Operational Resource Theory of Coherence

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Outline

Introduction

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“Operational"

Main results

Pure state transformations
Coherence distillation
Coherence formation
(Ir)reversibility and bound coherence

Summary

Resource theory

Ingredients

- Free states FS
- Resource states RS
- Free operations FO

Requirement

$FS \xmapsto{\text{FO}} FS$: Resource states cannot be created from free states under free operations.

Task

$RS \xmapsto{\text{FO}} RS$: The conversion between resource states under the restricted operations.

Coherence [Baumgratz/Cramer/Plenio, PRL(2014)]

Motivation: superposition principle + fixed basis $\{|i\rangle\}$

Ingredients

- Incoherent states: $\mathcal{I} = \{\rho = \sum_i p_i |i\rangle\langle i|\}$.
- Coherent states: $\rho \neq \sum_i p_i |i\rangle\langle i|$.
- Incoherent operations: $\mathcal{E} = \{K_\ell : K = \sum_i c(i)|j(i)\rangle\langle i|\}$. $K_\ell \mathcal{I} \subseteq \mathcal{I}$.

Composite system

Fixed basis=$\{|i\rangle \otimes |j\rangle\}$.

Example

- Phase unitary: $\sum e^{i\theta_i}|i\rangle\langle i|$.
- Permutation: $\sum |\pi(i)\rangle\langle i|$.
- Measurement: $\{|i\rangle\langle i|\}$.
- CNOT: $\sum |i \otimes (i \oplus j)\rangle\langle i \otimes j|$.

“Operational"

Usually resource monotone is employed to study the the conversion between resource state. Here by “operational", we mean

- it is a resource monotone: $f(IC(\rho)) \leq f(\rho)$,
- it has a physical interpretation.

How to construct the operational coherence theory?

We will recall entanglement theory to show what questions should be asked in a resource theory?

Two facts

- General task: $\rho \overset{\text{LOCC}}{\longmapsto} \sigma$.
- Maximally entangled states play a special role because usually they are used to circumvent the restriction on operations. (teleportation)

Pure states transformations

- Entanglement Concentration: from partial entangled state to MES $|\Phi_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$,

$$\psi^{\otimes n} \overset{\text{LOCC}}{\longmapsto} \overset{1-\epsilon}{\approx} \Phi_2^{\otimes nR} \text{ as } n \to \infty, \ \epsilon \to 0.$$

$$\sup R = E(\psi) = -\operatorname{Tr} \psi^A \log \psi^A.$$

- Entanglement Dilution: from Φ_2 to partial entangled state,

$$\Phi_2^{\otimes nR} \overset{\text{LOCC}}{\longmapsto} \overset{1-\epsilon}{\approx} \psi^{\otimes n} \text{ as } n \to \infty, \ \epsilon \to 0,$$

$$\inf R = E(\psi) = -\operatorname{Tr} \psi^A \log \psi^A.$$

Remark

Important! Reversible $\Longrightarrow$ the standard unit entanglement.

Basic transformations and basic measures

- Entanglement Distillation: from mixed to pure

$$\rho^{\otimes n} \stackrel{\text{LOCC}}{\longmapsto} \stackrel{1-\epsilon}{\approx} \Phi_2^{\otimes nR} \text{ as } n \to \infty, \ \epsilon \to 0,$$

  Distillable entanglement $E_d(\rho) := \sup R$.

- Entanglement formation: from pure to mixed

$$\Phi_2^{\otimes nR} \stackrel{\text{LOCC}}{\longmapsto} \stackrel{1-\epsilon}{\approx} \rho^{\otimes n} \text{ as } n \to \infty, \ \epsilon \to 0.$$

  Entanglement cost $E_c(\rho) := \inf R$.

Task: to evalute E_d and E_c.

Remark

- E_d and E_c have the operational interpretation.
- The evaluations are infeasible without additivity.

$$\lim_{n\to\infty} \frac{1}{n} E_f(\rho^{\otimes n}), \ E_f(\rho) := \min_{\rho=\sum p_i|\psi_i\rangle\langle\psi_i|} E(\psi_i).$$

(Ir)reversibility and bound entanglement

Basic question: (Ir)reversible?

- State reversible: $E_d(\rho) = E_c(\rho)$.
  Theory reversible: all states reversible.
- State irreversible: $E_d(\rho) < E_d(\rho)$.
  Theory irreversible: not all states reversible.

Bound entanglement?

Bound entanglement: $E_d = 0$ but $E_c > 0$.

What do we study in coherence theory?

- coherence concentration
- coherence dilution
- coherence distillation
- coherence formation
- (ir)reversible
- bound coherence?

Pure states transformations [Yuan et al arXiv:1505.04032]

Denote $|\Phi_2\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $\Delta(\rho) := \sum_i \langle i|\rho|i\rangle |i\rangle\langle i|$.

- Coherence concentration: from partial coherent state to $|\Phi_2\rangle$,

$$\psi^{\otimes n} \xmapsto{\text{IC}} \overset{1-\epsilon}{\approx} \Phi_2^{\otimes nR} \text{ as } n \to \infty,\ \epsilon \to 0.$$

- Coherence dilution: from Φ_2 to partial coherent state,

$$\Phi_2^{\otimes nR} \xmapsto{\text{IC}} \overset{1-\epsilon}{\approx} \psi^{\otimes n} \text{ as } n \to \infty,\ \epsilon \to 0.$$

Answer

$$C(\psi) = S(\Delta(\psi))$$

Remark

Reversible! $\Longrightarrow |\Phi\rangle_2$ is the unit coherence state.

two basic transformations and two basic measures

- Coherence distillation: from mixed to pure

$$\rho^{\otimes n} \xmapsto{\text{IC}} \overset{1-\epsilon}{\approx} \Phi_2^{\otimes nR} \text{ as } n \to \infty,\ \epsilon \to 0.$$

  Distillable coherence

$$C_d(\rho) := \sup R.$$

- Coherence formation: from pure to mixed

$$\Phi_2^{\otimes nR} \xmapsto{\text{IC}} \overset{1-\epsilon}{\approx} \rho^{\otimes n} \text{ as } n \to \infty,\ \epsilon \to 0.$$

  Coherence cost

$$C_c(\rho) := \inf R.$$

Answers

- Distillable coherence

$$C_d(\rho) = C_r(\rho) := \min_{\sigma \in \mathcal{I}} S(\rho \| \sigma) = S(\Delta(\rho)) - S(\rho),$$

- Coherence cost

$$C_c(\rho) = C_f(\rho) := \min_{\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|} \sum_i p_i C(\psi_i)$$

Remark

- Operational interpretation.
- Computable single-letter formulas.
- Additivity

$$C_r(\rho \otimes \sigma) = C_r(\rho) + C_r(\sigma),$$
$$C_f(\rho \otimes \sigma) = C_f(\rho) + C_f(\sigma).$$

(Ir)reversible

- State reversible: $C_d(\rho) = C_c(\rho)$.
  Theory reversible: all states reversible.
- State irreversible: $C_d(\rho) < C_c(\rho)$.
  Theory irreversible: not all states reversible.

Criterion of reversibility

$$\rho = \bigoplus_j p_j |\phi_j\rangle\langle\phi_j|,$$

$|\phi_j\rangle \in \mathcal{H}_j = \mathrm{span}\{|i_j\rangle\}$, and $\mathcal{H}_j \perp \mathcal{H}_k$ for $j \neq k$.

Remark

(1) It is simple to check.
(2) It completely characterizes all the reversible states.

Bound coherence?

Question

Does there exist bound coherent state s.t $C_d(\rho) = 0$ but $C_c(\rho) > 0$?

Answer: NO!

Reason

$C_d = C_r$ is faithful in the sense that $C_r = 0$ implies $C_f = 0$.

$C(\psi) = S(\Delta(\psi))$: sketched proof

$\psi \xmapsto{\text{IC}} \varphi \Longleftrightarrow \Delta(\psi) \prec \Delta(\varphi)$.[Du et al, arXiv:1503.09176]
$\Longrightarrow \Phi_d \xmapsto{\text{IC}} \rho$, where $rank(\rho) \leq d$.

Suppose $|\psi\rangle = \sum_{i=1}^{d} \sqrt{q_i}|i\rangle$, then $|\psi\rangle^{\otimes n} = \sum_{i^n} \sqrt{q_{i^n}}|i^n\rangle$.

Concentration: Perform type measurement $\{M_P\}$

$$M_P = \sum_{i_1 \cdots i_n \in T(P)} |i_1 \cdots i_n\rangle\langle i_1 \cdots i_n|,$$
$$2^{nS(P)} \geq |T(P)| \geq (n+1)^{-d} 2^{nS(P)},$$
$$P_{typ} \approx Q,$$

Dilution: Prepare the typical part,

$$|\psi\rangle^{\otimes n} = \sqrt{\Pr(T^n_{Q,\delta})}|\text{typ}\rangle + \sqrt{1 - \Pr(T^n_{Q,\delta})}|\text{atyp}\rangle.$$
$$|T^n_{Q,\delta}| \leq 2^{n(S(Q)+\delta)}.$$

$C_d = C_r$: sketched roof

Achievability: type measurement + operator covering lemma + proper IC + Uhlmann's theorem

The purification $|\phi\rangle^{AE} = \sum \sqrt{q_i}|i\rangle^A|\phi_i\rangle^E$

Step 1. Perform type measurement $\{M_P\}$, $P_{typ} \approx Q$,

$$
\begin{aligned}
|\phi(P)\rangle^{AE} &= \frac{1}{\sqrt{|T(P)|}} \sum_{i^n \in T(P)} |i^n\rangle^A |\phi_{i^n}\rangle^E, \\
&= \frac{1}{\sqrt{M}} \sum_m |m\rangle^{A_1} \frac{1}{\sqrt{S}} \sum_s |s\rangle^{A_2} |\phi_{ms}\rangle = \frac{1}{\sqrt{M}} \sum_m |m\rangle^{A_1} |\phi(P)\rangle_m^{A_2E}, \\
&= \frac{1}{\sqrt{(1-\epsilon)M}} \sum_{good\ m} |m\rangle^{A_1}|\phi(P)\rangle_m^{A_2E} + \frac{1}{\sqrt{\epsilon M}} \sum_{bad\ m} |m\rangle^{A_1}|\phi(P)\rangle_m^{A_2E},
\end{aligned}
$$

"good m": $\phi(P)_m^E \approx \phi(P)^E$,
"bad m": $\phi(P)_m^E \not\approx \phi(P)^E$.

Step 2. Perform measurement $\{P_{good\ m}^{A_1} \otimes I^{A_2}, P_{bad\ m}^{A_1} \otimes I^{A_2}\}$,

$$
\frac{1}{\sqrt{(1-\epsilon)M}} \sum_{good\ m} |m\rangle |\phi(P)\rangle_m^{A_2E}.
$$

For good m, by Uhlmann's theorem: $\exists \; U_m^{A_2}$,

$$U_m^{A_2} \otimes I^E |\phi(P)\rangle_m^{A_2E} \approx |\phi(P)\rangle_0^{A_2E}.$$

Step 3. Apply IC operation $\{K_s\}$

$$K_s = \sum_{good\ m} |m\rangle\langle m|^{A_1} \otimes |0\rangle\langle s| U_m^{A_2}.$$

$$\approx \frac{1}{\sqrt{(1-\epsilon)M}} \sum_{good\ m} |m\rangle |\phi(P)\rangle_0^{A_2E}.$$

Step 4. Count $|T(P)|$ and $S = S(\rho)$ [operator covering lemma].

Optimality: monotonicity + additivity + asymptotical continuity.

For any protocol $\mathcal{L}_n$ such that $\mathcal{L}_n(\rho^{\otimes n}) \approx |\Phi_2\rangle\langle\Phi_2|^{\otimes nR}$,

$$\begin{aligned}
nC_r(\rho) &= C_r(\rho^{\otimes n}) && (1)\\
&\geq C_r(\mathcal{L}_n(\rho^{\otimes n})) && (2)\\
&\geq C_r(\Phi_2^{\otimes nR}) - 2n\epsilon \log d - 2\eta(\epsilon) && (3)\\
&= nR - 2n\epsilon \log d - 2\eta(\epsilon), && (4)
\end{aligned}$$

$C_c = C_f$: sketched proof

Achievabilitiy: typicality + Caratheodory's Theorem

For an optimal decomposition, $\rho = \sum_i^M p_i \psi_i$, by Caratheodory's Theorem, $M \leq d^2$.

$$\rho^{\otimes n} = \sum_{i^n} p_{i^n} \psi_{i^n} = (1-\epsilon_1)\rho(\mathcal{T}) + \epsilon_1 \rho_0,$$

$\mathcal{T}$ is the set of typical sequences.
Prepare each state whose index sequence in the $\mathcal{T}$ with coherence concentration.

Error: $(1-\epsilon_1)(1-\epsilon_2)^M$.

Consumption of $|\Phi_2\rangle$: $n(p_j + \delta_1)(S(\Delta(\psi_j)) + \delta_2)$.

Optimality: monotonicity + additivity + asymptotical continuity

For any protocol $\mathcal{L}_n$ such that $\mathcal{L}(\Phi_2^{\otimes nR}) = \rho^{(n)} \approx \rho^{\otimes n}$.

$$
\begin{aligned}
nR &= C_f(\Phi_2^{\otimes nR}) && (5)\\
&\geq C_f(\rho^{(n)}) && (6)\\
&\geq C_f(\rho^{\otimes n}) - 10n\epsilon \log d - 2\eta(\epsilon) && (7)\\
&= nC_f(\rho) - 10n\epsilon \log d - 2\eta(\epsilon), && (8)
\end{aligned}
$$

Additivity

Additivity of C_r: $C_r(\rho) = S(\Delta\rho) - S(\rho)$.

Additivity of C_f:

$$\begin{aligned} C_f(\rho) &= E_f(\widetilde{\rho}), \\ \rho = \sum_{ij} \rho_{ij} |i\rangle\langle j| &\longleftrightarrow \widetilde{\rho} = \sum_{ij} \rho_{ij} |ii\rangle\langle jj|, \end{aligned}$$

$\widetilde{\rho}$ is a maximally correlated state (MCS).

$$E_f(\widetilde{\rho} \otimes \widetilde{\sigma}) = E_f(\widetilde{\rho}) + E_f(\widetilde{\sigma}). [\textit{Vidal etal}, \textit{PRL}(2002)]$$

Criterion: sketched proof

$$\rho = \sum_{ij} \rho_{ij} |i\rangle\langle j| \longleftrightarrow \widetilde{\rho} = \sum_{ij} \rho_{ij} |ii\rangle\langle jj|$$

$$\begin{aligned} C_r(\rho) &= S(\widetilde{\rho}^B) - S(\widetilde{\rho}^{AB}) \\ C_f(\rho) &= E_f(\widetilde{\rho}^{AB}). \end{aligned}$$

Apply a lemma [Winter and Yang, arXiv:1505.00907]:
If $S(\sigma^B) - S(\sigma^{AB}) = E_f(\sigma^{AB})$, then

$$\sigma^{AB} = \bigoplus_i p_i \rho_i^{B_i^L} \otimes |\phi_j\rangle\langle\phi_j|^{AB_i^R},$$

$$\mathcal{H}_B = \bigoplus_i \mathcal{H}_{B_i^L} \otimes \mathcal{H}_{B_i^R}.$$

Summary

- Pure states transformation: $C(\psi) = S(\Delta(\psi))$
- Distillation: $C_d(\rho) = C_r(\rho) = S(\Delta(\rho)) - S(\rho)$.
- Cost: $C_c(\rho) = C_f(\rho)$.
- Reversible: $\rho = \bigoplus_j p_j|\phi_j\rangle\langle\phi_j|$, $\Delta(\phi_i) \perp \Delta(\phi_j)$, $i \neq j$.
- No bound coherence.

Open questions

Q1.

$$\rho = \sum_{ij} \rho_{ij}|i\rangle\langle j| \longleftrightarrow \widetilde{\rho} = \sum_{ij} \rho_{ij}|ii\rangle\langle jj|,$$

Coherence theory $\longleftrightarrow$ Entanglement theory of MCS.
Incoherent operation $\longleftrightarrow$ LOCC ?
Q2. How to expand the restricted operations to recover reversible resource theory? [Brandao and Gour, arXiv:1502.03149.]

Thank you

for your attention !