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In 2007 all ECB publications feature a motif taken from the €20 banknote.

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Abstract

This paper shows how large-dimensional dynamic factor models are suitable for structural analysis. We establish sufficient conditions for identification of the structural shocks and the associated impulse-response functions. In particular, we argue that, if the data follow an approximate factor structure, the "problem of fundamentalness", which is intractable in structural VARs, can be solved provided that the impulse responses are sufficiently heterogeneous. Finally, we propose a consistent method (and n, T rates of convergence) to estimate the impulse-response functions, as well as a bootstrapping procedure for statistical inference.

JEL subject classification : E0, C1

Key words and phrases : Dynamic factor models, structural VARs, identification, fundamentalness

Non Technical Summary

Agents and policy makers have access to rich information, coming from data on different sectors of the economy. However, standard macro time series models are typically based on few selected variables. Recent econometric literature has introduced models that can exploit large data-sets and still retain simplicity (parsimony). These models - known in the literature as dynamics factor models - are based on the idea that the macroeconomy is driven by few shocks, common to all variables. Since a robust empirical characteristics of macroeconomic time series is that they exhibit strong co-movements, common shocks generate the bulk of the observed dynamics in macro variables.

Dynamic factor models have been shown to be successful to forecast macroeconomic variables, but only few applications have considered these models for identifying and estimating structural shocks, as, for example, it is done in the VAR literature.

The aim of this paper is to develop the estimation and identification theory needed to study structural shocks and their impulse response functions in dynamic factor models.

The analysis of the paper and the empirical application we present show that dynamic factor models are suitable for structural macroeconomic modelling and constitute an interesting alternative to structural VARs. In particular, if the information used by economic agents cannot be captured by the small set of variables considered in a typical VAR, an econometric model based on large information can recover the structural shocks while the small VAR cannot. The factor model framework is also useful when the aim is to study the effect of macro-shocks on many variables in the economy, possibly sectoral and regional, rather than studying the effect of these shocks to core macro variables only.

1 Introduction

Recent literature has shown that large-dimensional approximate (or generalized) dynamic factor models can be used successfully to forecast macroeconomic variables (Forni, Hallin, Lippi and Reichlin, 2005, Stock and Watson, 2002a, 2002b, Boivin and Ng, 2003, Giannone, Reichlin and Sala, 2005). These models assume that each time series in the dataset can be expressed as the sum of two orthogonal components: the "common component", capturing that part of the series which comove with the rest of the economy and the "idiosyncratic component" which is the residual. The vector of the common components is highly singular, i.e. is driven by a very small number (as compared to the number of variables) of shocks (the "common shocks" or "common factors") which generate comovements between macro series. Indeed, evidence based on different datasets points to the robust finding that few shocks explain the bulk of dynamics of macro data (see Sargent and Sims, 1977 and Giannone, Reichlin and Sala, 2002 and 2005). If the common component of the variable to be predicted is large, a forecasting method based on a projection on linear combinations of these shocks performs well because, while being parsimonious, it captures the relevant comovements in the economy.

The present paper argues that the scope of dynamic factor models goes beyond forecasting. Our aim is to open the black box of these models and show how statistical constructs such as factors can be related to macroeconomic shocks and their propagation mechanisms.

We define *macroeconomic* shocks those structural sources of variation that are cross-sectionally pervasive, i.e. that significantly affect most of the variables of the economy, while we call *idiosyncratic* the shocks that are specific to a single variable or a small group of variables, hence capturing either sectoral-local dynamics (let us say "micro" dynamics) or measurement error. This has a natural formalization within large-dimensional approximate factor models. More precisely, we assume that a q-dimensional vector of macroeconomic shocks drives the common components of a macroeconomic panel $\boldsymbol{x}_t$ of size n, with n very large with respect to q. Our aim is the identification of the macroeconomic shocks and of the impulse response function of the common components of the x's to $\boldsymbol{u}_t$, whereas the idiosyncratic components are disregarded.

Firstly, we claim that ideas and methods of structural VAR analysis can be fruitfully imported in dynamic factor models. We start with the estimate of an autoregression of the common-components vector. Thus an autoregression of dimension n, the size of the panel, with a residual vector of dimension q, the number of factors. Calling $\boldsymbol{v}_t$ the estimated residual vector, the vector of structural shocks, call it $\boldsymbol{u}_t$, is then obtained as in structural VAR analysis (SVAR) by linearly transforming $\boldsymbol{v}_t$ in order to fulfill restrictions that derive from economic theory. All the identification schemes proposed in the SVAR literature, such as long-run or impact effects can be imposed. The key difference is that the number of shocks is smaller than the number of variables.

Secondly, we show that the fundamentalness problem, a weakness of VAR analysis, finds a satisfactory solution within our approach. Let us recall that in SVAR analysis, even when economic theory is sufficient to determine just one linear transformation of the estimated residuals, still identification is achieved by arbitrarily assuming that the structural shocks are fundamental with respect to the variables included in the model, i.e. that they can be obtained as linear combinations of present and past values of such variables. This assumption cannot hold true if economic agents have larger information (on the fundamentalness issue see Hansen and Sargent, 1991, Lippi and Reichlin, 1993 and 1994 and, more recently, Chari, Kehoe and Mcgrattan, 2005, Fernandez-Villaverde, Rubio-Ramirez and Sargent, 2005, Giannone and Reichlin, 2006).

The fundamentalness problem depends on a somewhat artificial feature of the SVAR approach, namely that the number of variables used to estimate the structural vector $\boldsymbol{u}_t$ must be equal to the dimension of $\boldsymbol{u}_t$, so that the space spanned by present an past values of $\boldsymbol{x}_t$ can be "too small" to recover $\boldsymbol{u}_t$. This equal-dimension constraint is relaxed in the structural dynamic factor model proposed in this paper. We will argue that when the number of variables is large as compared to the number of structural shocks, non fundamentalness of the structural shocks is unlikely, since it would require economically meaningless homogeneity restrictions on the impulse-response functions. The economic intuition of this claim is that in the factor model present and past information used to recover $\boldsymbol{u}_t$ is not confined to q variables, as in VAR models, but ranges over the set of all available macroeconomic series, so that the "superior information" argument no longer holds (on the importance of this feature for monetary models, see Bernanke and Boivin, 2003 and Giannone, Reichlin and Sala, 2002 and 2005).

Our work is closely related to the recently introduced FAVAR model (Bernanke, Boivin and Eliasz, 2005). The FAVAR approach consists in augmenting the VAR by common factors precisely as a device to condition on a larger information set. We go one step further and give the factors themselves a structural interpretation.

The factor model employed here should be distinguished from what studied in the traditional factor literature (see Sargent and Sims, 1977, Geweke, 1977, Geweke and Singleton, 1981, Altug, 1989, Sargent, 1989, Giannone, Reichlin and Sala, 2006). Since our model is approximate and feasible for large panels we need less stringent assumptions to identify the common from the idiosyncratic component (we do not need to impose cross-sectional orthogonality of the idiosyncratic residuals).

The paper is organized as follows. In Section 2, we define the model and discuss the conditions needed to recover the common components from the panel. Section 3 develops the structural analysis by showing conditions needed for recovering fundamental shocks and identify them uniquely. Section 4 studies consistency and rates of convergence for the estimation of the shocks and the impulse response functions. Section 5 analyses an empirical example on US macroeconomic data which revisits the results of King et al. (1991) in light of our discussion on fundamentalness.

2 The Model

The dynamic factor model used in this paper is a special case of the generalized dynamic factor model of Forni, Hallin, Lippi and Reichlin (2000) and Forni and Lippi (2001). Such model, and the one used here, differs from the traditional dynamic factor model of Sargent and Sims (1977) and Geweke (1977), in that the number of cross-sectional variables is infinite and the idiosyncratic components are allowed to be mutually correlated to some extent, along the lines of Chamberlain (1983), Chamberlain and Rothschild (1983) and Connor and Korajczyk (1988). Closely related models have been recently studied by Stock and Watson (2002a, 2002b), Bai and Ng (2002) and Bai (2003).

Denote by $\boldsymbol{x}_n^T = (x_{it})_{i=1,\dots,n;\,t=1,\dots,T}$ an $n \times T$ rectangular array of observations. We make two preliminary assumptions:

PA1. $\boldsymbol{x}_n^T$ is a finite realization of a real-valued stochastic process

$$\boldsymbol{X} = \{x_{it}, i \in \mathbb{N}, t \in \mathbb{Z}\,,\; x_{it} \in L_2(\Omega, \mathcal{F}, P)\}$$

indexed by $\mathbb{N} \times \mathbb{Z}$, where the n-dimensional vector processes

$$\{\boldsymbol{x}_{nt} = (x_{1t}\ \cdots\ x_{nt})',\ t \in \mathbb{Z}\}, \quad n \in \mathbb{N},$$

are stationary, with zero mean and finite second-order moments $\boldsymbol{\Gamma}_{nk} = \mathrm{E}[\boldsymbol{x}_{nt}\boldsymbol{x}'_{n,t-k}]$, $k \in \mathbb{N}$.

PA2. For all $n \in \mathbb{N}$, the process $\{\boldsymbol{x}_{nt}, t \in \mathbb{Z}\}$ admits a Wold representation $\boldsymbol{x}_{nt} = \sum_{k=0}^{\infty} C_k^n \boldsymbol{w}_{n,t-k}$, where the full-rank innovations $\boldsymbol{w}_{nt}$ have finite moments of order four, and the matrices $C_k^n = (C_{ij,k}^n)$ satisfy $\sum_{k=0}^{\infty} |C_{ij,k}^n| < \infty$ for all $n, i, j \in \mathbb{N}$.

We assume that each variable x_{it} is the sum of two unobservable components, the *common component* χ_{it} and the *idiosyncratic component* ξ_{it}. The common component is driven by q *common shocks* $\boldsymbol{u}_t = (u_{1t}\ u_{2t}\ \cdots\ u_{qt})'$. Note that q is independent of n (and small as compared to n in empirical applications). More precisely:

FM0. (Dynamic-factor structure of the model) Defining $\boldsymbol{\chi}_{nt} = (\chi_{1t}\ \dots\ \chi_{nt})'$ and $\boldsymbol{\xi}_{nt} = (\xi_{1t}\ \dots\ \xi_{nt})'$, we suppose that

$$\begin{aligned} \boldsymbol{x}_{nt} &= \boldsymbol{\chi}_{nt} + \boldsymbol{\xi}_{nt} \\ &= B_n(L)\boldsymbol{u}_t + \boldsymbol{\xi}_{nt}, \end{aligned} \tag{2.1}$$

where $\boldsymbol{u}_t$ is a q-dimensional orthonormal white noise vector.

Moreover, we assume that

$$B_n(L) = A_n N(L), \tag{2.2}$$

where (i) $N(L)$ is an $r \times q$ absolutely summable matrix function of L, (ii) A_n is an $n \times r$ matrix, nested in A_m for $m > n$. Defining the $r \times 1$ vector $\boldsymbol{f}_t$ as

$$\boldsymbol{f}_t = N(L)\boldsymbol{u}_t, \tag{2.3}$$

(2.1) can be rewritten in the static form

$$\boldsymbol{x}_{nt} = A_n \boldsymbol{f}_t + \boldsymbol{\xi}_{nt} \tag{2.4}$$

In the sequel, we shall use the term *static factors* to denote the r entries of $\boldsymbol{f}_t$, whereas the common shocks $\boldsymbol{u}_t$ will be also referred to as *dynamic factors*.

Note that under (2.2) all the variables χ_{it}, $i = 1, \dots, \infty$, belong to the finite dimensional vector space spanned by $\boldsymbol{f}_t$.

The common shocks $\boldsymbol{u}_t$ are assumed to be *structural* sources of variation. Therefore the model (2.1), (2.3), (2.4) is a *structural factor model*. We will establish conditions under which $\boldsymbol{u}_t$ can be identified and estimated by means of the observable variables x_{it}. We start in this section by recalling the assumptions necessary for identification and estimation of the common components χ_{it}.

FM1. (Orthogonality of common and idiosyncratic components) $\boldsymbol{u}_t$ is orthogonal to $\xi_{i\tau}$, $i \in \mathbb{N}$, $t \in \mathbb{Z}$, $\tau \in \mathbb{Z}$.

Indicate by Γ^{χ}_{nk} and Γ^{ξ}_{nk} the k-lag covariance matrix of $\boldsymbol{\chi}_{nt}$ and $\boldsymbol{\xi}_{nt}$ respectively. Denote by μ^{χ}_{nj} and μ^{ξ}_{nj} the j-th eigenvalue, in decreasing order, of Γ^{χ}_{n0} and Γ^{ξ}_{n0} respectively.

FM2. (Pervasiveness of common dynamic and static factors)

(a) The matrix $N(e^{-i\theta})$ has (maximum) rank q for θ almost everywhere in $[-\pi \;\pi]$.

(b) There exists constants $\underline{c}_1, \overline{c}_1, \dots, \underline{c}_r, \overline{c}_r$ such that

$$0 < \underline{c}_r \leq \liminf_{n\to\infty} n^{-1}\mu^{\chi}_{nr} \leq \overline{c}_r < \dots < \underline{c}_1 \leq \liminf_{n\to\infty} n^{-1}\mu^{\chi}_{n1} \leq \overline{c}_1 < \infty$$

FM3. (Non-pervasiveness of the idiosyncratic components) There exists a real Λ such that $\mu^{\xi}_{n1} \leq \Lambda$ for any $n \in \mathbb{N}$.

FM3 limits the cross-correlation generated by the idiosyncratic shock. It includes the case in which the idiosyncratic components are mutually orthogonal with an upper bound for the variances. Mutual orthogonality is a standard, though highly unrealistic assumption in factor models. Condition FM3 relaxes such assumption by allowing for a limited amount of cross-correlation among the idiosyncratic components.

Assumption FM2 implies that each common shock u_{it} is pervasive in the sense that it affects all items of the cross-section as n increases. Precisely, denoting by $\lambda^{\chi}_{nk}(\theta)$, $k = 1, 2, \ldots, n$, the eigenvalues of the spectral density matrix $\Sigma^{\chi}_{n}(\theta)$, in decreasing order at each frequency, Assumption FM2 implies that $\lambda^{\chi}_{nq}(\theta) \to \infty$ as $n \to \infty$, for θ a.e. in $[-\pi\ \pi]$. This implies that (I) the common components χ_{it} are identified (see Chamberlain and Rothschild, 1983), (II) the number q is unique, i.e. a representation (2.1)-(2.4) with a different number of dynamic factors is not possible (see Forni and Lippi, 2001).

Note also that FM2(b) entails that, for n sufficiently large, $A'_n A_n/n$ has full rank r. This, jointly with identification of the common components χ_{it}, implies that the space spanned by the r static factors $\boldsymbol{f}_t$ is identified, or, equivalently, that the r static factors $\boldsymbol{f}_t$ are identified up to a linear contemporaneous transformation.

In conclusion, given a model of the form (2.1)-(2.4), then under FM0-FM3, the integers q and r, the components χ_{it} and ξ_{it}, and the space spanned by the static factors $\boldsymbol{f}_t$ are identified.

The following *rational specification* of model (2.1)-(2.4) provides a dynamic representation which is parsimonious and fairly general. Assume that the entries of $B_n(L)$ are rational functions and let $\phi_{jn}(L)$, $j = 1, \ldots, q$, be the least common multiple of the denominators of the entries on the j-th column of $B_n(L)$. Elementary polynomial and matrix algebra shows that

$$B_n(L) = C_n(L)\Psi_n(L),$$

where $C_n(L)$ is a finite moving average $n \times q$ matrix and $\Psi_n(L)$ is the $q \times q$ diagonal matrix having

$$\begin{pmatrix} \phi_{1n}(L)^{-1} & \phi_{2n}(L)^{-1} & \cdots & \phi_{qn}(L)^{-1} \end{pmatrix}$$

on the main diagonal. Further assumptions are needed to ensure that all the variables χ_{it} belong to a finite dimensional vector space. These are:
(a) $C_n(L) = C^n_0 + C^n_1 L + \cdots + C^n_s L^s$, i.e. there exists a maximum for the length of the moving averages,
(b) $\Psi_n(L)$ is independent of n and can therefore be denoted by $\Psi(L)$, with $\phi_j(L)^{-1}$ denoting its $(j,\ j)$ entry.
The rational specification of our model can then be written as

$$\boldsymbol{x}_{nt} = C_n(L)\Psi(L)\boldsymbol{u}_t + \boldsymbol{\xi}_{nt}.^1 \tag{2.5}$$

Model (2.5) can be tentatively put in the form (2.3)-(2.4) by setting $r = q(s+1)$, $A_n = (C_0^n \ C_1^n \ \cdots \ C_s^n)$, $\boldsymbol{f}_t = (\boldsymbol{u}_t' \ \boldsymbol{u}_{t-1}' \ \cdots \ \boldsymbol{u}_{t-s}')'$ and

$$N(L) = (\Psi(L)' \ \ \Psi(L)'L \ \cdots \ \Psi(L)'L^s)'.$$

FM2(a) is trivially fulfilled. However, FM2(b) requires that the first $q(s+1)$ eigenvalues μ_{nj}^{χ} diverge as $n \to \infty$. If no restrictions hold for the entries of the matrices C_h^n (assume for instance that they are independently drawn from the same distribution), then FM2(b) is fulfilled, otherwise r is smaller than $q(s+1)$ and the model for the static factors is less obvious. The following elementary specification of (2.5), will help to understand the interplay between assumption FM2(b) and the parameters q and r.

Example. Part A Suppose that $s = 1$, $q = 1$ and $\Psi = 1$, so that the common components in (2.5) can be written as:

$$\chi_{it} = a_i(1 - c_i L)u_t$$

The number of static factors r depends on the heterogeneity in the panel:

(i) Assume that the restriction $c_i = c$ holds. In this case FM2(b) is fulfilled by the first eigenvalue provided that

$$0 < \underline{a} \le \frac{1}{n}\sum_{i=1}^{n} a_i^2 \le \overline{a} < \infty$$

as $n \to \infty$, but not by the second. As a consequence $r = 1$, $f_t = (1 - cL)u_t$ and

$$A_n = (a_1 \ a_2 \ \cdots \ a_n)'.$$

(ii) If no restriction holds, then also the second eigenvalue fulfills FM2(b) provided that $c_i \ne c_j$ for infinitely many couples $(i, \ j)$. Thus $r = 2$, $\boldsymbol{f}_t = (u_t, u_{t-1})'$ and

$$A_n = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_1c_1 & a_2c_2 & \cdots & a_nc_n \end{pmatrix}'$$

Note that in case (i), with $r = q = 1$, though the static factor $f_t = (1 - cL)u_t$ is identified, identification of u_t would require an assumption on c. In Section

[1] We might assume that $\Psi(L) = \Phi(L)^{-1}$, where $\Phi(L)$ is any (not necessarily diagonal) invertible $q \times q$ finite order matrix polynomial. However, as $C_n(L)\Phi(L)^{-1} = [C_n(L)\Phi_{\rm ad}(L)] \ [I_q \det \Phi(L)^{-1}]$, which is (2.5) after simplifying some of the roots of $\det \Phi(L)$, no gain in generality would be achieved.

3 we will see that this difference between cases (i) and (ii) is crucial for the identification of the structural shocks.

Our short analysis of both model (2.5) and the example suggest that the more heterogeneous the dynamic responses of the χ's to $\boldsymbol{u}_t$, the bigger is r with respect to q, i.e. the bigger is the number of static factors which is necessary to transform representation (2.1) into (2.4).

To conclude this section, it only remains to observe that representation (2.3)-(2.4) is not unique under FM0-FM3. Identification of the structural shocks $\boldsymbol{u}_t$ and the coefficients of the filter $B_n(L)$ calls for further informational and economic assumptions and will be thoroughly discussed in the next section.

3 Identification of the structural shocks

3.1 Response heterogeneity, n large and fundamentalness

3.3.1 Let us begin by briefly recalling some basic notions on fundamental representations of stationary stochastic vectors. Assume that the n stochastic vector $\boldsymbol{\mu}_t$ admits a moving average representation, i.e. that there exist a q-dimensional white noise $\boldsymbol{v}_t$ and an $n \times q$, one-sided, square-summable filter $K(L)$, such that

$$\boldsymbol{\mu}_t = K(L)\boldsymbol{v}_t. \tag{3.6}$$

If $\boldsymbol{v}_t$ belongs to the space spanned by present and past values of $\boldsymbol{\mu}_t$ we say that representation (3.6) is *fundamental* and that $\boldsymbol{v}_t$ is fundamental for $\boldsymbol{\mu}_t$ (the condition defining fundamentalness is also referred to as the *miniphase assumption*; see e.g. Hannan and Deistler, 1988, p. 25). With no substantial loss of generality we can suppose that $q \leq n$ and that $\boldsymbol{v}_t$ is full rank. Moreover, for our purpose, we can suppose that the entries of $K(L)$ are rational functions of L and that the rank of $K(z)$ is maximal, i.e. q, except for a finite number of complex numbers. Then:

(F) Representation (3.6) is fundamental if and only if the rank of $K(z)$ is q for all z such that $|z| < 1$ (see Rozanov, 1967, Ch. 1, Section 10, and Ch. 2, p. 76).

Assuming that (3.6) is fundamental, all fundamental white-noise vectors $\boldsymbol{z}_t$ are linear transformations of $\boldsymbol{v}_t$, i.e. $\boldsymbol{z}_t = C\boldsymbol{v}_t$ (see Proposition 2 below). Non fundamental white-noise vectors result from $\boldsymbol{v}_t$ by means of linear filters that involve the so-called Blaschke matrices (see e.g. Lippi and Reichlin, 1994).

A fundamental white noise naturally arises with linear prediction. Precisely, the prediction error

$$\boldsymbol{w}_t = \boldsymbol{\mu}_t - \mathrm{Proj}(\boldsymbol{\mu}_t | \boldsymbol{\mu}_{t-1},\ \boldsymbol{\mu}_{t-2},\ \ldots)$$

is white noise and fundamental for $\boldsymbol{\mu}_t$. As a consequence, when estimating an ARMA with forecasting purposes, the MA matrix polynomial is always chosen to be invertible, which implies fundamentalness.

Fundamentalness plays also an important role for the identification of structural shocks in SVAR analysis. SVAR analysis starts with the projection of a full rank n-dimensional vector $\boldsymbol{\mu}_t$ on its past, thus producing an n-dimensional full rank fundamental white noise $\boldsymbol{w}_t$. The structural shocks are then obtained as a linear transformation $A\boldsymbol{w}_t$, the matrix A resulting from economic theory statements, which is tantamount to assuming that the structural shocks are fundamental. Fundamentalness has here the effect that the identification problem is enormously simplified. However, as pointed out in the literature mentioned in the introduction, economic theory, in general, does not provide support for fundamentalness, so that all representations that fulfill the same economic statements but are non fundamental are ruled out with no justification.

Our main point is that the situation changes dramatically if structural analysis is conducted assuming that $n > q$. Precisely, as we shall see below, non fundamentalness is a generic property for $n = q$, while it is non generic for $n > q$. Thus the question "why assuming fundamentalness?", which is legitimately asked when $n = q$, is replaced by "why should we care about non fundamentalness?" when $n > q$.

An easy and effective illustration can be obtained assuming that $q = 1$, that the entries of $K(L) = (K_1(L)\ K_2(L)\ \cdots\ K_n(L))'$ are polynomials whose degree does not exceed s, so that $K(L)$ is parameterized in $\mathbb{R}^{n(s+1)}$. In this case, if $n = q = 1$, non fundamentalness translates into the condition that no root of $K_1(z)$ has modulus smaller than unity. Continuity of the roots of $K_1(z)$ implies that non fundamentalness is generic, i.e. that if it holds for a point $\boldsymbol{\kappa}$ in the parameter space it holds also within a neighborhood of $\boldsymbol{\kappa}$.

On the other hand, if $n > q$, by (F), non fundamentalness implies that the polynomials $K_j(z)$ have a common root. As a consequence, their coefficients must fulfill $n - 1$ *equality constraints* (see e.g. van der Waerden, 1953, p. 83). Non fundamentalness is therefore non generic.

This analytic argument has a forceful economic counterpart. Suppose for example that our variables are driven by two macroeconomic shocks, a monetary and a technology shock, so that the structural white noise $\boldsymbol{v}_t$ is 2-dimensional. Let the first two variables in $\boldsymbol{\mu}_t$ be the common components of aggregate output and consumption. The fundamentalness problem is that, in general, we do not know if $\boldsymbol{v}_t$ can be recovered from present and past observations on output and consumption. However, if $\boldsymbol{\mu}_t$ contains other variables, say, the common components of investment, employment, industrial production, etc., then non fundamentalness of $\boldsymbol{v}_t$, with respect to $\boldsymbol{\mu}_t$, is possible only if the responses of all such variables to $\boldsymbol{v}_t$ are forced to follow very special patterns. Thus in a framework in which the number of variables is larger than the number of shocks, a reasonable *heterogeneity* in the way different variables respond to the shocks provides a sound motivation for the fundamentalness assumption and for its consequences on identification (see Section 3.2 for further details on this example).

3.1.2 The general discussion above will now be adapted to our specification of the dynamic factor model. We have seen in Section 2 that under FM0 heterogeneity of the dynamic responses implies that r is big as compared to q. Further analysis of heterogeneity in the example of Section 2 and the rational model (2.5) will provide support to the assumption that $N(L)$ is left invertible, i.e. there exists a one-sided square-summable $q \times r$ filter $G(L)$ such that $G(L)N(L) = I_q$.

Example. Part B Still assuming

$$\chi_{it} = a_i(1 - c_i L)u_t,$$

heterogeneity of the dynamic responses (no restrictions) implies $r = 2$. In this case $\boldsymbol{f}_t = N(L)u_t$ takes the form

$$\begin{pmatrix} u_t \\ u_{t-1} \end{pmatrix} = \begin{pmatrix} 1 \\ L \end{pmatrix} u_t.$$

Obviously $N(L)$ has the left inverse $(1 \;\; 0)$, so that u_t is fundamental for $\boldsymbol{f}_t$. Moreover, since $r = 2$, FM2 implies that for n large enough there must be a couple $(i,\; j)$ such that $a_i \neq 0$, $a_j \neq 0$ and $c_i \neq c_j$. Then

$$u_t = \frac{a_j c_j \chi_{it} - a_i c_i \chi_{jt}}{a_i a_j (c_j - c_i)},$$

so that u_t is fundamental for the whole set of the χ's (actually for the two-dimensional vector $(\chi_{it} \;\; \chi_{jt})$). Note that this result holds independently of the values taken by the coefficients c_i. It holds in particular even when $c_i > 1$ for all i, so that u_t is not fundamental for any of the χ's.

Conversely, the restriction $c_i = c$, i.e. homogeneity, implies $r = q = 1$ and $\boldsymbol{f}_t = N(L)u_t$ takes the form

$$f_t = (1 - cL)u_t.$$

Here we are precisely in the VAR situation. The system is square. Either some extra information is available to motivate the assumption that $|c| < 1$, or the assumption that $N(L)$ is invertible is ad hoc.

It is easily seen that the results obtained for the example, left invertibility of $N(L)$ in particular, generalize to model (2.5) in the case when no restrictions hold. In that case the dynamic responses are most heterogeneous and therefore $r = q(s+1)$. As already seen in Section 2, $N(L) = (\Psi(L)' \;\; \Psi(L)'L \;\cdots\; \Psi(L)'L^s)'$.

Setting $G(L) = (\Psi(L)^{-1} \;\; 0_q \cdots 0_q)$, where 0_q is a $q \times q$ matrix of zeros, we see that $G(L)N(L) = I_q$. If restrictions hold among the entries of $B_n(L)$, $C_n(L)$ in the rational case, obtaining $N(L)$ is less obvious. We do not need a detailed treatment of the problem. An example is the case $c_i = c$ above.

The above discussion motivates Assumption FM4 as a most likely consequence of the heterogeneity of the dynamic responses to $\boldsymbol{u}_t$. Proposition 1 shows that FM4, jointly with FM2, imply fundamentalness.

(FM4) (Fundamentalness) There exists a $q \times r$ one-sided filter $G(L)$ such that $G(L)N(L) = I_q$.

Proposition 1 If FM0-FM4 are satisfied, $\boldsymbol{u}_t$ is fundamental for $\boldsymbol{\chi}_{nt}$ for n sufficiently large and therefore fundamental for χ_{it}, $i = 1, \dots, \infty$. Moreover, $\boldsymbol{u}_t$ belongs to the space spanned by present and past values of x_{it}, $i = 1, \dots, \infty$, i.e. the shocks u_{ht} can be recovered as limits of linear combinations of the variables x_{it}.

Proof. As already observed, FM2 implies that $A_n'A_n$ is full rank for n sufficiently large. Setting, $S_n(L) = G(L)\left(A_n'A_n\right)^{-1} A_n'$, where $G(L)$ satisfies FM4, we have $S_n(L)\boldsymbol{x}_{nt} = S_n(L)\boldsymbol{\chi}_{nt} + Sn_(L)\boldsymbol{\xi}_{nt}$. Now

$$S_n(L)\boldsymbol{\chi}_{nt} = G(L)\left(A_n'A_n\right)^{-1} A_n'A_n\boldsymbol{f}_t = G(L)\boldsymbol{f}_t = G(L)N(L)\boldsymbol{u}_t = \boldsymbol{u}_t.$$

Therefore $\boldsymbol{u}_t$ lies in the space spanned by present and past values of $\boldsymbol{\chi}_{nt}$. Moreover, $S_n(L)\boldsymbol{\xi}_{nt} = G(L)\left(A_n'A_n\right)^{-1} A_n'\boldsymbol{\xi}_t$ converges to zero in mean square by assumptions FM2 and FM3. Q.E.D.

Consider now the orthogonal projection of $\boldsymbol{f}_t$ on the space spanned by its past values:

$$\boldsymbol{f}_t = \mathrm{Proj}(\boldsymbol{f}_t \mid \boldsymbol{f}_{t-1},\ \boldsymbol{f}_{t-2},\ \dots,\ ) + \boldsymbol{w}_t,$$

where $\boldsymbol{w}_t$ is the r-dimensional vector of the residuals. Under our assumptions, $\boldsymbol{w}_t$ has rank q. Moreover, by the same argument used to prove Proposition 2 (see the next subsection), $\boldsymbol{w}_t = R\boldsymbol{u}_t$, where R is a maximum-rank $r \times q$ matrix. It can be remarked that:

(a) For model (2.5), with $\Psi(L) = I_q$ and no restrictions, the projection above requires only one lag. The intuition is that when $r > q$ and the panel dynamics are very heterogenous, information contained in lagged values of f_{ht} can be substituted by cross-sectional information (just the same reason motivating fundamentalness).

(b) If we relax the assumption $\Psi(L) = I_q$, as the reader can easily check, the orthogonal projection requires only a finite number of lags, one lag being sufficient if the order of the polynomials appearing in the denominators of $\Psi(L)$ is not greater than $s + 1$.

As a consequence, a specification of FM4 as

$$\boldsymbol{f}_t = F_1 \boldsymbol{f}_{t-1} + \cdots + F_m \boldsymbol{f}_{t-m} + R\boldsymbol{u}_t$$

does not seem to cause a dramatic loss of generality, even when $m = 1$. In the sequel we will adopt the VAR(1) specification:

(FM4)$'$ (Fundamentalness: VAR(1) specification) The r-dimensional static factors $\boldsymbol{f}_t$ admit a VAR(1) representation

$$\boldsymbol{f}_t = F\boldsymbol{f}_{t-1} + R\boldsymbol{u}_t \tag{3.7}$$

where F is $r \times r$ and R is a maximum-rank matrix of dimension $r \times q$.

Summing up, a large n and heterogeneity of the dynamic responses of the χ's to $\boldsymbol{u}_t$ makes fundamentalness of $\boldsymbol{u}_t$ with respect to the χ's most plausible. In our model dynamic heterogeneity implies that $r > q$ and that, most likely, $N(L)$ is invertible, which implies fundamentalness. Lastly, with no significant loss of generality, the model for $\boldsymbol{f}_t$ can be written as a VAR(1).

3.2 Economic conditions for shocks identification

Proposition 1 ensures that under Assumptions FM0-FM4 $\boldsymbol{u}_t$ is fundamental for the common components χ_{it} and can be recovered by using past and present values of the observable variables x_{it}. Our next result shows that under the same assumptions $\boldsymbol{u}_t$ is identified up to a static rotation.

Proposition 2 Consider the common components of model (2.1):

$$\boldsymbol{\chi}_{nt} = B_n(L)\boldsymbol{u}_t. \tag{3.8}$$

If

$$\boldsymbol{\chi}_{nt} = C_n(L)\boldsymbol{v}_t \tag{3.9}$$

for any $n \in \mathbb{N}$, where $\boldsymbol{v}_t$ is a q-dimensional fundamental orthonormal white noise vector, then representation (3.9) is related to representation (3.8) by

$$\begin{aligned} C_n(L) &= B_n(L)H \\ \boldsymbol{v}_t &= H'\boldsymbol{u}_t, \end{aligned} \tag{3.10}$$

where H is a $q \times q$ unitary matrix, i.e. $HH' = I_q$.

Proof. Projecting $\boldsymbol{v}_t$ entry by entry on the linear space $\mathcal{U}_t$ spanned by the present and the past of u_{ht}, $h = 1, \ldots, q$ we get

$$\boldsymbol{v}_t = \sum_{k=0}^{\infty} H_k \boldsymbol{u}_{t-k} + \boldsymbol{r}_t, \tag{3.11}$$

where $\boldsymbol{r}_t$ is orthogonal to $\boldsymbol{u}_{t-k}$, $k \geq 0$. Now consider that $\mathcal{U}_t$ and the space spanned by present and past of the χ_{it}'s, call it $\mathcal{X}_t$, are identical, because the entries of $\boldsymbol{\chi}_{t-k}$, $k \leq 0$, belong to $\mathcal{U}_t$ by equation (3.8), while the entries of $\boldsymbol{u}_{t-k}$, $k \leq 0$, belong to $\mathcal{X}_t$ by condition FM4. The same is true for $\mathcal{X}_t$ and the space spanned by present and past of the v_{ht}'s, call it $\mathcal{V}_t$, so that $\mathcal{U}_t = \mathcal{V}_t$. Hence $\boldsymbol{r}_t = 0$. Moreover, serial non-correlation of the u_{ht}'s imply that $\sum_{k=1}^{\infty} H_k \boldsymbol{u}_{t-k}$ must be the projection of $\boldsymbol{v}_t$ on $\mathcal{U}_{t-1}$, which is zero because $\mathcal{U}_{t-1} = \mathcal{V}_{t-1}$. It follows that $\boldsymbol{v}_t = H_0 \boldsymbol{u}_t$. Orthonormality of $\boldsymbol{v}_t$ implies that H_0 is unitary $H_0 H_0' = I$. QED

Since fundamentalness of the structural shocks can be assumed in the dynamic factor model framework, identification is reduced to the choice of a matrix H such that economically motivated restrictions on the matrix $B_n(L)H$ are fulfilled. For instance, identification can be achieved by maximizing or minimizing an objective function involving $B_n(L)H$ (see, for example, Giannone, Reichlin and Sala, 2005). An alternative is to impose zero restrictions either on the impact effects $B_n(0)H$ or the long-run effects $B_n(1)H_0$ or both. In this case we have to impose $q(q-1)/2$ restrictions (since orthonormality entails $q(q+1)/2$ restrictions). Notice that, once the conditions FM0-FM4 are satisfied, the number of economic identification restrictions we need to identify the shocks depend on q and not on n. This is an advantage for structural analysis, since, provided q is small, we need few restrictions for identification while we are not limited on the informational assumptions (size of the panel).

A comparison with identification in SVAR analysis is in order here. To simplify the presentation, suppose, like in the example at the end of Section 3.3.1, that $q = 2$, that we are interested in the impulse-response functions of the first two common components to the structural shocks u_{1t} and u_{2t}, and that our economic restrictions are sufficient to identify the matrix H. We have $\boldsymbol{\chi}_{nt} = B_n(L)\boldsymbol{u}_t$, with

$$\begin{pmatrix} \chi_{1t} \\ \chi_{2t} \end{pmatrix} = B_2(L) \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix} \tag{3.12}$$

being the subsystem of interest. Now, $(u_{1t}\ u_{2t})'$ is fundamental with respect to $\boldsymbol{\chi}_{nt}$, but, as already noted in Section 3.1, is not necessarily fundamental with respect to $(\chi_{1t}\ \chi_{2t})'$, i.e. representation (3.12) is not necessarily fundamental. By contrast, if a VAR were estimated for the vector $(\chi_{1t}\ \chi_{2t})'$,

$$A(L) \begin{pmatrix} \chi_{1t} \\ \chi_{2t} \end{pmatrix} = \begin{pmatrix} v_{1t} \\ v_{2t} \end{pmatrix},$$

the resulting MA representation,

$$\begin{pmatrix} \chi_{1t} \\ \chi_{2t} \end{pmatrix} = A(L)^{-1} \begin{pmatrix} v_{1t} \\ v_{2t} \end{pmatrix},$$

would be fundamental by definition. As a consequence, if $B_2(L)$ were not fundamental, applying the same economic restrictions to rotate $(v_{1t}\ v_{2t})'$ would never allow recovering the structural shocks $(u_{1t}\ u_{2t})'$. This point is further illustrated in Section 5, where an important empirical example of non-fundamentalness of the subsystem of interest is presented.

4 Estimation

Going back to equation (2.4) it is easily seen that the static factors $\boldsymbol{f}_t$ are identified only up to pre-multiplication by a non-singular $r \times r$ matrix. Hence we cannot estimate $\boldsymbol{f}_t$. However, we can estimate the common-factor space, i.e. we can estimate an r-dimensional vector whose entries span the same linear space as the entries of $\boldsymbol{f}_t$. Such vector can be written as $\boldsymbol{g}_t = G\boldsymbol{f}_t$, were G is a non-singular matrix.

The static factor space can be consistently estimated by the first r principal components of the panel $\boldsymbol{x}_{nt}$ as in Stock and Watson, 2002a and 2002b[2].

Precisely, the estimated static factors will be

$$\hat{\boldsymbol{g}}_t = \frac{1}{\sqrt{n}} W_n^{T\prime} \boldsymbol{x}_{nt}, \tag{4.13}$$

where W_n^T is the $n \times r$ matrix having on the columns the eigenvectors corresponding to the first r largest eigenvalues of the sample variance-covariance matrix of $\boldsymbol{x}_{nt}$, say Γ_{n0}^{xT}. We do not normalize the factors to have unit variance. The estimated variance-covariance matrix of $\hat{\boldsymbol{g}}_t$ is the diagonal matrix having on the diagonal the normalized eigenvalues of Γ_{n0}^{xT} in descending order, $\frac{1}{n}\Lambda_n^T = \frac{1}{n} W_n^{T\prime} \Gamma_{n0}^{xT} W_n^T$. The corresponding estimate of the common components is obtained by regressing $\boldsymbol{x}_{nt}$ on the estimated factors to get

$$\boldsymbol{\chi}_{nt}^T = W_n^T W_n^{T\prime} \boldsymbol{x}_{nt}. \tag{4.14}$$

Having an estimate of $\boldsymbol{g}_t$, we have still to unveil the leading-lagging relations between its entries, in order to find out the underlying dynamic factors (or, better, a unitary transformation of such factors $\boldsymbol{v}_t = H\boldsymbol{u}_t$, with $HH' = I_q$). This can be done in our dynamic factor model by projecting $\boldsymbol{g}_t$ on its first lag. This approach is also followed in Giannone, Reichlin and Sala (2002, 2005).

4.1 Population formulas

By equation (3.7), any non-singular transformation of the common factors $\boldsymbol{g}_t = G\boldsymbol{f}_t$ has the VAR(1) representation

$$\boldsymbol{g}_t = GFG^{-1}\boldsymbol{g}_{t-1} + \boldsymbol{\epsilon}_t = D\boldsymbol{g}_{t-1} + \boldsymbol{\epsilon}_t. \tag{4.15}$$

[2]Alternative (n, T) consistent estimators proposed in the literature are Forni and Reichlin (1998), Boivin and Ng (2003) and Forni, Hallin, Lippi and Reichlin (2005).

Note that

$$D = \Gamma_1^g \left(\Gamma_0^g\right)^{-1}, \tag{4.16}$$

where $\Gamma_h^g = \mathrm{E}(\boldsymbol{g}_t\boldsymbol{g}'_{t-h})$, and

$$\mathrm{var}(\boldsymbol{\epsilon}_t) = \Gamma_0^g - D\Gamma_0^g D'. \tag{4.17}$$

By (3.7), the residual $\boldsymbol{\epsilon}_t$ can be written as

$$\boldsymbol{\epsilon}_t = GR\boldsymbol{u}_t = (GRH')\,H\boldsymbol{u}_t = KMH\boldsymbol{u}_t, \tag{4.18}$$

where

(i) M is the diagonal matrix having on the diagonal the square roots of the first q largest eigenvalues of the variance-covariance matrix of $\boldsymbol{\epsilon}_t$, i.e. the matrix $GRR'G' = \Gamma_0^g - D\Gamma_0^g D'$, in descending order.

(ii) K is the $r \times q$ matrix whose columns are the eigenvectors corresponding to such eigenvalues.

(iii) H is a $q \times q$ unitary matrix;

By inverting the VAR we get

$$\boldsymbol{g}_t = (I - DL)^{-1}KMH\boldsymbol{u}_t.$$

On the other hand, by equations (2.1) and (2.4)

$$\boldsymbol{\chi}_{nt} = B_n(L)\boldsymbol{u}_t = A_n\boldsymbol{f}_t = A_nG^{-1}\boldsymbol{g}_t = Q_n\boldsymbol{g}_t, \tag{4.19}$$

where

$$Q_n = \mathrm{E}(\boldsymbol{\chi}_{nt}\boldsymbol{g}'_t) = \mathrm{E}(\boldsymbol{x}_{nt}\boldsymbol{g}'_t). \tag{4.20}$$

Hence, we have

$$\begin{aligned}\boldsymbol{\chi}_{nt} &= B_n(L)\boldsymbol{u}_t \\ &= Q_n(I - DL)^{-1}KMH\boldsymbol{u}_t \\ &= Q_n(I + DL + D^2L^2 + \cdots)KMH\boldsymbol{u}_t.\end{aligned} \tag{4.21}$$

4.2 Estimators

By substituting $\hat{\boldsymbol{g}}_t = \frac{1}{\sqrt{n}}W_n^{T\prime}\boldsymbol{x}_{nt}$ for $\boldsymbol{g}_t$, it is quite natural to estimate Q_n by $\frac{1}{\sqrt{n}}\Gamma_0^{xT}W_n^T$ (see equation (4.20)). Moreover, Γ_0^g, the variance-covariance matrix of $\boldsymbol{g}_t$, can be estimated by $\frac{1}{n}W_n^{T\prime}\Gamma_{n0}^{xT}W_n^T = \frac{1}{n}\Lambda_n^T$, and Γ_1^g by $\frac{1}{n}W_n^{T\prime}\Gamma_{n1}^{xT}W_n^T$, so that, basing on equation (4.16), we estimate D_n by $D_n^T = W_n^{T\prime}\Gamma_{n1}^{xT}W_n^T(\Lambda_n^T)^{-1}$. Finally, to estimate the eigenvectors and eigenvalues in K_n and M_n we estimate the variance-covariance matrix of $\boldsymbol{\epsilon}_t$ by $\Sigma_n^T = \frac{1}{n}(\Lambda_n^T - D_n^T \Lambda_n^T D_n^{T\prime})$ (see equation (4.17)).

Summing up, in analogy with (4.21) we propose to estimate the impulse-response functions by

$$B_n^T(L) = Q_n^T \left(I + D_n^T L + (D_n^T)^2 L^2 + \cdots \right) K_n^T M_n^T H, \tag{4.22}$$

where

(i) $Q_n^T = \frac{1}{\sqrt{n}} \Gamma_{n0}^{xT} W_n^T$, where Γ_{n0}^{xT} is the sample variance-covariance matrix of $\boldsymbol{x}_{nt}$ and W_n^T the $n \times r$ matrix having on the columns the eigenvectors corresponding to the first r largest eigenvalues of Γ_{n0}^{xT};

(ii) $D_n^T = W_n^{T\prime} \Gamma_{n1}^{xT} W_n^T (\Lambda_n^T)^{-1}$, where Γ_{n1}^{xT} is the sample covariance matrix of $\boldsymbol{x}_{nt}$ and $\boldsymbol{x}_{nt-1}$;

(iii) M_n^T is the diagonal matrix having on the diagonal the square roots of the first q largest eigenvalues of the the matrix $\frac{1}{n}(\Lambda_n^T - D_n^T \Lambda_n^T D_n^{T\prime})$, in descending order;

(iv) K_n^T is the $r \times q$ matrix whose columns are the eigenvectors corresponding to such eigenvalues.

(v) H is a unitary matrix to be fixed by the identifying restrictions.

In order to render operative the above procedure we need to set values for r and q. Unfortunately, there are no criteria in the literature to fix jointly q and r. Bai and Ng (2002) propose some consistent criteria to determine r. As regards the number of dynamic factors, we can follow a decision rule like that proposed in Forni, Hallin, Lippi and Reichlin (2000) i. e., we go on to add factors until the additional variance explained by the last dynamic principal component is less than a pre-specified fraction, say 5% or 10%, of total variance.

4.3 Consistency

Consistency of (4.22) as estimator of the impulse-response functions for large cross-sections and large sample size ($n, T \to \infty$) is shown in Proposition 3 below.

Proposition 3 Under assumptions PA1-2, FM1-3, we have, as $\min(n, T) \to \infty$:

$$\sqrt{\delta_{nt}} |b_{ni}^T(L) - b_i(L)| = O_p(1), i = 1, ..., n.$$

where $\delta_{nt} = \min(n, T)$, $b_{ni}^T(L)$ and $b_i(L)$ denote the ith row of $B_n^T(L)$ and $B_n(L)$ respectively, *Proof.* See Appendix 1.

Proposition 3 shows that consistency is achieved along any path for (n, T) with T and n both tending to infinity. The consistency rate is given by min $\left(\sqrt{T}, \sqrt{n}\right)$. This implies that if the cross-section dimension n is large relative to the sample size T $(T/n \to 0)$ the rate of consistency is $\sqrt{T}$, the same we would obtain if the common components were observed, i.e. if the variables were not contaminated by idiosyncratic component. On the other hand, if $n/T \to 0$, then the consistency rate is $\sqrt{n}$ reflecting the fact that the common components are not observed but have to be estimated[3].

4.4 Standard errors and confidence bands

To obtain confidence bands and standard errors we propose the following bootstrap procedure.

Firstly, compute $\boldsymbol{\chi}_{nt}^T$ and $B_n^T(L)$ according to (4.14) and (4.22), and $\boldsymbol{\xi}_{nt}^T = \boldsymbol{x}_{nt} - \boldsymbol{\chi}_{nt}^T$.

Secondly, for each one of the estimated idiosyncratic components, estimate the univariate autoregressive model

$$a_j(L)\xi_{jt}^T = \sigma_j \omega_{jt}, \qquad j = 1, \ldots, n,$$

whose order can be fixed by the Schwarz criterion, and take the estimated coefficients $a_j^T(L)$ and σ_j^T and the unit variance residuals ω_{jt}^T.

Thirdly, generate new simulated series for the shocks, say $\boldsymbol{u}_t^*$ and ω_{jt}^*, $j = 1, \ldots, n$, by drawing from the standard normal. Use these new series to construct $\boldsymbol{\chi}_{nt}^* = B_n^T(L)\boldsymbol{u}_t^*$, $\xi_{jt}^* = a_j^T(L)^{-1}\sigma_j^T\omega_{jt}^*$, $j = 1, \ldots, n$, and $\boldsymbol{x}_{nt}^* = \boldsymbol{\chi}_{nt}^* + \boldsymbol{\xi}_{nt}^*$.

Finally, compute new estimates of the impulse-response functions $B_n^*(L)$ starting from $\boldsymbol{x}_{nt}^*$.

By repeating the two last steps N times we get a distribution of estimated values which can be used to obtain standard errors and confidence bands. Note that the estimates will in general be biased, since the estimation procedure involves implicitly the estimation of a VAR. An estimate of such bias is provided by the difference between the point estimate $B_n^T(L)$ and the average of the N estimates $B_n^*(L)$.

5 Empirical application

We illustrate our proposed structural factor model by revisiting a seminal work in the structural VAR literature, i.e. King *et al.*, 1991 (KPSW from now on). To this

[3]It should be pointed out that, under the model assumptions of Stock and Watson (2002a and 2002b) or Bai and Ng (2002), an alternative proof of consistency has been proposed by Giannone, Reichlin and Sala(2002).

end, we constructed a panel of macroeconomic series including the series used by KPSW, with the same sampling period. Just like KPSW, we identify a long-run shock by imposing long-run neutrality of all other shocks on per-capita output. The data are well described by three common shocks, so that the comparison with the three-variable exercise of KPSW is particularly appropriate. Having the same data, the same identification scheme and the same number of shocks, different results can only be due to the additional information coming from the other series in the panel.

5.1 The data

The data set was constructed by downloading mainly from the FRED II database of the Federal Reserve Bank of St. Louis and Datastream. The original data of KPSW have been downloaded from Mark Watson's home page. We collected 89 series, including data from NIPA tables, price indeces, productivity, industrial production indeces, interest rates, money, financial data, employment, labor costs, shipments, and survey data. A larger n would be desirable, but we were constrained by both the scarcity of series starting from 1949 (like in KPSW) and the need of balancing data of different groups. In order to use Datastream series we were forced to start from 1950:1 instead of 1949:1, so that the sampling period is 1950:1 - 1988:4. Monthly data are taken in quarterly averages. All data have been transformed to reach stationarity according to the ADF(4) test at the 5% level. Finally, the data were taken in deviation from the mean as required by our formulas, and divided by the standard deviation to render results independent of the units of measurement. A complete description of each series and the related transformations is reported in Appendix 2.

5.2 The choice of r and the number of common shocks

As a first step we have to set r and q. Let us begin with r. We computed the six consistent criteria suggested by Bai and Ng (2002) with $r = 1, \dots, 30$. The criteria IC_{p1} and IC_{p3} do not work, since they do not reach a minimum for $r < 30$; IC_{p2} has a minimum for $r = 12$. To compute PC_{p1}, PC_{p2} and PC_{p3} we estimated $\hat{\sigma}^2$ with $r = 15$ since with $r = 30$ none of the criteria reaches a minimum for $r < 30$. PC_{p1} gives $r = 15$, PC_{p2} gives $r = 14$ and PC_{p3} gives $r = 20$. Below we report results for $r = 12$, $r = 15$ and $r = 18$, with more detailed statistics for $r = 15$. With $r = 15$, the common factors explain on average 79.7% of total variance. With reference to the variables of interest in KPSW, the common factors explain 85.6% of total variance for output, 84.4% for investment and 89.4% for consumption.

Regarding the choice of q, for comparison with the three variable VAR of KPSW we set $q = 3$. This choice is consistent with the decision rule proposed in Forni, Hallin, Lippi and Reichlin (2000), since, with Bartlett lag window size 18, the overall variance explained by the third dynamic principal component is larger than 10% (10.2%), whereas the variance explained by the fourth one is less than 10% (6.8%). Given the illustrative purpose of this application, we do not use the more formal criteria for the choice of q proposed in recent literature (Bai and Ng, 2005, Hallin and Liska, 2006 or Stock and Watson, 2005).

5.3 Fundamentalness

Now let us focus on the 3×3 impulse-response function system for the three variables of KPSW, i.e. per capita consumption, per capita income and per capita investment. As observed at the end of Section 3, we can compute the roots of the determinant of this system to check whether it is invertible or not.[4] Figure 1 plots the moduli of the two smallest roots of the above determinant as a function of r, for r varying over the range 3-30. Note that for $r = 3$ all roots must be larger than one in modulus, since they stem from a three-variate VAR. This is in fact the case for $r = 3$ and $r = 4$, but for $r \geq 5$ the smallest root is declining and lies always within the unit circle. For $r \geq 22$ the second smallest root becomes smaller than one in modulus.

Figure 1: **The moduli of the first and the second smallest roots as functions of r**

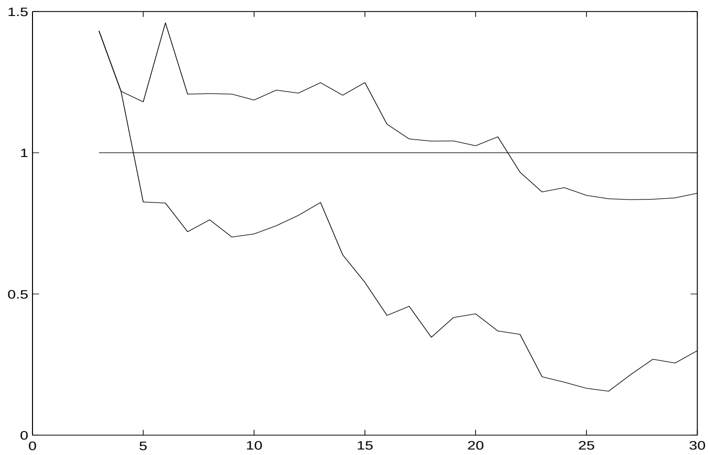
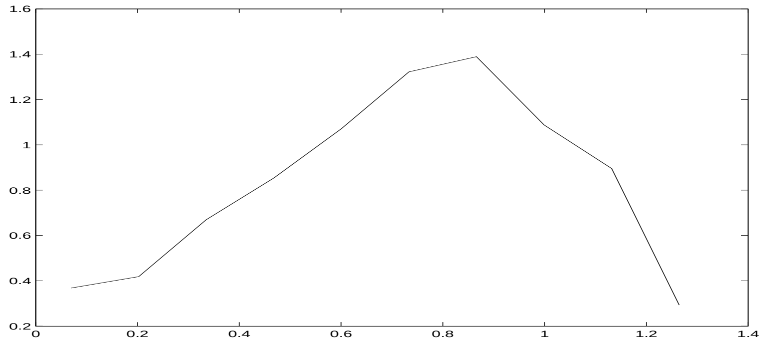


Figure 2 reports the distribution of the modulus of the smallest root for $r = 15$ across 1000 bootstrapping replications. The mean value is 0.71, indicating a non-negligible upward bias, since our point estimate for $r = 15$ is 0.54. We shall come back to the estimation bias below. Here we limit ourselves to observe that if the smallest root is overestimated on average, the true value could be even smaller

[4]Note that these roots (and therefore fundamentalness) are independent of the identification rule adopted and the rotation matrix H.

than 0.54. Without any bias correction, the probability of an estimated value larger than one in modulus is less than 22%.

Figure 2: **Frequency distribution of the modulus of the smallest root**



We conclude that the true, structural impulse-response function system for the common components associated with these three variables is probably non-fundamental. As a consequence, such impulse response functions, as well as the associated structural shocks, cannot be recovered by estimating a three-dimensional VAR.

5.4 Impulse-response functions and variance decomposition

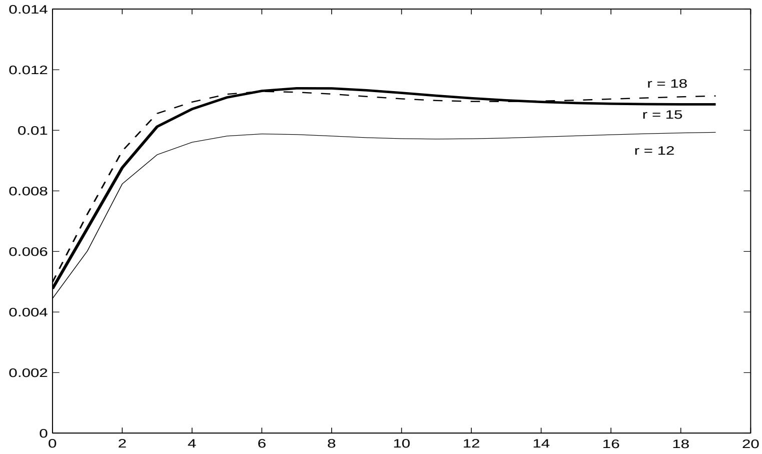
Coming to the impulse-response functions, as anticipated above we impose long-run neutrality of two shocks on per-capita output, like in KPSW. This is sufficient to reach a partial identification, i.e. to identify the long-run shock and its response functions on the three variables.

Figure 3 shows the response functions of per capita output for $r = 12, 15, 18$. The general shape does not change that much with r. The productivity shock has positive effects declining with time on the output level. The response function reach its maximum value after 6-8 quarters with only negligible effects after two years. It should be observed that this simple distributed-lag shape is different from the one in KPSW, where there is a sharp decline during the second and the third year, which drives the overall effect back to the impact value.

In Figure 4 we concentrate on the case $r = 15$. We report the response functions with 90% confidence bands for output, consumption and investment respectively. Confidence bands are obtained with the procedure explained above (with 1000 replications). The shapes are similar for the three variables, with a positive impact effect followed by important, though declining, positive lagged effects.

Note that confidence bands are not centered around the point estimate, especially for consumption, suggesting the existence of a non-negligible bias. This

Figure 3: **The impulse response function of the long-run shock on output for** $r = 12, 15, 18$



is not surprising, since formula (4.22) implicitly involves estimation of a VAR, where in addition the variable involved (the static factors) contain errors (a residual idiosyncratic term). Figure 5 shows the point estimate along with the mean of the bootstrap distribution for the output. Such a large bias is probably due to the small cross-sectional dimension. We have evidence of a much smaller bias for the larger data set of Giannone, Reichlin and Sala (2002). We do not make any attempt here to correct for the bias, but a procedure like the one suggested in Kilian (1998) could be appropriate.

Table 1 reports the fraction of the forecast-error variance attributed to the permanent shock for output, consumption and investment at different horizons. For ease of comparison we report the corresponding numbers obtained with the (restricted) VAR model and reported in Table 4 of KPSW.

At horizon 1, our estimates are smaller. The difference is important for consumption: only 0.30 according to the factor model as against 0.88 according to the KPSW model. But at horizons larger than or equal to 8 quarters our estimates are greater and the difference is very large for investment. At horizon 20 (5 years) the permanent shock explains 46% of investment variance according to KPSW as against 86% with the factor model. This result is interesting in that it solves a typical puzzle of the VAR literature: the finding that technological and other supply shocks explain a small fraction of investment variations even in the medium-long run.

6 Conclusions

In this paper we have argued that dynamic factor models are suitable for structural macroeconomic modeling and constitute an interesting alternative to structural VARs.

Figure 4: **The impulse response function of the long-run shock on output, consumption and investment for** $r = 15$

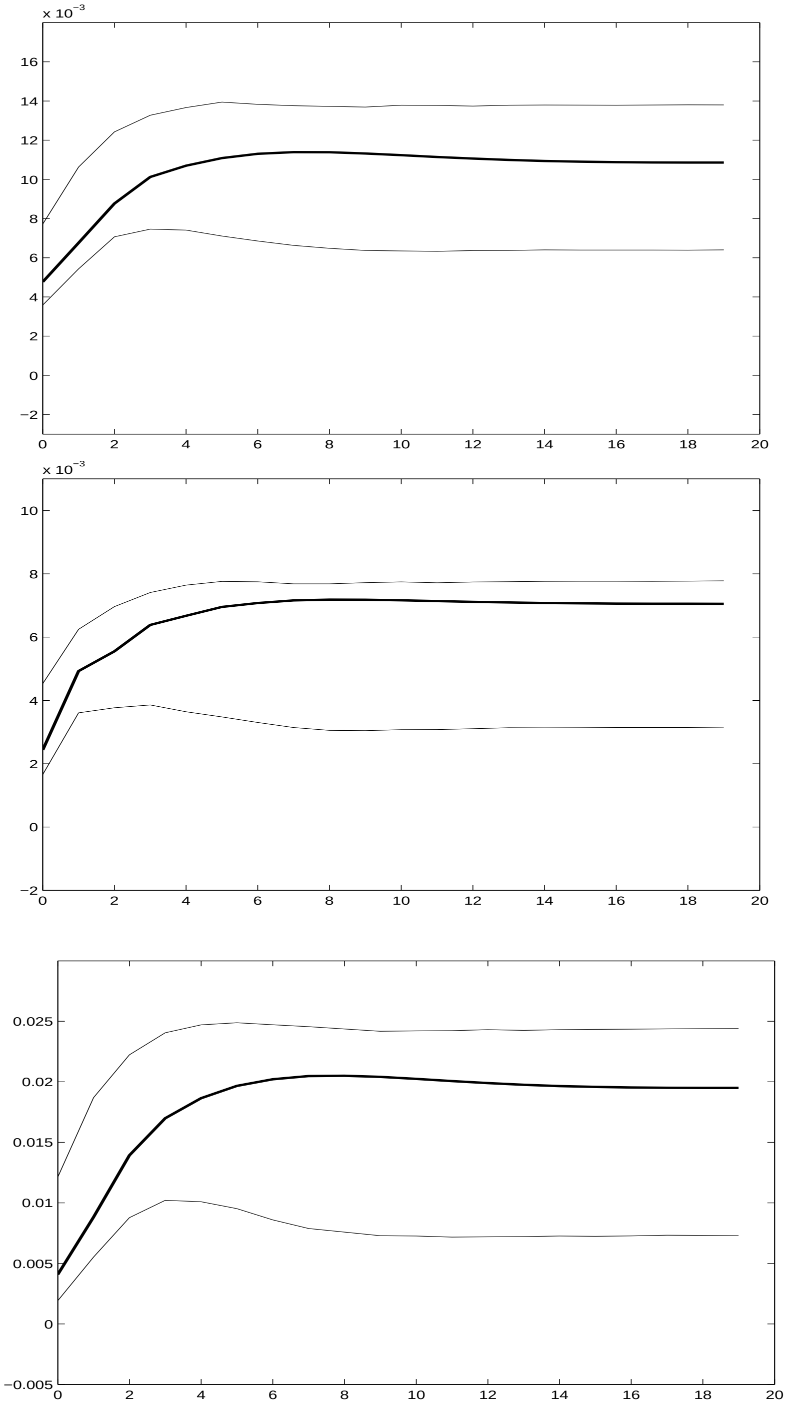
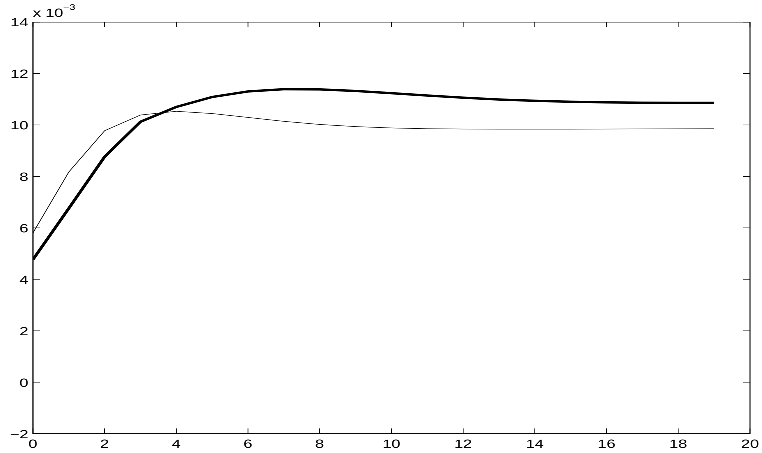


Figure 5: **Estimation bias**



We have shown that large information and a small number of shocks generating the comovement of many variables, allow the econometrician to recover the structural shocks driving the economy under the mild assumption that the structure of leads and lags is rich enough so that the cross-section can convey information on dynamic relations. Thus the fundamentalness problem, which has no solution in the VAR framework, where n shocks must me recovered using present and past values of n variables, becomes easily tractable when the number of variables exceeds the number of shocks.

Having established sufficient conditions for identification, we have proposed a procedure to estimate the impulse response functions. Moreover, we have shown consistency of such a procedure and have suggested a bootstrapping method for the construction of confidence bands and inference purposes.

In the empirical application, we have revisited the seminal paper by King*et al.* (1991, KPSW). We have designed a large data set including output, consumption and investment (the data analysed by KPSW) on the same sample period. We have estimated a large factor model with a three-shock specification and, after having identified the shocks as in KPSW, we have analysed impulse response functions on the three variables of interest: output, consumption and investment. We find that the smallest root of the determinant of the impulse-response functions formed by the three variables sub-system is non-fundamental and therefore could have not been obtained by estimating a VAR on these three variables alone. These impulse response functions imply a larger effect of the permanent shock on output and investment than those found by KPSW.

Table 1: **Fraction of the forecast-error variance due to the long-run shock**

	Dynamic factor model			KPSW vector ECM		
Horizon	Output	Cons.	Inv.	Output	Cons.	Inv.
1	0.37	0.30	0.07	0.45	0.88	0.12
	(0.18)	(0.21)	(0.19)	(0.28)	(0.21)	(0.18)
4	0.57	0.77	0.42	0.58	0.89	0.31
	(0.12)	(0.12)	(0.19)	(0.27)	(0.19)	(0.23)
8	0.78	0.87	0.72	0.68	0.83	0.40
	(0.07)	(0.11)	(0.16)	(0.22)	(0.18)	(0.18)
12	0.86	0.90	0.80	0.73	0.83	0.43
	(0.05)	(0.11)	(0.16)	(0.19)	(0.18)	(0.17)
16	0.89	0.91	0.83	0.77	0.85	0.44
	(0.04)	(0.11)	(0.16)	(0.17)	(0.16)	(0.16)
20	0.91	0.92	0.86	0.79	0.87	0.46
	(0.03)	(0.11)	(0.16)	(0.16)	(0.15)	(0.16)

Appendix 1: Proof of Proposition 3

Let $\mathbf{A}$ and $\mathbf{E}$ be two $n \times n$ symmetric matrices and denote by $\sigma_j(\cdot), j = 1, \ldots, n$ the eigenvalues in decreasing order of magnitude. Throughout this section we will use the following inequalities due to Weyl (cfr. Stewart and Sun, 1990):

$$|\sigma_j(\mathbf{A}+\mathbf{E}) - \sigma_j(\mathbf{A})| \leq \sqrt{\sigma_1(\mathbf{E}^2)} \leq \sqrt{\mathrm{trace}(\mathbf{E}^2)}$$

Denote by Λ_n and Λ_n^T, the $r \times r$ diagonal matrices having on the diagonal elements the first r largest eigenvalues of Γ_{n0}^{χ} and Γ_{n0}^{x}, respectively. Writing W_n and W_n^T for the $n \times r$ matrices having on the columns the corresponding eigenvectors, we have, by definition:

$$\Gamma_{n0}^{\chi} W_n = W_n \Lambda_n$$

$$\Gamma_{n0}^{xT} W_n^T = W_n^T \Lambda_n^T$$

Let us recall here our notation for the eigenvalues of the relevant matrices:

$$\mu_{nj}^{x} := \sigma_j(\Gamma_{n0}^{x}), \;\; \mu_{nj}^{xT} := \sigma_j(\Gamma_{n0}^{xT}), \;\; \mu_{nj}^{\chi} := \sigma_j(\Gamma_{n0}^{\chi}), \;\; \mu_{nj}^{\xi} := \sigma_j(\Gamma_{n0}^{\xi}), \;\; j = 1, ..., n$$

we have $\Lambda_n = \mathrm{diag}(\mu^{\chi}_{n1}, ..., \mu^{\chi}_{nr})$ and $\Lambda^T_n = \mathrm{diag}(\mu^{xT}_{n1}, ..., \mu^{xT}_{nr})$

Using the following non-singular transformation of the common factors, $\mathbf{g}_t = G_n \mathbf{f}_t$ where $G_n = \frac{1}{\sqrt{n}} W'_n A_n$, we have (cfr. Section 4.1):

$$Q_n = \frac{1}{\sqrt{n}} \Gamma^{\chi}_{n0} W_n, D_n = W'_n \Gamma^{\chi}_{n1} W_n \Lambda^{-1}_n \text{ and } \Sigma_n = \frac{1}{n}\Lambda_n - \frac{1}{n} D_n \Lambda_n D'_n$$

Lemma 1 Under assumptions PA1-2, FM1-3, as $n, T \to \infty$, we have:

(i) $\mathrm{trace}\left[(\Gamma^{xT}_{kn} - \Gamma^{x}_{kn})^2\right] = O_p\left(\frac{n^2}{T}\right)$, $k = 0, 1$

(ii) $\frac{1}{n}\mu^{xT}_{nj} = \frac{1}{n}\mu^{\chi}_{nj} + O\left(\frac{1}{n}\right) + O_p\left(\frac{1}{\sqrt{T}}\right)$ for $k = 1, ..., n$

Proof. By assumption PA2, there exists a positive constant $K \leq \infty$, such that for all $T \in \mathbb{N}$ and $i, j \in \mathbb{N}$

$$T\mathrm{E}[(\hat{\gamma}^{xT}_{0ij} - \gamma^{x}_{0ij})^2] < K$$

as $T \to \infty$, where γ^{xT}_{0ij} and γ^{x}_{0ij} denote the i, jth entries of Γ^{xT}_{0n} and Γ^{x}_{0n} respectively.

We have:

$$\mathrm{trace}\left[(\Gamma^{xT}_{0n} - \Gamma^{x}_{0n})^2\right] = \sum_{i=1}^{n}\sum_{j=1}^{n}(\gamma^{xT}_{0ij} - \gamma^{x}_{0ij})^2$$

Taking expectations, we obtain:

$$\mathrm{E}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}(\gamma^{xT}_{0ij} - \gamma^{x}_{0ij})^2\right] = \sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{E}\left[(\gamma^{xT}_{0ij} - \gamma^{x}_{0ij})^2\right] = O_p\left(\frac{n^2}{T}\right)$$

Result (i), for $k = 0$, follows from the Markov inequality. The result for $k = 1$ can be easily proved using the same arguments.

Turning to (ii), from the Weyl inequality, we have:

$$\left(\mu^{xT}_{nj} - \mu^{x}_{nj}\right)^2 \leq \mathrm{trace}\left[(\Gamma^{xT}_{0n} - \Gamma^{x}_{0n})^2\right]$$

moreover, from assumption FM0-3:

$$\frac{1}{n}\mu^{x}_{nj} \leq \frac{1}{n}\mu^{\chi}_{nj} + \frac{1}{n}\mu^{\xi}_{n1} = \frac{1}{n}\mu^{\chi}_{nj} + O\left(\frac{1}{n}\right)$$

The desired result follows. *Q.E.D.*

Corollary 1 Under assumptions PA1-2, FM1-3, as $n, T \to \infty$, we have:

(i) $\frac{1}{n}\Lambda_n^T = \frac{1}{n}\Lambda_n + O_p(\frac{1}{\sqrt{T}}) + O_p\left(\frac{1}{n}\right)$

(ii) $W_n' W_n^T = I_r + O_p\left(\frac{1}{n}\right) + O_p\left(\frac{1}{\sqrt{T}}\right)$

Proof. Result (i) trivially follows from Lemma 1. Turning to (ii), we have the following decomposition:

$$\frac{1}{n}\Lambda_n^T = \frac{1}{n}W_n^{T\prime}\Gamma_{n0}^{xT}W_n^T = \frac{1}{n}W_n^{T\prime}W_n\Lambda_n W_n' W_n^T + \frac{1}{n}W_n^{T\prime}\Gamma_{n0}^{\xi T}W_n^T + \frac{1}{n}W_n^{T\prime}\left(\Gamma_{n0}^{xT} - \Gamma_{n0}^{\chi}\right)W_n^T$$

From results Lemma 1 (i) we get:

$$\frac{1}{n}W_n^{T\prime}\left(\Gamma_{n0}^{xT} - \Gamma_{n0}^{\chi}\right)W_n^T \leq \frac{1}{n}\sqrt{\text{trace}\left[(\Gamma_{0n}^{xT} - \Gamma_{0n}^{x})^2\right]} = O\left(\frac{1}{\sqrt{T}}\right)$$

Moreover, $W_n^{T\prime}\Gamma_{n0}^{\xi T}W_n^T \leq \mu_{n1}^{\xi} = O_p(1)$ by assumption FM3. The desired result follows. *Q.E.D.*.

Lemma 2 Under assumption PA1-2, FM1-FM3, as $n, T \to \infty$, we have:

(i) $Q_{ni}^T - Q_{ni} = O_p\left(\frac{1}{\sqrt{n}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right)$

(ii) $D_n^T - D_n = O_p\left(\frac{1}{\sqrt{n}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right)$

(iii) $\Sigma_n^T - \Sigma_n = O_p\left(\frac{1}{\sqrt{n}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right)$

where Q_{ni}^T and Q_{ni} denote the ith row of Q_n^T and Q_n, respectively.

Proof. Let us start from result (i). We have the following decomposition

$$Q_n^T = \frac{1}{\sqrt{n}}\Gamma_{n0}^{xT}W_n^T = \frac{1}{\sqrt{n}}\Gamma_{n0}^{\chi}W_n^T + \frac{1}{\sqrt{n}}\Gamma_{n0}^{\xi}W_n^T + \frac{1}{\sqrt{n}}\left(\Gamma_{n0}^{xT} - \Gamma_{n0}^{xT}\right)W_n^T$$

Write $\mathbf{1}_{ni}$ for the n dimensional vector with entries equal to zero at the ith position and zero for the rest. Consequently:

$$Q_{ni}^T = \mathbf{1}_{ni}'Q_n^T = \frac{1}{\sqrt{n}}\mathbf{1}_{ni}'\Gamma_{n0}^{xT}W_n^T = \frac{1}{\sqrt{n}}\mathbf{1}_{ni}'\Gamma_{n0}^{\chi}W_n^T + \frac{1}{\sqrt{n}}\mathbf{1}_{ni}'\Gamma_{n0}^{\xi}W_n^T + \frac{1}{\sqrt{n}}\mathbf{1}_{ni}'\left(\Gamma_{n0}^{xT} - \Gamma_{n0}^{xT}\right)W_n^T$$

Let us study separately each term of the right hand side. For the first term, Corollary 1 (ii), imply:

$$\frac{1}{\sqrt{n}}\mathbf{1}'_{ni}\Gamma^{\chi}_{n0}W^T_n = \frac{1}{\sqrt{n}}\mathbf{1}'_{ni}\Gamma^{\chi}_{n0}W_nW'_nW^T_n = Q_{ni}W'_nW^T_n = Q_{n1}+O_p\left(\frac{1}{n}\right)+O_p\left(\frac{1}{\sqrt{T}}\right)$$

since $W_nW'_nA_n = A_n$ by Assumption FM0.

For the second term, we have:

$$\frac{1}{\sqrt{n}}\mathbf{1}'_{ni}\Gamma^{\xi}_{n0}W^T_n \leq \frac{1}{\sqrt{n}}\sqrt{\mathbf{1}'_{ni}\Gamma^{\xi}_{n0}\mathbf{1}_{ni}}\sqrt{W^{T\prime}_n\Gamma^{\xi}_{n0}W^T_n} \leq \frac{1}{\sqrt{n}}\mu^{\xi}_{n1} = O_p\left(\frac{1}{\sqrt{n}}\right)$$

from assumption FM3.

Writing w^T_{jh} for the entry of W^T_n in the jth row and the hth columns, the third term can be written as:

$$\begin{aligned}\tfrac{1}{\sqrt{n}}\left|\mathbf{1}'_{ni}\left(\Gamma^{xT}_{n0}-\Gamma^{xT}_{n0}\right)W^T_n\right| &\leq \tfrac{1}{\sqrt{n}}\textstyle\sum_{h=1}^r\left|\sum_{j=1}^n(\gamma^{xT}_{0ij}-\gamma^{x}_{0ij})w^T_{jh}\right| \\ &\leq \tfrac{1}{\sqrt{n}}\textstyle\sum_{h=1}^r\sqrt{\sum_{j=1}^n(\gamma^{xT}_{0ij}-\gamma^{x}_{0ij})^2}\sqrt{\sum_{j=1}^n(w^T_{jh})^2} = \tfrac{1}{\sqrt{n}}\sum_{h=1}^r\sqrt{\sum_{j=1}^n(\gamma^{xT}_{0ij}-\gamma^{x}_{0ij})^2}\end{aligned}$$

since W^T_n is orthonormal. Because $\mathrm{E}\left[\sum_{j=1}^n(\gamma^{xT}_{0ij}-\gamma^{x}_{0ij})^2\right] = O_p\left(\frac{n}{T}\right)$, from the Markov's inequality, we get

$$\frac{1}{\sqrt{n}}\mathbf{1}'_{ni}\left(\Gamma^{xT}_{n0}-\Gamma^{xT}_{n0}\right)W^T_n = O_p\left(\frac{1}{\sqrt{T}}\right)$$

This proves result (i).

Turning to (ii), we have:

$$\frac{1}{n}D^T_n\Lambda^T_n = \frac{1}{n}W^{T\prime}_n\Gamma^{xT}_{n1}W^T_n = \frac{1}{n}W^{T\prime}_n\Gamma^{\chi}_{n1}W^T_n + \frac{1}{n}W^{T\prime}_n\Gamma^{\xi}_{n1}W^T_n + \frac{1}{n}W'_n(\Gamma^{xT}_{n1}-\Gamma^{x}_{n1})W_n$$

From result (ii) of Corollary 1, we have:

$$\frac{1}{n}W^{T\prime}_n\Gamma^{\chi}_{n1}W^T_n = \frac{1}{n}(W^{T\prime}_nW_n)W'_n\Gamma^{\chi}_{n1}W_n(W'_nW^T_n) = \frac{1}{n}D_n\Lambda_n + O_p\left(\frac{1}{n}\right) + O_p\left(\frac{1}{\sqrt{T}}\right)$$

since $W_nW'_nA_n = A_n$ by Assumption FM0.

By assumptions PA1-2 and FM3, $W^{T\prime}_n\Gamma^{\xi}_{n1}W^T_n = O_p(1)$. Moreover, Lemma 1 (i) implies that: $\frac{1}{n}W'_n(\Gamma^{xT}_{n1}-\Gamma^{x}_{n1})W_n = O_p(\frac{1}{\sqrt{T}})$. Result (ii), hence, follows from Corollary 1 (i) and Assumption FM2.

Finally, result (iii) is an immediate consequence of Lemma 1 (i) and result (ii) above.
Q.E.D.

Proof of Proposition 3
Note that the matrix Σ_n is of fixed dimension r. Because of continuity of the eigenvalues and eigenvectors with respect to the matrix entries, by Lemma 2 (iii) and the continuous mapping theorem we have

$$M_n^T = M_n + O_p\left(\frac{1}{\sqrt{n}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right) \quad \text{as } \ n, T \to \infty$$

and

$$K_n^T = K_n + O_p\left(\frac{1}{\sqrt{n}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right) \quad \text{as } \ n, T \to \infty$$

Continuity of the matrix product (notice that D_n has fixed dimension r), implies:

$$\left(D_n^T\right)^h = (D_n)^h + O_p\left(\frac{1}{\sqrt{n}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right) \quad \text{as } \ n, T \to \infty$$

Result (i) is hence an immediate consequence of Lemma 2 (i) and (ii).
Q.E.D.

Appendix 2: Data description and data treatment

	Database	Original Source	Variable Description	ID Code in the Database	Units	Orig. Freq.	Seas. Adj.	Treatment
1	MW	Citibase	Per Capita Real Consumption Expenditure					DLOG
2	MW	Citibase	Per Capita Gross Private Domestic Fixed Investment					DLOG
3	MW	Citibase	Per Capita Private Gross National product					DLOG
4	MW	Citibase	Per Capita Real M2 (M2 divided by P)					DLOG
5	MW	Citibase	3-Month Treasury Bill Rate					D
6	MW	Citibase	Implicit Price Deflator for Private GNP					DDLOG
7	Fred II	BEA	Real Gross Domestic Product, 1 Decimal	GDPC1	Bil. of Ch. 1996 $	Q	YES	DLOG
8	Fred II	BEA	Real Final Sales of Domestic Product, 1 Decimal	FINSLC1	Bil. of Ch. 1996 $	Q	YES	DLOG
9	Fred II	BEA	Real Gross Private Domestic Investment, 1 Decimal	GPDIC1	Bil. of Ch. 1996 $	Q	YES	DLOG
10	Fred II	BEA	Real State & Local Cons. Expend. & Gross Inv., 1 Dec.	SLCEC1	Bil. of Ch. 1996 $	Q	YES	DLOG
11	Fred II	BEA	Real Private Residential Fixed Investment, 1 Dec.	PRFIC1	Bil. of Ch. 1996 $	Q	YES	DLOG
12	Fred II	BEA	Real Private Nonresidential Fixed Investment, 1 Dec.	PNFIC1	Bil. of Ch. 1996 $	Q	YES	DLOG
13	Fred II	BEA	Real Nonresidential Inv.: Equipment & Software, 1 Dec.	NRIPDC1	Bil. of Ch. 1996 $	Q	YES	DLOG
14	Fred II	BEA	Real Imports of Goods & Services, 1 Decimal	IMPGSC1	Bil. of Ch. 1996 $	Q	YES	DLOG
15	Fred II	BEA	Real Federal Cons. Expend. & Gross Investment, 1 Dec.	FGCEC1	Bil. of Ch. 1996 $	Q	YES	DLOG
16	Fred II	BEA	Real Government Cons. Expend. & Gross Inv., 1 Dec.	GCEC1	Bil. of Ch. 1996 $	Q	YES	DLOG
17	Fred II	BEA	Real Fixed Private Domestic Investment, 1 Decimal	FPIC1	Bil. of Ch. 1996 $	Q	YES	DLOG
18	Fred II	BEA	Real Exports of Goods & Services, 1 Decimal	EXPGSC1	Bil. of Ch. 1996 $	Q	YES	DLOG
19	Fred II	BEA	Real Change in Private Inventories, 1 Decimal	CBIC1	Bil. of Ch. 1996 $	Q	YES	NONE
20	Fred II	BEA	Real Personal Cons. Expenditures: Nondurable Goods	PCNDGC96	Bil. of Ch. 1996 $	Q	YES	DLOG
21	Fred II	BEA	Real State & Local Government: Gross Investment	SLINVC96	Bil. of Ch. 1996 $	Q	YES	DLOG
22	Fred II	BEA	Real Personal Consumption Expenditures: Services	PCESVC96	Bil. of Ch. 1996 $	Q	YES	DLOG
23	Fred II	BEA	Real Personal Cons. Expenditures: Durable Goods	PCDGCC96	Bil. of Ch. 1996 $	Q	YES	DLOG
24	Fred II	BEA	Real Personal Consumption Expenditures	PCECC96	Bil. of Ch. 1996 $	Q	YES	DLOG
25	Fred II	BEA	Real National Defense Gross Investment	DGIC96	Bil. of Ch. 1996 $	Q	YES	DLOG
26	Fred II	BEA	Real Federal Nondefense Gross Investment	NDGIC96	Bil. of Ch. 1996 $	Q	YES	DLOG
27	Fred II	BEA	Real Disposable Personal Income	DPIC96	Bil. of Ch. 1996 $	Q	YES	DLOG
28	Fred II	BEA	Personal Cons. Expenditures: Chain-type Price Index	PCECTPI	Index 1996 = 100	Q	YES	DDLOG
29	Fred II	BEA	Gross Domestic Product: Chain-type Price Index	GDPCTPI	Index 1996 = 100	Q	YES	DDLOG
30	Fred II	BEA	Gross Domestic Product: Implicit Price Deflator	GDPDEF	Index 1996 = 100	Q	YES	DDLOG
31	Fred II	BEA	Gross National Product: Implicit Price Deflator	GNPDEF	Index 1996 = 100	Q	YES	DDLOG
32	Fred II	BEA	Gross National Product: Chain-type Price Index	GNPCTPI	Index 1996 = 100	Q	YES	DDLOG
33	Fred II	BLS	Nonfarm Business Sector: Unit Labor Cost	ULCNFB	Index 1996 = 100	Q	YES	DLOG
34	Fred II	BLS	Nonfarm Business Sector: Real Compensation Per Hour	COMPRNFB	Index 1992 = 100	Q	YES	DLOG
35	Fred II	BLS	Nonfarm Bus. Sector: Output Per Hour of All Persons	OPHNFB	Index 1992 = 100	Q	YES	DLOG
36	Fred II	BLS	Nonfarm Business Sector: Compensation Per Hour	COMPNFB	Index 1992 = 100	Q	YES	DLOG
37	Fred II	BLS	Manufacturing Sector: Unit Labor Cost	ULCMFG	Index 1992 = 100	Q	YES	DLOG
38	Fred II	BLS	Manufacturing Sector: Output Per Hour of All Persons	OPHMFG	Index 1992 = 100	Q	YES	DLOG
39	Fred II	BLS	Business Sector: Output Per Hour of All Persons	OPHPBS	Index 1992 = 100	Q	YES	DLOG
40	Fred II	BLS	Business Sector: Compensation Per Hour	HCOMPBS	Index 1992 = 100	Q	YES	DLOG
41	Fred II	St. Louis	St. Louis Adjusted Reserves	ADJRESSL	Bil. of $	M	YES	DLOG
42	Fred II	St. Louis	St. Louis Adjusted Monetary Base	AMBSL	Bil. of $	M	YES	DLOG
43	Fred II	Moody's	Moody's Seasoned Aaa Corporate Bond Yield	AAA	%	M	NO	D
44	Fred II	Moody's	Moody's Seasoned Baa Corporate Bond Yield	BAA	%	M	NO	D
45	Fred II	FR	Bank Prime Loan Rate	MPRIME	%	M	NO	D
46	Fred II	FR	3-Month Treasury Bill: Secondary Market Rate	TB3MS	%	M	NO	D
47	Fred II	FR	Currency in Circulation	CURRCIR	Bil. of $	M	NO	DD4LOG
48	Fred II	FR	Currency Component of M1	CURRSL	Bil. of $	M	YES	DDLOG
49	Fred II	BLS	CPI for All Urban Consumers: All Items Less Food	CPIULFSL	Ind. 1982-84 = 100	M	YES	DDLOG
50	Fred II	BLS	Consumer Price Index for All Urban Consumers: Food	CPIUFDSL	Ind. 1982-84 = 100	M	YES	DDLOG
51	Fred II	BLS	CPI For All Urban Consumers: All Items	CPIAUCSL	Ind. 1982-84 = 100	M	YES	DDLOG
52	Fred II	BLS	CPI: Intermediate Materials: Supplies & Components	PPIITM	Index 1982 = 100	M	YES	DDLOG
53	Fred II	BLS	Producer Price Index: Industrial Commodities	PPIIDC	Index 1982 = 100	M	NO	DDLOG
54	Fred II	BLS	PPI: Fuels & Related Products & Power	PPIENG	Index 1982 = 100	M	NO	DDLOG
55	Fred II	BLS	PPI Finished Goods: Capital Equipment	PPICPE	Index 1982 = 100	M	YES	DDLOG
56	Fred II	BLS	Producer Price Index: Finished Goods	PPIFGS	Index 1982 = 100	M	YES	DDLOG
57	Fred II	BLS	Producer Price Index: Finished Consumer Goods	PPIFCG	Index 1982 = 100	M	YES	DDLOG
58	Fred II	BLS	Producer Price Index: Finished Consumer Foods	PPIFCF	Index 1982 = 100	M	YES	DDLOG
59	Fred II	BLS	PPI: Crude Materials for Further Processing	PPICRM	Index 1982 = 100	M	YES	DDLOG
60	Fred II	BLS	Producer Price Index: All Commodities	PPIACO	Index 1982 = 100	M	NO	DLOG
61	Fred II	FR	Commercial and Industrial Loans at All Comm. Banks	BUSLOANS	Bil. of $	M	YES	DLOG
62	Fred II	FR	Total Loans and Leases at Commercial Banks	LOANS	Bil. of $	M	YES	DLOG
63	Fred II	FR	Total Loans and Investments at All Commercial Banks	LOANINV	Bil. of $	M	YES	DLOG
64	Fred II	FR	Total Consumer Credit Outstanding	TOTALSL	Bil. of $	M	YES	DLOG
65	Fred II	FR	Real Estate Loans at All Commercial Banks	REALLN	Bil. of $	M	YES	DLOG
66	Fred II	FR	Other Securities at All Commercial Banks	OTHSEC	Bil. of $	M	YES	DLOG
67	Fred II	FR	Consumer (Individual) Loans at All Comm. Banks	CONSUMER	Bil. of $	M	YES	DLOG
68	Fred II	BLS	All Employees: Construction	USCONS	Thous.	M	YES	DLOG
69	Fred II	BLS	Total Nonfarm Payrolls: All Employees	PAYEMS	Thous.	M	YES	DLOG
70	Fred II	BLS	Employees on Nonfarm Payrolls: Manufacturing	MANEMP	Thous.	M	YES	DLOG
71	Fred II	BLS	Unemployed: 16 Years & Over	UNEMPLOY	Thous.	M	YES	DLOG
72	Fred II	BLS	Civilian Unemployment Rate	UNRATE	%	M	YES	DLOG
73	Fred II	BLS	Civilian Participation Rate	CIVPART	%	M	YES	DLOG
74	Fred II	BLS	Civilian Labor Force	CLF16OV	Thous.	M	YES	DLOG
75	Fred II	BLS	Civilian Employment: Sixteen Years & Over	CE16OV	Thous.	M	YES	DLOG
76	Fred II	BLS	Civilian Employment-Population Ratio	EMRATIO	%	M	YES	DLOG

	Database	Original Source	Variable Description	ID Code in the Database	Units	Orig. Freq.	Seas. Adj.	Treatment
77	EconStats	FR	Industrial Production: total	Index		M	YES	DLOG
78	EconStats	FR	Industrial Production: Manufacturing (SIC-based)	Index		M	YES	DLOG
79	Datastream	ISM	ISM Manufacturers Survey: Supplier Delivery Index	USNAPMDL	Index	M	YES	NONE
80	Datastream	ISM	Chicago Purchasing Manager Business Barometer	USPMCUBB	%	M	NO	NONE
81	Datastream	ISM	ISM Manufacturers Survey: New Orders Index	USNAPMNO	Index	M	YES	NONE
82	Datastream	ISM	ISM Manufacturers Survey: Employment Index	USNAPMIV	Index	M	YES	NONE
83	Datastream	ISM	ISM Manufacturers Survey: Production Index	USNAPMEM	Index	M	YES	NONE
84	Datastream	ISM	ISM Purchasing Managers Index (MFG Survey)	USNAPMPR	Index	M	YES	NONE
85	Datastream	BC	Manufacturing Shipments - Total	USMNSHIPB	Bil. of $	M	YES	DLOG
86	Datastream	BC	Shipments of Durable Goods	USSHDURGB	Bil. of $	M	YES	DLOG
87	Datastream	BC	Shipments of Non-Durable Goods	USSHNONDB	Bil. of $	M	YES	DLOG
88	Datastream	S&P	Standard & Poor's 500 (monthly average)	US500STK	Index	M	NO	DLOG
89	Datastream	FT	Dow Jones Industrial Share Price Index	USSHRPRCF	Index	M	NO	DLOG

Abbreviations:
MW: Mark Watson's home page (http://www.wws.princeton.edu/ mwatson/publi.html)
Fred II: Fred II database of the Federal Reserve Bank of St. Louis
BEA: Bureau of Economic Analysis
BLS: Bureau of Labor Statistics
FR: Federal Reserve Board
St Louis: Federal Reserve Bank of St. Louis
ISM: Institute for Supply Management
BC: Bureau of Census
S&P: Standard & Poors'
FT: Financial Times
Q: Quarterly
M: Monthly (we take quarterly averages)

References

[1] Altug, S. (1989). Time-to-Build and Aggregate Fluctuations: Some New Evidence, *International Economic Review*, **30**, pp.889-920.

[2] Bai, J., and S. Ng (2002). Determining the number of factors in approximate factor models. *Econometrica* **70**, 191-221.

[3] Bai, J. (2003). Inferential Theory for Factor Models of Large Dimensions, *Econometrica*, 71, 135-171.

[4] Bernanke, B. S., and J. Boivin (2003). Monetary Policy in a Data Rich environment, *Journal of Monetary Economics* **50**, pp. 525-546.

[5] Bernanke, B. S., J. Boivin and P. Eliasz (2005). Measuring Monetary Policy: A Factor Augmented Autoregressive (FAVAR) Approach, *Quarterly Journal of Economics* **120**, pp.387-422.

[6] Boivin, J. and S. Ng, (2003), Are more data always better for factor analysis?, NBER Working Paper no. 9829. *Journal of Econometrics* forthcoming.

[7] Chamberlain, G. (1983). Funds, factors, and diversification in arbitrage pricing models. *Econometrica* **51**, 1281-1304.

[8] Chamberlain, G., and M. Rothschild (1983). Arbitrage, factor structure and mean-variance analysis in large asset markets. *Econometrica* **51**, 1305-1324.

[9] Chari, V. V., P. J. Kehoe and E. R. Mcgrattan (2005). A Critique of Structural VARs Using Real Business Cycle Theory, Federal Reserve Bank of Minneapolis Working no. 631.

[10] Connor, G. and R.A. Korajczyk (1988). Risk and return in an equilibrium APT. Application of a new test methodology. *Journal of Financial Economics* **21**, pp.255-89.

[11] Fernandez-Villaverde, J., J. Rubio-Ramirez and T. J. Sargent (2005). A, B, C's (and D)'s for Understanding VARs. NBER Technical Working Papers no. 0308.

[12] Forni, M., M. Hallin, M. Lippi, and L. Reichlin (2000). The generalized dynamic factor model: identification and estimation. *The Review of Economics and Statistics* **82**, 540-554.

[13] Forni, M., M. Hallin, M. Lippi, and L. Reichlin (2005). The generalized factor model: one-sided estimation and forecasting. *Journal of the American Statistical Association* **100** 830-40.

[14] Forni, M., and M. Lippi (2001). The generalized dynamic factor model: representation theory. *Econometric Theory* **17**, 1113-41.

[15] Forni, M. and L. Reichlin (1998). Let's get real: a factor analytical approach to disaggregated business cycle dynamics. *Review of Economic Studies*, **65**, 453-473.

[16] Geweke, J. (1977). The dynamic factor analysis of economic time series. In D.J. Aigner and A.S. Goldberger, Eds., *Latent Variables in Socio-Economic Models*, North Holland, Amsterdam.

[17] Geweke J. F., and K. J. Singleton (1981). Maximum Likelihood "Confirmatory" Factor Analysis of Economic Time Series, *International Economic Review*, **22**, pp.37-54.

[18] Giannone, D. and Reichlin, L. (2006). Does information help recovering structural shocks from past observations? *Journal of the European Economic Association, Papers and Procedings*, **4**, 455-65.

[19] Giannone, D., L. Reichlin and L. Sala (2002). Tracking Greenspan: Systematic and Nonsystematic Monetary Policy Revisited, CEPR Discussion Paper no. 3550.

[20] Giannone, D., L. Reichlin and L. Sala (2005). Monetary Policy in Real Time. In M. Gertler and K. Rogoff, Eds., *NBER Macroeconomic Annual, 2004*, MIT Press.

[21] Giannone, D., Reichlin, L. and L. Sala (2006). VAR's, Factor Models and the Empirical Validation of Equilibrium Business Cycle Models, *Journal of Econometrics*, **132,** 257-79.

[22] Hannan, E.J., and M. Deistler (1988). *The Statistical Theory of Linear Systems,* Wiley & Sons: New York.

[23] Hansen, L.P., and T.J. Sargent (1991) Two problems in interpreting vector autoregressions. In *Rational Expectations Econometrics*, L.P. Hansen and T.J. Sargent, eds. Boulder: Westview, pp.77-119.

[24] Kilian, L. (1998). Small-Sample Confidence Intervals for Impulse Response Functions *Review of Economics and Statistics* **80**, pp.218-30.

[25] King, R.G., C. I. Plosser, J. H. Stock and M.W. Watson (1991). Stochastic Trends and Economic Fluctuations *American Economic Review*, **81**, pp.819-40.

[26] Lippi, M., and L. Reichlin (1993). The dynamic effects of aggregate demand and supply disturbances: Comment. *American Economic Review* **83**, pp.644-52.

[27] Lippi, M., and L. Reichlin (1994). VAR analysis, non fundamental representation, Blaschke matrices. *Journal of Econometrics* **63**, pp.307-25.

[28] Quah, D., and Sargent, T. J. (1993) A Dynamic Index Model for Large Cross Sections, in J. Stock and M. Watson, Eds., *Business Cycle, Indicators and Forecasting*, University of Chicago Press and NBER, Chicago.

[29] Rozanov, Yu. (1963), *Stationary Random processes*, San Francisco: Holden Day.

[30] Rudebush, G.D. (1998) Do measures of monetary policy in a VAR make sense? *International Economic Review* **39**, pp.907-31.

[31] Sargent, T. J. (1989). Two Models of Measurements and the Investment Accelerator *The Journal of Political Economy*, **97**, pp.251-287.

[32] Sargent, T.J., C.A. Sims (1977). Business cycle modelling without pretending to have too much *a priori* economic theory. In C.A. Sims, Ed., *New Methods in Business Research*, Federal Reserve Bank of Minneapolis, Minneapolis.

[33] Stewart, G. W., and Ji-Guang Sun (1990), *Matrix Perturbation Theory*. Academic Press, Inc., San Diego.

[34] Stock, J.H., and M.W. Watson (2002a) Macroeconomic Forecasting Using Diffusion Indexes. *Journal of Business and Economic Statistics* **20**, pp.147-162.

[35] Stock, J.H., and M.W. Watson (2002b) Forecasting Using Principal Components from a Large Number of Predictors, *Journal of the American Statistical Association* **97**, pp.1167-79.

[36] Waerden, van der, B.L., (1953). *Modern Algebra*, Frederick Ungar: New York.

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