bar{N}^{rs}_{mn}e^{m\tau_r+n\tau'_s}\cos(m\sigma_r)\cos(n\sigma'_s). \qquad (\text{B}\cdot 2)$$

The Neumann function and its derivative with respect to ρ are continuous at the interaction time $\tau_r = 0$, provided that we use momentum conservation for the zero mode parts. In other words, in order to maintain its continuity, it is necessary to multiply the zero mode terms $2\delta_{rs}\max(\tau_r, \tau'_s)$ and $\bar{N}^{rs}_{n0}$ by the factor $\sum_s p^{(s)}$.

Using the continuity of the Neumann function, we find the identity

$$\begin{aligned}
0 &= \sum_{t=1}^{N}\int_0^{\pi}d\sigma''_t N(i\sigma''_t, \rho_r)\frac{\partial N}{\partial\tau''_t}(i\sigma''_t, \rho'_s)\\
&= -2\pi\delta_{rs}\sum_{n\geq 1}\frac{1}{n}e^{n\tau_r+n\tau'_s}\cos(n\sigma_r)\cos(n\sigma'_s)\\
&\quad +2\pi\sum_{n\geq 1}\bar{N}^{rs}_{n0}e^{n\tau_r}\cos(n\sigma_r) + 2\pi\sum_{n\geq 1}\bar{N}^{rs}_{0n}e^{n\tau'_s}\cos(n\sigma'_s) + 4\pi\bar{N}^{rs}_{00}\\
&\quad +2\pi\sum_{m,l\geq 1}\left(\sum_{n\geq 1}\sum_{t=1}^{N}\bar{N}^{rt}_{mn}n\bar{N}^{ts}_{nl}\right)e^{m\tau_r+l\tau'_s}\cos(m\sigma_r)\cos(l\sigma'_s)\\
&\quad +2\pi\sum_{m\geq 1}\left(\sum_{n\geq 1}\sum_{t=1}^{N}\bar{N}^{rt}_{mn}n\bar{N}^{ts}_{n0}\right)e^{m\tau_r}\cos(m\sigma_r)\\
&\quad +2\pi\sum_{l\geq 1}\left(\sum_{n\geq 1}\sum_{t=1}^{N}\bar{N}^{rt}_{0n}n\bar{N}^{ts}_{nl}\right)e^{l\tau'_s}\cos(l\sigma'_s). \qquad (\text{B}\cdot 3)
\end{aligned}$$

From this identity, we can obtain (3·11).

To obtain (3·12), we take the limit $\rho'_s \to \rho_r$ in the Neumann function (B·1):

$$N(\rho_r, \rho_r+\delta) = \ln\delta - \sum_{n\geq 1}\frac{1}{n}\cos(2n\sigma_r) + 2\tau_r$$

$$+2\sum_{m,n\geq 0}\bar{N}^{rs}_{mn}e^{(m+n)\tau_r}\cos(m\sigma_r)\cos(n\sigma_r)+O(\delta^2). \qquad \text{(B·4)}$$

From the continuity at the time of the interaction, we can obtain (3·12) as

$$0=\sum_{r=1}^{N}\int_0^{\pi}d\sigma_r\frac{\partial}{\partial\tau_r}N(i\sigma_r,i\sigma_r+\delta)=2\pi\sum_{n\geq 1}\sum_{r=1}^{N}n\bar{N}^{rr}_{nn}, \qquad \text{(B·5)}$$

where we can make $\sum_{r=1}^{N}\partial/\partial\tau_r\,\tau_r$ vanish due to momentum conservation.

These arguments and identities can also be established for light-cone-type vertices with one interaction time.

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