TAIWANESE JOURNAL OF MATHEMATICS
Vol. 25, No. 6, pp. 1241–1260, December 2021
DOI: 10.11650/tjm/210801

Open Subsets in a Stein Space with Singularities

Jing Zhang

Abstract. Serre proved that a domain Y in $\mathbb{C}^n$ is Stein if and only if $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$. Laufer showed that if Y is an open subset of a Stein manifold of dimension n and $H^i(Y, \mathcal{O}_Y)$ is a finite dimensional complex vector space for every $i > 0$, then Y is Stein. Vâjâitu generalized these theorems to singular Stein space of dimension n. In this paper, we consider singular Stein spaces X with arbitrary dimension and give necessary and sufficient conditions for an open subset Y in X to be Stein. We show that if Y is an open subset of a reduced Stein space X with arbitrary dimension and singularities, then Y is Stein if and only if $H^i(Y, \mathcal{O}_Y)$ is a finite dimensional complex vector space for every $i > 0$. Without cohomology condition, if $X - Y$ is a closed subspace of X, then we show that the geometric condition of the boundary $X - Y$ determines the Steinness of Y. More precisely, we show that if X is normal and the boundary $X - Y$ is the support of an effective $\mathbb{Q}$-Cartier divisor, or $X - Y$ is of pure codimension 1 and does not contain any singular points of X, then Y is Stein.

1. Introduction

We work over the field $\mathbb{C}$ of complex numbers.

Let X be a Hausdorff topological space. $(X, \mathcal{O}_X)$ is a complex space if every point of X has a neighborhood U such that $(U, \mathcal{O}_U)$ is isomorphic to a closed complex subspace $(A, \mathcal{O}_A)$ of a domain $D \subset \mathbb{C}^m$ for some $m \in \mathbb{N}$, where A is the support of the analytic coherent $\mathcal{O}_D$ sheaf $\mathcal{O}_A = \mathcal{O}_D/\mathcal{I}|_A$, and $\mathcal{I} \subset \mathcal{O}_D$ is an analytic coherent ideal sheaf.

A complex space Y is Stein if it is both holomorphically convex and holomorphically separable [8, pp. 293—294, Theorem 63.2]. We say that Y is holomorphically convex if for any discrete sequence $\{y_n\} \subset Y$, there is a holomorphic function f on Y such that the supremum of the set $\{|f(y_n)|\}$ is ∞. Y is holomorphically separable if for every pair $x, y \in Y$, $x \neq y$, there is a holomorphic function f on Y such that $f(x) \neq f(y)$. By Cartan's Theorem B, a complex space Y is Stein if and only if $H^i(Y, \mathcal{F}) = 0$ for every analytic coherent sheaf $\mathcal{F}$ on Y and all positive integers i [4, p. 124].

Serre proved that a domain Y in $\mathbb{C}^n$ is Stein if and only if $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$ [17], where $\mathcal{O}_Y$ is the analytic structure sheaf of Y. Laufer generalized Serre's

Received January 30, 2021; Accepted August 8, 2021.
Communicated by Min Ru.
2020 *Mathematics Subject Classification.* 32E10, 14A15, 14C20.
Key words and phrases. active germ, analytic hypersurface, analytic structure sheaf, Cěch cohomology, Stein space.

result to Stein manifolds of dimension m (see [4, pp. 159–160] or [10]). In algebraic geometry, Neeman showed that a quasi-compact Zariski open subset Y of an affine scheme $X = \operatorname{Spec} A$ (with singularities) is affine if and only if $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$ [14], where $\mathcal{O}_Y$ is the algebraic structure sheaf of Y and A may not be noetherian.

In [10], Laufer also proved: If Y is an n-dimensional Riemann domain over a Stein manifold such that $\mathcal{O}_Y$ separates points, and for all $i > 0$, $H^i(Y, \mathcal{O}_Y)$ is a finite dimensional complex vector space, then Y is a Stein manifold. A dimension n Stein manifold can be biholomorphicall mapped onto a closed complex submanifold of $\mathbb{C}^{2n+1}$ [7, Ch. 5]. Laufer's proof heavily relies on the local coordinates of a complex manifold which cannot be applied to a singular Stein space.

For an open subset of a Stein space with singularities, we use complex algebraic geometry approach to avoid dealing with singularities directly and show the following result.

Theorem 1.1. *If Y is an open subset of a reduced Stein space X of dimension n, then Y is Stein if and only if $H^i(Y, \mathcal{O}_Y)$ is a finite dimensional complex vector space for every $i > 0$.*

A complex space X is locally of finite dimension but globally its dimension may not be finite. If a nonempty complex space X is irreducible, then there is a nonnegative integer $n \geq 0$ such that $\dim_x X = n$ for all $x \in X$ [5, p. 106] and n is the dimension of X. If X has infinitely many irreducible components, then the dimension of X: $\dim X = \sup_{x \in X} \dim_x X$ [5, p. 94] may not be finite and there are connected complex spaces with dimension ∞ [8, p. 190]. Without the finite dimension condition, we have

Theorem 1.2. *Let Y be an open subset of a reduced Stein space X with arbitrary dimension and singularities. Then Y is Stein if and only if $H^i(Y, \mathcal{O}_Y)$ is a finite dimensional vector space over $\mathbb{C}$ for all $i > 0$.*

The idea of proof of Theorem 1.2 is the following[1]. First, by Sard and Remmert's theorems, we can construct countably many holomorphic functions $f_1, f_2, \dots$ such that each f_i defines a (smooth) hypersurface F_i with disconnected components, $G_i = F_i \cap Y$ is an open subset in F_i and $1, f_1, f_2, \dots$ are linearly independent (see Lemmas 2.9, 2.11, and 3.4). Here every G_i is Stein, using the fact that it has disconnected components, Mayer–Vietoris sequence and Theorem 1.1 (Lemma 3.7). Then we can show that for all $i > 0$, $H^i(Y, \mathcal{O}_Y) = 0$ by Lemma 3.7 and the dimension counting method of vector spaces due to

[1]The author was informed by an anonymous referee that Theorem 1.1 was proved by Vâjâitu for complex spaces (non-reduced) of dimension n in 2010 and by modifying this proof and not using mathematical induction, the results in [21] hold for complex spaces with arbitrary dimension. The key idea in [21] to prove holomorphic convexity is to generalize an estimate of Fornæss and Narasimhan and use Wiegmann's construction to get a proper surjective morphism from a hypersurface to an n-dimensional Stein space.

Goodman and Hartshorne [3] (see Lemmas 2.13 and 3.8). Finally, by Nagel's theorem on finitely generated property of the ring of holomorphic functions on Y [12], we show that Y is holomorphically convex (see Theorems 2.3 and 3.9).

In [22–24], we investigated a question raised by J.-P. Serre [17]: Let Y be a complex manifold with $H^i(Y, \Omega^j_Y) = 0$ for all $j \geq 0$ and $i > 0$ (where Ω^j_Y is the sheaf of holomorphic j-forms), then what is Y? Is Y Stein? If Y is an algebraic manifold (i.e., an irreducible nonsingular algebraic variety defined over $\mathbb{C}$) and Ω^j_Y is the sheaf of regular j-forms, we know that Y is not an affine variety in general. In fact, if the dimension of Y is d and X is a smooth projective variety containing Y, then X may have $d-j$ algebraically independent nonconstant rational functions which are regular on Y, where $j = 1, 3, \ldots, d-2, d$ if d is odd or $j = 0, 2, \ldots, d-2, d$ if d is even. But the Steinness question is still open except for the trivial case when the dimension is one. By Theorem 1.1, we have

Corollary 1.3. *If Y is a nonsingular open subset of a Stein space X with dimension n such that $H^i(Y, \Omega^j_Y) = 0$ for all $j \geq 0$ and $i > 0$, then Y is Stein.*

Simha proved that an open subset of a normal Stein surface obtained by removing a closed analytic subspace of pure codimension one is a Stein surface [18]. This result does not hold for higher dimensional complex spaces with singularities (see an example in Section 3). For a normal Stein space, we have

Theorem 1.4. *Let Y be an open subset of a normal Stein space X such that the complement $X - Y$ is a closed analytic subspace of X.*

(1) *If $X - Y$ is the support of an effective $\mathbb{Q}$-Cartier divisor, then Y is Stein.*

(2) *If $X - Y$ is of pure codimension 1 and does not contain any singular points of X, then Y is Stein.*

In order to prove Theorem 1.2 in Section 3, we first prove Theorem 1.1 in Section 2 by algebraic geometry approach. In Section 2, we also prove Theorem 2.3 for an open subset in a Stein space with arbitrary dimension and several lemmas which will be used in the proof of Theorem 1.2 in Section 3.

2. Preparations

A ring R is local if it has exactly one maximal ideal $\mathcal{M}$. Every stalk $\mathcal{O}_x$ of the structure sheaf $\mathcal{O}_X$ of a complex space X is a local ring: the maximal ideal $\mathcal{M}_x \subset \mathcal{O}_x$ consists of all germs at x which can be represented in a neighborhood of x by a holomorphic function. In fact, $\mathcal{O}_x$ is a local $\mathbb{C}$-algebra: the composition

$$\phi\colon \mathbb{C} \cdot 1 \to \mathcal{O}_x \to \mathcal{O}_x/\mathcal{M}_x$$

is an isomorphism of fields (see [5, pp. 5–7] or [8, p. 66, p. 97]).

Let $\mathcal{C}_X$ be the sheaf of germs of complex valued continuous functions on a Hausdorff topological space X. Then $\mathcal{C}_X$ is a local $\mathbb{C}$-algebra [5, p. 5]. Since the stalk

$$\mathcal{O}_x = \mathbb{C} \oplus \mathcal{M}_x,$$

every germ $f_x \in \mathcal{O}_x$ can be uniquely written in the form

$$f_x = c_x + m_x,$$

where c_x is the complex value of f_x at x and $m_x \in \mathcal{M}_x$ [5, p. 8]. For a holomorphic function f on an open subset U of X, define a function $[f]\colon U \to \mathbb{C}$ by $[f](x) = c_x$ for $x \in U$. Then f induces a continuous function $[f] \in \mathcal{C}_X(U)$ [5, p. 9].

We need a theorem of Nagel [12].

Let Y be a topological space, and let $\mathcal{A}$ be a sheaf of local $\mathbb{C}$-algebras on Y. We assume:

(a) For every $y \in Y$, the maximal ideal of the stalk $\mathcal{A}_y$ is $\mathcal{M}_y$, and the composition

$$\phi\colon \mathbb{C} \cdot 1 \to \mathcal{A}_y \to \mathcal{A}_y/\mathcal{M}_y$$

is an isomorphism.

(b) For every global section $f \in \Gamma(Y, \mathcal{A})$, the associated complex valued function $[f]$ is continuous, where $[f](y)$ is the residue class of the germ of f at y in $\mathcal{A}_y/\mathcal{M}_y$.

(c) For all $i > 0$, $H^i(Y, \mathcal{O}_Y) = 0$.

Lemma 2.1 (Nagel)**.** *Let $\mathcal{A}$ be a sheaf of local $\mathbb{C}$-algebras on Y such that the above three conditions are satisfied. Suppose that I is an ideal in $\Gamma(Y, \mathcal{A})$, and that there is a finite subset $\{f_1, f_2, \ldots, f_m\} \subset I$, so that for every $y \in Y$, there is an f_j such that $f_j(y) \neq 0$. Then $I = (f_1, f_2, \ldots, f_m) = \Gamma(Y, \mathcal{A})$.*

Lemma 2.2. *Let Y be an open subset of a Stein space X such that $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$. If $h \in H^0(Y, \mathcal{O}_Y)$ is not a zero divisor of the stalk $\mathcal{O}_y$ at every point $y \in Y$, then $H^i(Z, \mathcal{O}_Z) = 0$ for all $i > 0$, where $Z = \{y \in Y, h(y) = 0\}$ is the hypersurface defined by the holomorphic function h.*

Proof. If h is a unit in $H^0(Y, \mathcal{O}_Y)$, then h does not vanish on Y and Z is an empty set. We assume that h is not a unit on Y. The multiplication by h defines an injective map from $\mathcal{O}_Y$ to itself. Z is a hypersurface of pure codimension 1 on Y [5, p. 100] and we have a short exact sequence

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_Y/h\mathcal{O}_Y \cong \mathcal{O}_Z \longrightarrow 0,$$

where the first map is defined by the non-zero divisor (a holomorphic function) h on X. Since $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$, the corresponding long exact sequence gives

$$0 \longrightarrow H^0(Y, \mathcal{O}_Y) \longrightarrow H^0(Y, \mathcal{O}_Y) \longrightarrow H^0(Y, \mathcal{O}_Z) \longrightarrow 0,$$

and $H^i(Z, \mathcal{O}_Z) = 0$. □

A holomorphic map $f\colon X \to X'$ is proper if for every compact subset $K \in X'$, the inverse image $f^{-1}(K) \subset X$ is a compact subset in X. Remmert's Proper Mapping Theorem states that for any proper holomorphic map $f\colon X \to X'$ between complex spaces, the image $f(X)$ is an analytic subset of X' [5, p. 213].

In Theorem 2.3, the dimension of the Stein space X is arbitrary.

Theorem 2.3. *If Y is an open subset of a Stein space X such that every accumulation point $P_0 \in X - Y$ of a discrete sequence in Y is the only common zero of finitely many holomorphic functions $f_1, \ldots, f_m$ on X, then Y is Stein if and only if $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$.*

Proof. If Y is Stein, then for any coherent analytic sheaf $\mathcal{F}$ on Y and all $i > 0$, by Theorem B, $H^i(Y, \mathcal{F}) = 0$. By Oka's theorem, the structure sheaf $\mathcal{O}_Y$ is coherent [5, p. 60] so $H^i(Y, \mathcal{O}_Y) = 0$. We only need to show that if $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$, then Y is Stein. Y is holomorphically separable since holomorphic functions on X separate points on the open subset Y. We will show that Y is holomorphically convex.

Let $S = \{P_1, P_2, \ldots, P_k, \ldots\}$ be a discrete sequence in Y. If S has no accumulation points in X, then there is a holomorphic function f on X such that f is not bounded on S. We are done. We may assume that S has an accumulation point $P_0 \in X - Y$. Since there are finitely many holomorphic functions $f_1, \ldots, f_m$ on X such that $P_0 \notin Y$ is their only common zero, for every point $y \in Y$, at least one f_j does not vanish at y. By Lemma 2.1, $f_1, f_2, \ldots, f_m$ generate the ring $H^0(Y, \mathcal{O}_Y) = \Gamma(Y, \mathcal{O}_Y)$. Particularly, there are $g_1, g_2, \ldots, g_m \in H^0(Y, \mathcal{O}_Y)$ such that

$$f_1 g_1 + f_2 g_2 + \cdots + f_m g_m = 1$$

on Y.

Every holomorphic function f_i is continuous on X and $f_i(P_0) = 0$, $i = 1, 2, \ldots, m$, so its limit at $P_0 \in X - Y$ is 0. By the equation, at least one g_j has limit infinity at P_0. This implies that g_j is not bounded on the discrete sequence S since $P_0 \in X - Y$ is an accumulation point of $S \subset Y$.

We show that Y is holomorphically convex so it is Stein. □

Lemma 2.4. *Let Y be an open subset of a Stein space X with dimension n such that $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$. Then Y is holomorhically convex.*

Proof. Since X is a Stein space of dimension n, there is a one-to-one, proper holomorphic map from X to $\mathbb{C}^{2n+1}$ [13].

Let $S = \{P_1, P_2, \ldots, P_k, \ldots\}$ be a discrete sequence in Y. As in the proof of Theorem 2.3, we may assume that S has an accumulation point $P_0 \in X - Y$. By Narasimhan's theorem, let $\psi\colon X \to \mathbb{C}^{2n+1}$ be a one-to-one, proper holomorphic map which is regular at every uniformizable point [13]. By Remmert's Proper Mapping Theorem, $\psi(X)$ is a closed subspace of $\mathbb{C}^{2n+1}$. By affine algebraic geometry, there are m polynomials $f_1, f_2, \ldots, f_m$ in $\mathbb{C}^{2n+1}$ such that the point $\psi(P_0)$ in $\psi(X)$ (not in $\psi(Y)$) is the only point in the zero set

$$\{x \in \psi(X), f_1(x) = f_2(x) = \cdots = f_m(x) = 0\}.$$

Pull these polynomials back to X by the proper injective holomorphic map ψ, we receive m holomorphic functions (still denoted by f_i for simplicity) $f_1, f_2, \ldots, f_m$ in X such that their only common zero is the point $P_0 \in X - Y$. So for every point $y \in Y$, at least one f_j does not vanish at y. By Lemma 2.1, $f_1, f_2, \ldots, f_m$ generate $\Gamma(Y, \mathcal{O}_Y)$ and the rest of the proof is the same as proof of Theorem 2.3.

We show that Y is holomorphically convex so it is Stein. □

By Lemma 2.4, we have

Theorem 2.5. *If Y is an open subset of a Stein space X of dimension n, then Y is Stein if and only if $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$.*

Definition 2.6. (1) If I is an ideal of a ring R, then the set

$$\sqrt{I} = \{r \in R, r^j \in I, j \in \mathbb{N}\}$$

is an ideal of R called the radical of I in R.

(2) An element $r \in R$ is an nilpotent element if there is a positive integer n such that $r^n = 0$.

(3) The radical $N = \sqrt{0}$ is called the nilradical of R.

(4) The ideal I in a commutative ring R is reduced if for $r \in R$, there is an integer $m \in \mathbb{N}$, $r^m \in I$, then $r \in I$.

Definition 2.7. The radical sheaf $\mathcal{N} = \sqrt{0}$ of the zero ideal in the structure sheaf $\mathcal{O}_X$ of a complex space X is called the nilradical of $\mathcal{O}_X$.

By Definition 2.7, the stalk $\mathcal{N}_x$ is the ideal of all nilpotent germs in the stalk $\mathcal{O}_x$. For a complex space X, the nilradical $\mathcal{N}$ is a coherent ideal sheaf of $\mathcal{O}_X$ [5, p. 86]. A complex space X is reduced at a point $x_0 \in X$ if the stalk $\mathcal{O}_{x_0}$ is reduced: $\mathcal{O}_{x_0}$ has no nilpotent elements. X is reduced if for all points $x \in X$, all stalks $\mathcal{O}_x$ are reduced rings, i.e., if $f_x \in \mathcal{O}_x$ and $f_x^m = 0$ for some $m \in \mathbb{N}$ (m relies on the point x and the function f), then $f_x = 0$ [8, p. 151].

Definition 2.8. A germ $f_x \in \mathcal{O}_x$ at a point x in a complex space X is active if for every $g_x \in \mathcal{O}_x$, with $f_x g_x \in \mathcal{N}_x$, we have $g_x \in \mathcal{N}_x$.

The set of points of a complex space X where X is not reduced is an analytic subset of X [5, p. 88]. A holomorphic function f on X is active at a point $x \in X$ if there is an open neighborhood U of x such that f does not vanish at every irreducible component of X in U [5, p. 98].

Lemma 2.9. *Let $\{1, f_1, f_2, \ldots, f_m, \ldots\} \subset V$ be linearly independent in a vector space V over $\mathbb{C}$. Then for any constant $a_i \in \mathbb{C}$, $\{1, f_1 - a_1, f_2 - a_2, \ldots, f_m - a_m, \ldots\}$ is also linearly independent in V.*

Proof. For any $m \in \mathbb{N}$, let $c_i \in \mathbb{C}$, $i = 0, 1, 2, \ldots, m$ and

$$c_0 + c_1(f_1 - a_1) + c_2(f_2 - a_2) + \cdots + c_m(f_m - a_m) = 0.$$

Then

$$(c_0 - c_1a_1 - c_2a_2 - \cdots - c_ma_m) + c_1f_1 + c_2f_2 + \cdots + c_mf_m = 0.$$

Since $1, f_1, f_2, \ldots, f_m$ are linearly independent, we have

$$c_0 - c_1a_1 - c_2a_2 - \cdots - c_ma_m = c_1 = c_2 = \cdots = c_m = 0.$$

So $c_0 = c_1 = c_2 = \cdots = c_m = 0$ and $1, f_1 - a_1, f_2 - a_2, \ldots, f_m - a_m$ are linearly independent. Similarly, we can show that any finite subset of $\{1, f_1 - a_1, f_2 - a_2, \ldots, f_m - a_m, \ldots\}$ is linearly independent. Therefore, $\{1, f_1 - a_1, f_2 - a_2, \ldots, f_m - a_m, \ldots\}$ is linearly independent. □

Lemma 2.10. *If X is a Stein space of dimension at least 1, then the dimension $h^0(X, \mathcal{O}_X)$ of the vector space $H^0(X, \mathcal{O}_X)$ over $\mathbb{C}$ is not finite.*

Proof. We will construct infinitely many holomorphic functions on X which are linearly independent.

Let C be an irreducible analytic curve in X and $\mathcal{I}_C$ be its ideal sheaf in X such that $C \cap Y$ is an open subset of C and contains smooth points in X. Then $\mathcal{I}_C$ is coherent analytic sheaf on X [5, p. 84]. We have a short exact sequence

$$0 \to \mathcal{I}_C \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}_C = \mathcal{O}_C \to 0.$$

Since X is Stein, $H^1(X, \mathcal{I}_C) = 0$ and we have a surjective map $H^0(X, \mathcal{O}_X) \to H^0(C, \mathcal{O}_C)$.

Let P_0 be a smooth point of the curve C and X and z be the local coordinate at P_0 on C. Now C is a Stein curve so there is a nonconstant holomorphic function f on C such that $f(z) = z + z^2 g(z)$ near P_0 [4, p. 151], where $g(z)$ is holomorphic near P_0. Then $1, f, f^2, \dots, f^m, \dots$ are holomorphic functions on C and linearly independent over $\mathbb{C}$. Since $h^0(C, \mathcal{O}_C) = \infty$, we have $h^0(X, \mathcal{O}_X) = \infty$. □

Lemma 2.11. *If X is a reduced Stein space of arbitrary dimension, then there are infinitely many holomorphic functions $1, f_1 - a_1, f_2 - a_2, \dots, f_m - a_m, \dots$ on X such that they are linearly independent in $H^0(Y, \mathcal{O}_Y)$ and each of $f_1 - a_1, f_2 - a_2, \dots, f_m - a_m, \dots$ defines a reduced hypersurface on Y.*

Proof. By Lemma 2.10, there are holomorphic functions $1, f_1, f_2, \dots, f_m, \dots$ on X such that they are linearly independent in the vector space $H^0(X, \mathcal{O}_X)$ over $\mathbb{C}$. Each function f_i gives a nonconstant holomorphic map from X to $\mathbb{C}$. By open mapping theorem, if f_i is not a constant near a point $p \in X$, then the map $f_i \colon X \to \mathbb{C}$ is open near p [5, p. 109]. So the image $f_i(X)$ contains an open subset V of $\mathbb{C}$. By the Sard type theorem, there is a countable subset $B \subset \mathbb{C}$ such that for every point $a_i \in \mathbb{C} - B$, the fiber (hypersurface) $X_{a_i} = f_i^{-1}(a_i)$ is reduced [11]. We may choose suitable a_i such that each $f_i - a_i$ defines a reduced hypersurface in Y.

By the construction in the proof of Lemma 2.10, we may choose these holomorphic functions so that they are linearly independent on an irreducible curve C (i.e., linearly independent in $H^0(C, \mathcal{O}_C)$) in X such that $C \cap Y$ is an open subset of C, then they are linearly independent in $H^0(Y, \mathcal{O}_Y)$. This is because if we have

$$c_0 + c_1 f_1 + c_2 f_2 + \cdots + c_m f_m = 0,$$

on Y, then $c_0 + c_1 f_1 + c_2 f_2 + \cdots + c_m f_m = 0$ on the curve $C \cap Y$. By the Identity Theorem [5, p. 170], the equation holds on the irreducible curve C. But these functions are linearly independent on C, we have $c_0 = c_1 = \cdots = c_m = 0$ and they are linearly independent on Y. By Lemma 2.9, $\{1, f_1 - a_1, f_2 - a_2, \dots, f_m - a_m, \dots\}$ is linearly independent in $H^0(Y, \mathcal{O}_Y)$. □

Lemma 2.12. *Let Y be an open subset of a Stein space X of arbitrary dimension such that $H^i(Y, \mathcal{O}_Y)$ is a finite dimensional vector space over $\mathbb{C}$ for all $i > 0$, then the dimension $h^i(Z, \mathcal{O}_Z) < \infty$ for every hypersurface Z defined by a holomorphic function h on Y which is not a zero divisor of $\mathcal{O}_y$ at every point $y \in Y$.*

Proof. Since h is not a zero divisor on Y, we have a short exact sequence

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_Y / h\mathcal{O}_Y \cong \mathcal{O}_Z \longrightarrow 0,$$

where the first map is defined by h. The corresponding long exact sequence gives

$$0 \longrightarrow H^0(Y,\mathcal{O}_Y) \longrightarrow H^0(Y,\mathcal{O}_Y) \longrightarrow H^0(Z,\mathcal{O}_Z) \longrightarrow H^1(Y,\mathcal{O}_Y) \longrightarrow H^1(Y,\mathcal{O}_Y)$$
$$\xrightarrow{\alpha} H^1(Z,\mathcal{O}_Z) \xrightarrow{\beta} H^2(Y,\mathcal{O}_Y) \longrightarrow H^2(Y,\mathcal{O}_Y) \longrightarrow H^2(Z,\mathcal{O}_Z) \longrightarrow \cdots .$$

The sequence is exact at $H^1(Z,\mathcal{O}_Z)$ so the relationship between the image of α and kernel of β is

$$\operatorname{im}(\alpha) = \alpha(H^1(Y,\mathcal{O}_Y)) = \ker(\beta) \subset H^1(Z,\mathcal{O}_Z)$$

and

$$\dim_{\mathbb{C}} \ker(\beta) = \dim_{\mathbb{C}} \operatorname{im}(\alpha) \leq h^1(Y,\mathcal{O}_Y) < \infty.$$

β is a homomorphism from the vector space $H^1(Z,\mathcal{O}_Z)$ to the vector space $H^2(Y,\mathcal{O}_Y)$ [16, pp. 627–629], so the image vector space $\operatorname{im}(\beta)$ is a subspace of $H^2(Y,\mathcal{O}_Y)$. This implies

$$\dim_{\mathbb{C}} \operatorname{im}(\beta) \leq h^2(Y,\mathcal{O}_Y) < \infty.$$

By the rank theorem in linear algebra, these two inequalities give $h^1(Z,\mathcal{O}_Z) < \infty$. Using the fact that the sequence is exact at $H^i(Z,\mathcal{O}_Z)$ and $h^i(Y,\mathcal{O}_Y) < \infty$ for all $i > 0$, $h^i(Z,\mathcal{O}_Z) < \infty$ can be similarly proved. □

Lemma 2.13. *Let Y be an open subset of a reduced Stein space X of dimension n such that $H^i(Y,\mathcal{O}_Y)$ is a finite dimensional vector space over $\mathbb{C}$ for all $i > 0$, then $H^i(Y,\mathcal{O}_Y) = 0$.*

Proof. For every holomorphic function $f \in H^0(Y,\mathcal{O}_Y)$, the multiplication by f induces a homomorphism:

$$f^{*i} \colon H^i(Y,\mathcal{O}_Y) \longrightarrow H^i(Y,\mathcal{O}_Y)$$

and the map $f \to f^{*i}$ is a $\mathbb{C}$-homomorphism (see [3] or [16, pp. 627–629])

$$H^0(Y,\mathcal{O}_Y) \longrightarrow \operatorname{End}_{\mathbb{C}}(H^i(Y,\mathcal{O}_Y)),$$

where $\operatorname{End}_{\mathbb{C}}(V)$ is the set of all vector homomorphisms (linear transformations) from a vector space V over $\mathbb{C}$ to itself. Since $H^i(Y,\mathcal{O}_Y)$ is a finite dimensional vector space over $\mathbb{C}$ for all $i > 0$, $\operatorname{End}_{\mathbb{C}}(H^i(Y,\mathcal{O}_Y))$ is also a finite dimensional vector space over $\mathbb{C}$ for all $i > 0$.

By Lemma 2.11, there are infinitely many holomorphic functions $1, f_1, f_2, \ldots, f_m, \ldots$ on X such that they are linearly independent and each of them defines a reduced hypersurface on X. Each f_j defines a homomorphism f_j^{*i} from the vector space $H^i(Y,\mathcal{O}_Y)$ to itself. But $\operatorname{End}_{\mathbb{C}}(H^i(Y,\mathcal{O}_Y))$ is a finite dimensional vector space over $\mathbb{C}$ for all $i > 0$, so for each i, there is an $f_{l_i} \subset \{f_1, f_2, \ldots, f_m, \ldots\}$ such that it induces a zero map from $H^i(Y,\mathcal{O}_Y)$ to itself. By the choice of the functions, f_{l_i} defines a reduced hypersurface Z_{l_i}.

We will use mathematical induction on the dimension of X to show that for all $i > 0$, $H^i(Y, \mathcal{O}_Y) = 0$.

If X is a Stein curve, then Y is an open subset of X and for any coherent sheaf $\mathcal{F}$ on Y and all $i > 0$, $H^i(Y, \mathcal{F}) = 0$ [19] so $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$.

If X is a Stein surface and $h^i(Y, \mathcal{O}_Y) < \infty$ for all $i > 0$, then the hypersurface Z defined by a holomorphic function (non-zero divisor) f on Y is Stein [19] so $H^1(Z, \mathcal{O}_Z) = 0$ and $H^2(Z, \mathcal{O}_Z) = 0$. Since Y is an open surface, $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 1$ [19].

By the above long exact sequence,

$$f^{*1} \colon H^1(Y, \mathcal{O}_Y) \to H^1(Y, \mathcal{O}_Y)$$

is surjective. Now for all $j \in \mathbb{N}$, we have infinitely many surjective group homomorphisms f_j^{*1} of a finite dimensional vector space $H^1(Y, \mathcal{O}_Y)$

$$f_j^{*1} \colon H^1(Y, \mathcal{O}_Y) \to H^1(Y, \mathcal{O}_Y)$$

induced by each $f_j \in \{f_1, f_2, \ldots, f_m, \ldots\}$. Because $\mathrm{End}_{\mathbb{C}}(H^i(Y, \mathcal{O}_Y))$ is a finite dimensional vector space over $\mathbb{C}$ for all $i > 0$, by counting the dimensions of vector spaces, we see that there is a $k \in \mathbb{N}$ such that $f_k^{*1} = 0$ [3]. But f_k^{*1} is a surjective map from $H^1(Y, \mathcal{O}_Y)$ to itself. We see $H^1(Y, \mathcal{O}_Y) = 0$.

We receive $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$.

By mathematical induction, we may assume that if dimension of X is $n - 1$, and $h^i(Y, \mathcal{O}_Y) < \infty$ for all $i > 0$, then $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$.

Let X be a Stein space of dimension n in the lemma and $h^i(Y, \mathcal{O}_Y) < \infty$ for all $i > 0$. By Lemma 2.12, any reduced hypersurface Z defined by a holomorphic function satisfies $h^i(Z, \mathcal{O}_Z) < \infty$ for all $i > 0$. By inductive assumption, $H^i(Z, \mathcal{O}_Z) = 0$ for all $i > 0$. Using the long exact sequence, for every $f_j \in \{f_1, f_2, \ldots, f_m, \ldots\}$ we have infinitely many surjective maps

$$f_j^{*1} \colon H^1(Y, \mathcal{O}_Y) \to H^1(Y, \mathcal{O}_Y)$$

and isomorphisms

$$f_j^{*i} \colon H^i(Y, \mathcal{O}_Y) \to H^i(Y, \mathcal{O}_Y)$$

for $i > 1$. By counting the dimensions of the vector spaces, we see that for all $i > 0$, $H^i(Y, \mathcal{O}_Y) = 0$ because $\mathrm{End}_{\mathbb{C}}(H^i(Y, \mathcal{O}_Y))$ is a finite dimensional vector space over $\mathbb{C}$ for all $i > 0$. □

Lemma 2.14. *Let Y be an open subset of a reduced Stein space X of dimension n such that $H^i(Y, \mathcal{O}_Y)$ is a finite dimensional vector space over $\mathbb{C}$ for all $i > 0$, then the ring of holomorphic functions on Y is finitely generated: there are holomorphic functions $f_1, f_2, \ldots, f_m$ on Y such that they generate $H^0(Y, \mathcal{O}_Y)$.*

Proof. By Lemma 2.13, $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$. By Lemma 2.1 and proof of Lemma 2.4, $H^0(Y, \mathcal{O}_Y)$ is generated by finitely many holomorphic functions $f_1, f_2, \ldots, f_m$ on Y. □

Theorem 2.15. *Let Y be an open subset of a reduced Stein space X with dimension n. Then Y is Stein if and only if $H^i(Y, \mathcal{O}_Y)$ is a finite dimensional complex vector space for every $i > 0$.*

Proof. By Lemma 2.13, if $H^i(Y, \mathcal{O}_Y)$ is a finite dimensional complex vector space for every $i > 0$, then $H^i(Y, \mathcal{O}_Y) = 0$. By Theorem 2.5, Y is Stein. □

In Section 3, we will prove that Theorem 2.15 holds if the dimension of X is not finite.

3. Spaces with arbitrary dimension

We will first prove Theorem 1.2 in this section.

By Remmert's Proper Mapping Theorem (see [5, p. 213] or [15, Satz 23]), if $f\colon X \to Y$ is a proper holomorphic map, then the image of any analytic set in X is again analytic in Y. If f is not proper, this is not true.

Definition 3.1. A subset A of a complex space X is said to be analytically meagre if $A \subset \bigcup_{i\in\mathbb{N}} A_i$, where each A_i is a locally analytic subset of X with codimension at least 1.

An analytically meagre subset of a curve is a countable set [11]. If f is not proper, Remmert proved (see [11] or [15, Satz 20]).

Lemma 3.2 (Remmert)**.** *If $f\colon X \to Y$ is a holomorphic map between complex spaces and Z is an analytic subset of X, then $f(Z)$ is a countable union of locally analytic subsets of Y. In particular, if the interior of $f(Z)$ is empty, then $f(Z)$ is analytically meagre.*

An analytic subset Z in a complex space X is always nowhere dense in X if Z is at least 1 codimensional in X and Z contains interior points of X if Z contains an irreducible component of X [5, pp. 102–103].

Lemma 3.3 (Sard)**.** *If X is a complex manifold and $f\colon X \to \mathbb{C}$ is a holomorphic function, then there exists a countable subset $A \subset \mathbb{C}$ such that for each $c \in \mathbb{C} - A$, the fiber $X_c = f^{-1}(c)$ is a manifold.*

The following construction is inspired by Remmert and Sard's theorems.

Lemma 3.4. *Let Y be an open subset of a reduced Stein space X, then there is a holomorphic function h on X such that for any $a \in \mathbb{C} - A$, the hypersurface defined by $h - a$ on X is a complex manifold $H = H_1 \cup H_2 \cup \cdots$ of codimension 1 in X and for all $i \neq j$, $H_i \cap H_j = \emptyset$, where A is a countable subset in $\mathbb{C}$.*

Proof. Since X is reduced, the singular locus X_{sing} of X is a nowhere dense analytic subset of X (i.e., for every open subset U in X, $U \cap X_{\mathrm{sing}}$ is not dense in U) such that the local dimension at x: $\dim_x X_{\mathrm{sing}} < \dim_x X$ [5, p. 117]. Let $X = X_1 \cup X_2 \cup \cdots$ be the decomposition of X into irreducible components [4, p. 19]. The singular locus X_{sing} consists of all intersection points of $X_i \cap X_j$, $i \neq j$ and the singular points of each component X_i [5, p. 117].

First, we claim that there is a holomorphic function h on X such that it is not a constant on Y and $h(X_{\mathrm{sing}})$ is nowhere dense in $\mathbb{C}$. In fact, let C be a holomorphic curve in X such that $C \cap Y$ is an open subset of C and $C \cap X_{\mathrm{sing}}$ is empty (that is, C contains no singular points of X). Let $B = C \cup X_{\mathrm{sing}}$, then B is a closed subspace of X locally defined by holomorphic functions [5, p. 15]. Let $\mathcal{I}_B$ be the ideal sheaf of B, then we have a short exact sequence

$$0 \longrightarrow \mathcal{I}_B \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{C \cup X_{\mathrm{sing}}} \longrightarrow 0.$$

Since X is Stein and $\mathcal{I}_B$ is a coherent ideal sheaf, $H^1(X, \mathcal{I}_B) = 0$. We have a surjective map $H^0(X, \mathcal{O}_X) \twoheadrightarrow H^0(B, \mathcal{O}_{C \cup X_{\mathrm{sing}}})$. By the fact that C and X_{sing} are disconnected, we may construct a holomorphic function h on B such that h is not a constant on C and $h(X_{\mathrm{sing}})$ is nowhere dense in $\mathbb{C}$ (for example, we may choose h such that it is a constant on every connected component of X_{sing}).

By Lemma 3.2, $h(X_{\mathrm{sing}}) = A_1$ is a countable union of locally analytic subsets so is a countable subset of $\mathbb{C}$. For any $a \in \mathbb{C} - A_1$, the fiber $X_a = h^{-1}(a)$ has no intersection points with the singular locus X_{sing} of X. But the hypersurface $X_a \subset X - X_{\mathrm{sing}}$ may have singular points as a closed subspace of X. Now the restriction $h\colon X - X_{\mathrm{sing}} \to \mathbb{C}$ is a holomorphic function on the complex manifold $X - X_{\mathrm{sing}}$. By Sard's Theorem, there exists a countable subset $A_2 \subset \mathbb{C}$ such that for each $c \in \mathbb{C} - A_2$, the fiber

$$X_c \cap (X - X_{\mathrm{sing}}) = h^{-1}(c) \cap (X - X_{\mathrm{sing}}) \subset X - X_{\mathrm{sing}}$$

is a manifold. h may be a constant at some irreducible component of X. Since X has at most a countably many irreducible components [4, p. 19], there is a countable subset $A_3 \subset \mathbb{C}$ such that for every $a \in \mathbb{C} - A_3$, the fiber $X_a = h^{-1}(a)$ is of pure codimension 1 in X. Let $A = A_1 \cup A_2 \cup A_3$, then A is the union of three countable subsets so is a countable subset of $\mathbb{C}$. For all $a \in \mathbb{C} - A$, the fiber $X_a = h^{-1}(a)$ is of pure codimension 1 in X, smooth and can be decomposed into the union of disjoint complex manifolds. Therefore $H = X_a = H_1 \cup H_2 \cup \cdots$ is a smooth hypersurface in X, each H_i is irreducible and for all $i \neq j$, $H_i \cap H_j = \emptyset$. □

Lemma 3.5. *Let Y be an open subset of a reduced Stein space X such that $H^i(Y, \mathcal{O}_Y)$ is a finite dimensional vector space over $\mathbb{C}$ for all $i > 0$. In above lemma, for every irreducible* *component H_j of hypersurface H, let $Z = Y \cap H$ and $Z_j = Y \cap H_j$, then Z_j is an open subset in H_j and*

$$h^i(Z_j, \mathcal{O}_Z) < \infty.$$

Proof. We may assume that X contains no isolated points since X is Stein then every connected component of X is Stein [4, p. 125]. Let $Z = H \cap Y = Z_1 \cup Z_2 \cup \cdots$, where $Z_i = Y \cap H_i$ is either empty or an open subset in H_i (the subspace topology on H_i is induced from the topology on X since Y is open in X and H_i is a closed subspace of X).

If Z_j is an empty set or a set of points, then the inequality is true. We may assume that the dimension of Z_j is at least one. Let $Z = Z_j \cup Z'_j$, where $Z'_j = Z - Z_j$ is the complement of Z_j in Z. By the construction in Lemma 3.4, $Z_j \cap Z'_j$ is empty and by Mayer–Vietoris sequence [1, p. 30], we have

$$\begin{aligned}
0 &\longrightarrow H^0(Z, \mathcal{O}_Z) \longrightarrow H^0(Z_j, \mathcal{O}_Z) \oplus H^0(Z'_j, \mathcal{O}_Z) \longrightarrow H^0(Z_j \cap Z'_j, \mathcal{O}_Z) \\
&\longrightarrow H^1(Z, \mathcal{O}_Z) \longrightarrow H^1(Z_j, \mathcal{O}_Z) \oplus H^1(Z'_j, \mathcal{O}_Z) \longrightarrow H^1(Z_j \cap Z'_j, \mathcal{O}_Z) \\
&\longrightarrow H^2(Z, \mathcal{O}_Z) \longrightarrow H^2(Z_j, \mathcal{O}_Z) \oplus H^2(Z'_j, \mathcal{O}_Z) \longrightarrow H^2(Z_j \cap Z'_j, \mathcal{O}_Z) \longrightarrow \cdots .
\end{aligned}$$

Since $Z_j \cap Z'_j = \emptyset$, $H^i(Z_j \cap Z'_j, \mathcal{O}_Z) = 0$ for all $i \geq 0$, we have

$$H^i(Z, \mathcal{O}_Z) \cong H^i(Z_j, \mathcal{O}_Z) \oplus H^i(Z'_j, \mathcal{O}_Z).$$

By Lemma 2.12, for all $i > 0$,

$$h^i(Z, \mathcal{O}_Z) < \infty,$$

so $h^i(Z_j, \mathcal{O}_Z) < \infty$, and $h^i(Z'_j, \mathcal{O}_Z) < \infty$. □

Lemma 3.6. *In above lemma, for every irreducible component H_j of hypersurface H such that $Z_j = Y \cap H_j \neq \emptyset$, Z_j is a Stein subset in H_j.*

Proof. The hyersurface H in X is Stein [4, p. 125]. Since $H = H_1 \cup H_2 \cup \cdots$ and for all $i \neq j$, $H_i \cap H_j = \emptyset$, every irreducible (thus connected) component H_i is Stein [4, p. 125]. For each irreducible component H_i, its dimension is a constant [5, p. 169] even though the dimension of H may not be finite. By Lemma 3.5 and Theorem 2.15, the nonempty open subset Z_j in H_j is a Stein open subset in H_j. □

Lemma 3.7. *In above lemmas, the hypersurface $Z = H \cap Y$ in the open subset Y is holomorphically convex therefore is Stein.*

Proof. Let $S = \{P_1, P_2, \ldots, P_k, \ldots\}$ be a discrete sequence in $Z = Z_1 \cup Z_2 \cup \cdots$, where $Z_i = H_i \cap Y$. As in the proof of Theorem 2.3, we may assume that it has an accumulation point P_0 in X.

If there is an irreducible hypersurface $Z_j \subset H_j \subset H$ such that $Z_j \cap S \subset H_j \cap S$ contains infinitely many points of S, then there is a holomorphic function f on Z_j such that f is not bounded on $Z_j \cap S$. By Mayer–Vietoris sequence,

$$H^i(Z, \mathcal{O}_Z) \cong H^i(Z_j, \mathcal{O}_Z) \oplus H^i(Z'_j, \mathcal{O}_Z),$$

we can extend f to the complement Z'_j of Z_j in Z by zero since Zj and Z'_j are disconnected. In this way, we receive a holomorphic function f on Z such that it is not bounded on S.

Now we assume that every nonempty component Z_i only contains finitely many points of S and S has an accumulation point P_0 in X. Choose a subsequence $\{P_{n_i}\}_{i=1}^{\infty}$ in S such that $P_0 \in X - Z$ is its limit point. Let the holomorphic function h define the hypersurface H in X. Since $h(P_i) = 0$ for all i, we have $h(P_0) = 0$. This implies that P_0 is a point on some irreducible component H_k of H. By Lasker–Noether Decomposition Theorem, there is an open subset $U \ni P_0$ in X such that in U, H has only finitely many components: $H \cap U = H_{i_1} \cup H_{i_2} \cup \cdots \cup H_{i_m}$ [5, pp. 78–79]. But each irreducible component in $H \cap U$ contains only finitely many points of S, P_0 cannot be an accumulation point of S. The contradiction implies that if $S = \{P_1, P_2, \ldots, P_k, \ldots\} \subset Z$ has an accumulation point in X, then there is a component H_j such that $H_j \cap S$ is not a finite set. By the above proof, we show that there is a holomorphic function f on Z such that it is not bounded on S. So the hypersurface Z in the open subset Y is holomorphically convex. □

Lemma 3.8. *Let Y be an open subset of a reduced Stein space X such that $H^i(Y, \mathcal{O}_Y)$ is a finite dimensional vector space over $\mathbb{C}$ for all $i > 0$, Then*

$$H^i(Y, \mathcal{O}_Y) = 0.$$

Proof. By the construction in Lemmas 2.11 and 3.4, let $f_1, f_2, \ldots, f_m, \ldots$ be holomorphic functions on X such that $1, f_1, f_2, \ldots, f_m, \ldots$ are linearly independent in $H^0(Y, \mathcal{O}_Y)$ and for every i, each image $f_i(X_{\mathrm{sing}})$ in $\mathbb{C}$ is nowhere dense. By Lemma 3.4, choose $a_j \in \mathbb{C}$ such that each fiber $X_{a_j} = f_i^{-1}(a_j)$ defines a pure codimension 1 complex manifold X_{a_j} in X. By Lemma 2.9, $1, f_1 - a_1, f_2 - a_2, \ldots, f_m - a_m, \ldots$ are linearly independent in $H^0(Y, \mathcal{O}_Y)$. By Lemmas 3.4–3.7, each $Y_{a_j} = Y \cap X_{a_j}$ is a smooth Stein hypersurface on Y, so $h^i(Y_{a_j}, \mathcal{O}_{Y_{a_j}}) = 0$ for all $i > 0$. Using the idea of the proof of Lemma 2.13, multiplicating by each $f_j - a_j$ from $\mathcal{O}_Y$ to itself for all $j \in \mathbb{N}$, we have infinitely many surjective $\mathbb{C}$-homomorphisms $(f_j - a_j)^{*1}$ of a finite dimensional vector space $H^1(Y, \mathcal{O}_Y)$

$$(f_j - a_j)^{*1} \colon H^1(Y, \mathcal{O}_Y) \to H^1(Y, \mathcal{O}_Y)$$

and infinitely many $\mathbb{C}$-isomorphisms

$$(f_j - a_j)^{*i} \colon H^i(Y, \mathcal{O}_Y) \to H^i(Y, \mathcal{O}_Y),$$

which are induced by each $f_j - a_j \in \{f_1 - a_1, f_2 - a_2, \ldots, f_m - a_m, \ldots\}$ for $i > 1$ [3]. Comparing the dimensions of vector spaces, for all $i > 0$, we have $H^i(Y, \mathcal{O}_Y) = 0$. $\square$

Theorem 3.9. *Let Y be an open subset of a reduced Stein space X such that $H^i(Y, \mathcal{O}_Y)$ is a finite dimensional vector space over $\mathbb{C}$ for all $i > 0$. Then Y is holomorphically convex therefore is Stein.*

Proof. Let $S = \{P_1, P_2, \ldots, P_k, \ldots\}$ be a discrete sequence in Y and P_0 be its accumulation in X. Since X is Stein, X is holomorphically spreadable, that is, there exist finitely many holomorphic functions $f_1, f_2, \ldots, f_m$ on X such that P_0 is an isolated point in the zero set $A = \{x \in X, f_1(x) = f_2(x) = \cdots = f_m(x) = 0\}$ [8, pp. 293–294]. We can write $A = B \cup \{P_0\}$ then $B \cap \{P_0\}$ is an empty set. Let $\mathcal{I}_A$ be the ideal generated by $f_1, f_2, \ldots, f_m$ in X, then we have a short exact sequence

$$0 \longrightarrow \mathcal{I}_A \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathcal{I}_A = \mathcal{O}_A \longrightarrow 0.$$

The ideal sheaf $\mathcal{I}_A$ is coherent on the Stein space X [5, p. 84]. The long exact sequence and $H^1(X, \mathcal{I}_A) = 0$ give

$$0 \longrightarrow H^0(X, \mathcal{I}_A) \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow H^0(A, \mathcal{O}_A) \longrightarrow 0.$$

Let $f_{m+1} \in H^0(A, \mathcal{O}_A)$ such that $f_{m+1}(P_0) = 0$ and $B \cap \{x \in X, f_{m+1}(x) = 0\} = \emptyset$. Then there is a holomorphic function (still denoted by f_{m+1}) on X such that it vanishes at P_0 and does not vanish at every point of B.

Now $f_1, f_2, \ldots, f_m, f_{m+1}$ are holomorphic on X and have a unique common zero P_0 on X. They have no common zeros on Y. By Lemma 3.8, for all $i > 0$, $H^i(Y, \mathcal{O}_Y) = 0$. By Theorem 2.3, Y is Stein. $\square$

We have proved

Theorem 3.10. *Let Y be an open subset of a reduced Stein space X with arbitrary dimension and singularities. Then Y is Stein if and only if $H^i(Y, \mathcal{O}_Y)$ is a finite dimensional vector space over $\mathbb{C}$ for all $i > 0$.*

Next we will prove Theorem 1.4.

Definition 3.11. A Weil divisor on a reduced complex space X is a locally finite linear combination with integral coefficients of irreducible reduced analytic subspaces of codimension 1 in X such that every subspace is not contained in the singular locus of X.

The set of all Weil divisors form an abelian group. If D is a Weil divisor, then we can write $D = \sum_{i=1}^{\infty} n_i D_i$, where $n_i \in \mathbb{Z}$ and each D_i is an irreducible reduced analytic subspace of codimension 1 in X which is not contained in the singular locus of X (see [2], [4, pp. 139–140], [6, pp. 130–143], or [20, pp. 35–36]).

The support of a Weil divisor D is the union of all closed subspaces D_i such that $n_i \neq 0$. D is an effective divisor, written $D > 0$, if every coefficient $n_i \geq 0$ and D is not a zero divisor. Two Weil divisors $D \geq D'$ if $D - D' \geq 0$, i.e., $D - D'$ is an effective divisor or a zero divisor in the space. When every coefficient $n_i = 1$, $D = \sum D_i$ is called a reduced divisor.

A reduced point $x \in X$ is a normal point of X if the stalk $\mathcal{O}_x$ is integrally closed in its quotient ring. A reduced complex space is normal if every point in the space is a normal point [5, p. 8]. If X is a compact normal reduced complex space, then a Weil divisor D is a finite sum on X: $D = \sum_{i=1}^{N} n_i D_i$ [20, p. 35].

If X is normal, then the singular locus of X is a closed subspace of codimension at least 2 in X [5, p. 128]. A Weil divisor is well-defined as a linear combination of irreducible codimension one closed subspaces on a normal complex space X.

A Cartier divisor D on a complex space X is a global section of the sheaf $\mathcal{M}_X^*/\mathcal{O}_X^*$, where $\mathcal{M}_X^*$ is the sheaf of germs of not identically vanishing meromorphic functions on X and $\mathcal{O}_X^*$ is the sheaf of germs of nowhere vanishing holomorphic functions on X. A Cartier divisor D on a complex space X can be described by an appropriate open cover $\{U_i\}_{i\in I}$ of X and a collection of meromorphic functions f_i on U_i, $i \in I$ such that on $U_i \cap U_j \neq \emptyset$, $\frac{f_i}{f_j}$ and $\frac{f_j}{f_i}$ are holomorphic (see [4, p. 138] or [20, p. 30]). D is an effective Cartier divisor, written $D > 0$, if every f_i is a holomorphic function and at least one of them has zeros [20, p. 31].

Every Cartier divisor on a normal reduced complex space X defines a Weil divisor and if X is nonsingular, then every Weil divisor is Cartier, i.e., locally it is defined by one equation. But if X is not a complex manifold, then the Weil divisor D is not a Cartier divisor in general, i.e., it is not locally defined by one equation [20, p. 36].

A Weil divisor D is $\mathbb{Q}$-Cartier if there is an $n \in \mathbb{N}$ such that nD is a Cartier divisor, i.e., nD is locally defined by one equation.

Example 3.12. Let $X \in \mathbb{C}^4$ be a quadric threefold defined by

$$X = \{z = (z_1, z_2, z_3, z_4) \in \mathbb{A}_k^4, p(z) = z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\}.$$

The structure sheaf

$$\mathcal{O}_X = \mathcal{O}_{\mathbb{C}^4}/p(z)\mathcal{O}_{\mathbb{C}^4}.$$

X is a normal Stein variety with a unique isolated singularity at 0. Let H be a hypersurface through 0 defined by

$$H = \{z = (z_1, z_2, z_3, z_4) \in X, z_1 = iz_2, z_3 = iz_4\}.$$

H cannot be defined by a single holomorphic function and $X - H$ is not Stein [4, p. 130].

Example 3.12 shows that the open subset $Y = X - A$ in a normal Stein space X obtained by removing a pure codimension 1 subspace A of X is not Stein in general if the dimension of X is at least 3. We give a sufficient condition:

Theorem 3.13. *If Y is an open subset of a normal Stein space X such that the complement $X - Y$ is a closed analytic subspace of X and the support of an effective $\mathbb{Q}$-Cartier divisor, then Y is Stein.*

Proof. Let D' be the effective $\mathbb{Q}$-Cartier divisor with support $X - Y$ on X. Then there is an $n \in \mathbb{N}$ such that $D = nD'$ is an effective Cartier divisor with support $X - Y$ on X [20, pp. 36–38].

Let $\{U_i\}_{i\in I}$ be a Stein open cover of X and let f_i be the holomorphic function on U_i defining $D|_{U_i}$. Then for every point $x \in U_i$, the stalk of the invertible sheaf (coherent) $\mathcal{O}_X(D)$ is defined by [20, p. 30]

$$\mathcal{O}_X(D)_x = \frac{1}{f_i}\mathcal{O}_x \cong \mathcal{O}_x.$$

Let $S = \{P_1, P_2, \ldots\} \subset Y$ be a discrete sequence on Y with an accumulation point $P_0 \in (X - Y) \cap U_i$ for some $i \in I$. Since $\mathcal{O}_X(D)$ is a coherent sheaf on X, by Cartan's Theorem A [4, p. 124], the module of global sections $H^0(X, \mathcal{O}_X(D))$ generates every stalk $\mathcal{O}_X(D)_x$. There is a meromorphic function $f \in H^0(X, \mathcal{O}_X(D))$ (holomorphic on Y with poles in $X - Y$) and a local holomorphic function $g \in \mathcal{O}_{P_0}$ such that near P_0 [4, p. 129],

$$\frac{1}{f_i} = fg.$$

Now f_i is a holomorphic function on $U_i \cap Y$ and vanishes at P_0. So $f(f_i g) = 1$ near P_0 in Y. From $f_i(P_0) = 0$, we see that f is not bounded near P_0 on the sequence S in Y. We show that Y is holomorphically convex therefore is Stein. □

Remark 3.14. A Stein open subset of an algebraic affine variety is not an algebraic affine variety in general. For example, let $X = \mathbb{C}^n$, let Z be the closed analytic subvariety of X defined by $f(z) = \sin z_1$, where $(z_1, z_2, \ldots, z_n)$ are coordinates in $\mathbb{C}^n$. Then $Y = X - Z$ is Stein but not an algebraic variety.

Surprisingly, Neeman constructed an example: there is a scheme U of finite type over $\mathbb{C}$ such that U is a Zariski open subset of an affine scheme and the associated analytic complex space U' of U is a Stein space, but U is not an affine scheme [14].

Theorem 3.15. *If Y is an open subset of a Stein space X such that the complement $X - Y$ is a closed analytic subspace of X with pure codimension 1 and $X - Y$ does not contain any singular points of X, then Y is Stein.*

Proof. By Reduction Theorem [4, p. 154], X is Stein if and only if its reduction is Stein. The normalization of a reduced complex space is a finite surjective holomorphic map [4, p. 22]. So a complex space is Stein if and only if its normalization is Stein [8, p. 313, Prop. 73.1]. The normalization $\widetilde{X}$ of X is a disjoint union of irreducible components and it is Stein if and only if every irreducible component is Stein [8, p. 308, Cor. 71.14]. Therefore we may assume that X is an irreducible normal (reduced) Stein space.

Let X_{sing} be the set of singular points of X. Then X_{sing} is of codimension at least 2 in X [5, p. 128], and $X - Y \subset X - X_{\text{sing}}$ is a closed subspace of pure codimension 1 in the complex manifold $X - X_{\text{sing}}$. Since every point in $X - Y$ is smooth in X, $(X - Y) \cap X_{\text{sing}}$ is an empty set. So $X - Y$ is support of an effective Cartier divisor D in the complex manifold $X - X_{\text{sing}}$ [20, p. 36].

Let $\{(U_i, f_i)\}_{i \in I}$ be a representive of D in the complex manifold $X - X_{\text{sing}}$, where $\{U_i\}_{i \in I}$ is a Stein open cover of the complex manifold $X - X_{\text{sing}}$, each f_i is a holomorphic function on U_i, at least one f_i has zeros, and f_i/f_j is a holomorphic function on $U_i \cap U_j$ for all $i, j \in I$.

Let $\{V_j\}_{j \in J}$ be a Stein open cover of X_{sing} in Y: $X_{\text{sing}} \subset \cup_j V_j \subset Y$. On each $V_j \cap U_i \neq \emptyset$, $f_i|_{V_j \cap U_i}$ is nowhere zero. In particular, on every $V_j - V_j \cap X_{\text{sing}}$, we have [20, p. 36, Thm. 4.13]

$$\mathcal{O}_{V_j - V_j \cap X_{\text{sing}}}(D) \cong \mathcal{O}_{V_j - V_j \cap X_{\text{sing}}}.$$

Now the codimension of $V_j \cap X_{\text{sing}}$ is at least 2 in V_j, therefore the invertible sheaf $\mathcal{O}_{V_j - V_j \cap X_{\text{sing}}}(D)$ can be extended to V_j uniquely [9]. This implies that we have an invertible sheaf $\mathcal{O}_X(D)$ on X, i.e., D is an effective Cartier divisor on X [6, p. 144, Prop. 6.13]. By Theorem 3.13, Y is Stein. □

Corollary 3.16. *If Y is an open subset of a Stein manifold X such that the complement $X - Y$ is a closed subspace of X with pure codimension 1, then Y is Stein.*

References

[1] D. Arapura, *Complex Algebraic Varieties and their Cohomology*, Lecture Notes, 2003, https://www.math.purdue.edu/∼arapura/preprints/sheaves.pdf.

[2] E. Artal Bartolo, J. Martín-Morales and J. Ortigas-Galindo, *Cartier and Weil divisors on varieties with quotient singularities*, Internat. J. Math. **25** (2014), no. 11, 1450100, 20 pp.

[3] J. Goodman and R. Hartshorne, *Schemes with finite-dimensional cohomology groups*, Amer. J. Math. **91** (1969), 258–266.

[4] H. Grauert and R. Remmert, *Theory of Stein Spaces*, Grundlehren der Mathematischen Wissenschaften **236**, Springer-Verlag, Berlin, 1979.

[5] ———, *Coherent Analytic Sheaves*, Grundlehren der Mathematischen Wissenschaften **265**, Springer-Verlag, Berlin, 1984.

[6] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathmatics **52**, Springer-Verlag, New York, 1977.

[7] L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, Second revised edition, North-Holland Mathematical Library **7**, North-Holland Publishing Co., Amsterdam, 1973.

[8] L. Kaup and B. Kaup, *Holomorphic Functions of Several Variables: An introduction to the fundamental theory*, De Gruyter Studies in Mathematics **3**, Walter de Gruyter & Co., Berlin, 1983.

[9] J. Kollár, *The structure of algebraic threefolds: An introduction to Mori's program*, Bull. Amer. Math. Soc. (N.S.) **17** (1987), no. 2, 211–273.

[10] H. B. Laufer, *On sheaf cohomology and envelopes of holomorphy*, Ann. of Math. (2) **84** (1966), 102–118.

[11] M. Manaresi, *Sard and Bertini type theorems for complex spaces*, Ann. Mat. Pura Appl. (4) **131** (1982), 265–279.

[12] A. Nagel, *Cohomology, maximal ideals, and point evaluations*, Proc. Amer. Math. Soc. **42** (1974), 47–50.

[13] R. Narasimhan, *Imbedding of holomorphically complete complex spaces*, Amer. J. Math. **82** (1960), 917–934.

[14] A. Neeman, *Steins, affines and Hilbert's fourteenth problem*, Ann. of Math. (2) **127** (1988), no. 2, 229–244.

[15] R. Remmert, *Holomorphe und meromorphe Abbildungen komplexer Räume*, Math. Ann. **133** (1957), 328–370.

[16] O. Riemenschneider, *Singular Points of Complex Analytic Surface: An introduction to the local analysis of complex analytic spaces*, Chapter 19, Supplement: A short introduction to complex analytic spaces. Retrieved from https://www.math.uni-hamburg.de/home/riemenschneider/bookchap30.pdf

[17] J.-P. Serre, *Quelques problèmes globaux relatifs aux variétés de Stein*, in: *Colloque sur les fonctions de plusieurs variables, tenu à Bruxelles*, 57–68, Georges Thone, Liège, Paris, 1953.

[18] R. R. Simha, *On the complement of a curve on a Stein space of dimension two*, Math. Z. **82** (1963), 63–66.

[19] Y.-t. Siu, *Analytic sheaf cohomology groups of dimension n of n-dimensional complex spaces*, Trans. Amer. Math. Soc. **143** (1969), 77–94.

[20] K. Ueno, *Classification Theory of Algebraic Varieties and Compact Complex Spaces*, Lecture Notes in Mathematics **439**, Springer-Verlag, Berlin, 1975.

[21] V. Vâjâitu, *A cohomological Steinness criterion for holomorphically spreadable complex spaces*, Czechoslovak Math. J. **60** (2010), no. 3, 655–667.

[22] J. Zhang, *Threefolds with vanishing Hodge cohomology*, Trans. Amer. Math. Soc. **357** (2005), no. 5, 1977–1994.

[23] ______, *On the existence of nontrivial 3-folds with vanishing Hodge cohomology*, Michigan Math. J. **54** (2006), no. 2, 447–467.

[24] ______, *Complex manifolds with vanishing Hodge cohomology*, Houston J. Math. **43** (2017), no. 3, 807–827.

Jing Zhang
1 University Parkway, Department of Mathematics, Division of Science, Mathematics and Technology, College of Arts and Sciences, Governors State University, University Park, IL 60484, USA
E-mail address: `jzhang@govst.edu`