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Vlastimil Pták

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# OPENNESS OF LINEAR MAPPINGS IN *LF*-SPACES

VLASTIMIL PTÁK, Praha

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In a recent paper we have introduced and discussed the notion of orthogonality for pairs of subspaces of a locally convex space. There are many ways of characterizing this notion; these have been discussed at some length in [3]. For our purposes and for the particular case of Fréchet spaces, the following two will be sufficient.

Let $Y$ and $R$ be two closed subspaces of a Fréchet space $E$. We shall say that $Y$ and $R$ are *orthogonal* if one of the following two equivalent conditions is satisfied:

1° the natural mapping of $R \oplus Y$ onto $R + Y$ is open,

2° given two continuous linear functionals on $Y$ and $R$ which coincide on $R \cap Y$, there exists a continuous linear functional on $E$ which is their common extension.

Clearly the relation of orthogonality is symmetric; we shall write simply $Y \perp R$ or $R \perp Y$.

In the present note we intend to introduce, for $LF$-spaces, a notion which describes the position of a subspace with respect to the spaces of a defining sequence. If $R$ is a sequentially closed subspace of an $LF$-space $F$ such that – roughly speaking – $R$ is orthogonal to each element of a defining sequence for $F$, it turns out that $R$ has many nice properties. In particular, this notion enables us to formulate a simple condition for a sequentially open mapping to be open. This condition – given in section two – is essentially equivalent to condition (4,4) of [2]. However, using the notion of orthogonality, both the statement and the proof of the theorem become exceedingly simple and transparent.

Some remarks concerning terminology and notation. If $E$ is a locally convex space, we denote by $\mathsf{U}(E)$ the system of all closed absolutely convex neighbourhoods of zero in $E$. If $Y$ is a subspace of $E$, we denote by $P(Y)$ the operator which assigns to each $x' \in E'$ its restriction to $Y$. If $R$ is a subspace of an $LF$-space $F$, we say that $R$ is sequentially closed in $F$ if $x_n \in R$ and $\lim x_n = x$ implies $x \in R$. A continuous linear mapping of an $LF$-space $E$ into an $LF$-space $F$ is said to be sequentially open if its range is sequentially closed in $F$. For other equivalent descriptions of this notion see Proposition (3,2) of [2].

**(1,1) Proposition.** *If $R$ is a sequentially closed subspace of an LF-space $F$, the following conditions are equivalent:*

1° *there exists a defining sequence $F_j$ such that $F_j \perp (R \cap F_k)$ for each $k > j$;*

2° *for each defining sequence $F_j$ there exists a sequence $p(j)$ of natural numbers, $p(j) \geqq j$, with the following property: given $k > p(j)$, $y' \in F'_{p(j)}$ and $r' \in (R \cap F_k)'$ which coincide on $R \cap F_{p(j)}$, there exists an $x' \in F'_k$ such that $P(F_j)\, x' = P(F_j)\, y'$ and $P(R \cap F_k)\, x' = r'$;*

3° *for each defining sequence $F_j$ there exists another defining sequence $P_j \supset F_j$ such that $P_j \perp (R \cap P_k)$ for each $k > j$.*

*If one of these conditions is fulfilled we shall say that $R$ is orthogonal in $F$.*

Proof. Assume 1° and consider a defining sequence $H_j$. Given $j$, there exist indices $m, r$ such that

$$H_j \subset F_m \subset H_r .$$

Suppose that $k > r$ and consider a $y' \in H'_r$ and an $r' \in (R \cap H_k)'$ such that $y'$ and $r'$ coincide on $R \cap H_r$.

Take an $F_s \supset H_k$ so that $s > m$ and consider an extension $p'$ of $r'$ to $R \cap F_s$. Since $F_m \subset H_r$ the functionals $P(F_m)\, y'$ and $p'$ coincide on $R \cap F_m$. Condition 1° being satisfied, there exists an $x' \in F'_s$ such that $P(F_m)\, x' = P(F_m)\, y'$ and

$$P(R \cap F_s)\, x' = p' .$$

Put $z' = P(H_k)\, x'$ so that $z' \in H'_k$. Since $H_k \subset F_s$, we have

$$P(R \cap H_k)\, z' = P(R \cap H_k)\, P(H_k)\, x' = P(R \cap H_k)\, x' =$$
$$= P(R \cap H_k)\, P(R \cap F_s)\, x' = P(R \cap H_k)\, p' = r'$$

on the other hand, since $H_j \subset F_m \subset H_k$

$$P(H_j)\, z' = P(H_j)\, P(H_k)\, x' = P(H_j)\, x' = P(H_j)\, P(F_m)\, x' =$$
$$= P(H_j)\, P(F_m)\, y' = P(H_j)\, y' .$$

Hence it suffices to take $p(j) = r$ and condition 2° is satisfied.

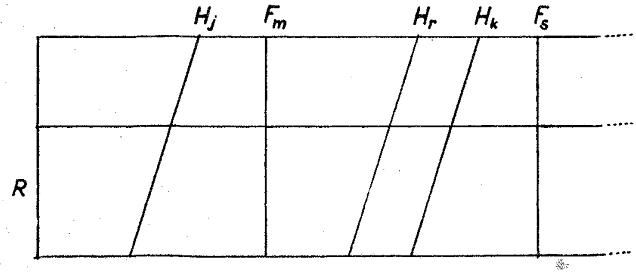


Fig. 1.

Suppose now that we are given a defining sequence $F_j$ which satisfies condition 2°. Define first an increasing sequence of natural numbers $z(n)$, $n \in N$, as follows: $z(1) = 1$ and $z(n + 1) = p(z(n)) + 1$. It follows that $z(n + 1) > z(n)$ and $p(z(n + 1)) \geqq z(n + 1) > p(z(n))$. Define $P_j$ as the closure in $F_{p(z(j))}$ of the space

$$F_{z(j)} + (R \cap F_{p(z(j))})\,;$$

it follows that

$$F_{z(j)} \subset P_j \subset F_{p(z(j))} \tag{1}$$

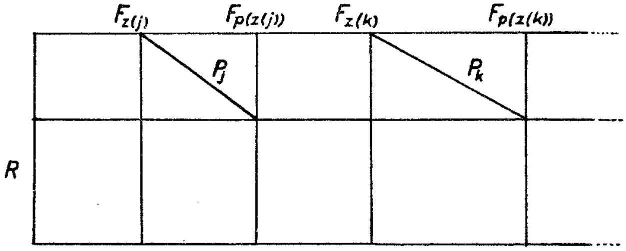


Fig. 2.

whence $R \cap P_j \subset R \cap F_{p(z(j))}$. Since, on the other hand, $P_j$ contains, by its definition, the intersection $R \cap F_{p(z(j))}$, we have

$$R \cap P_j = R \cap F_{p(z(j))}\,. \tag{2}$$

Let $k > j$ and consider a pair of functionals $y' \in P'_j$ and $r' \in (R \cap P_k)'$ such that

$$P(R \cap P_j)\, y' = P(R \cap P_j)\, r'\,. \tag{3}$$

Since $P_j \subset F_{p(z(j))}$, there exists a $u' \subset F'_{p(z(j))}$ such that $P(P_j)\, u' = y'$. We have $p(z(k)) > p(z(j))$ and two functionals $u' \in F'_{p(z(j))}$ and $r' \in (R \cap P_k)' = (R \cap F_{p(z(k))})'$ which coincide on $R \cap P_j = R \cap F_{p(z(j))}$. It follows from condition 2° that there exists an $x' \in F'_{p(z(k))}$ with the following properties

$$P(F_{z(j)})\, x' = P(F_{z(j)})\, u' \tag{4}$$

$$P(R \cap F_{p(z(k))})\, x' = r'\,. \tag{5}$$

Since $P_k \subset F_{p(z(k))}$, we may form $z' = P(P_k)\, x' \in P'_k$. According to (4), we have

$$P(F_{z(j)})\, z' = P(F_{z(j)})\, P(P_k)\, x' = P(F_{z(j)})\, x' = P(F_{z(j)})\, u' = \\ = P(F_{z(j)})\, P(P_j)\, u' = P(F_{z(j)})\, y'\,.$$

It follows from (5) and (3) that

$$P(R \cap F_{p(z(j))})\, z' = P(R \cap F_{p(z(j))})\, x' = P(R \cap F_{p(z(j))})\, r' =$$
$$= P(R \cap P_j)\, r' = P(R \cap P_j)\, y' .$$

It follows that $z' \in P'_k$ coincides with $y'$ on $F_{z(j)} + (R \cap F_{p(z(j))})$ and hence on $P_j$. Further, again by (5),

$$P(R \cap P_k)\, z' = P(R \cap F_{p(z(k))})\, x' = r' .$$

This completes the proof of the orthogonality of $P_j$ and $(R \cap P_k)$. Since 3° implies 1° immediately, the proof is complete.

We shall need the following simple lemma.

**(1,2)** *Let $H$ be an absolutely convex neighbourhood of zero in a locally convex space $E$. If $0 < \xi < 1$, then $\xi \bar{H} \subset H$.*

Proof. If $x \in \xi \bar{H}$, the set $x + (1 - \xi) H$ is a neighbourhood of $x$ and hence intersects $\xi H$. It follows that $x + (1 - \xi) h_1 = \xi h_2$ for suitable $h_1, h_2 \in H$. The set $H$ being absolutely convex, $h_3 = -h_1 \in H$ so that $x = \xi h_2 + (1 - \xi) h_3 \in H$.

## 2. OPEN MAPPINGS

**(2,1) Theorem.** *Let $E$ and $F$ be two LF-spaces and let $T$ be a continuous linear mapping of $E$ into $F$. Suppose that the following two conditions are satisfied:*

1° *the mapping $T$ is sequentially open,*

2° *the range of $T$ is orthogonal in $F$.*

*Then $T$ is open.*

Proof. Denote by $R$ the range of $T$ so that $R$ is sequentially closed by proposition (3,2) of [2]. Let $U \in \mathbf{U}(E)$ and let us show that there exists a $V \in \mathbf{U}(F)$ such that $V \cap R \subset TU$. We shall apply lemma (4,2) of [2]. Since $R$ is orthogonal in $F$ there exists, by (1,1), a defining sequence $F_j$ of $F$ such that $F_j \perp R_k$ for each $k > j$ where $R_k = R \cap F_k$. Since $T$ is sequentially open, there exists, by (3,2) of [2], for each $n$ a $H_n \subset \mathbf{U}(F)$ such that $H_n \cap R_n \subset TU$. Suppose now we are given $j$, $\varepsilon$, $V_j$ with the following properties: $0 < \varepsilon < 1$, $V_j \in \mathbf{U}(F)$ and $V_j \cap R_j \subset TU$. We intend to construct a $V_{j+1} \in \mathbf{U}(F)$ such that

$$(1) \qquad V_{j+1} \cap R_{j+1} \subset TU$$

$$(2) \qquad (1 - \varepsilon)(V_j \cap F_j) \subset V_{j+1} .$$

Take a $H_{j+1} \in \mathbf{U}(F)$ such that $H_{j+1} \cap R_{j+1} \subset TU$. Since $F_j \perp R_{j+1}$, the mapping

$F_j \oplus R_{j+1} \to F_j + R_{j+1}$ is open so that $(V_j \cap F_j) + (H_{j+1} \cap R_{j+1})$ is a neighbourhood of zero in $F_j + R_{j+1}$. Accordingly, there exists a $W \in \mathbf{U}(F)$ such that

$$W \cap (F_j + R_{j+1}) \subset (V_j \cap F_j) + (H_{j+1} \cap R_{j+1}) . \tag{3}$$

With $\sigma = \frac{1}{2}\varepsilon$ define

$$H = \operatorname{conv}\left((1 - \sigma)(V_j \cap F_j), \sigma W\right) .$$

Let us show now that $H \cap R_{j+1} \subset TU$. Suppose that $p \in H \cap R_{j+1}$ so that $p$ may be written in the form

$$p = \lambda \sigma w + (1 - \lambda)(1 - \sigma) z$$

with $0 \leqq \lambda \leqq 1$, $w \in W$, $z \in V_j \cap F_j$. If $\lambda = 0$, we have $p = (1 - \sigma) z \in V_j \cap F_j$ and $p \in R$ so that $p \in V_j \cap R_j \subset TU$. Hence we may suppose $\lambda > 0$. It follows that

$$\lambda \sigma w = -(1 - \lambda)(1 - \sigma) z + p \in F_j + R_{j+1}$$

whence $w \in F_j + R_{j+1}$ so that, according to (3), the vector $w$ may be written in the form $w = z_0 + Tu_0$ for a suitable $z_0 \in V_j \cap F_j$ and $u_0 \in U$. Since $0 < \varepsilon < 1$, we have $0 < \sigma/(1 - \sigma) < 1$ so that $z_{00} = [\sigma/(1 - \sigma)] z_0 \in V_j \cap F_j$ as well. We have thus

$$p = (1 - \lambda)(1 - \sigma) z + \lambda \sigma z_0 + \lambda \sigma T u_0 =$$

$$= (1 - \lambda)(1 - \sigma) z + \lambda(1 - \sigma) z_{00} + \lambda \sigma T u_0 = (1 - \sigma) z_{000} + \lambda \sigma T u_0$$

with $z_{000} \in V_j \cap F_j$. It follows that

$$z_{000} \in V_j \cap F_j \cap R \subset TU \text{ so that } z_{000} = Tu \text{ for some } u \in U.$$

Hence

$$p = T[(1 - \sigma) u + \sigma \lambda u_0] \in TU .$$

This proves the inclusion $H \cap R_{j+1} \subset TU$.

To sum up: we have constructed an absolutely convex set $H$ with the following properties:

(4) $H$ is a neighbourhood of zero if $F$ (since $H \supset \sigma W$)

$$(1 - \sigma)(V_j \cap F_j) \subset H . \tag{5}$$

$$H \cap R_{j+1} \subset TU \tag{6}$$

We intend to show now that it suffices to take $V_{j+1} = \xi \overline{H}$ where $\xi = (1 - \varepsilon)/(1 - \sigma)$. First of all, the inclusion $V_{j+1} \in \mathbf{U}(F)$ is obvious. Since $V_{j+1} = \xi \overline{H} \subset H$, we have $V_{j+1} \cap R_{j+1} \subset H \cap R_{j+1} \subset TU$ and

$$(1 - \varepsilon)(V_j \cap F_j) = \xi(1 - \sigma)(V_j \cap F_j) \subset \xi H \subset \xi \overline{H} = V_{j+1} .$$

The proof is complete.

**(2,2) Corollary.** *Let $F_j$ be an increasing sequence of Fréchet spaces such that the topology of $F_{j+1}$ induces the topology of $F_j$ on $F_j$. Let $(F, u)$ be the inductive* *limit of the sequence $F_j$. Let $R$ be a sequentially closed subspace of $F$. Let $(R, v)$ be the inductive limit of the sequence $R \cap F_j$ so that $v$ is finer than the restriction $u_R$ of $u$ to $R$. If $R$ is orthogonal in $F$ then $v = u_R$.*

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*Author's address:* Praha 1, Žitná 25, ČSSR (Matematický ústav ČSAV).