AD-786 371

OPERATING CHARACTERISTICS FOR DETECTION OF A FADING SIGNAL IN M ALTERNATIVE LOCATIONS WITH D-FOLD DIVERSITY

Albert H. Nuttall

Naval Underwater Systems Center
New London, Connecticut

20 August 1974

PREFACE

This report was prepared under NUSC Project No. TA-176-92, "IACS Signal Design," Principal Investigator, Dr. A. J. Van Woerkom (Code TC), sponsoring activity NAVSEA (SEA-06H), Program Manager, J. F. Calabrese; and also under NUSC Project No. A-752-05, "Applications of Statistical Communication Theory to Acoustic Signal Processing," Principal Investigator, Dr. A. H. Nuttall (Code TC), Navy Subproject No. ZF-61-112-001, sponsoring activity Chief of Naval Material, Program Manager, Dr. J. H. Huth.

The Technical Reviewer for this report was Dr. A. J. Van Woerkom (Code TC).

REVIEWED AND APPROVED: 20 August 1974

W. A. Von Winkle
Director, Science & Technology

The author of this report is located at the New London Laboratory, Naval Underwater Systems Center, New London, Connecticut 06320

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER TR 4793	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER AD 786 371
4. TITLE (and Subtitle) OPERATING CHARACTERISTICS FOR DETECTION OF A FADING SIGNAL IN M ALTERNATIVE LOCATIONS WITH D-FOLD DIVERSITY		5. TYPE OF REPORT & PERIOD COVERED 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Albert H. Nuttall		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Underwater Systems Center New London Laboratory New London, Connecticut 06320		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS TA-176-92, A-752-05, ZF 61-112-001
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Sea Systems Command (SEA 06H) and Chief of Naval Material Washington, D. C. 20360		12. REPORT DATE 20 August 1974 13. NUMBER OF PAGES 68
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Detection of fading signal
Multiple alternatives
Diversity
Correct-decision probability
False-alarm probability
Optimum energy fractionalization

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The probabilities of correct decision and false alarm for a fading signal in M alternative locations with D-fold diversity are derived and numerically evaluated over the range of values M = 1, 2, 4, 16, 64, 256, 1024 and D = 1, 2, 3, 4, 6, 8 12, 16, 24, 32, 48, 64 for a wide range of signal-energy-to-noise-density-level ratios (ENR). These results, which apply to the optimum receiver processor, have relevance to synchronization as well as to multiple alternative communica-

20. (Cont'd).

tion with thresholding. It is found that the optimum order of diversity does not correspond to the familiar 5 dB ENR per branch, but rather to ENR branch values as large as 14 dB for some cases of small false-alarm probability and moderate correct-decision probability. However, for error probabilities approximately equal to the false-alarm probability, the optimum ENR per branch approaches 5 dB. The required total received signal energy increases very slowly with M; for example, for a wide range of values of correct-decision and false-alarm probabilities, a 1-dB increase suffices for the change from M = 32 to M = 1024.

TABLE OF CONTENTS

ib

LIST OF ILLUSTRATIONS

LIST OF ILLUSTRATIONS (Cont'd)

LIST OF ABBREVIATIONS

Prob	Probability
max	Maximum
ENR	Signal-energy-to-noise-density-level ratio

LIST OF SYMBOLS

M	Number of alternative signal locations
D	Order of diversity
x_k	k-th decision variable, $1 \leq k \leq M$
Λ	Threshold
P_{FA}	Probability of false alarm
$\|N$	Conditioned on noise alone
P_{MD}	Probability of missed decision
$\|S + N$	Conditioned on signal and noise present
P_{ID}	Probability of incorrect decision
P_{CD}	Probability of correct decision
$P_e(S+N)$	Probability of error for signal and noise present
$P_e(N)$	Probability of error for noise alone
$p_0(\cdot)$	Probability density function of decision variable for noise alone
$p_1(\cdot)$	Probability density function of decision variable for signal and noise present
$P_0(\cdot)$	Cumulative distribution function of decision variable for noise alone
$P_1(\cdot)$	Cumulative distribution function of decision variable for signal and noise present
y_{md}	Envelope sample on m-th alternative, d-th diversity branch
$\mathbf{z}$	Observation vector
$\text{Prob}\{k\|\mathbf{z}\}$	A posteriori probability of alternative k
$\text{Prob}\{k\}$	A priori probability of alternative k
$p(\mathbf{z}\|k)$	Conditional probability density function of $\mathbf{z}$

LIST OF SYMBOLS (Cont'd)

$p(\mathbf{z})$	Unconditional probability density function of $\mathbf{z}$
σ_n^2	Common noise power level in all branches
σ_s^2	Common signal power level in D occupied branches
Q	A priori probability of alternative $k = 0$, no signal
R	σ_s^2/σ_n^2, $\bar{E}_1/N_0$
$\bar{E}_1$	Average received signal energy per branch
$\bar{E}_T$	Average total received signal energy on all D branches
N_0	Single-sided noise power density level
E	Received signal energy for nonfluctuating signal
d	$(2E/N_0)^{1/2}$
I_0	Modified Bessel function of order zero
Q	Q-function

OPERATING CHARACTERISTICS FOR DETECTION OF A FADING SIGNAL IN M ALTERNATIVE LOCATIONS WITH D-FOLD DIVERSITY

INTRODUCTION

Before the transfer of information between a transmitter and receiver can take place, synchronization is necessary; that is, the receiver must ascertain the time delay and frequency shift of the received signal before correct decisions about the information content of an ensuing message can be made. For large a priori uncertainties about the transmitter's range and relative velocity, the receiver must conduct a search — during this alert phase — of the numerous possible locations (time-delay frequency-shift cells) to determine the precise time delay and frequency shift of the received signal, and to determine, in fact, whether a signal is present at all. Furthermore, in order to combat the possibility of signal fading, the transmitter may employ diversity by dividing the available signal energy into a number of branches. The receiver must then know the order and pattern of diversity and combine the energies of the appropriate diversity branches before making a decision of absence or presence and location of a possible signal. The number of locations that the received signal may occupy, if present, is denoted by M; the order of diversity employed is denoted by D. The objective of this report is to determine the performance (in terms of appropriate probabilities of various types of errors) of the synchronization procedure as a function of M, D and the received signal and noise power levels. Additionally, the optimum order of diversity and optimum fractionalization of available signal energy for specified performance are to be determined.

If the arrival angle of the potential signal is not known to the receiver, a search must be conducted in this variable as well as in time delay and frequency shift. If the parameter M is increased to include this additional uncertainty, the case is subsumed under the earlier framework, where M denotes the number of potential signal locations, whether they be cells in time delay, frequency shift, or arrival angle. More generally, M is simply the number of locations in which a signal can be found, if present, regardless of the physical cause of uncertainty.

Although the discussion has been couched in a detection context, the framework also covers multiple alternative communication with thresholding. For example, suppose that either no signal is transmitted or one of M signals is transmitted. Then, for fading signals and D-fold diversity, the results to be presented are directly applicable to determining the error probabilities. As particular cases, $M = 1$ corresponds to on-off (binary) communication with diversity; $M = 2$ and a zero threshold corresponds to binary frequency-shift-keying with diversity; and $M > 2$ and a zero threshold corresponds to M-ary communication with diversity, as previously analyzed[1] and numerically evaluated.[2] For $M = 1$, some results on this problem are available[3] (when D here is identified with M there).

To develop the subject, two types of probabilities of mistakes are defined, and generic expressions for their values are presented. Then three different signal and noise models — all of which yield the same statistics for the observed variables — are described, and the optimum processor is derived. Finally, the actual evaluation of the correct-decision and false-alarm probabilities is undertaken and plotted, and relevant observations and conclusions are extracted from the numerical results.

DEFINITIONS OF ERROR PROBABILITIES

For a signal that can be found in one of M locations, let $x_1, \dots, x_M$ be the M decision variables upon which a decision must be reached as to whether signal is absent or whether a particular signal is present. If D-fold diversity is employed, the variable x_k is an appropriate combination (as yet undefined) of the D branch outputs utilized by signal number k. If threshold Λ is utilized, it is decided that

$$\left\{\begin{array}{l}\text{signal is absent if } \max(x_1, \dots, x_M) < \Lambda \\ \text{signal } k \text{ is present if } \max(x_1, \dots, x_M) = x_k > \Lambda\end{array}\right\}. \quad (1)$$

Then the probability of false-alarm is given by

$$P_{FA} = 1 - \text{Prob}\left\{\max(x_1, \dots, x_M) < \Lambda \,|\, N\right\}, \quad (2)$$

where $|N$ denotes conditioning on noise-alone present (that is, no signal present). In words, (2) is the probability that one or more of the M decision variables exceeds the threshold for noise-alone present.

When a signal is present, two types of mistakes can occur. In order to express these quantities, without loss of generality, let signal number 1 be transmitted. The probability of a missed decision is defined as

$$P_{MD} = \text{Prob}\left\{\max(x_1, \dots, x_M) < \Lambda \,|\, S+N\right\}, \quad (3)$$

where $|S+N$ denotes conditioning on signal-plus-noise present. That is, it is erroneously decided that no signal is present because the threshold was not exceeded.

On the other hand, the probability of an incorrect decision is defined as

$$P_{ID} = \sum_{k=2}^{M} \text{Prob}\left\{\max(x_1, \ldots, x_M) = x_k > \Lambda \,|\, S+N\right\} \quad . \tag{4}$$

That is, one of the wrong decision variables dominates and exceeds the threshold, thereby leading to the wrong decision about which signal was transmitted.*

Finally, the probability of a correct decision is defined as

$$P_{CD} = \text{Prob}\left\{\max(x_1, \ldots, x_M) = x_1 > \Lambda \,|\, S+N\right\} . \tag{5}$$

This is the probability that the correct decision variable is largest and exceeds the threshold. Since one of the events described in (3), (4), or (5) must happen for signal present, we have

$$P_{MD} + P_{ID} + P_{CD} = 1 \quad . \tag{6}$$

An alternative form of (6) is

$$P_{MD} + P_{ID} = 1 - P_{CD} \quad , \tag{7}$$

where the left side of (7) is the sum of the probabilities of the two types of mistakes when signal is present.

An average error probability could be defined (if desired) in the following manner: an error, for a signal transmitted, occurs with probability

$$P_e\,(S+N) = P_{MD} + P_{ID} = 1 - P_{CD} \quad . \tag{8}$$

An error, for no signal transmitted, occurs with probability

$$P_e\,(N) = P_{FA} \quad . \tag{9}$$

*An on-off communications system corresponds to $M = 1$; in this case, P_{ID} is not defined (that is, it has no meaning).

Then the average error probability is

$$P_e = P_e(S+N)\,\text{Prob}\,\{S+N\} + P_e(N)\,\text{Prob}\,\{N\}$$
$$= (1 - P_{CD})\,\text{Prob}\,\{S+N\} + P_{FA}\,\text{Prob}\,\{N\}\,, \tag{10}$$

where Prob $\{S+N\}$ and Prob $\{N\}$ are the a priori probabilities.

INTEGRAL EXPRESSIONS FOR PROBABILITIES

Analytic evaluation of the probabilities in (2) through (5) is generally impossible unless the variables $x_1, \ldots, x_M$ are all statistically independent of each other. From this point on, this crucial assumption of independence will be adopted. It is also assumed that the random variables $\{x_k\}$ not containing a signal have identical probability density functions $p_o(\cdot)$. The random variable containing a signal will have probability density function $p_1(\cdot)$. The cumulative distribution functions are denoted by $P_o(\cdot)$ and $P_1(\cdot)$, respectively.

The quantities in (2) through (5) then follow immediately:

$$P_{FA} = 1 - \left[\int_{-\infty}^{\Lambda} dx\; p_o(x)\right]^M = 1 - \left[P_o(\Lambda)\right]^M\,, \tag{11}$$

$$P_{MD} = \int_{-\infty}^{\Lambda} dx\; p_1(x) \left[\int_{-\infty}^{\Lambda} dy\; p_o(y)\right]^{M-1} = P_1(\Lambda)\left[P_o(\Lambda)\right]^{M-1}\,, \tag{12}$$

$$P_{ID} = (M-1)\int_{\Lambda}^{\infty} dx\; p_o(x) \int_{-\infty}^{x} dy\; p_1(y) \left[\int_{-\infty}^{x} dz\; p_o(z)\right]^{M-2}$$
$$= (M-1)\int_{\Lambda}^{\infty} dx\; p_o(x)\; P_1(x) \left[P_o(x)\right]^{M-2}\,, \tag{13}$$

and

$$P_{CD} = \int_{\Lambda}^{\infty} dx\; p_1(x) \left[\int_{-\infty}^{x} dy\; p_o(y)\right]^{M-1} = \int_{\Lambda}^{\infty} dx\; p_1(x) \left[P_o(x)\right]^{M-1}. \tag{14}$$

(As a check, notice that

$$P_{ID} + P_{CD} = \int_{\Lambda}^{\infty} dx \frac{d}{dx} \left\{ P_1(x) \left[P_o(x) \right]^{M-1} \right\} = 1 - P_1(\Lambda) \left[P_o(\Lambda) \right]^{M-1} = 1 - P_{MD}, \quad (15)$$

which agrees with (6).)

Of the four probabilities defined in (2) through (5), only three are independent; see (6). Furthermore, P_{FA} and P_{MD} can be found very simply from (11) and (12) once the cumulative probability distributions $P_0(\cdot)$ and $P_1(\cdot)$ are known. If (14) is numerically integrated in order to determine P_{CD}, P_{ID} is immediately available by use of (6). The discussion, then, will be concentrated on the two probabilities P_{FA} and P_{CD}; these are also the only two probabilities needed to evaluate the average error probability defined in (10). Therefore, only one numerical integration, that in (14), need be conducted. An error analysis for (14) and some interrelationships of the probabilities in (11) through (14) are presented in appendix A.

SIGNAL AND NOISE MODELS AND THE OPTIMUM PROCESSOR

In the previous section, the M decision variables $x_1, \ldots, x_M$ were presumed to be available for comparison with a threshold. Now we will backtrack to ascertain how these decision variables can be determined in the first place. The first model to consider is that in which the received signal consists of narrowband Gaussian processes on D statistically independent branches. The particular D branches occupied by the transmission of one of the M alternatives do not overlap any of the (M - 1) D branches utilized for the other signal alternatives. All MD branches are subject to additive narrowband Gaussian noise.

Let y_{md} denote a sample of the envelope of the received narrowband process on the m-th alternative signal location and the d-th diversity branch (after narrowband filtering to the expected signal bandwidth), where $1 \leq m \leq M$ and $1 \leq d \leq D$. It is presumed that all MD random variables $\{y_{md}\}$ are independent of each other. Then for the observation vector

$$\mathbf{z} = \left[y_{11} \cdots y_{1D} \cdots y_{M1} \cdots y_{MD} \right], \quad (16)$$

the a posteriori probability of alternative k (after observation of $\mathbf{z}$) is

$$\text{Prob}\,\{k\,|\,\mathbf{z}\} = \frac{\text{Prob}\,\{k\}\, p(\mathbf{z}\,|\,k)}{p(\mathbf{z})}, \quad 0 \leq k \leq M, \quad (17)$$

where alternative $k = 0$ denotes no signal, Prob $\{k\}$ is the a priori probability of alternative k, and $p(\mathbf{z}|k)$ and $p(\mathbf{z})$ are the conditional and unconditional probability density functions, respectively, of $\mathbf{z}$. The denominator of (17) can be expressed as

$$p(\mathbf{z}) = \sum_{k=0}^{M} \text{Prob}\ \{k\}\ p(\mathbf{z}|k) \tag{18}$$

if desired.

Under the conditions described above, one has

$$p(\mathbf{z}|0) = \prod_{m=1}^{M} \prod_{d=1}^{D} \left\{ \frac{y_{md}}{\sigma_n^2} \exp\left(-\frac{y_{md}^2}{2\sigma_n^2}\right) \right\}, \quad y_{md} > 0 , \tag{19}$$

where σ_n^2 is the common noise power level in all MD branches. And, for $k \geq 1$,

$$p(\mathbf{z}|k) = \prod_{\substack{m=1 \\ m \neq k}}^{M} \prod_{d=1}^{D} \left\{ \frac{y_{md}}{\sigma_n^2} \exp\left(-\frac{y_{md}^2}{2\sigma_n^2}\right) \right\} \prod_{d=1}^{D} \left\{ \frac{y_{kd}}{\sigma_s^2 + \sigma_n^2} \exp\left(-\frac{y_{kd}^2}{2(\sigma_s^2 + \sigma_n^2)}\right) \right\}, \tag{20}$$

where σ_s^2 is the common signal power level in the D occupied branches. A more convenient form of (20) that is available is (by use of (19))

$$p(\mathbf{z}|k) = p(\mathbf{z}|0) \left(\frac{\sigma_n^2}{\sigma_s^2 + \sigma_n^2} \right)^D \exp\left[\frac{1}{2} \frac{\sigma_s^2}{\sigma_s^2 + \sigma_n^2} \sum_{d=1}^{D} y_{kd}^2 \right], \quad k \geq 1 . \tag{21}$$

Now it is presumed that the a priori probabilities in (17) satisfy the following rules:

$$\text{Prob}\ \{0\} = Q; \quad \text{Prob}\ \{k\} = \frac{1-Q}{M}, \quad 1 \leq k \leq M ; \tag{22}$$

that is, the a priori probabilities of all the signal alternatives are equal. Then selection of the maximum a posteriori probability in (17) is tantamount to picking the maximum:

$\max_{0 \le k \le M}$ Prob $\{k \mid \mathbf{z}\}$ corresponds to

$$\max\left(Q, \left\{\frac{1-Q}{M}\left(\frac{\sigma_n^2}{\sigma_s^2+\sigma_n^2}\right)^D \exp\left[\frac{1}{2}\frac{\sigma_s^2}{\sigma_s^2+\sigma_n^2}\sum_{d=1}^{D} y_{kd}^2\right]\right\}_1^M\right). \quad (23)$$

But this is equivalent to

$$\max_{1 \le k \le M} \sum_{d=1}^{D} y_{kd}^2 \gtrless \Lambda , \quad (24)$$

where threshold Λ absorbs the a priori probabilities, the signal and noise levels, and D and M. Satisfaction of the upper inequality in (24) leads to a decision of signal present on the particular alternative that dominates the left side of (24); satisfaction of the lower inequality in (24) leads to the statement that no signal is present in any of the M alternative locations.* Physically, (24) indicates that the squared envelopes on all D diversity branches for a particular alternative should be summed, and the largest sum (out of the M alternatives) should be compared with a threshold. These sums are the decision variables mentioned in (1) through (5). For a specified false-alarm probability P_{FA}, only σ_n^2 need be known in order to assign a value to Λ; σ_s^2 need not be known (but, of course, the detection probabilities do depend on σ_s^2).

It should be noted that if the cost of mistaking signal location k for signal location j is independent of k and j, and if the cost of mistaking signal location k for no signal is independent of k, the minimum average cost results when the identical test (24) is utilized. Of course, threshold Λ then involves these costs also.

A second signal and noise model that leads to the same results as those above is that of envelopes of narrowband deterministic (except for phase) signals subject to slow Rayleigh fading and additive narrowband Gaussian noise. For example, $\{y_{md}\}$ could be samples of the envelopes of the outputs of narrowband matched filters that are nonoverlapping in time or nonoverlapping in

*This derivation parallels that in reference 1, section IV, very closely.

frequency. The signal strengths in the D branches are assumed to fade independently, and the signal strength changes only slightly in a time interval equal to a narrowband-filter impulse-response duration. The quantity σ_s^2 is the matched-filter output signal power per branch (averaged over the Rayleigh fading statistics). The relevant statistics are identical to those given in (19) and (20) [see, for example, reference 4, section 7.5].

A third signal model is afforded by samples of a zero-mean Gaussian process (rather than by envelopes). If the number of samples on each alternative is even, such as 2D, then it may be shown that the maximum a posteriori probability processor takes on exactly the form of the test in (24), where each y_{kd}^2 is equal to a sum of squares of two zero-mean Gaussian variates.

Thus test (24) is appropriate for three different signal and noise models; its performance is determined in the next section.

PERFORMANCE RESULTS

The false-alarm and correct-decision probabilities are derived in appendix B and are given by

$$P_{FA} = 1 - \left[1 - \exp(-\Lambda) \sum_{k=0}^{D-1} \Lambda^k/k!\right]^M \tag{25}$$

and

$$P_{CD} = \int_{\Lambda}^{\infty} dx \frac{x^{D-1}}{(D-1)!\,(1+R)^D} \exp\left(-\frac{x}{1+R}\right)\left[1 - \exp(-x) \sum_{k=0}^{D-1} x^k/k!\right]^{M-1}, \tag{26}$$

where the signal-to-noise ratio per diversity branch is

$$R = \frac{\sigma_s^2}{\sigma_n^2} . \tag{27}$$

[The result in (26) is a generalization of equation (19) in reference 1 in that the lower limit of integration is not zero.]

The parameter R in (27) can be interpreted differently for the second signal model of slow Rayleigh fading signals. For matched filtering of each of the D diversity branches, it is shown in appendix C that

$$R = \frac{\bar{E}_1}{N_o} = \frac{\bar{E}_T/N_o}{D}, \tag{28}$$

where $\bar{E}_1$ is the average received signal energy on one branch, $\bar{E}_T$ is the average total received signal energy on all D branches, and N_o is the single-sided noise power density level. The curves to follow will employ the signal-energy-to-noise-density-level ratio (ENR) parameters in (28).

In figures 1 through 35,* the probability of correct-decision P_{CD} is plotted versus $\bar{E}_1/N_o$ in dB,† with diversity D as a parameter (solid curves), for $M = 1, 2, 4, 16, 64, 256, 1024$ and $P_{FA} = 10^{-n}$, $n = 2, 3, 4, 6, 8$. The dashed curves on each figure connect points of equal total signal-energy-to-noise-density-level ratio $\bar{E}_T/N_o$. The first point to observe from these detection characteristics is that, for a given available $\bar{E}_T/N_o$, there is an optimum order of diversity to attain the maximum value of P_{CD} (for a specified M and P_{FA}), as exemplified by the peaks of the dashed curves. However, the maximum P_{CD} is not always realized when the ENR per branch is 5 dB,[2] but, rather, it can be significantly larger than this value. For example, in figure 5 for $P_{FA} = 10^{-8}$, $M = 1$, the optimum order of diversity for $\bar{E}_T/N_o$ = 13 dB is $D = 1$; therefore the optimum ENR per branch is 13 dB, realizing a value of $P_{CD} = 0.42$. And in figure 35 for $P_{FA} = 10^{-8}$, $M = 1024$, the same behavior occurs for $\bar{E}_T/N_o$ = 14 dB, yielding $P_{CD} = 0.38$.

The feature of large ENRs per branch being optimum is characteristic of the smaller values of P_{CD} (that is, $P_{CD} < 0.5$). For larger values of $\bar{E}_T/N_o$, where values of P_{CD} near unity are attainable, the optimum ENR per branch more nearly approaches 5 dB, and the optimum order of diversity increases. However, in figure 35 for $P_{FA} = 10^{-8}$, even as large a value of P_{CD} as 0.9999 still requires an optimum ENR per branch of $\bar{E}_1/N_o$ = 7 dB, not 5 dB. Only when the two "transition" probabilities — $(1 - P_{CD})$ and P_{FA} of (8) and (9), respectively — are approximately equal does the optimum ENR per branch approach 5 dB. But for large M (such as 1024) the optimum ENR per branch is still larger than 5 dB, even for $1 - P_{CD} \approx P_{FA}$.

---

*All figures are grouped together beginning on page 13.
†That is, $10 \log_{10} (\bar{E}_1/N_o)$.

In figures 36 through 38, the minimum required values of $\bar{E}_T/N_o$ in dB are plotted versus M, with P_{FA} as a parameter, for $P_{CD} = 0.5$, 0.9, and 0.99. The results are extracted from figures 1 through 35; results for other values of P_{CD}, such as 0.9999, can also be obtained. The slight irregularities in the curves are caused by an inability to determine the dB levels from figures 1 through 35 more accurately than 0.1 dB. The integer numbers under each curve denote the approximate optimum order of diversity at which the minimum $\bar{E}_T/N_o$ is realized.

One of the most striking features of figures 36 through 38 is the slow rate of increase with M of the minimum $\bar{E}_T/N_o$ required. (It must be remembered that M is the number of alternative signal locations, not the number of signal observations; this latter quantity is represented by D, the order of diversity.) For example, at the high-quality performance level of $P_{FA} = 10^{-8}$, $P_{CD} = 0.99$, a 1 dB increase suffices over the range of M from 1 to 1024. At the more moderate performance level of $P_{FA} = 10^{-4}$, $P_{CD} = 0.5$, a 2.5 dB increase in $\bar{E}_T/N_o$ is required over the same range of M.

The optimum order of diversity depends very strongly on the desired value of P_{CD} but is somewhat less dependent on the specified P_{FA}. Figures 36 through 38 show that the optimum diversity is 1 to 2 for $P_{CD} = 0.5$, 5 to 10 for $P_{CD} = 0.9$, and 11 to 19 for $P_{CD} = 0.99$. If the optimum order of diversity is not attainable, due perhaps to limited available bandwidth, figures 1 through 35 indicate the additional amount of signal energy necessary to realize the specified level of performance.

A sample program for the generation of the detection characteristics in figures 1 through 35 is furnished in appendix D. Extension to other ranges of P_{FA}, D, or M is possible by modification of this program.

DISCUSSION

For nonfluctuating signals received under phase-incoherent conditions and with D = 1,* the probability of correct decision is given [reference 5, (78), with crosscorrelation coefficient $\lambda = 0$] by

$$P_{CD} = \int_{\Lambda}^{\infty} dx\, x \exp\left(-\frac{x^2 + d^2}{2}\right) I_o(dx) \left[1 - \exp\left(-\frac{x^2}{2}\right)\right]^{M-1} , \qquad (29)$$

*No diversity is needed for nonfluctuating signals.

where

$$d = \sqrt{\frac{2E}{N_o}} \tag{30}$$

and E is the received signal energy. The false-alarm probability is [reference 5, (77)]

$$P_{FA} = 1 - \left[1 - \exp(-\Lambda^2/2)\right]^M . \tag{31}$$

(These results are identical in form to (14) and (11), respectively; only the densities $p_1(\cdot)$ and $p_0(\cdot)$ have changed.) Now, (31) could be solved explicitly for Λ in terms of P_{FA} and M, and then (29) could be numerically integrated to evaluate P_{CD}. Alternatively, the bracketed term in (29) can be expanded in a binomial series, yielding

$$P_{CD} = \frac{1}{M} \exp\left(-\frac{d^2}{2}\right) \sum_{n=1}^{M} (-1)^{n-1} \binom{M}{n} \exp\left(\frac{d^2}{2n}\right) Q\left(\frac{d}{\sqrt{n}}, \Lambda\sqrt{n}\right), \tag{32}$$

where $Q(\cdot,\cdot)$ is the Q function [reference 4, (4-55)]. However, the alternating sequence of large numbers in (32) will suffer loss of significance, and (32) may be useless for large M. As a final alternative, the upper and lower bounds of (A-10) could be used for evaluation of P_{CD}.

A comparison of the performance of the nonfluctuating signal results, (29) through (32), with the present results has not been pursued any further except for M = 1. In this case, (31) and (32) reduce to

$$P_{FA} = \exp(-\Lambda^2/2) \tag{33}$$

and

$$P_{CD} = Q(d, \Lambda) = Q\left(\sqrt{\frac{2E}{N_o}}, \sqrt{-2 \ln P_{FA}}\right). \tag{34}$$

If these results are superposed on figures 1 through 5, where the abscissa is interpreted as $10 \log_{10}(E/N_o)$, it is found that (33) cuts across the D = 1 curve at approximately $P_{CD} = 0.25$ for $P_{FA} = 10^{-2}$ and cuts across at approximately $P_{CD} = 0.36$ for $P_{FA} = 10^{-8}$. That is, the performance for fluctuating signals is better than for nonfluctuating signals for small correct-decision probabilities (< 0.4); the occasionally large signal amplitudes occurring in fading are actually helpful when the noise level is high. Of course, for larger ENR, where P_{CD} approaches 1, the nonfluctuating signal case yields better performance by several dB. For example, $P_{FA} = 10^{-8}$, $P_{CD} = 0.99$ requires an ENR of 15.4 dB for no fading versus 19.1 dB for fading, a difference of 3.7 dB. The same difference also prevails approximately at $P_{FA} = 10^{-2}$.

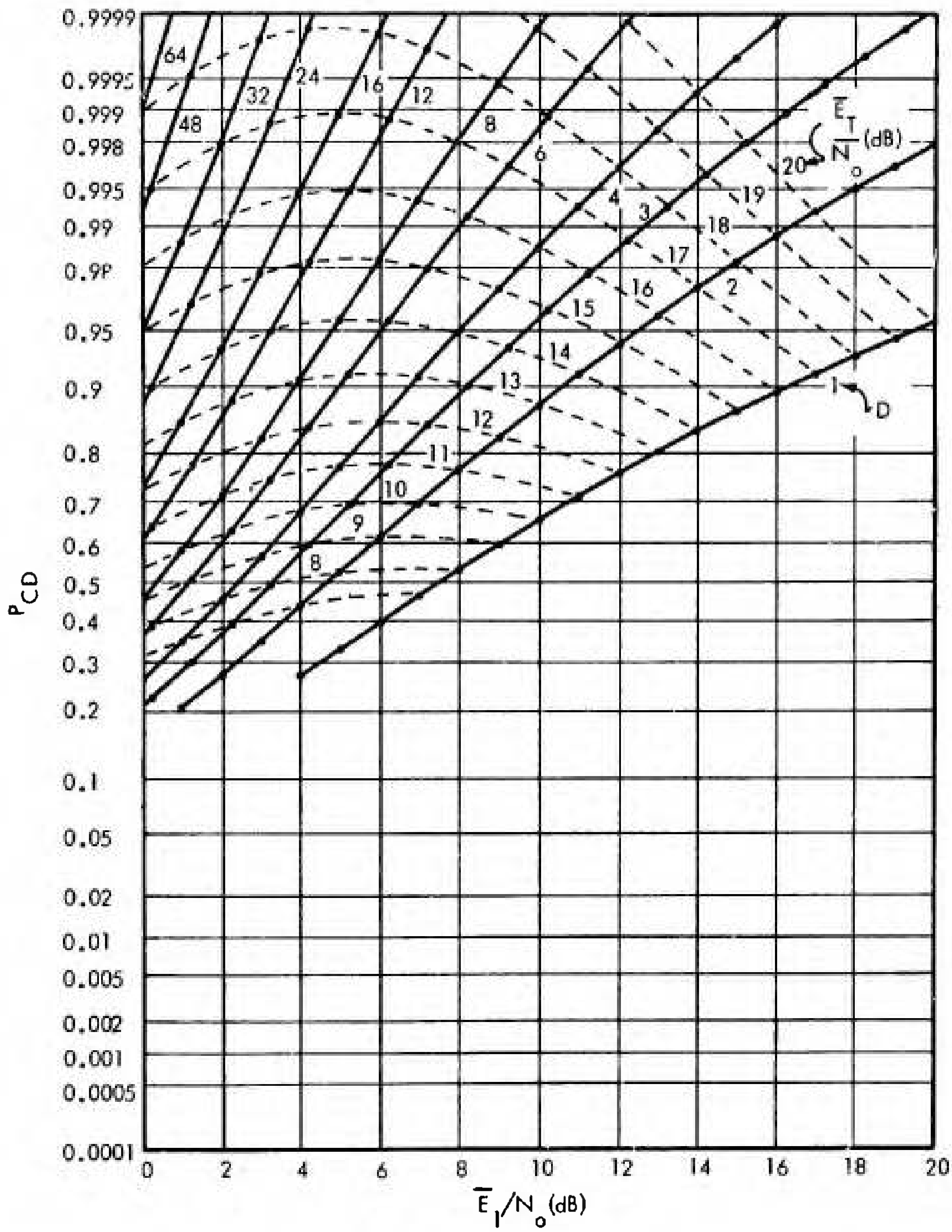

Figure 1. Detection Characteristics for $P_{FA} = 10^{-2}$, M = 1

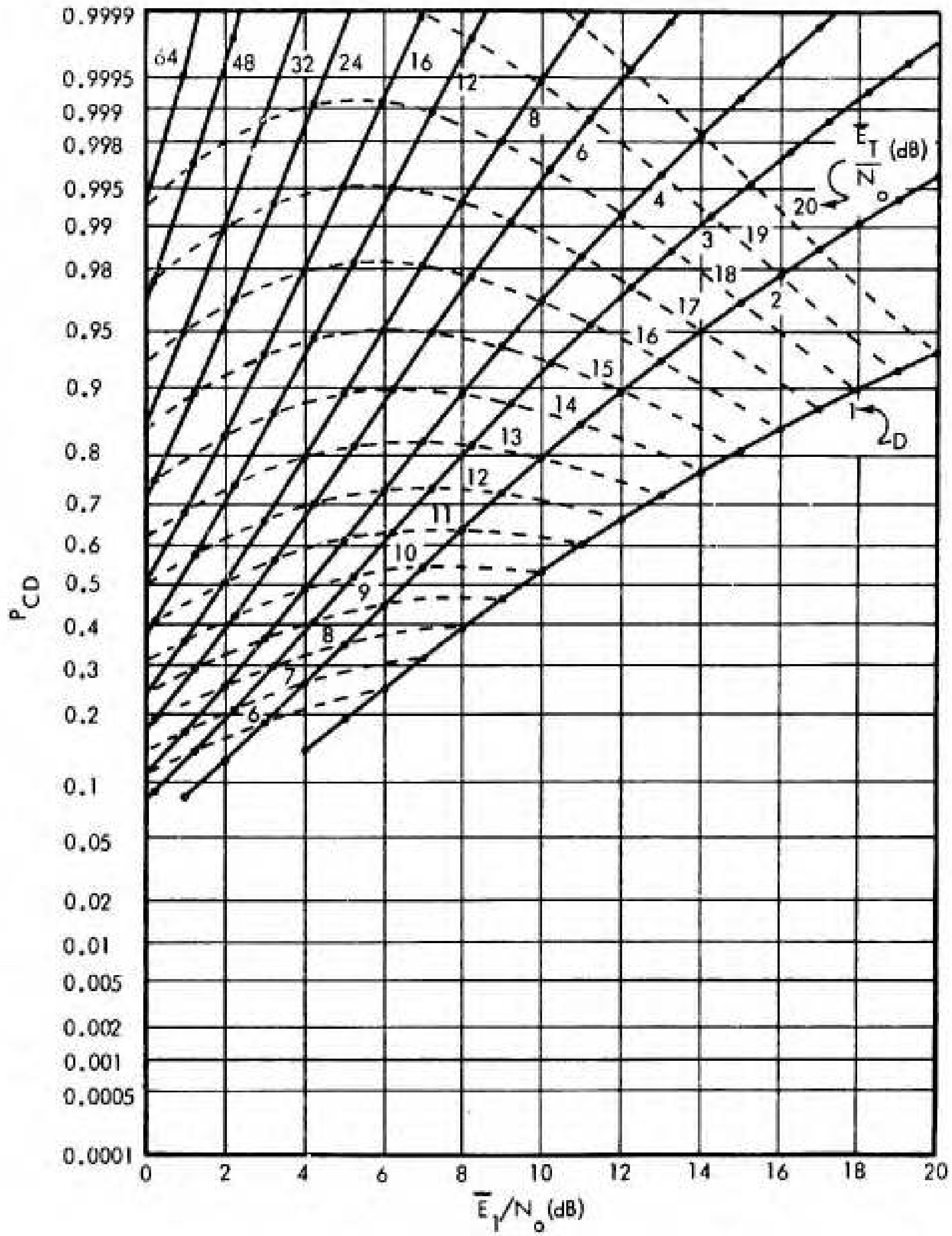

Figure 2. Detection Characteristics for $P_{FA} = 10^{-3}$, M = 1

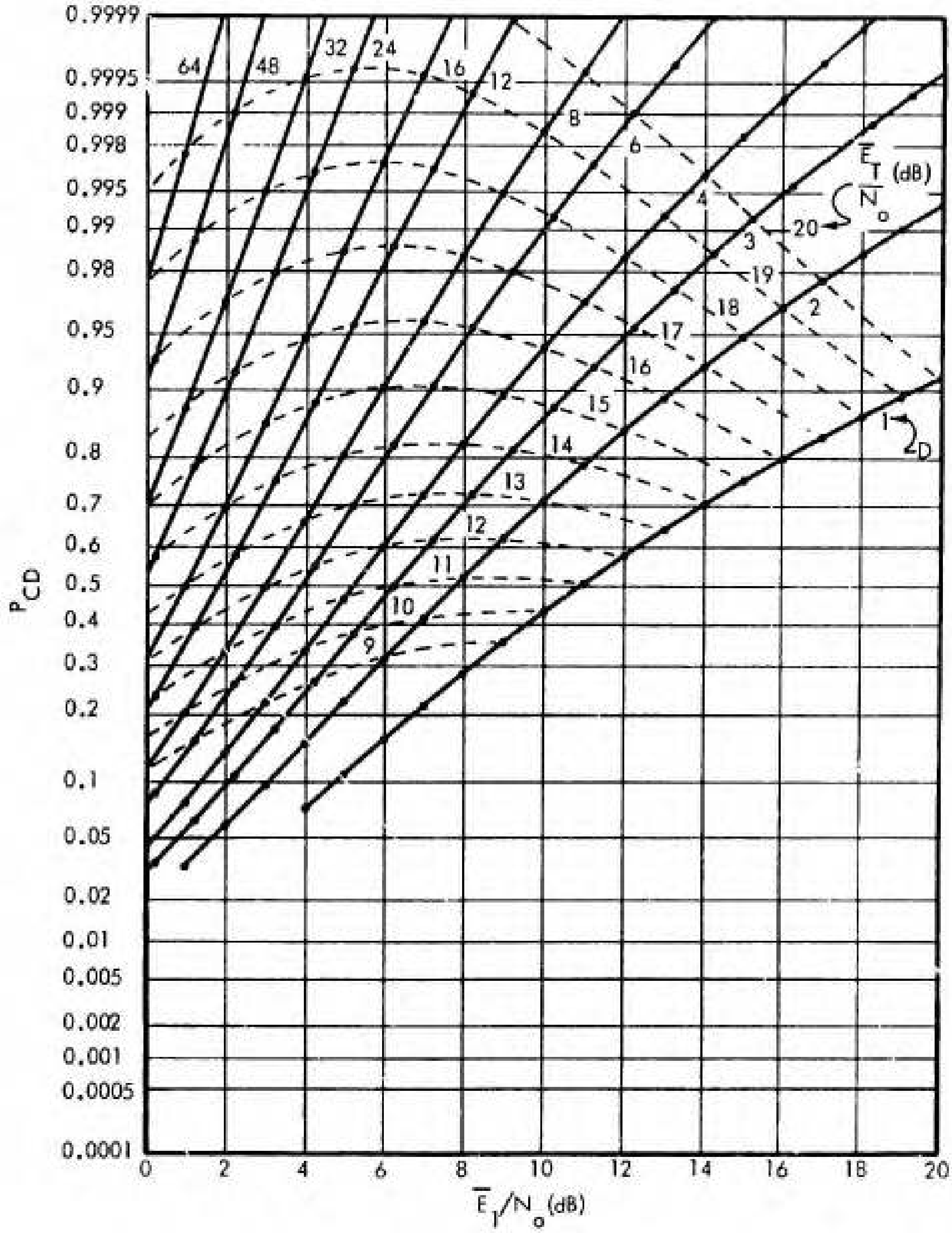

Figure 3. Detection Characteristics for $P_{FA} = 10^{-4}$, M = 1

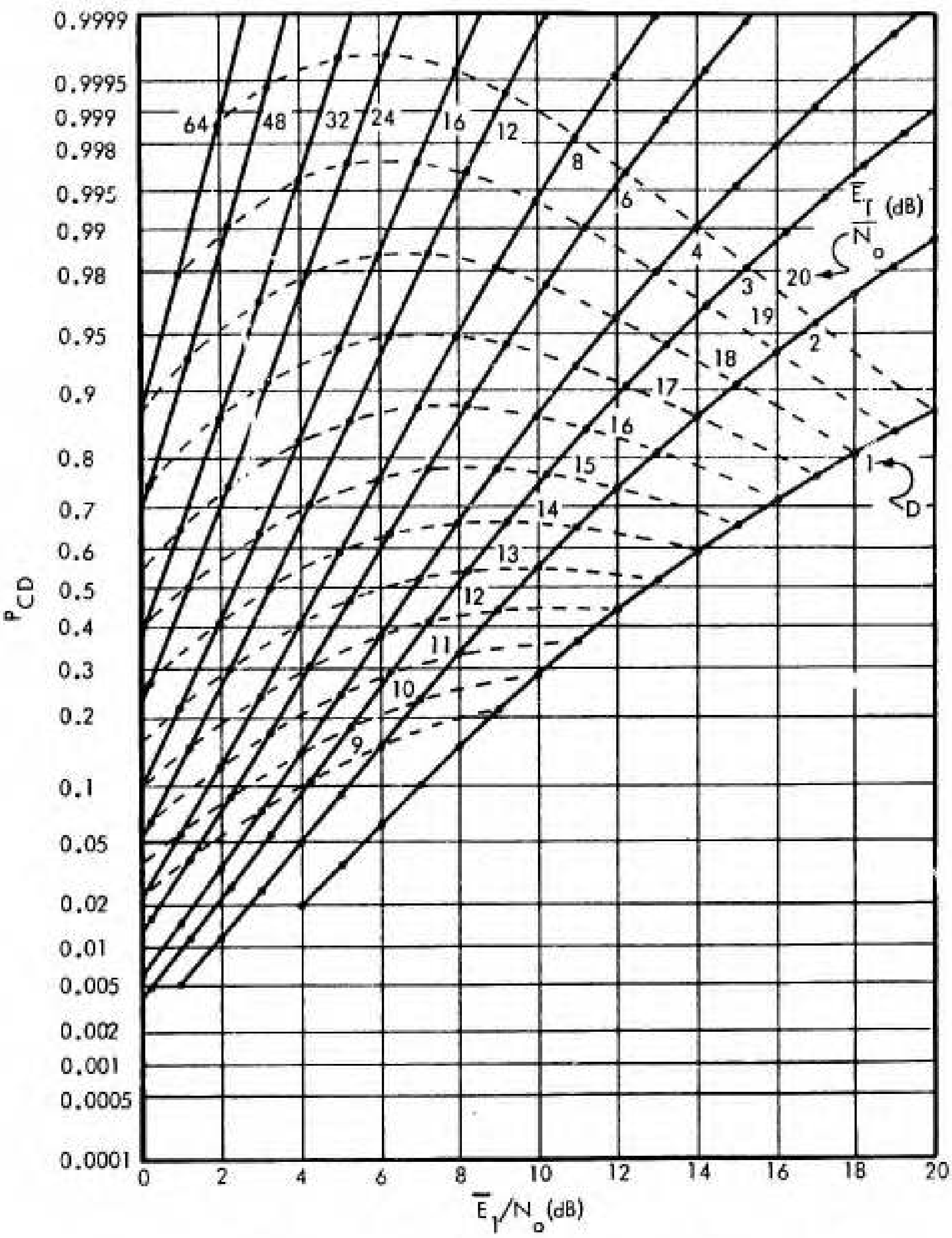

Figure 4. Detection Characteristics for $P_{FA} = 10^{-6}$, M = 1

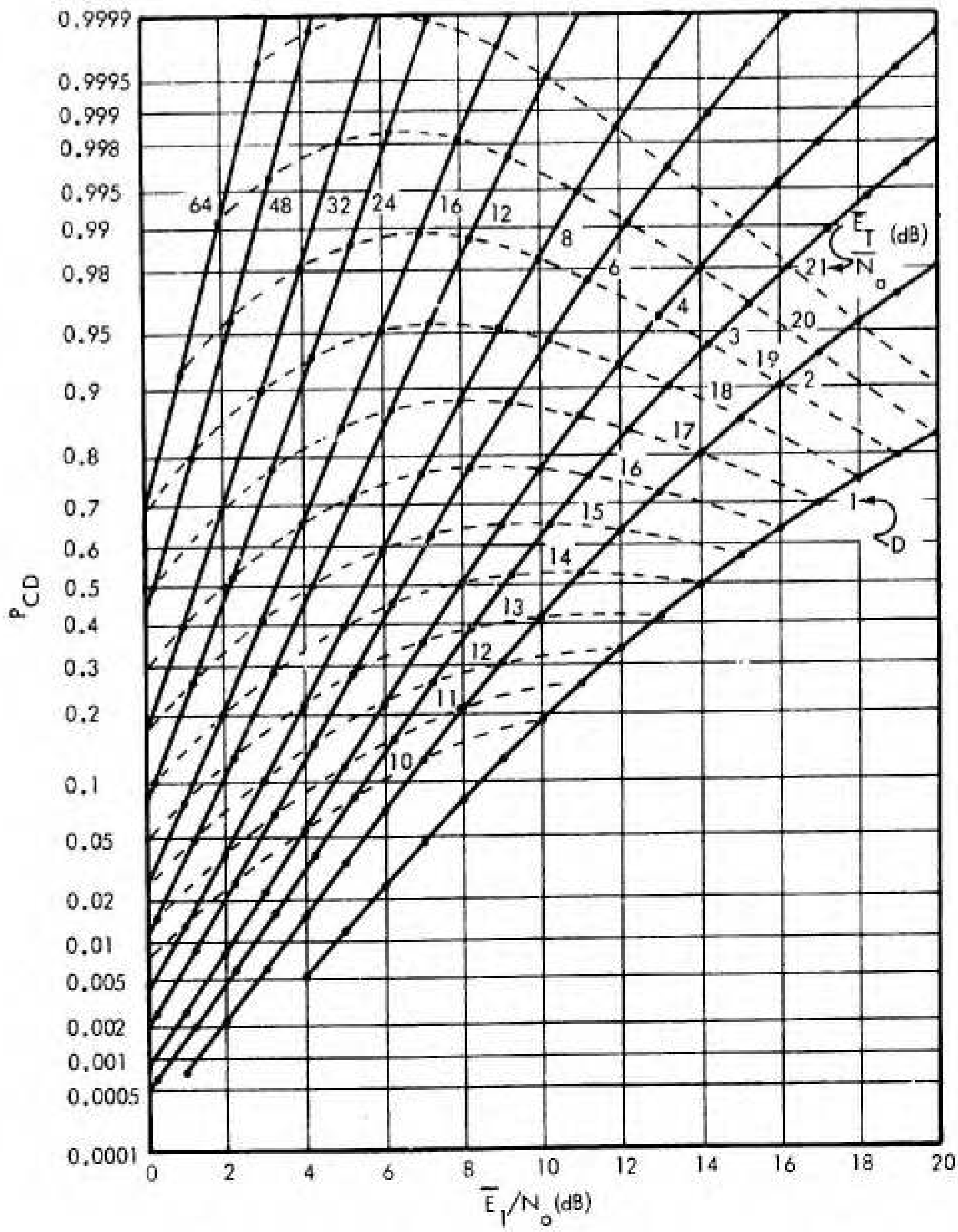

Figure 5. Detection Characteristics for $P_{FA} = 10^{-8}$, M = 1

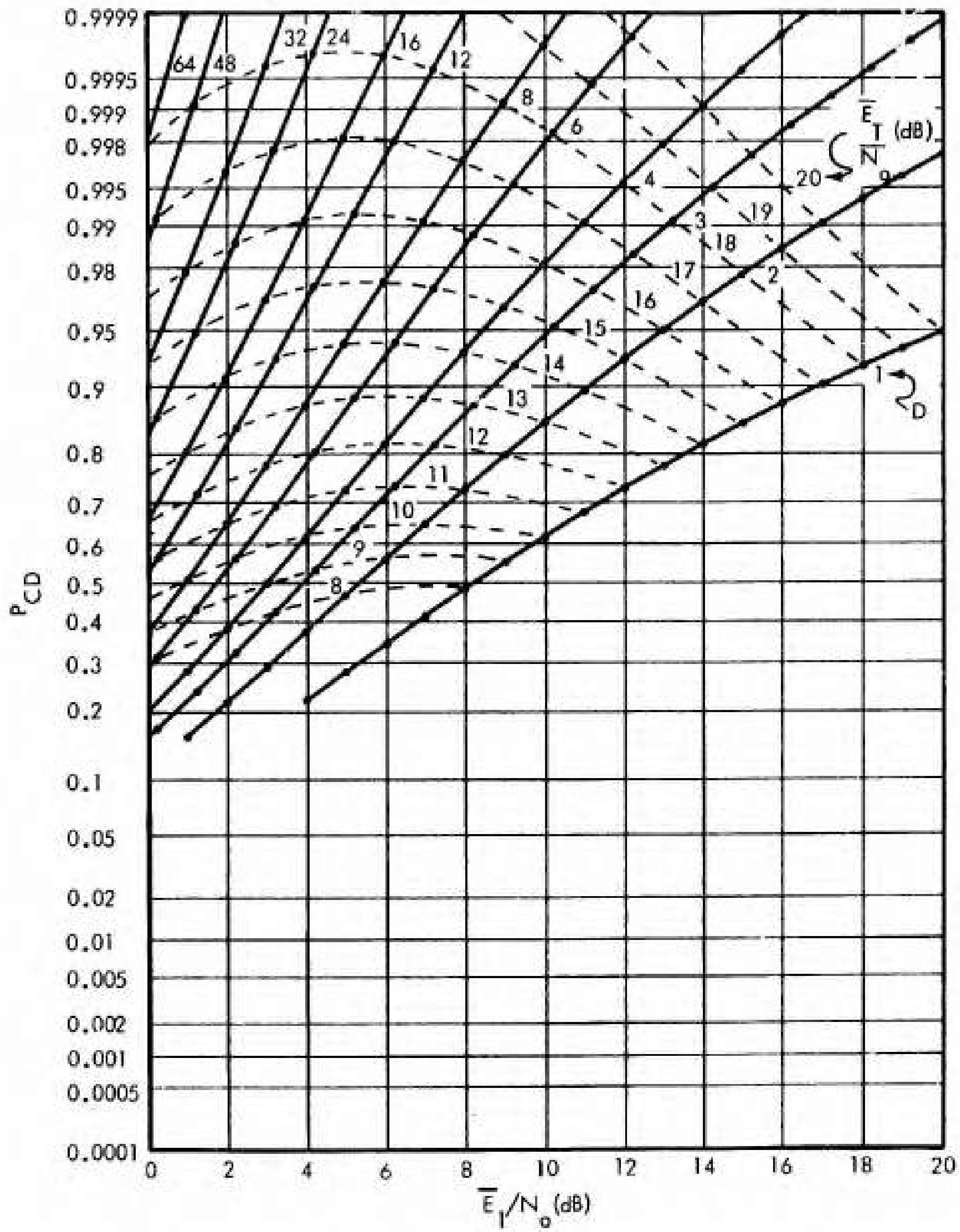

Figure 6. Detection Characteristics for $P_{FA} = 10^{-2}$, M = 2

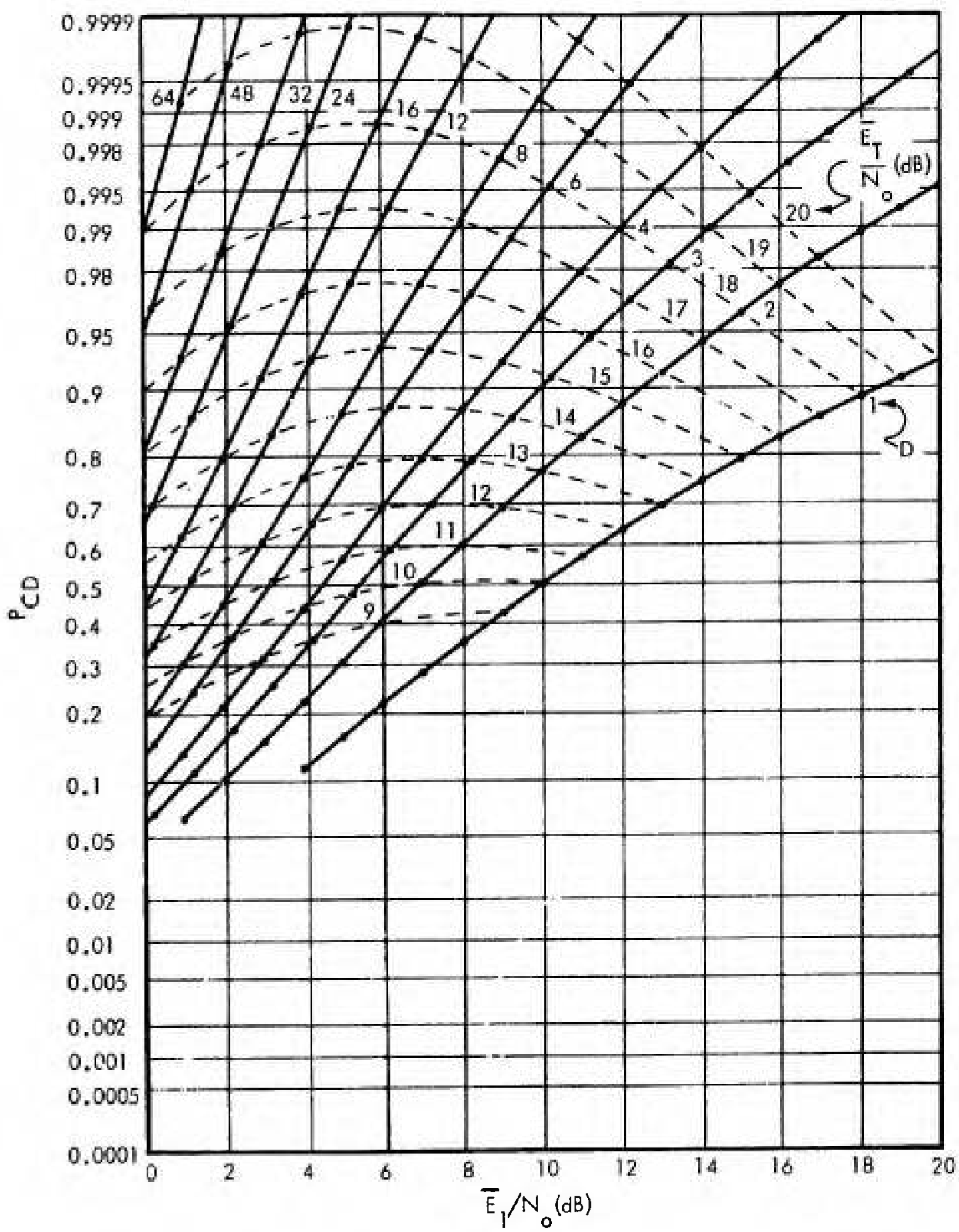

Figure 7. Detection Characteristics for $P_{FA} = 10^{-3}$, M = 2

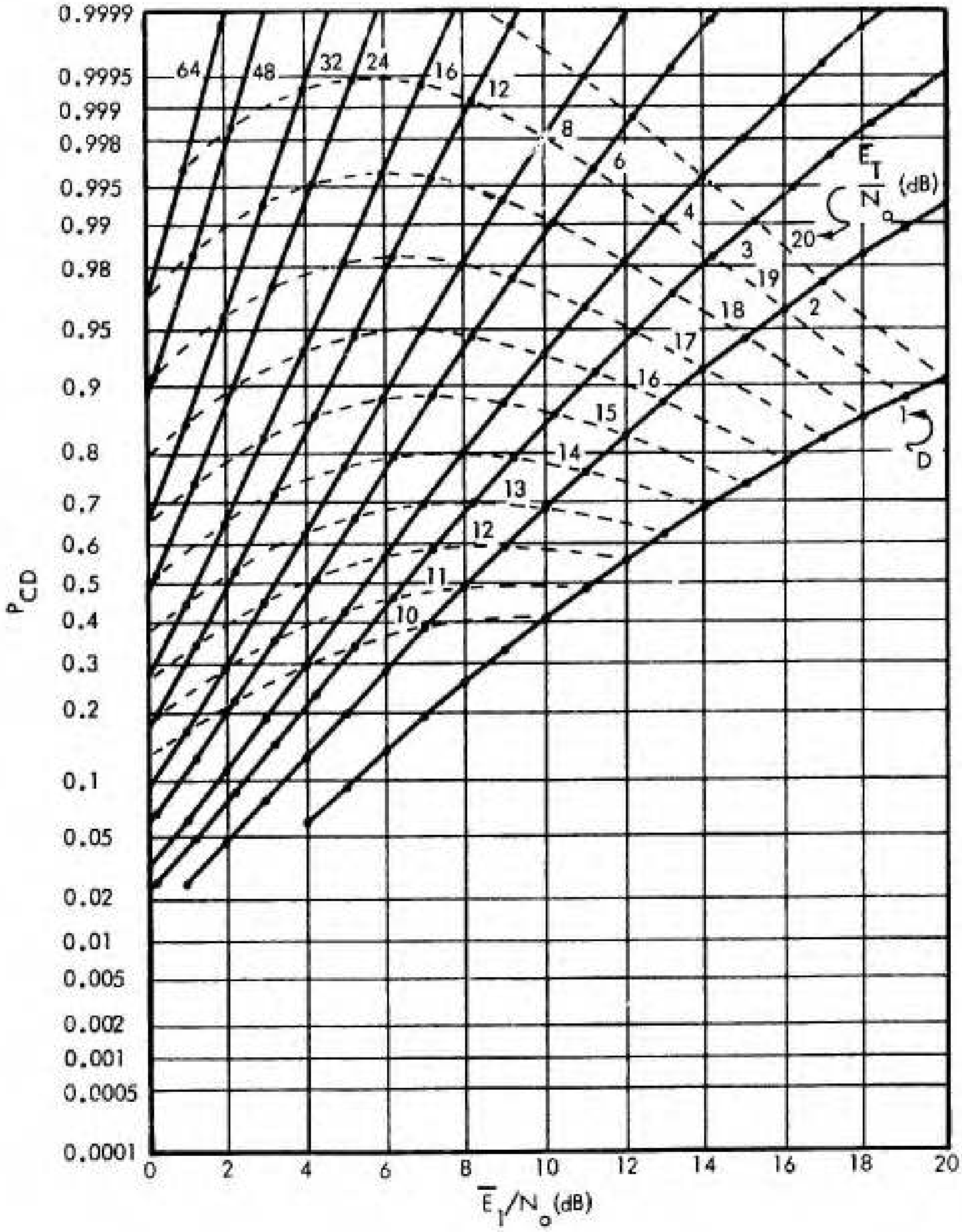

Figure 8. Detection Characteristics for $P_{FA} = 10^{-4}$, M = 2

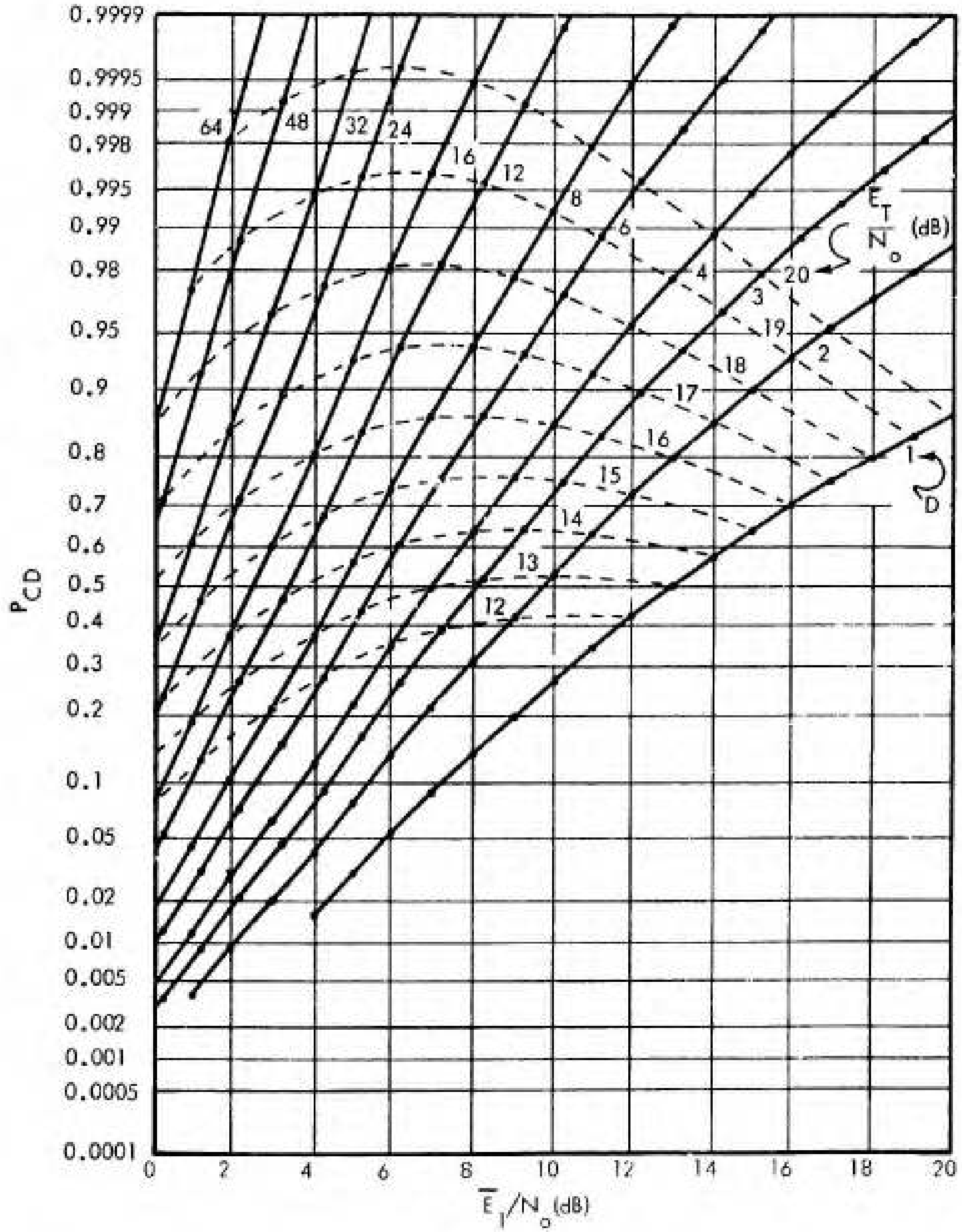

Figure 9. Detection Characteristics for $P_{FA} = 10^{-6}$, M = 2

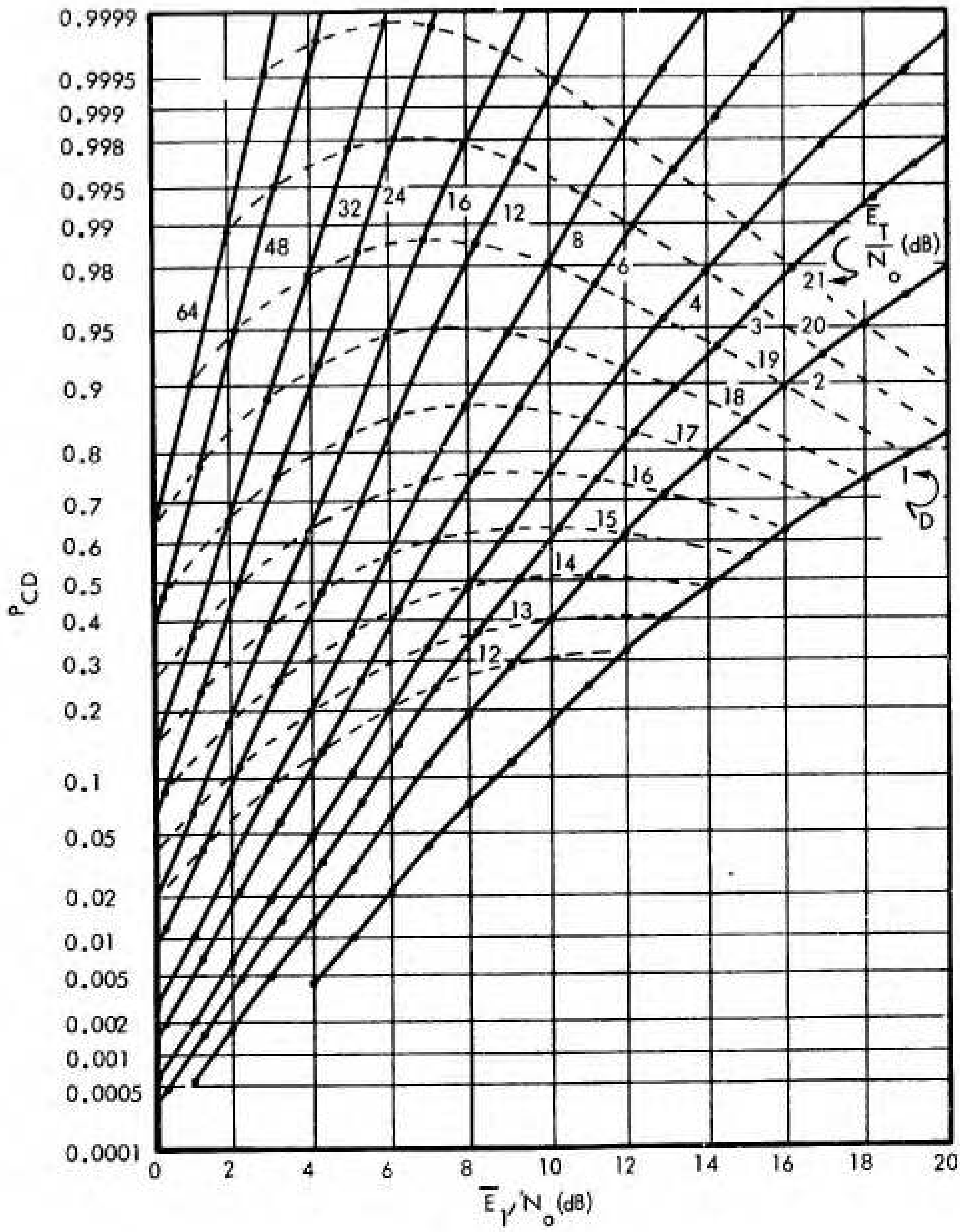

Figure 10. Detection Characteristics for $P_{FA} = 10^{-8}$, M = 2

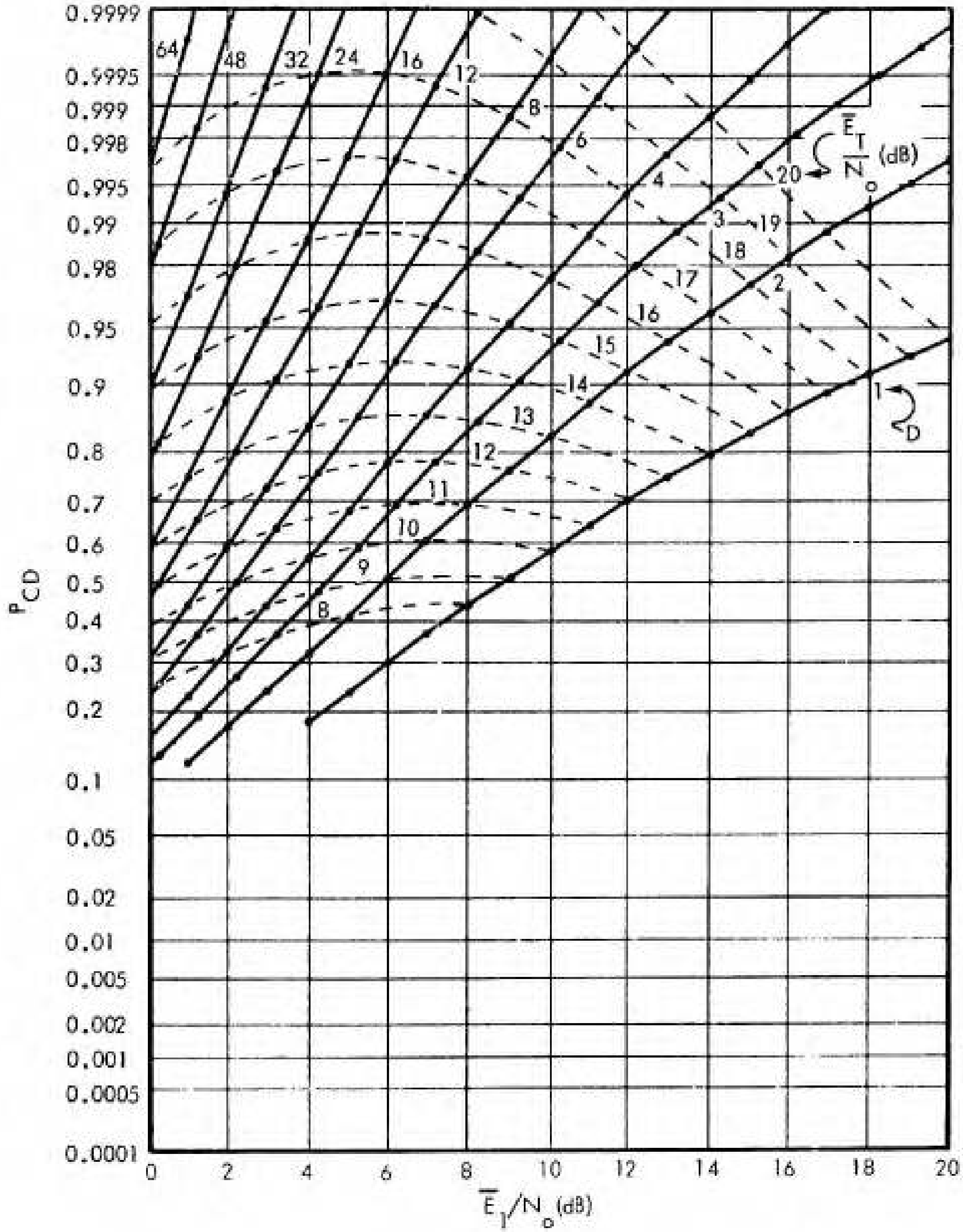

Figure 11. Detection Characteristics for $P_{FA} = 10^{-2}$, M = 4

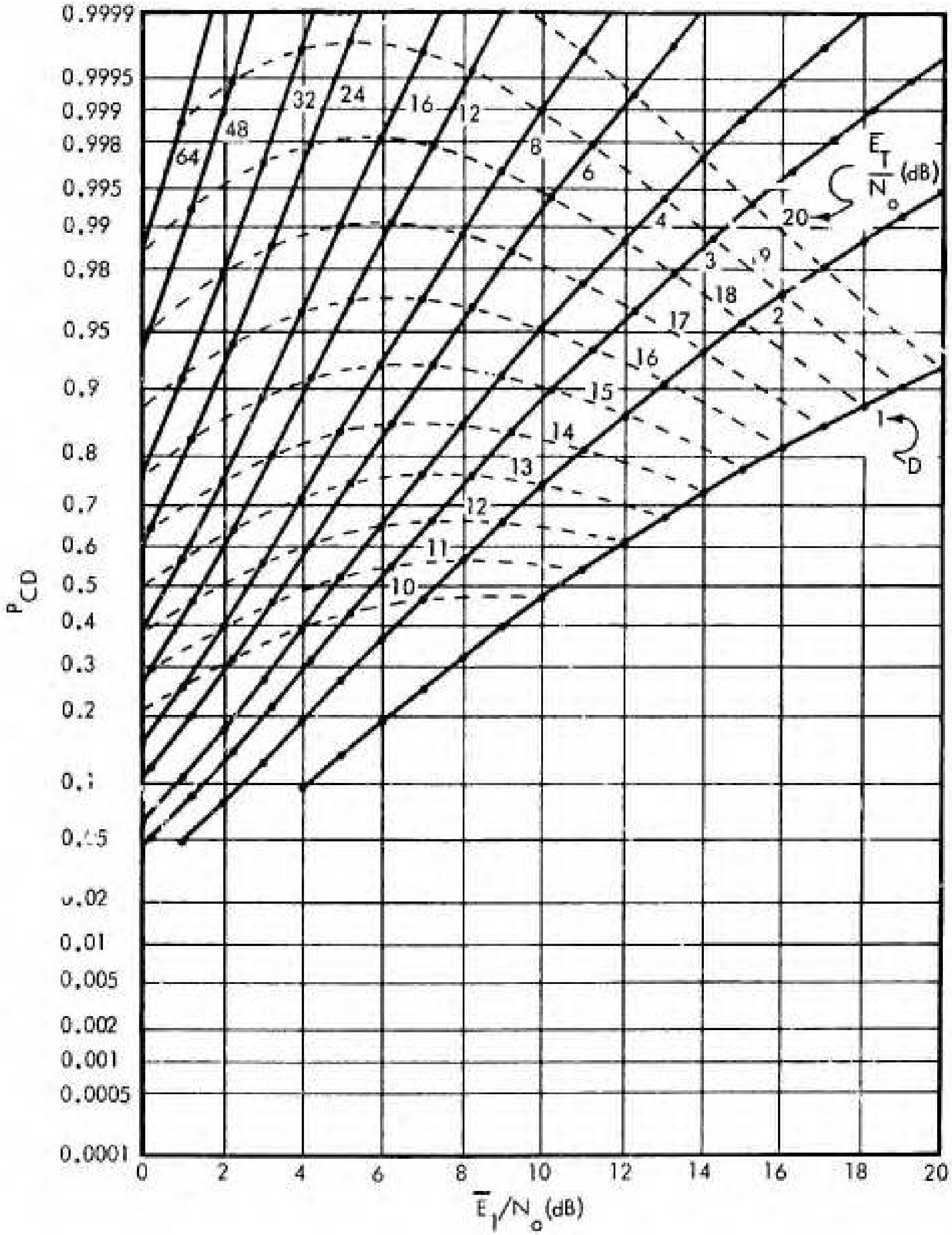

Figure 12. Detection Characteristics for $P_{FA} = 10^{-3}$, M = 4

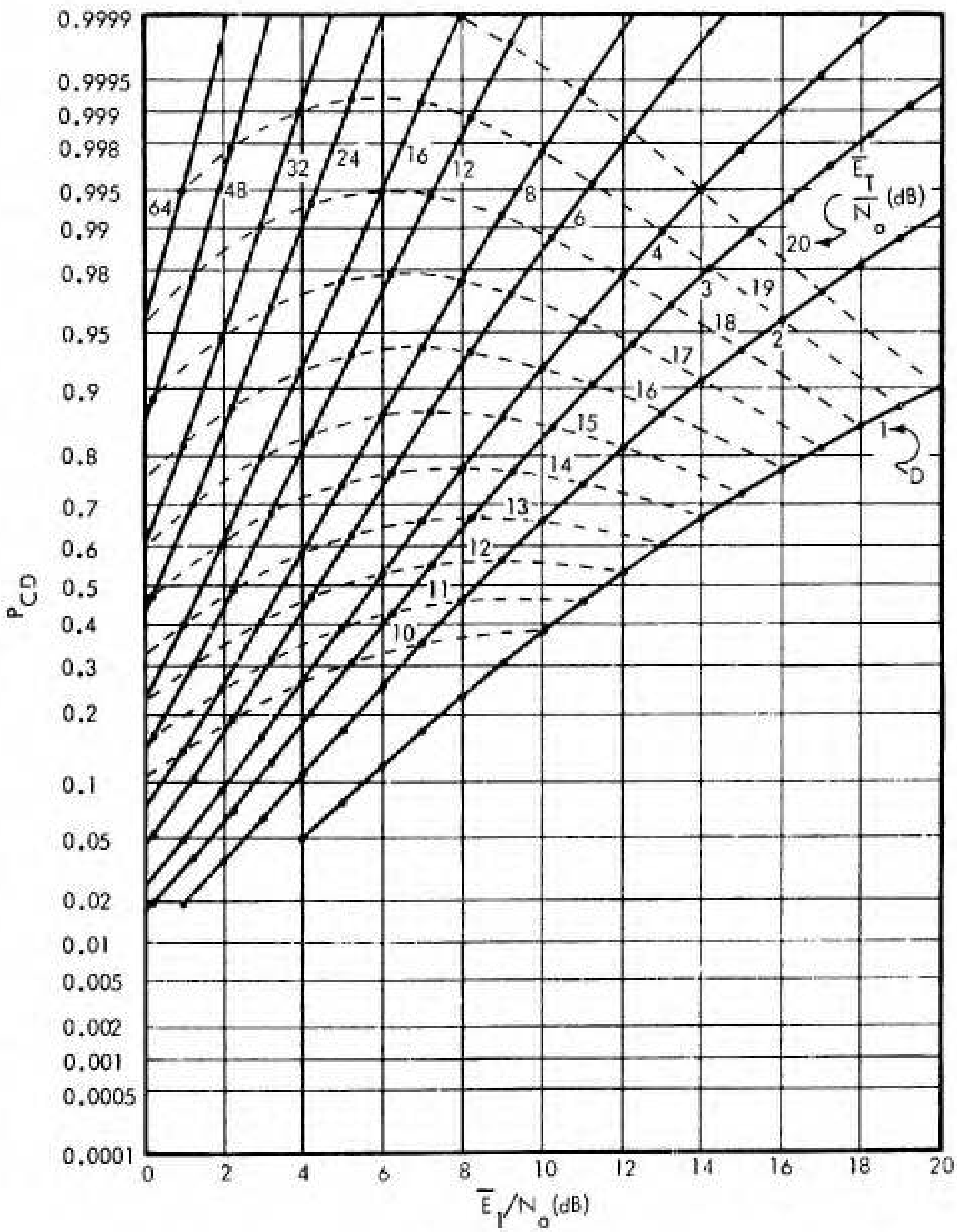

Figure 13. Detection Characteristics for $P_{FA} = 10^{-4}$, M = 4

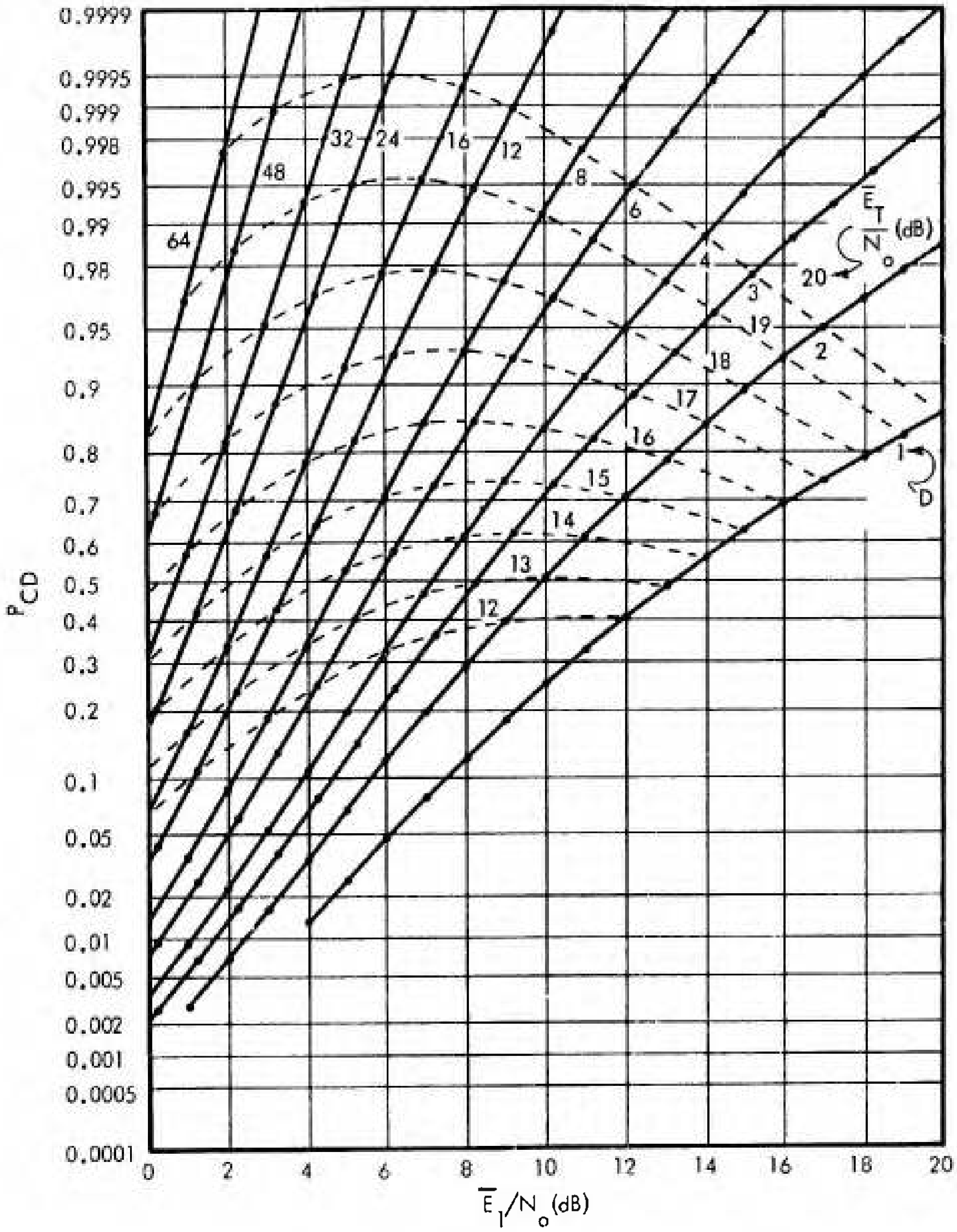

Figure 14. Detection Characteristics for $P_{FA} = 10^{-6}$, M = 4

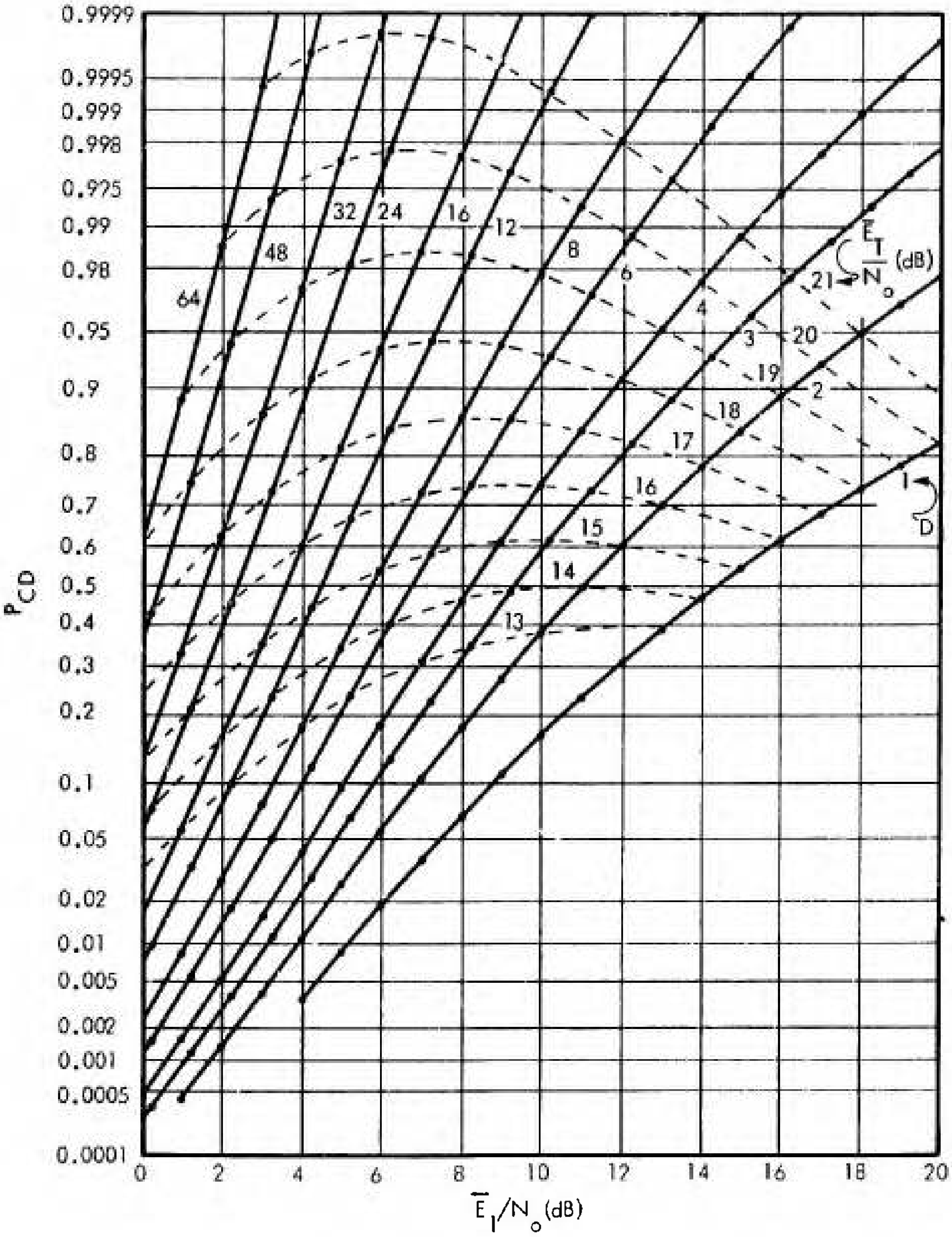

Figure 15. Detection Characteristics for $P_{FA} = 10^{-8}$, M = 4

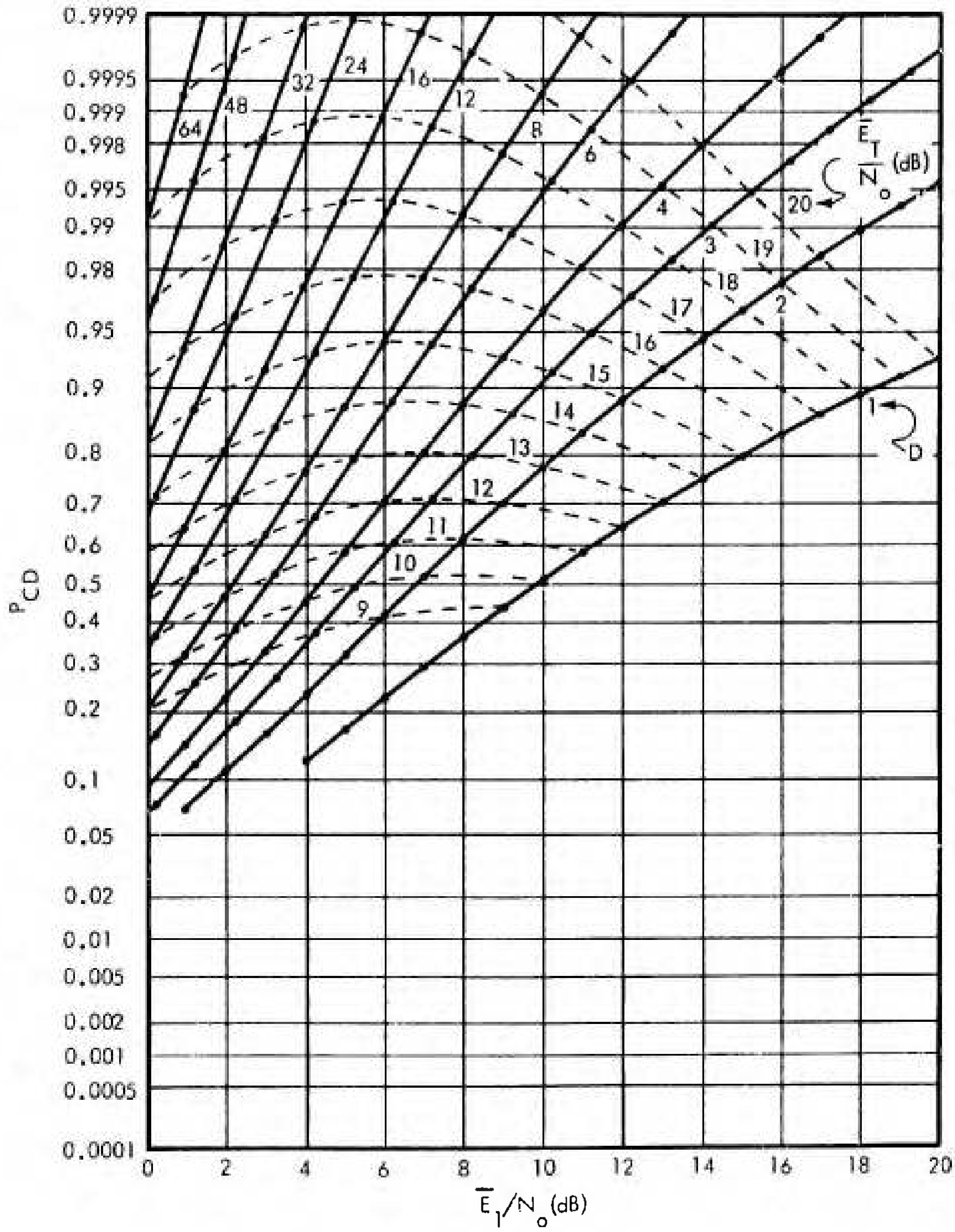

Figure 16. Detection Characteristics for $P_{FA} = 10^{-2}$, M = 16

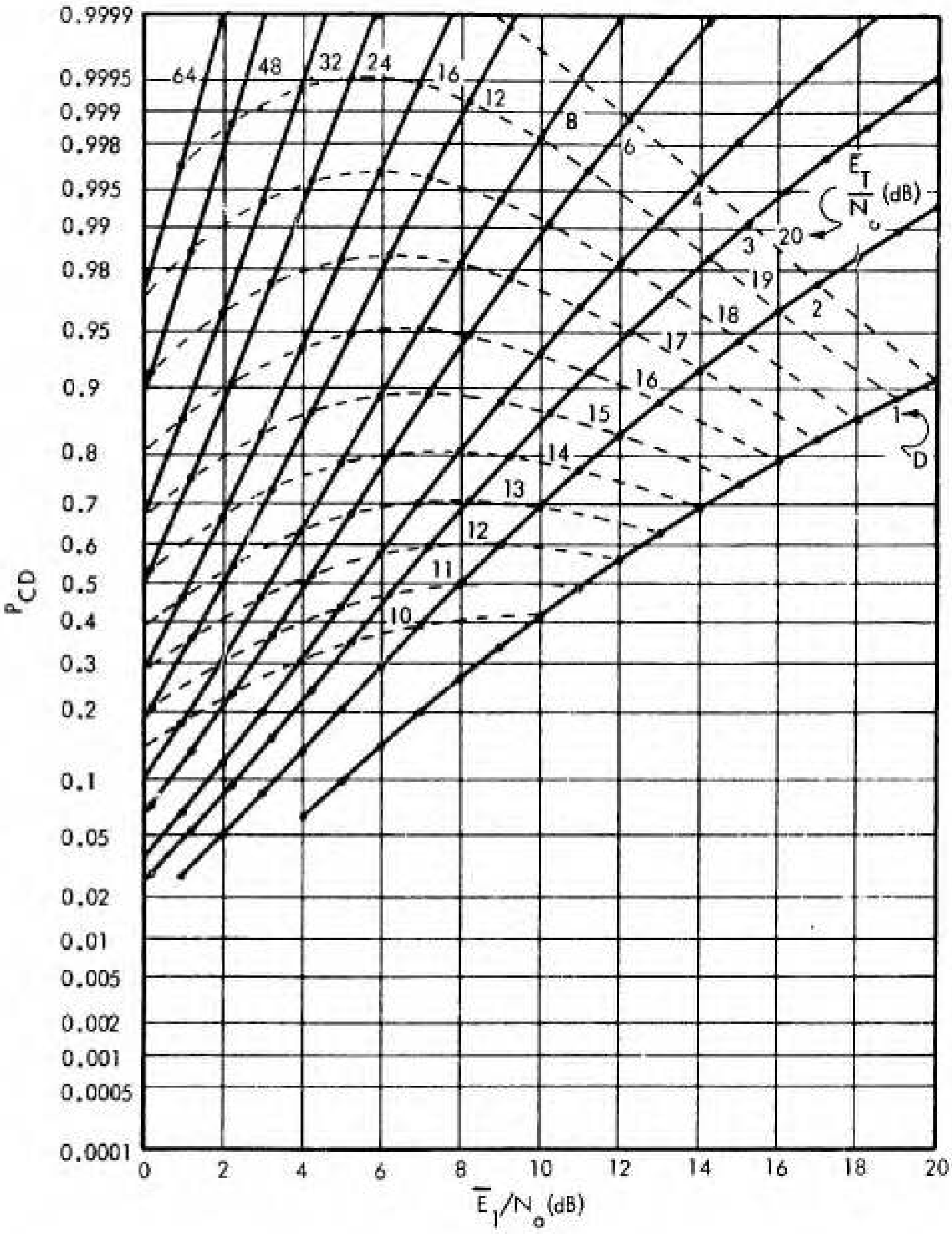

Figure 17. Detection Characteristics for $P_{FA} = 10^{-3}$, M = 16

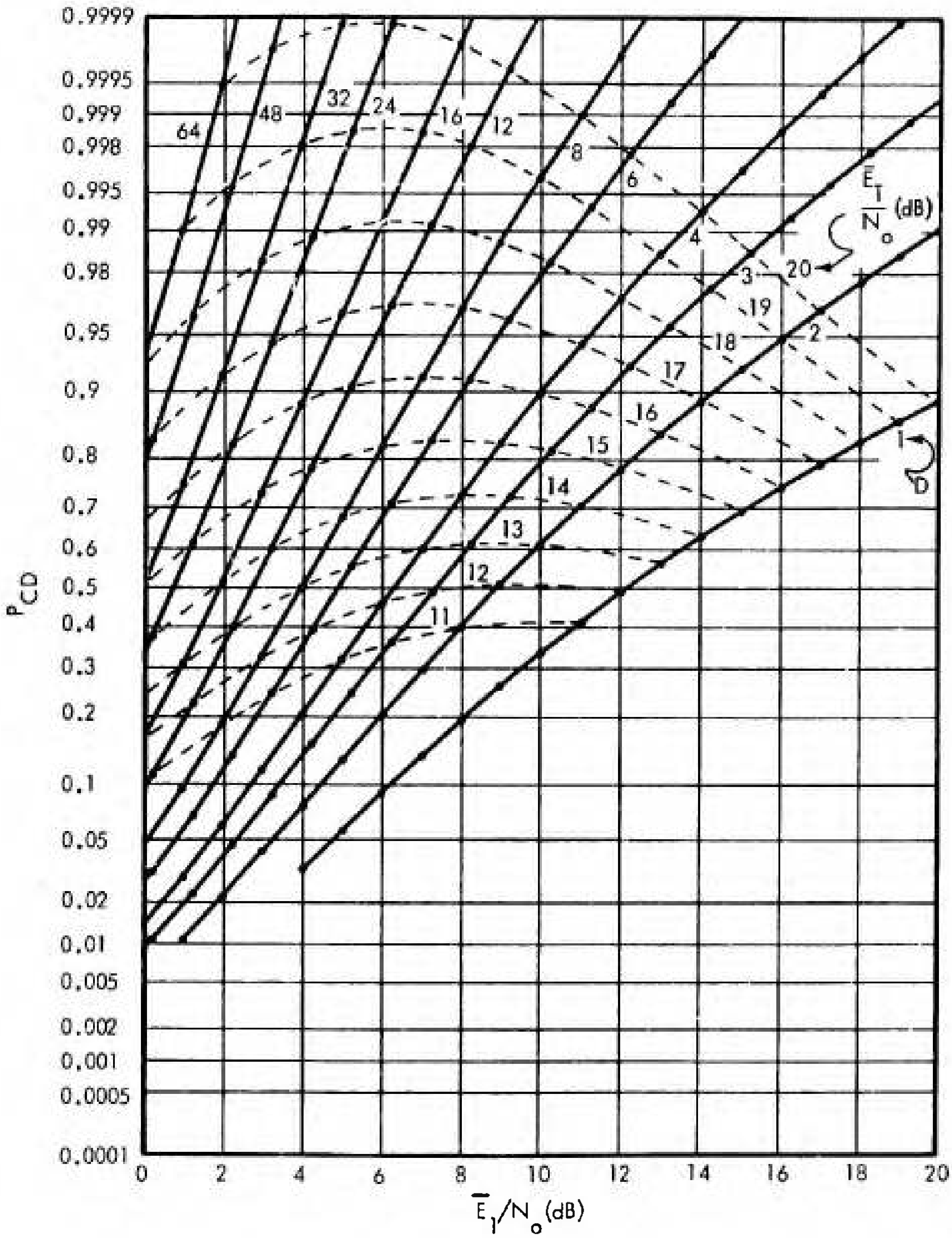

Figure 18. Detection Characteristics for $P_{FA} = 10^{-4}$, M = 16

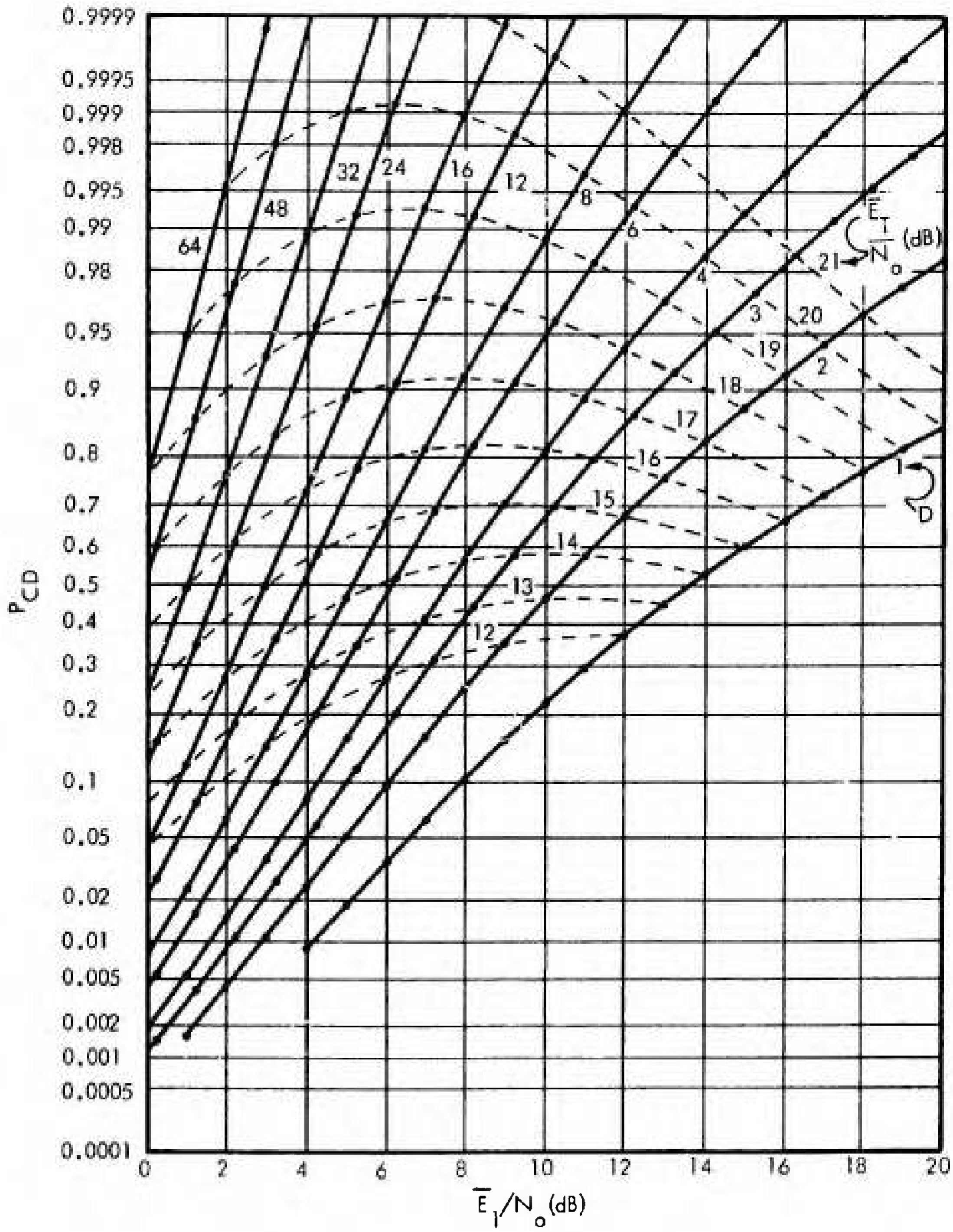

Figure 19. Detection Characteristics for $P_{FA} = 10^{-6}$, M = 16

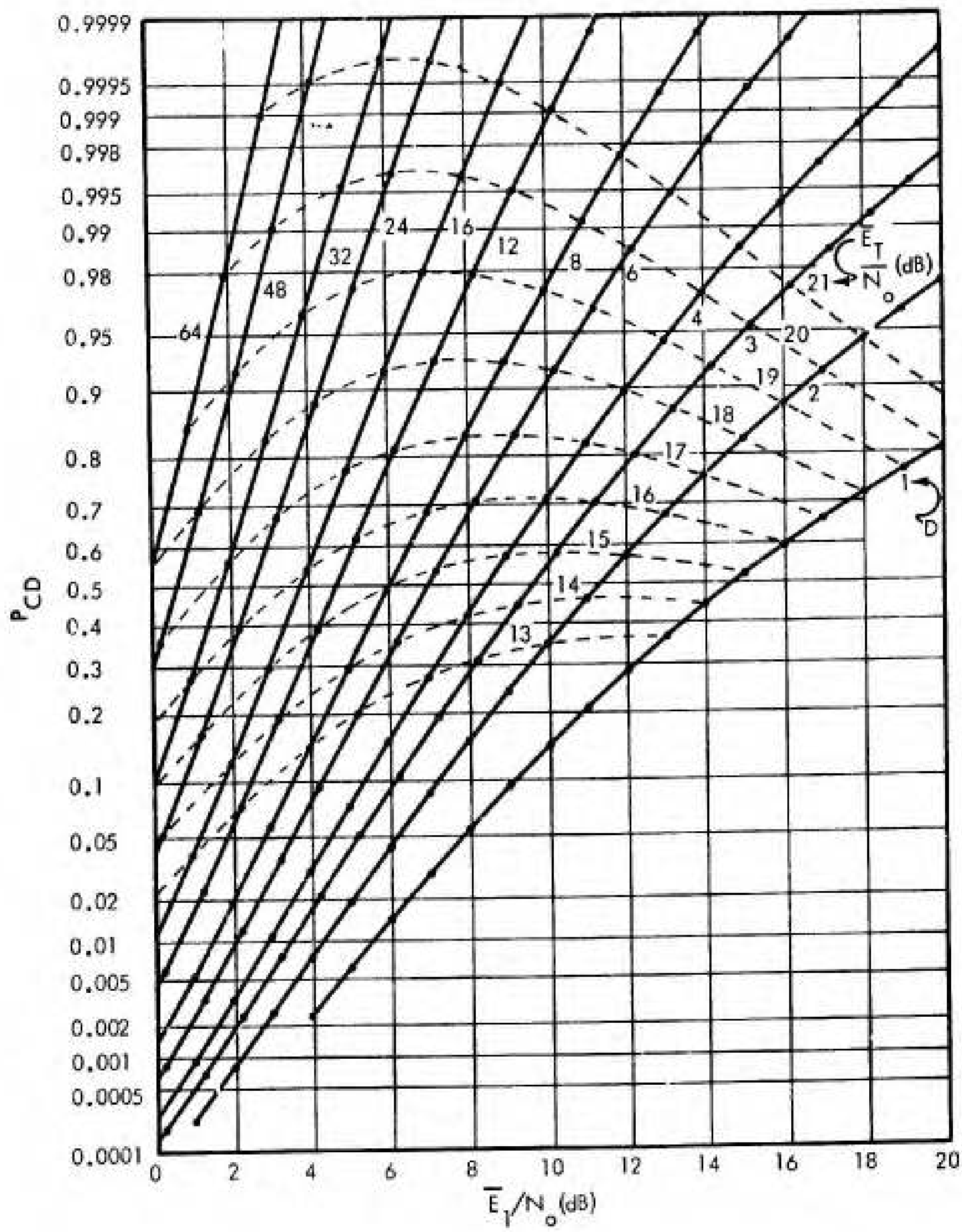

Figure 20. Detection Characteristics for $P_{FA} = 10^{-8}$, M = 16

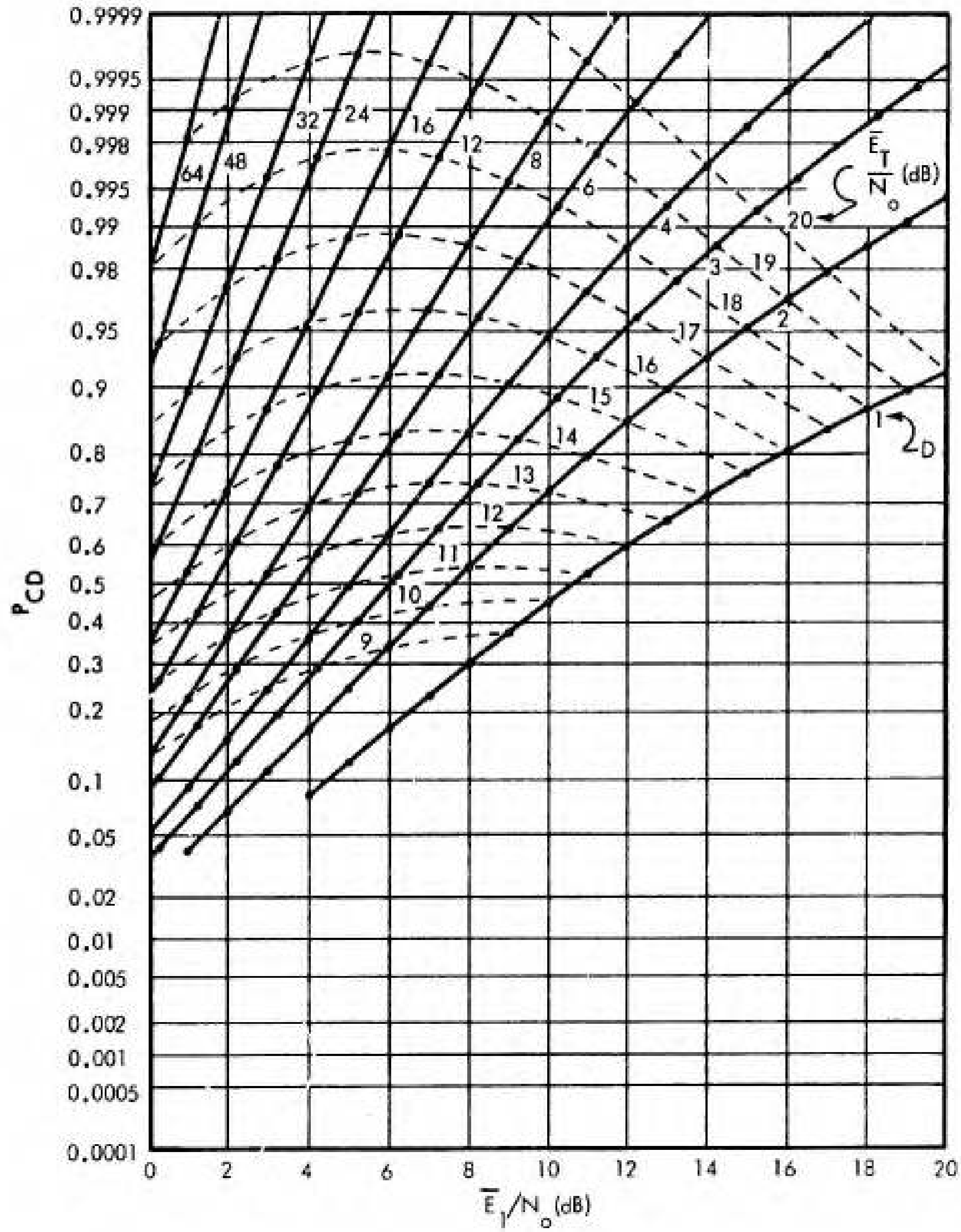

Figure 21. Detection Characteristics for $P_{FA} = 10^{-2}$, M = 64

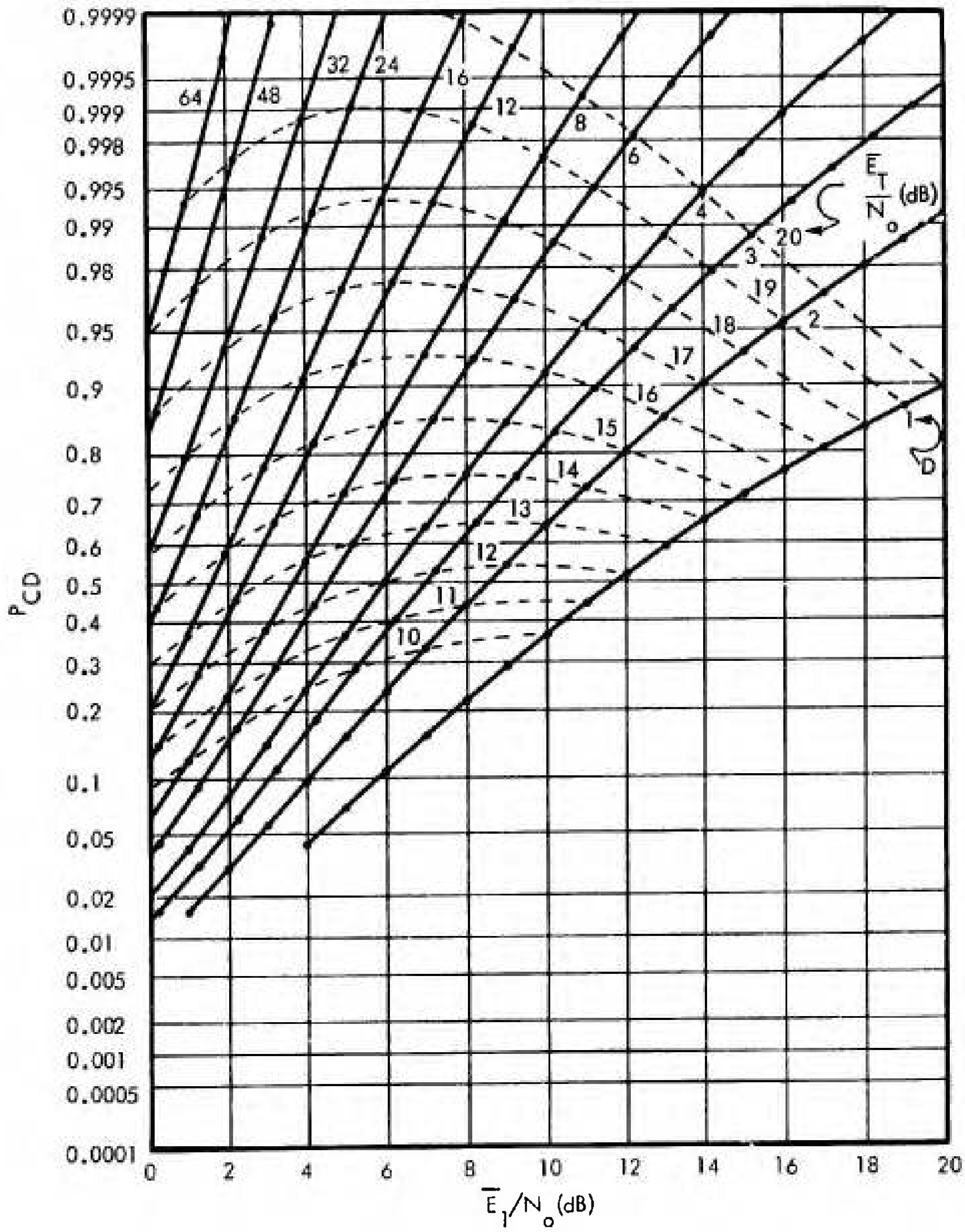

Figure 22. Detection Characteristics for $P_{FA} = 10^{-3}$, M = 64

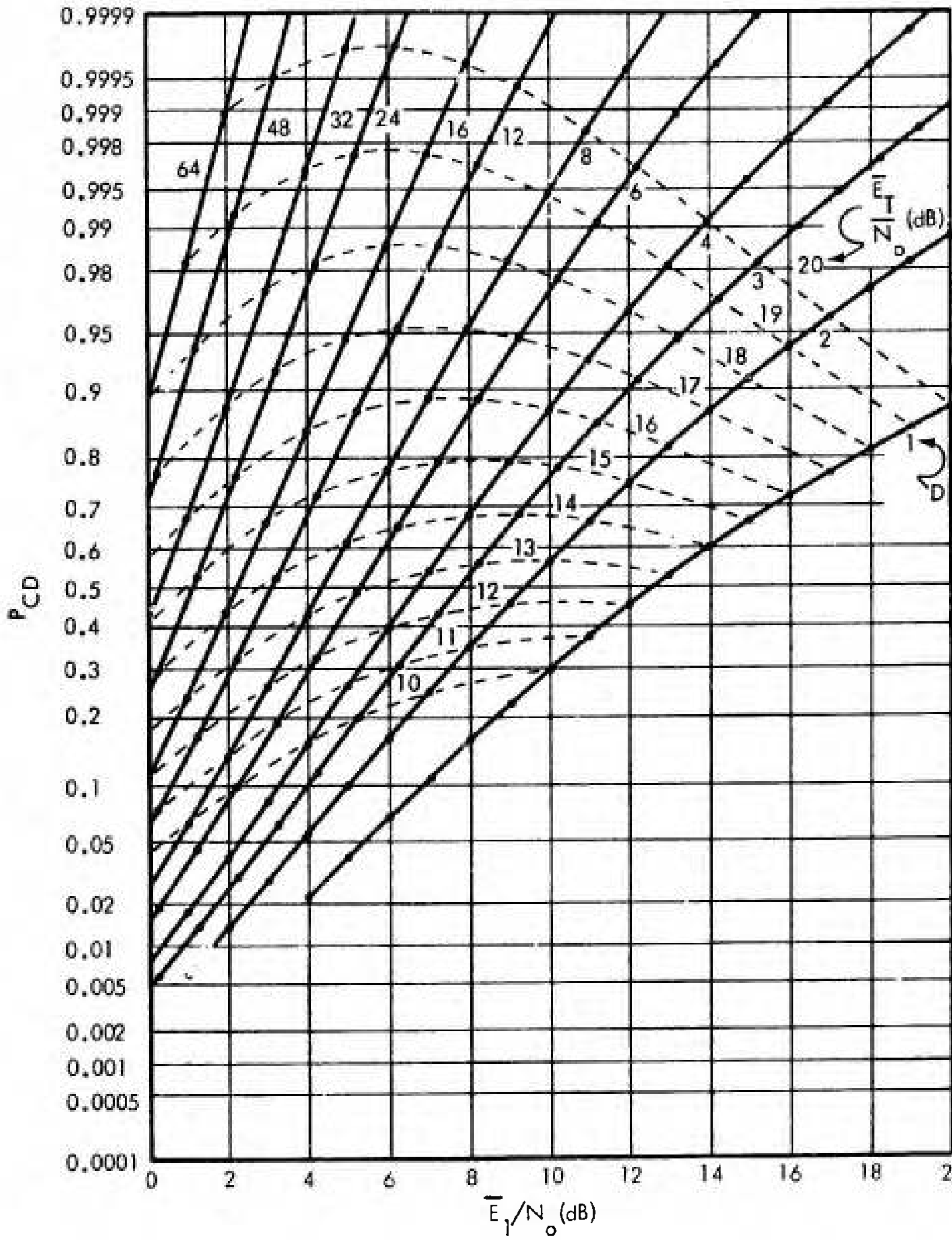

Figure 23. Detection Characteristics for $P_{FA} = 10^{-4}$, $M = 64$

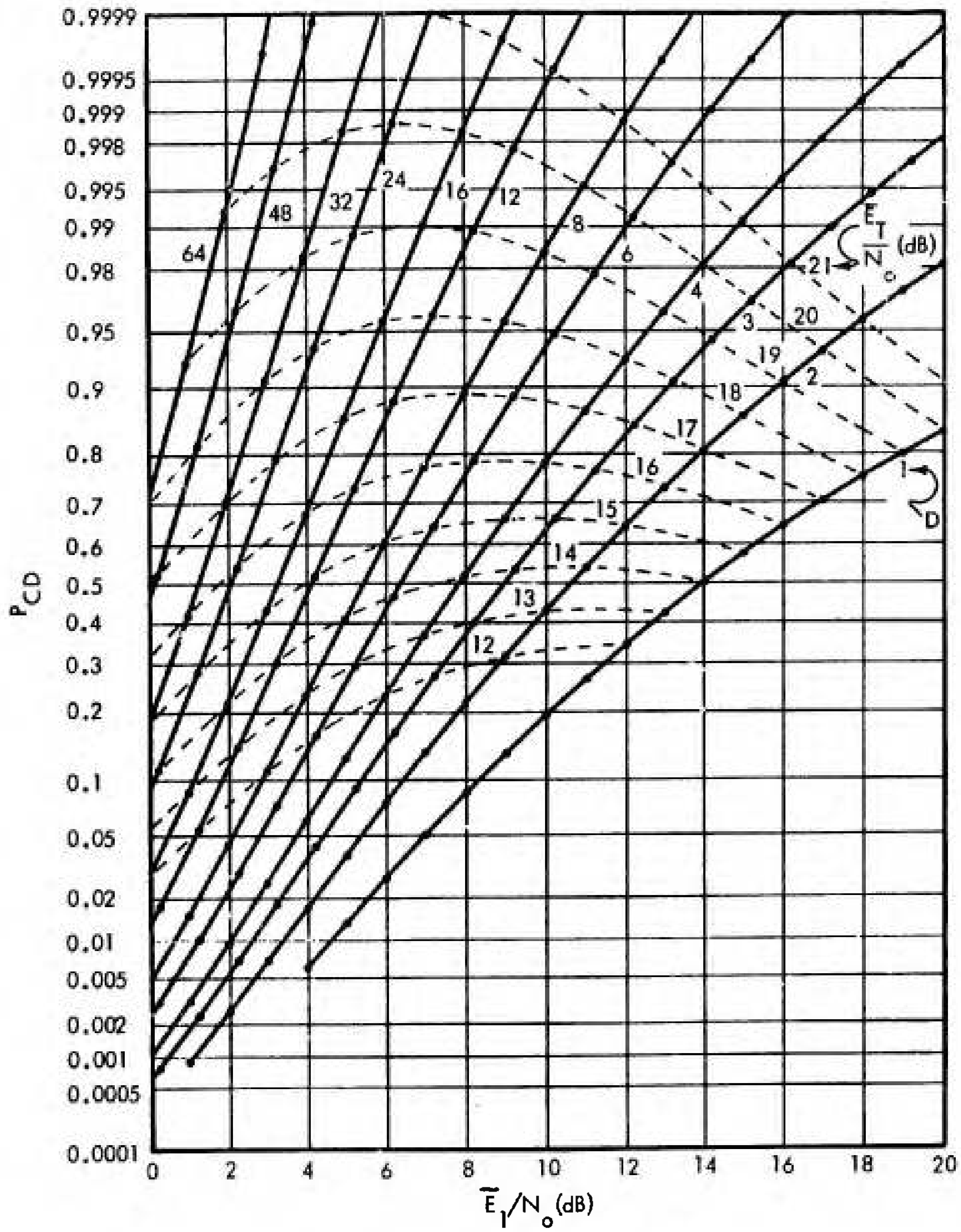

Figure 24. Detection Characteristics for $P_{FA} = 10^{-6}$, M = 64

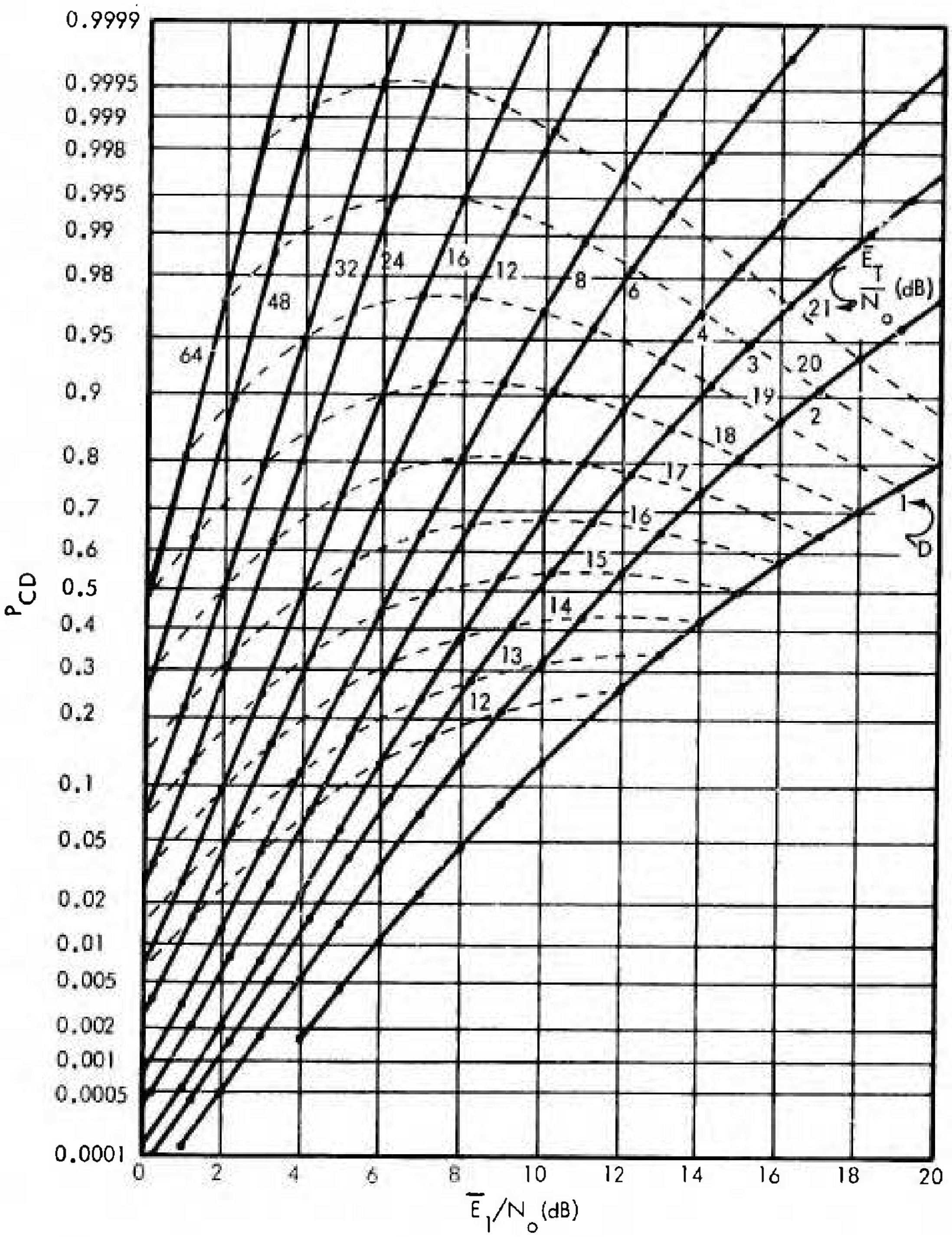

Figure 25. Detection Characteristics for $P_{FA} = 10^{-8}$, M = 64

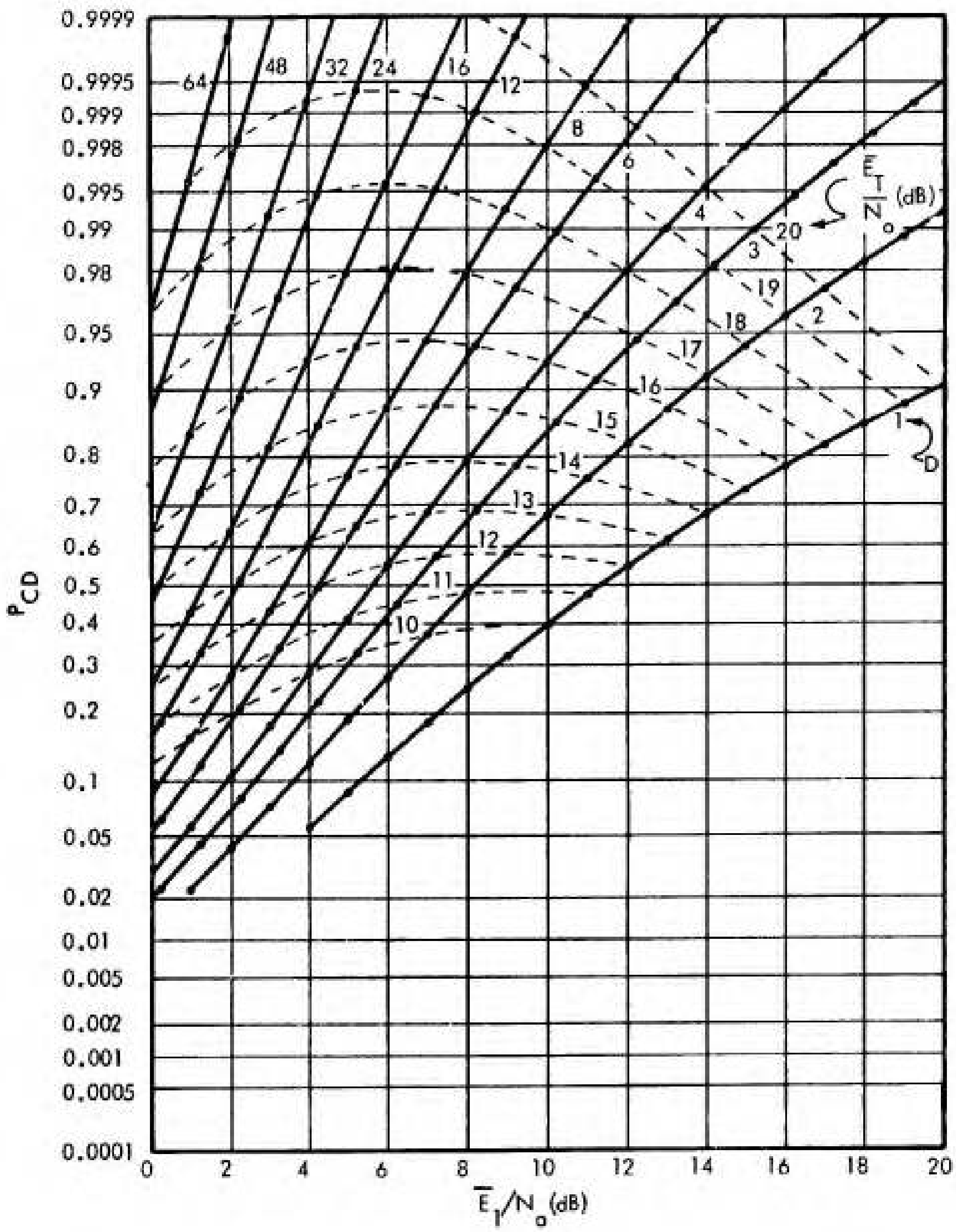

Figure 26. Detection Characteristics for $P_{FA} = 10^{-2}$, M = 256

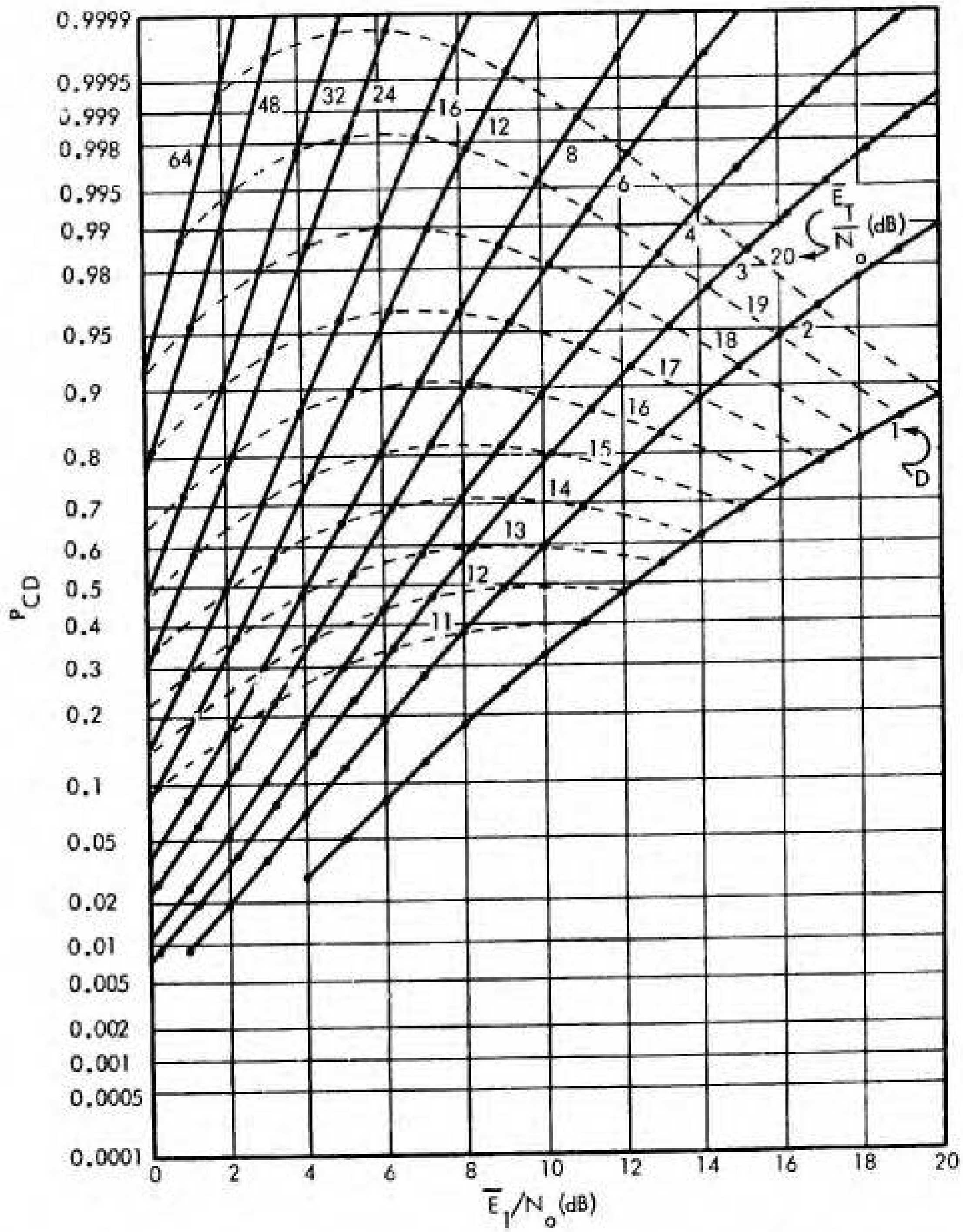

Figure 27. Detection Characteristics for $P_{FA} = 10^{-3}$, M = 256

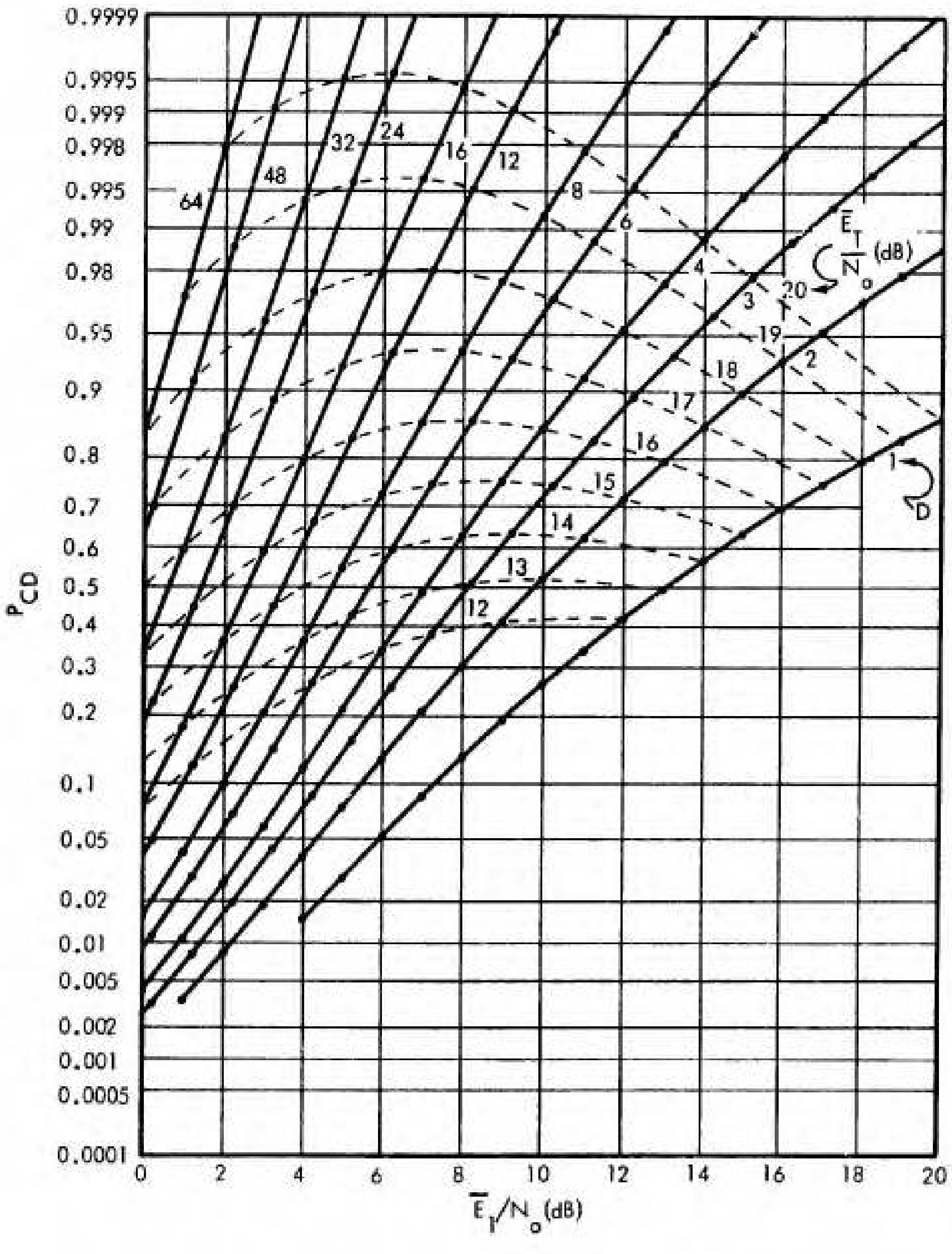

Figure 28. Detection Characteristics for $P_{FA} = 10^{-4}$, M = 256

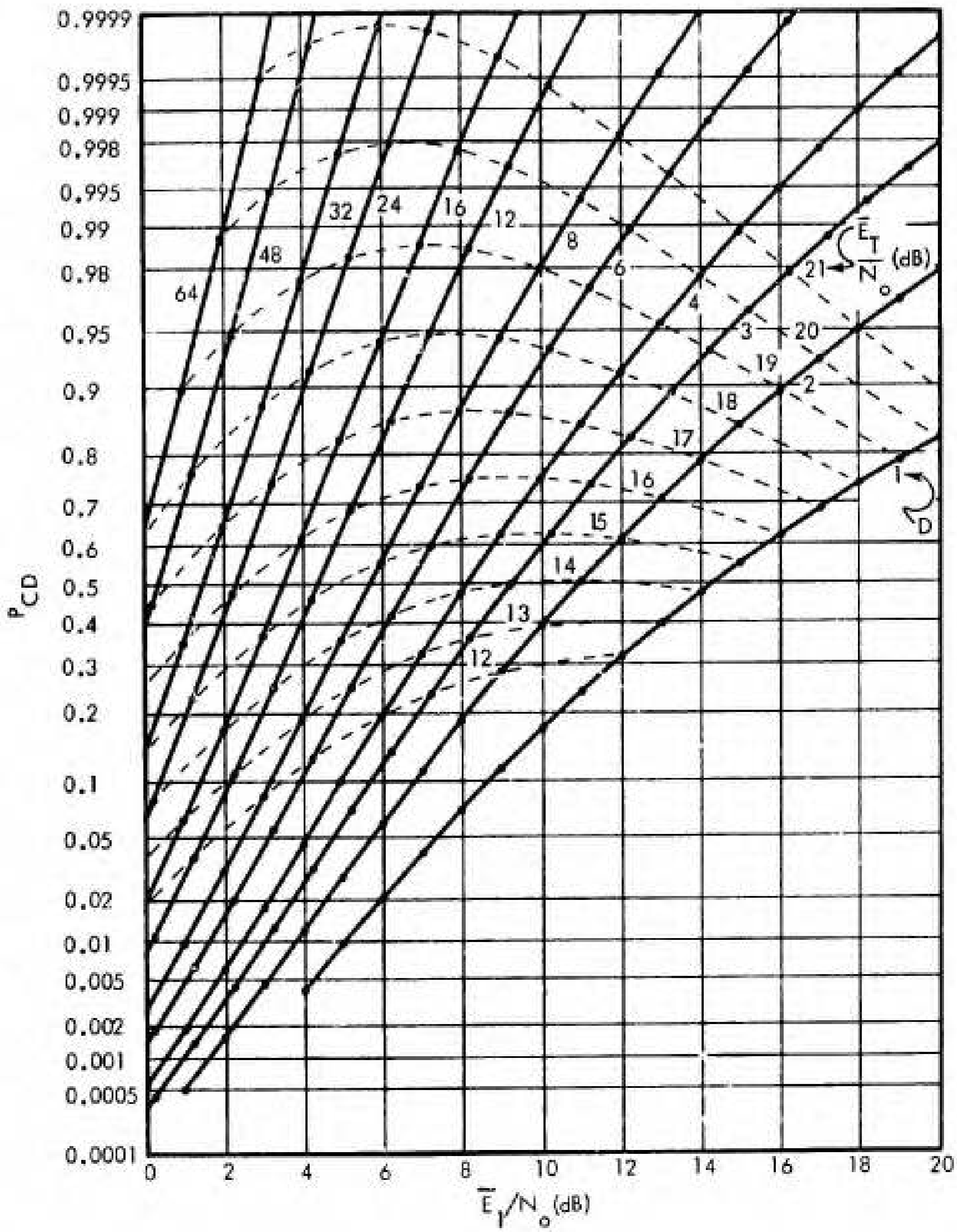

Figure 29. Detection Characteristics for $P_{FA} = 10^{-6}$, M = 256

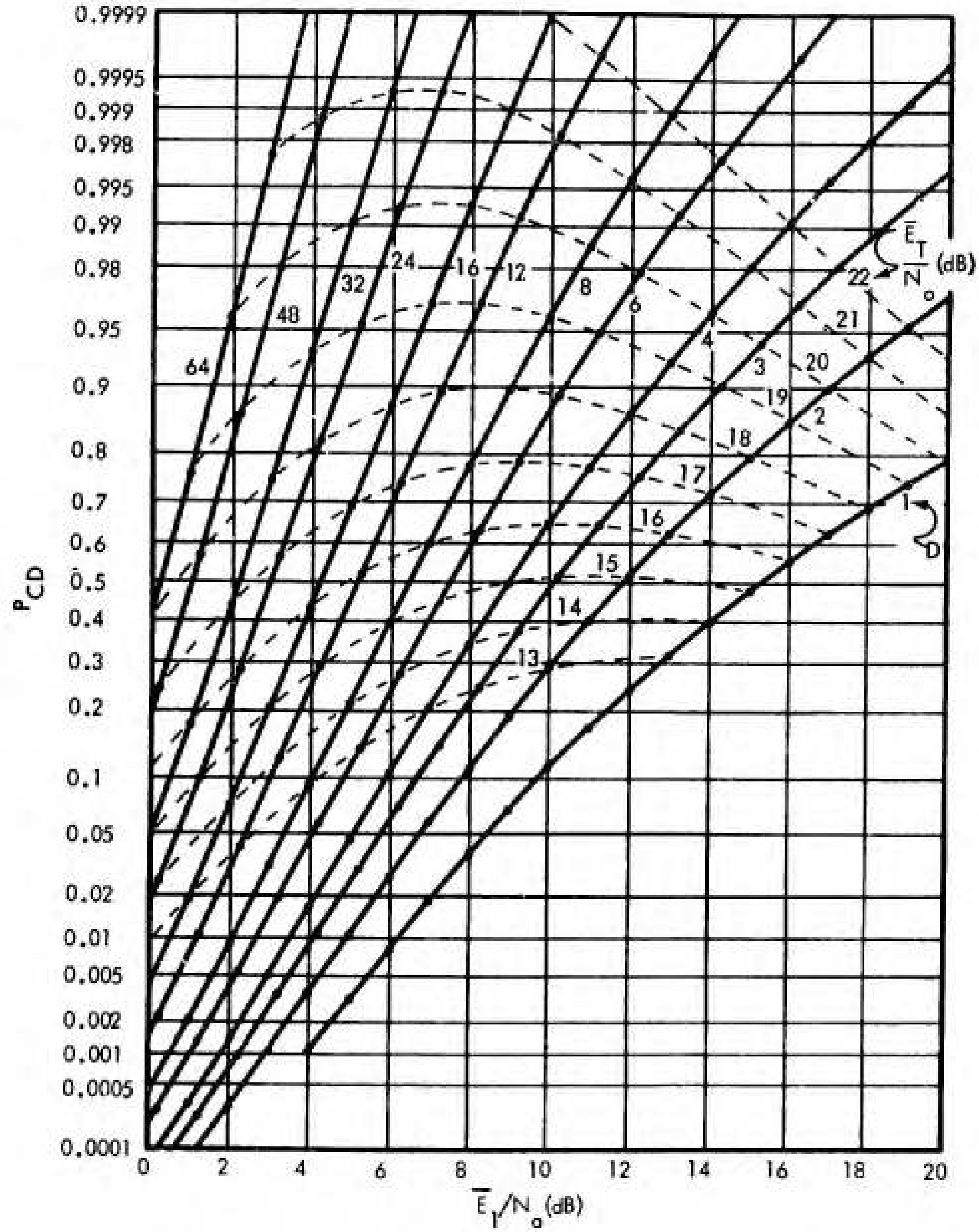

Figure 30. Detection Characteristics for $P_{FA} = 10^{-8}$, M = 256

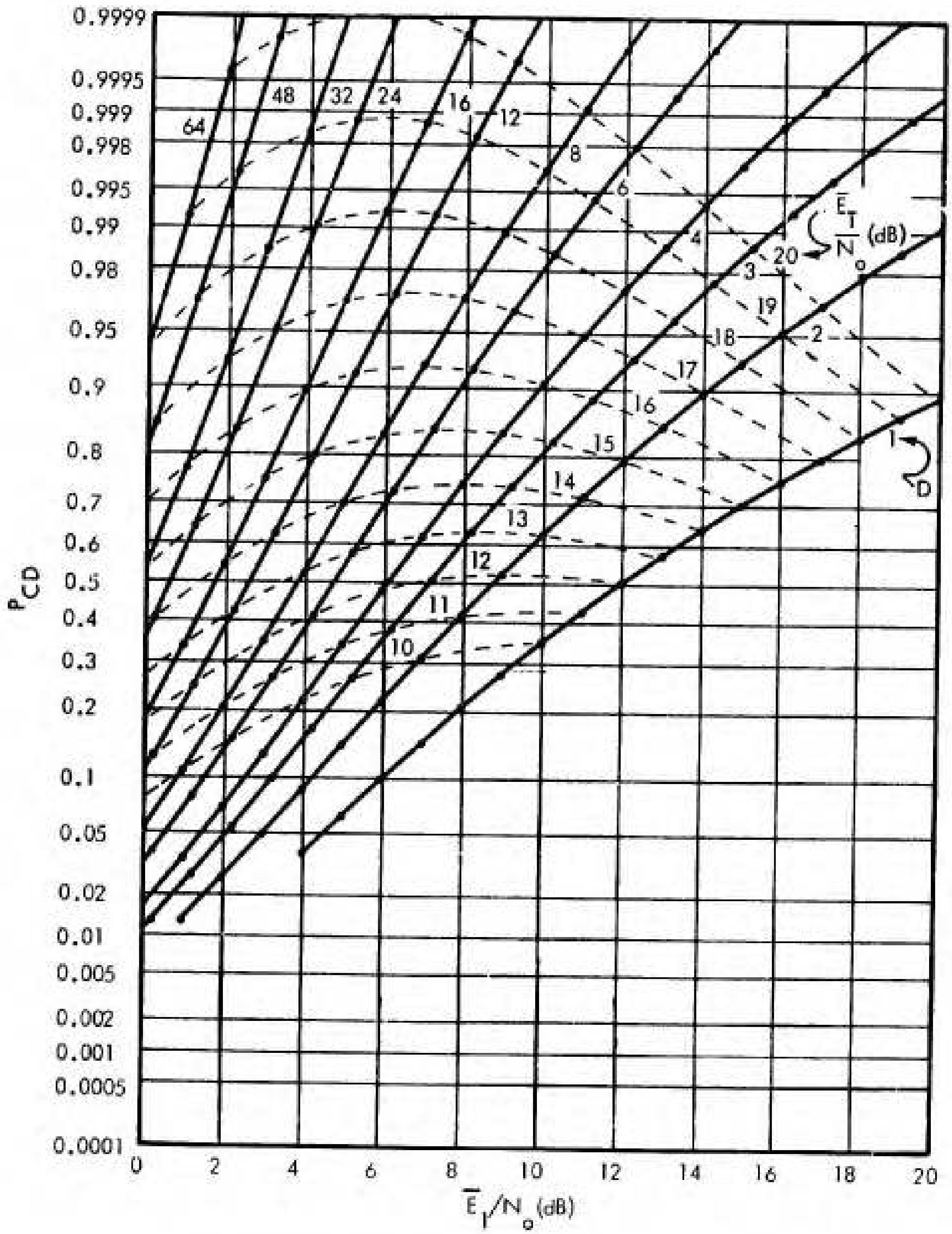

Figure 31. Detection Characteristics for $P_{FA} = 10^{-2}$, M = 1024

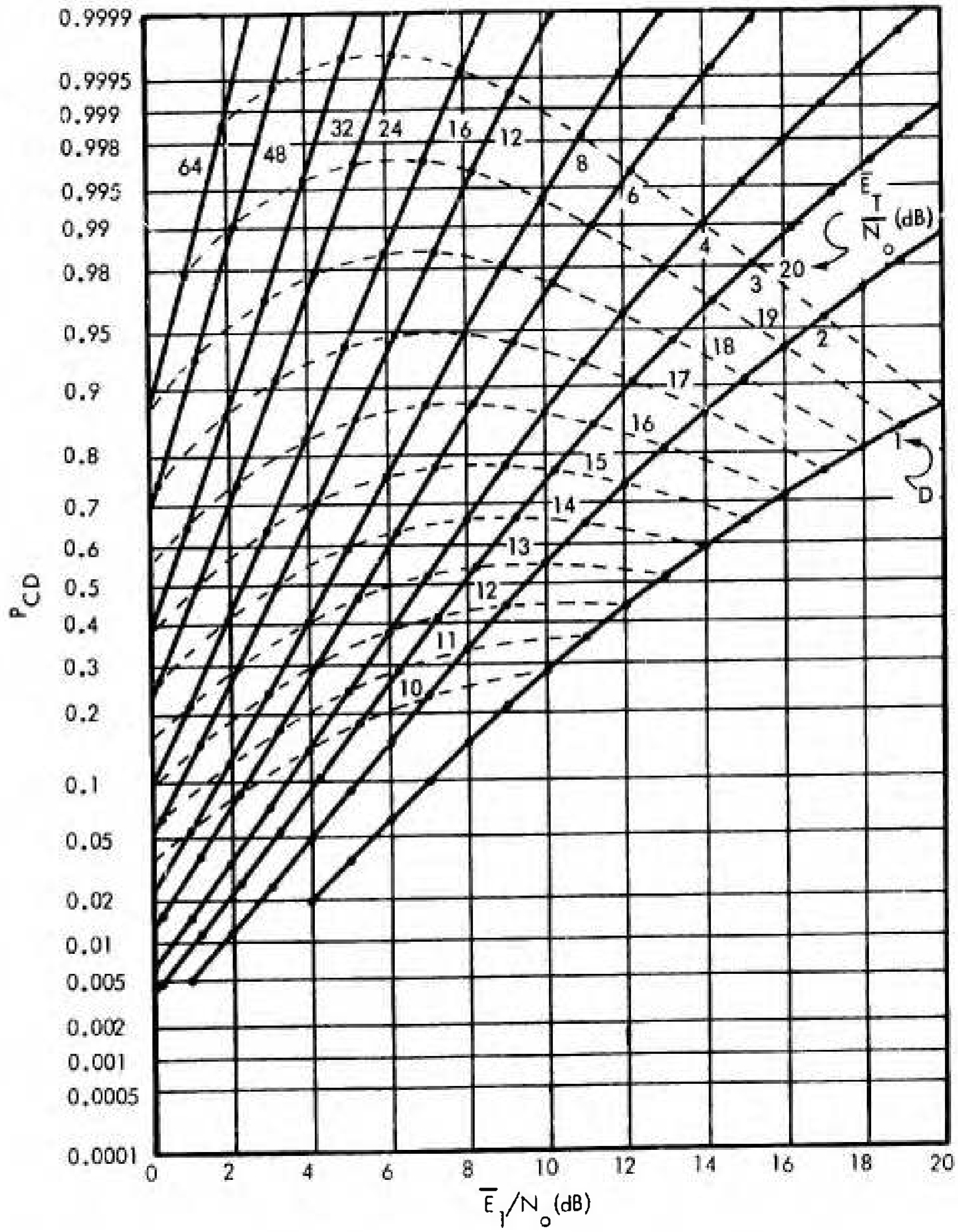

Figure 32. Detection Characteristics for $P_{FA} = 10^{-3}$, M = 1024

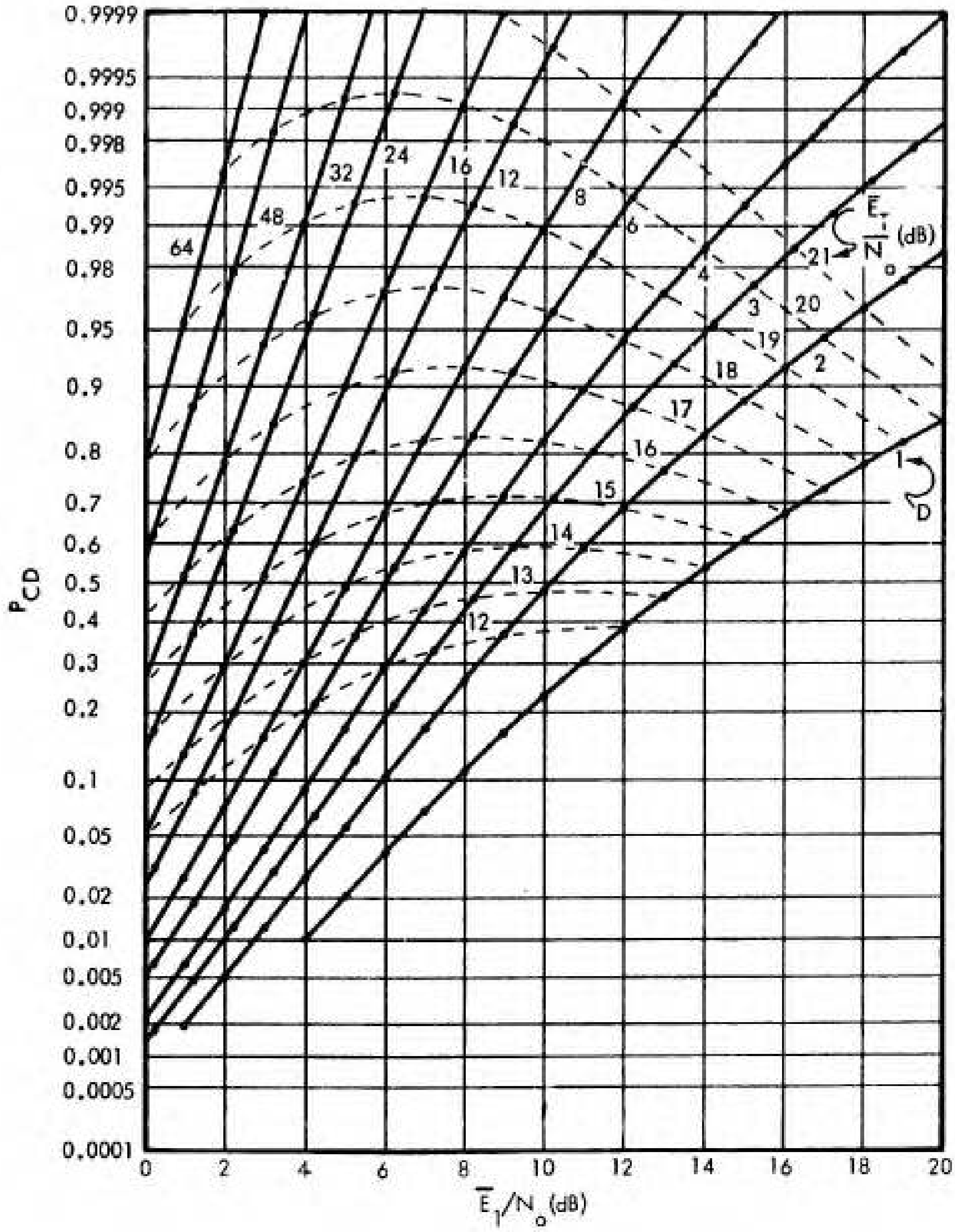

Figure 33. Detection Characteristics for $P_{FA} = 10^{-4}$, M = 1024

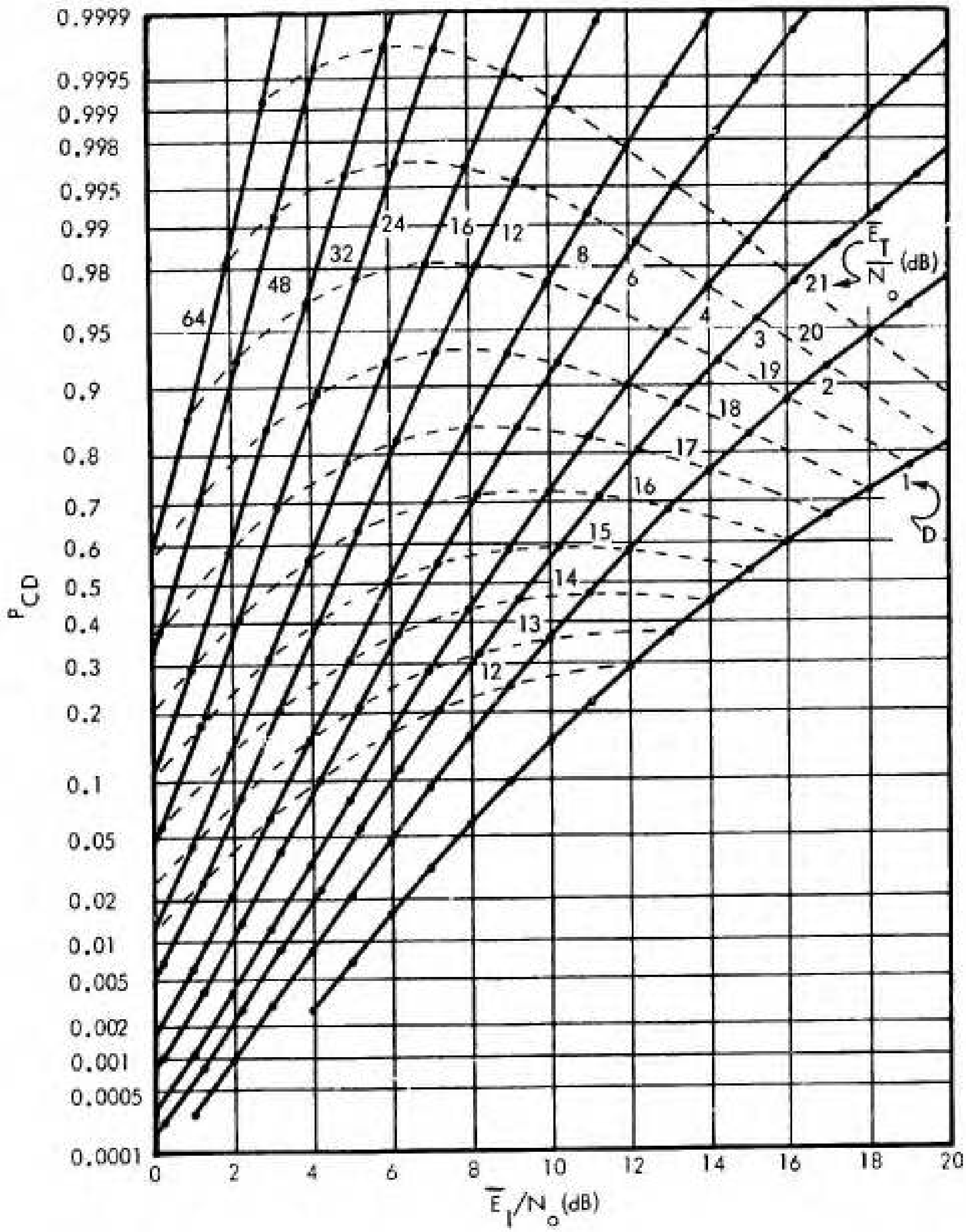

Figure 34. Detection Characteristics for $P_{FA} = 10^{-6}$, M = 1024

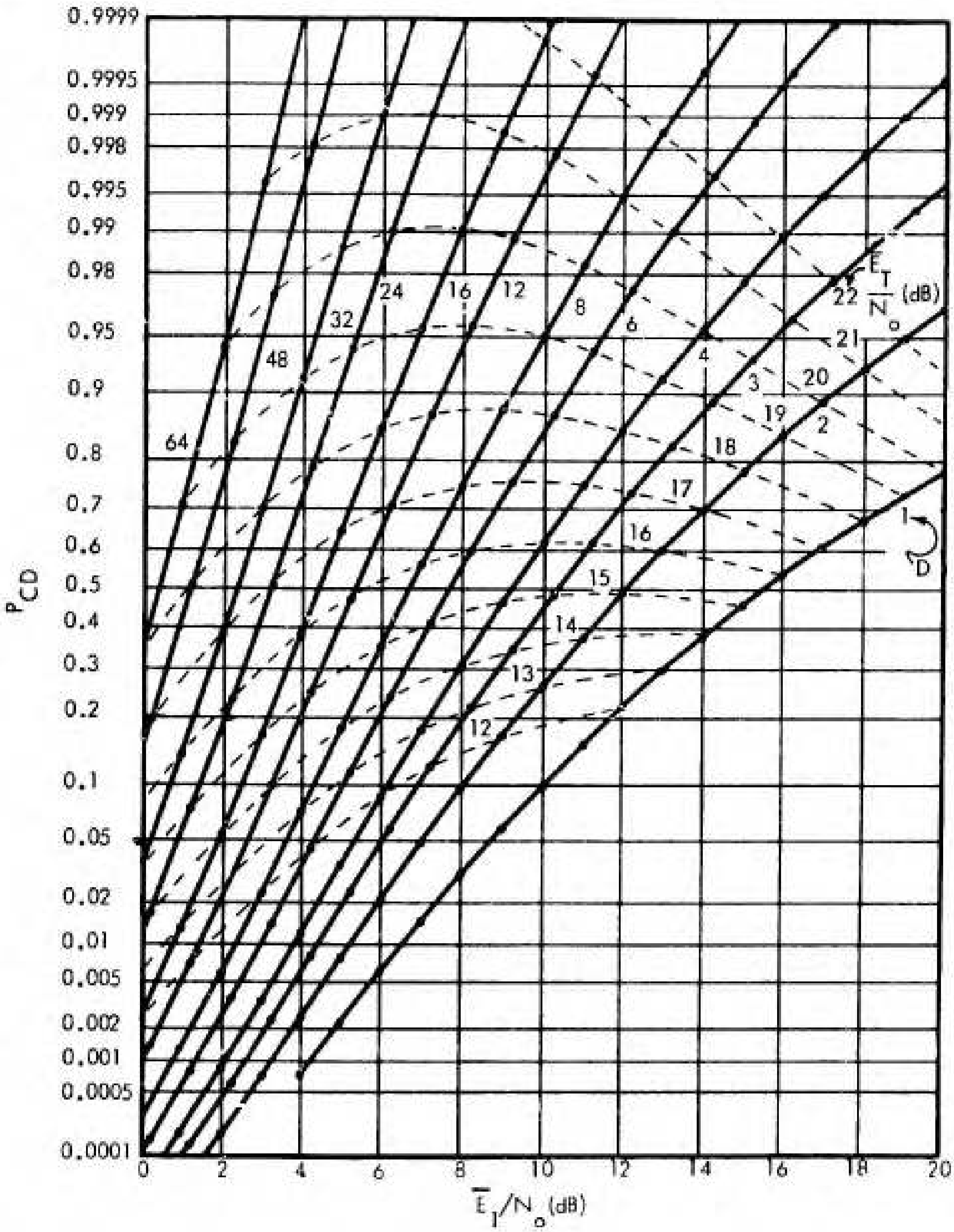

Figure 35. Detection Characteristics for $P_{FA} = 10^{-8}$, M = 1024

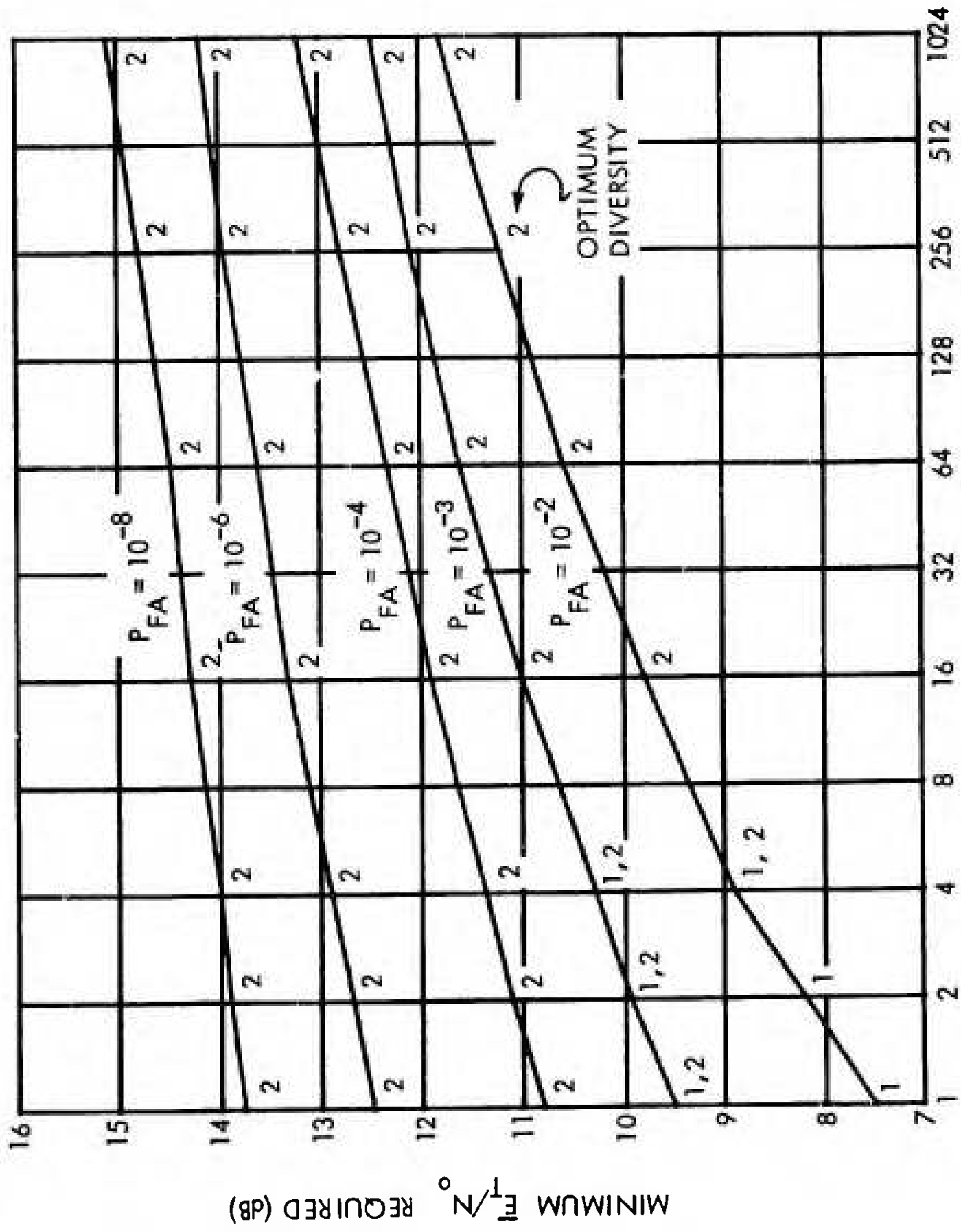

Figure 36. Minimum $\bar{E}_T/N_o$ Required for $P_{CD} = 0.5$

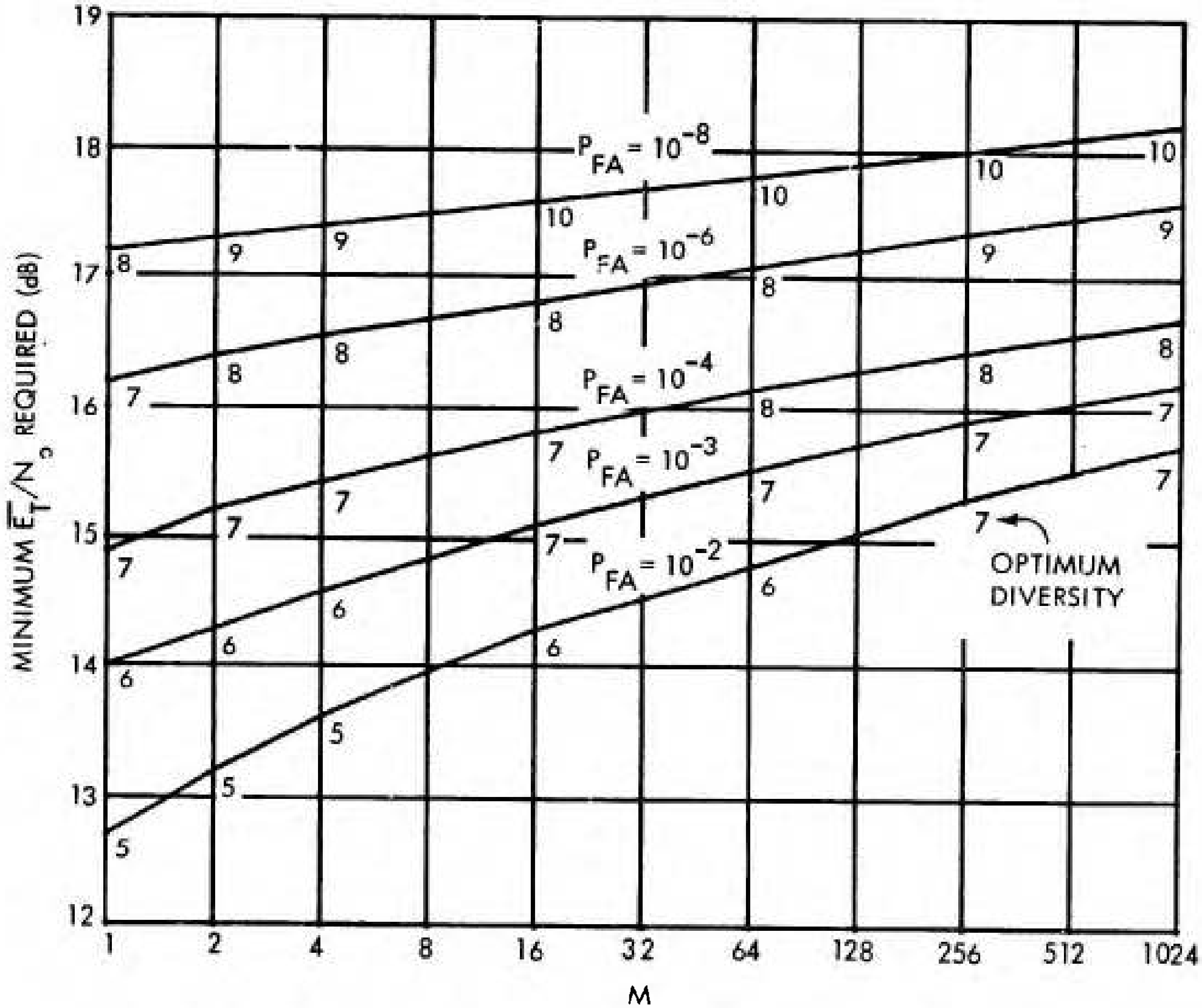

Figure 37. Minimum $\overline{E}_T/N_o$ Required for $P_{CD} = 0.9$

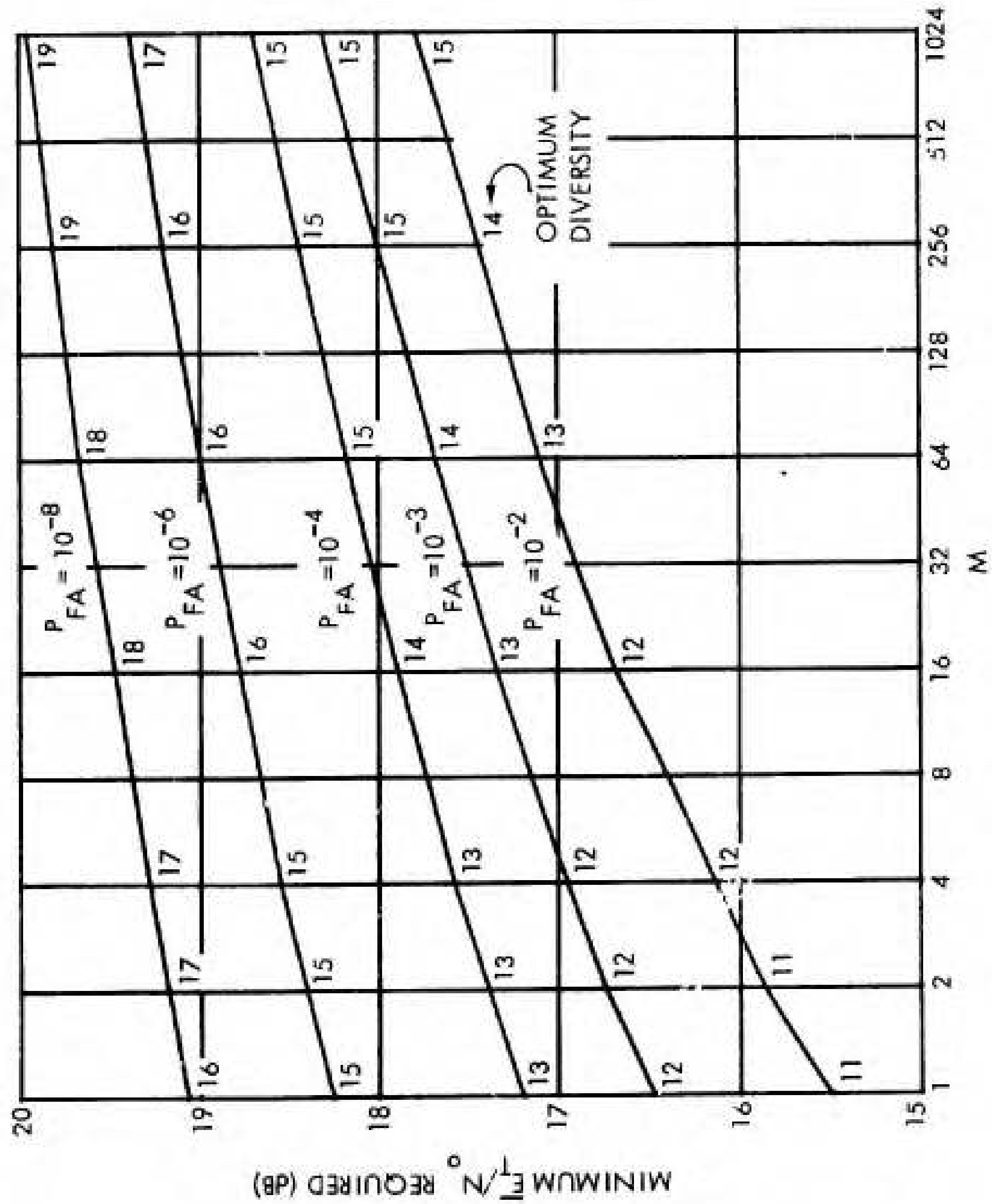

Figure 38. Minimum $\bar{E}_T/N_o$ Required for $P_{CD} = 0.99$

Appendix A

ERROR ANALYSIS AND INTERRELATIONSHIPS

ERROR ANALYSIS

The probability of correct decision, which is given in (14) of the main text as

$$P_{CD} = \int_{\Lambda}^{\infty} dx\, p_1(x)\, [P_o(x)]^{M-1} , \qquad \text{(A-1)}$$

will be approximated by

$$\widetilde{P}_{CD} \equiv \int_{\Lambda}^{A} dx\, p_1(x)\, [P_o(x)]^{M-1} + \int_{A}^{\infty} dx\, p_1(x) . \qquad \text{(A-2)}$$

The error is then

$$\begin{aligned} 0 \le \text{Error} \equiv \widetilde{P}_{CD} - P_{CD} &= \int_{A}^{\infty} dx\, p_1(x) \left\{1 - [P_o(x)]^{M-1}\right\} \\ &\le \left\{1 - [P_o(A)]^{M-1}\right\} \left\{1 - P_1(A)\right\} . \end{aligned} \qquad \text{(A-3)}$$

Now we must choose A large enough that Error is sufficiently small. One way of accomplishing this is to solve for A_o and A_1 in

$$\begin{aligned} 1 - [P_o(A_o)]^{M-1} &= \text{Error} \\ 1 - P_1(A_1) &= \text{Error} \end{aligned} \qquad \text{(A-4)}$$

and then choose $A = \text{minimum } (A_o, A_1)$. Only the first integral in (A-2) need be numerically evaluated inasmuch as the second integral is given by $1 - P_1(A)$.

INTERRELATIONSHIPS

From (11) and (12), we have

$$P_{MD} \le 1 - P_{FA} . \qquad \text{(A-5)}$$

This is a very weak inequality in most practical cases. However, in the limit of small signal-to-noise ratios, $1 - P_{MD}$ approaches P_{FA}.

From (13), we have

$$P_{ID} \leq (M-1)\int_{\Lambda}^{\infty} dx\, p_o(x)\, \left[P_o(x)\right]^{M-1} = \frac{M-1}{M}\left\{1 - \left[P_o(\Lambda)\right]^{M}\right\} = \frac{M-1}{M} P_{FA}; \quad \text{(A-6)}$$

that is, the incorrect-decision probability is always less than the false-alarm probability.

Upper and lower bounds on P_{CD} are also attainable: from (5) we have

$$P_{CD} \leq \text{Prob}\left\{x_1 > \Lambda \mid S{+}N\right\} = 1 - P_1(\Lambda) , \quad \text{(A-7)}$$

whereas from (14) we have

$$P_{CD} \geq \left[P_o(\Lambda)\right]^{M-1} \int_{\Lambda}^{\infty} dx\, p_1(x) = \left[P_o(\Lambda)\right]^{M-1} \left[1 - P_1(\Lambda)\right] . \quad \text{(A-8)}$$

Therefore it holds that

$$\left[P_o(\Lambda)\right]^{M-1} \left[1 - P_1(\Lambda)\right] \leq P_{CD} \leq 1 - P_1(\Lambda) . \quad \text{(A-9)}$$

By employing (11), this can be expressed as

$$\left[1 - P_{FA}\right]^{\frac{M-1}{M}} \left[1 - P_1(\Lambda)\right] \leq P_{CD} \leq 1 - P_1(\Lambda) , \quad \text{(A-10)}$$

which is a very useful and tight bound for $P_{FA} \ll 1$. In fact, the use of (A-10) requires no numerical integrations at all.

Appendix B

DERIVATION OF FALSE-ALARM AND CORRECT-DECISION PROBABILITIES

From the section on definitions of error probabilities and equation (24), we see that the M decision variables can be expressed as (proportional to)

$$x_k = \frac{1}{2\sigma_n^2} \sum_{d=1}^{D} y_{kd}^2 , \quad 1 \le k \le M . \tag{B-1}$$

When the probability density function for noise-alone given in (19) and (20) is utilized, the probability density function $p_o(\cdot)$ in (11) through (14) is given by

$$p_o(x) = \frac{x^{D-1} \exp(-x)}{(D-1)!} , \quad x > 0 . \tag{B-2}$$

Similarly, the probability density function $p_1(\cdot)$ is available from (20) as

$$p_1(x) = \frac{x^{D-1} \exp\left(-\frac{x}{1+R}\right)}{(D-1)!\,(1+R)^D} , \quad x > 0 , \tag{B-3}$$

where $R = \sigma_s^2/\sigma_n^2$ is the signal-to-noise ratio per diversity branch. Then the cumulative distribution functions are

$$P_o(x) = 1 - \exp(-x) \sum_{k=0}^{D-1} x^k/k! , \quad x > 0 \tag{B-4}$$

and

$$P_1(x) = 1 - \exp\left(-\frac{x}{1+R}\right) \sum_{k=0}^{D-1} \frac{1}{k!} \left(\frac{x}{1+R}\right)^k , \quad x > 0 . \tag{B-5}$$

The false-alarm and assorted decision probabilities follow upon substitution of (B-2) through (B-5) in (11) through (14).

Appendix C

RELATIONSHIP OF SIGNAL-TO-NOISE RATIO TO SIGNAL-ENERGY-TO-NOISE-DENSITY-LEVEL RATIO

For transmitted signal $R(t)\ \cos\left[\omega_o t + \theta(t)\right]$, the received waveform on one diversity branch in slow fading is given by

$$w(t) = A\ R(t)\ \cos\left[\omega_o t + \theta(t) + \phi\right] + n(t)\ , \tag{C-1}$$

where A and ϕ are the instantaneous amplitude and phase of the fading. The output of a synchronized matched filter (prior to envelope detection) is

$$y = \int dt\ R(t)\ \cos\left[\omega_o t + \theta(t)\right]\ w(t) \equiv y_s + y_n\ , \tag{C-2}$$

where

$$y_s \cong \frac{1}{2} A\ \cos\phi \int dt\ R^2(t) \tag{C-3}$$

and

$$y_n = \int dt\ R(t)\ \cos\left[\omega_o t + \theta(t)\right]\ n(t)\ . \tag{C-4}$$

Then, for uniform signal phase ϕ and white noise $n(t)$, filter output powers are

$$\begin{aligned} \sigma_s^2 &\equiv \overline{y_s^2} = \frac{1}{4} A^2 \frac{1}{2}\left[\int dt\ R^2(t)\right]^2 \\ \sigma_n^2 &\equiv \overline{y_n^2} = \frac{N_o}{2}\frac{1}{2} \int dt\ R^2(t)\ . \end{aligned} \tag{C-5}$$

Now, the instantaneous received signal energy per diversity branch is, from (C-1),

$$E_1 \cong \frac{1}{2} A^2 \int dt\ R^2(t)\ . \tag{C-6}$$

Therefore it holds that

$$\overline{E}_1 = \frac{1}{2} \overline{A^2} \int dt\ R^2(t)\ . \tag{C-7}$$

By combining (C-5) and (C-7), one obtains

$$R = \frac{\sigma_s^2}{\sigma_n^2} = \frac{\bar{E}_1}{N_o} \, . \qquad \text{(C-8)}$$

[See also reference 1, (22).]

Notice that nowhere in the above derivation has it been necessary to specify the time-bandwidth product of the component pulse; therefore (C-8) is applicable to any time-bandwidth product, provided the fading is slow and is frequency-non-selective relative to the pulse duration and bandwidth.

Appendix D

PROGRAM FOR GENERATION OF DETECTION CHARACTERISTICS

```
      DOUBLE PRECISION F,V1,V2
      INTEGER D(12),D1
      EXTERNAL F
      DIMENSION Z(200),G(12),FAC(12),PROB(23,12),YNORM(25)
      COMMON D1
      DATA D/1,2,3,4,6,8,12,16,24,32,48,64/
      DATA YNORM/-3.71902,-3.29053,-3.09023,-2.87816,-2.57583,-2.32635,
     $-2.05375,-1.64485,-1.28155,-.84162,-.52440,-.25335,0.,.25335,.5244
     $,.84162,1.28155,1.64485,2.05375,2.32635,2.57583,2.87816,3.09023,
     $3.29053,3.71902/
      DO 21 IPF=2,8,2
      PF=10.**(-IPF)
      DO 22 IM=2,4,2
      M=IM
      M1=M-1
      OM=1./M
      FAC(1)=0.
      FAC(2)=0.
      FAC(3)=LOG(2.)
      FAC(4)=LOG(6.)
      FAC(5)=LOG(120.)
      FAC(6)=LOG(5040.)
      FAC(7)=LOG(3.99168E7)
      FAC(8)=LOG(1.307674370E12)
      FAC(9)=LOG(2.5852016 7E22)
      FAC(10)=LOG(8.22283865E33)
      FAC(11)=LOG(2.58623242)+59.*LOG(10.)
      FAC(12)=LOG(1.98260832)+87.*LOG(10.)
      S=0.
      T=-1.
      DO 1 N=1,100
      T=T*(N-1.-OM)*PF/N
      S=S+T
    1 IF(ABS(T).LT.1.E-9*ABS(S)) GO TO 2
    2 G(1)=-LOG(S)
      DO 3 ID=2,12
      D1=D(ID)-1
      V1=2.*G(ID-1)+13.
      V2=V1+1.D0
      CALL ROOT(F,DBLE(S),V1,V2)
      G(ID)=V2
    3 PRINT 4, D(ID),G(ID)
    4 FORMAT(' D=',I3,'     THRESHOLD=',E15.9)
      DO 5 ID=1,12
      D1=D(ID)-1
      DO 6 IE=1,23
      ETNODB=IE+3
      R=10.**(.1*ETNODB)/D(ID)
      Q=1./(1.+R)
      Q=(1.+R)*(33.+1.38*D(ID)-15.*EXP(-.092*D1))
```

```
      COMP=D(ID)*LOG(1.+R)+FAC(ID)
      V=SIMPS(H,G(ID),W,128)
      PRINT 7, D(ID),ETNODB,V
7     FORMAT(I10,E10.2,E20.8)
      V=SIMPS2(0)
      PRINT 8,V
8     FORMAT(E40.8)
      V=MIN(V,.999999)
      PROB(IE,ID)=MAX(V,1.E-6)
6     CONTINUE
5     CONTINUE
      CALL MODESC(Z,0)
      CALL SUBJEG(Z,0.,YNORM(1),20.,YNORM(25))
      CALL OBJCTG(Z,1150.,300.,2850.,2700.)
      CALL SETSMG(Z,30,1.)
      DO 9 J=2,18,2
      CALL LINESG(Z,C,FLOAT(J),YNORM(1))
9     CALL LINESG(Z,1,FLOAT(J),YNORM(25))
      DO 10 J=2,24
      CALL LINESG(Z,0, 0.,YNORM(J))
10    CALL LINESG(Z,1,20.,YNORM(J))
      CALL SETSMG(Z,30,2.)
      CALL LINESG(Z,0, 0.,YNORM(1))
      CALL LINESG(Z,1, 0.,YNORM(25))
      CALL LINESG(Z,1,20.,YNORM(25))
      CALL LINESG(Z,1,20.,YNORM(1))
      CALL LINESG(Z,1, 0.,YNORM(1))
      CALL SETSMG(Z,84,'4')
      DO 11 ID=1,12
      DDB=10.*LOG10(D(ID))
      P=TINORM(PROB(1,ID),$30)
      CALL LINESG(Z,0,4.-DDB,P)
      CALL POINTG(Z,1,4.-DDB,P)
      DO 11 IE=2,23
      P=TINORM(PROB(IE,ID),$30)
      CALL LINESG(Z,1,IE+3-DDB,P)
11    CALL POINTG(Z,1,IE+3-DDB,P)
      CALL PAGEG(Z,0,4,1)
22    CONTINUE
21    CONTINUE
      CALL EXITG(Z)
30    STOP
      FUNCTION H(X)
      H=EXP(-X)
      IF(D1.EQ.0) GO TO 2
      T=H
      DO 1 K=1,D1
      T=T*X/K
1     H=H+T
2     H=EXP(D1*LOG(X)-0*X+M1*LOG(1.-H)-COMP)
      RETURN
      END
```

```
      SUBROUTINE ROOT(F,D,V1,V2)
      DOUBLE PRECISION F,D,V1,V2,X(100),DA(100),B,BA
      X(1)=V1
      X(2)=V2
      DA(1)=F(X(1))
      DA(2)=F(X(2))
      N1=2
      DO 4 N=3,100
      IF(ABS(DA(N-1)-D).LE.1.D- 9*ABS(D)) GO TO 5
      IF(DA(N-1)-DA(N-2)) 7,6,7
6     X(N)=X(N-1)*1.1D0
      GO TO 8
7     X(N)=(X(N-1)*(D-DA(N-2))+X(N-2)*(DA(N-1)-D))/(DA(N-1)-DA(N-2))
      B=X(N)-X(N-1)
      BA=ABS(B)
      X(N)=X(N-1)+MIN(BA,X(N-1)*.1D0)*SIGN(1.D0,B)
      X(N)=MAX(X(N),0.D0)
8     N1=N
4     DA(N)=F(X(N))
5     PRINT 9, X(N1),DA(N1),N1
9     FORMAT(/2D30.18,I10)
      V2=X(N1)
      RETURN
      END

      FUNCTION SIMPS(F,X1,X2,NN)
      DOUBLE PRECISION ACE,ACO
      N=((MAX(ABS(NN),1)+1)/2)*2
      H=(X2-X1)/N
      D=2.*H/3.
      S=.5*(F(X1)+F(X2))
      ACE=0.D0
      IF(N.EQ.2) GO TO 3
      LIM=N-2
      DO 2 I=2,LIM,2
      T=F(X1+H*I)
2     ACE=ACE+T
3     ACO=0.D0
      LIM=N-1
      DO 1 I=1,LIM,2
      T=F(X1+H*I)
1     ACO=ACO+T
      SIMPS=(S+2.*SNGL(ACO)+SNGL(ACE))*D
      RETURN
      ENTRY SIMPS2(IDUMMY)
      N=N*2
      H=H*.5
      D=D*.5
      ACE=ACE+ACO
      GO TO 3
      END
```

```
      DOUBLE PRECISION FUNCTION F(X)
      DOUBLE PRECISION X,T
      INTEGER D1
      COMMON D1
      F=EXP(-X)
      IF(D1.EQ.0) RETURN
      T=F
      DO 1 K=1,D1
      T=T*X/K
1     F=F+T
      RETURN
      END
```

REFERENCES

1. P. M. Hahn, "Theoretical Diversity Improvement in Multiple Frequency Shift Keying," IRE Transactions on Communications Systems, June 1962, pp. 177-184.

2. A. H. Nuttall, Error Probability Characteristics for Orthogonal Multiple Alternative Communication with D-Fold Diversity, NUSC Technical Report 4769, 19 June 1974.

3. A. H. Nuttall and A. F. Magaraci, Signal-to-Noise Ratios Required for Short-Term Narrowband Detection of Gaussian Processes, NUSC Technical Report 4417, 20 October 1972.

4. A. D. Whalen, Detection of Signals in Noise, Academic Press, New York, 1971.

5. A. H. Nuttall, "Error Probabilities for Equicorrelated M-ary Signals Under Phase-Coherent and Phase-Incoherent Reception," IRE Transactions on Information Theory, vol. IT-8, no. 4, July 1962, pp. 305-314.