Operational State Complexity under Parikh Equivalence

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Subset construction [Rabin&Scott '59]

NFA		DFA
n states	$\Longrightarrow$	2^n states
L		L

Moreover, this state bound cannot be reduced

[Meyer&Fischer '71, Moore '71]

What happens if we do not care of the order of symbols in the strings?

This problem is related to the concept of *Parikh equivalence*

[Parikh '66]

Standard equivalence: NFAs vs DFAs

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[Parikh '66]

Parikh equivalence: preliminaries

- $\Sigma = \{a_1, \ldots, a_m\}$ alphabet of m symbols
- $|w|_a$ be the number of occurrences of a in $w \in \Sigma^*$

Parikh map

The *Parikh map* $\psi : \Sigma^* \to \mathbb{N}^m$ associates with a word $w \in \Sigma^*$ the m-dimensional nonnegative vector $(|w|_{a_1}, |w|_{a_2}, \ldots, |w|_{a_m})$.

Parikh image

The *Parikh image of a language L* is $\psi(L) = \{\psi(w) \mid w \in L\}$.

- $w_1 =_\pi w_2$ iff $\psi(w_1) = \psi(w_2)$
- $L_1 =_\pi L_2$ iff $\psi(L_1) = \psi(L_2)$

Parikh equivalence: Parikh's theorem

Theorem ([Parikh '66])

For each context-free language $L \subseteq \Sigma^*$, there exists a Parikh equivalent regular language $R \subseteq \Sigma^*$.

Example ($L =_\pi R$)

$$L = \{a^n b^n \mid n \geq 0\} \quad \text{and} \quad R = (ab)^*$$

have the same Parikh image, namely the set

$$\{(n, n) \mid n \geq 0\}$$

From NFAs to Parikh equivalent DFAs

We have the following Parikh equivalent conversion:

Theorem (NFA to DFA)

NFA		DFA
n *states*	$\Longrightarrow_\pi$	$e^{O(\sqrt{n \cdot \ln n})}$ *states*
L_1		L_2

Moreover, this cost is tight.

Quite surprisingly:

Polynomial conversion

If the given NFA accepts only nonunary strings then the cost reduces to

a *polynomial* in n.

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Our Goal

We investigate, under Parikh equivalence, the state complexity of some language operations which preserve regularity $(\cup, \cap, ^c, \cdot, ^*, \sqcup\!\sqcup, ^R, P_{\Sigma_0})$.

Problem (DFAs to DFA)

A, B DFAs		C DFA
n_1, n_2 *states*	$\Longrightarrow_\pi$	$L(C) =_\pi L$
$L(A), L(B)$		*how many states?*

where:

- $L = L(A) \cup L(B)$
- $L = L(A) \cap L(B)$
- $L = L(A)L(B)$
- ...

Standard equivalence: concatenation

A, B DFAs
n_1, n_2 states
$L(A)L(B)$

$\Longrightarrow$

C DFA
$2^{n_1+n_2}$ states
$L(C) = L(A)L(B)$

In the worst case: $(2n_1 - 1)2^{n_2-1}$ states

[Yu '00]

Under Parikh equivalence we reduce this bound.

Standard equivalence: concatenation

A, B DFAs n_1, n_2 states $L(A)L(B)$	$\Longrightarrow$	C DFA $2^{n_1+n_2}$ states $L(C) = L(A)L(B)$

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Concatenation under Parikh equivalence

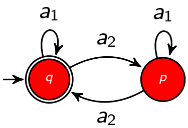
One of our contribution

Problem (DFAs to DFA)

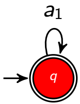
A, B DFAs		C DFA
n_1, n_2 *states*	$\Longrightarrow_\pi$	$L(C) =_\pi L$
$L = L(A)L(B)$		*how many states?*

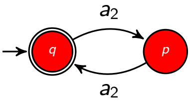
- Upper bound: $e^{\sqrt{n \cdot \ln n}}$, where $n = n_1 + n_2$
  by Parikh equivalent conversion
- Lower bound: $n_1 n_2$ states
  by unary case [Yu '00]

Unary and *nonunary* parts of a language

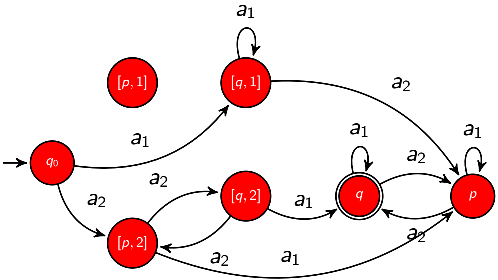


Unary parts:





Nonunary part:



$$L(A) = \bigcup_{i=0}^{m} L(A_i)$$

Concatenation under Parikh equivalence: proof idea

DFAs A, B
n_1, n_2 states
$L = L(A)L(B)$
$\Sigma = \{a_1, \ldots, a_m\}$

Concatenation under Parikh equivalence: proof idea

$\forall i = 1 \ldots m$
$A_i, O(n_1)$ states
$B_i, O(n_2)$ states

unary

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$\forall i = 1 \dots m$
$A_i, O(n_1)$ states
$B_i, O(n_2)$ states

$\Longrightarrow$

$\forall i = 1 \dots m$
DFA M_i
$L(M_i) = L(A_i)L(B_i)$
$O(n_1 n_2)$ states

unary

[Yu '00]

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n_1, n_2 states
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$B_i, O(n_2)$ states

$\Longrightarrow$ [Yu '00]

$\forall i = 1 \ldots m$
DFA M_i
$L(M_i) = L(A_i)L(B_i)$
$O(n_1 n_2)$ states

$\Longrightarrow$

DFA M'
$L(M') = \bigcup_{i=1}^{m} L(M_i)$
$poly(n_1, n_2)$ states

Concatenation under Parikh equivalence: proof idea

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n_1, n_2 states
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nonunary

NFA M
$L(M) = L$
$n_1 + n_2$ states

Concatenation under Parikh equivalence: proof idea

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unary

$\forall i = 1 \ldots m$
$A_i, O(n_1)$ states
$B_i, O(n_2)$ states

$\Longrightarrow$ [Yu '00]

$\forall i = 1 \ldots m$
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$L(M_i) = L(A_i)L(B_i)$
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$n_1 + n_2$ states

$\Longrightarrow$

NFA M_0
$L \setminus L(M')$
$(n_1+n_2)(m+1)+1$ states

Concatenation under Parikh equivalence: proof idea

DFAs A, B
n_1, n_2 states
$L = L(A)L(B)$
$\Sigma = \{a_1, \ldots, a_m\}$

unary

$\forall i = 1 \ldots m$
$A_i, O(n_1)$ states
$B_i, O(n_2)$ states

$\Longrightarrow$ [Yu '00]

$\forall i = 1 \ldots m$
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Parikh equivalent conversion $\Longrightarrow_\pi$

DFA M'_0
$poly(n_1, n_2)$ states

Concatenation under Parikh equivalence: proof idea

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$L = L(A)L(B)$
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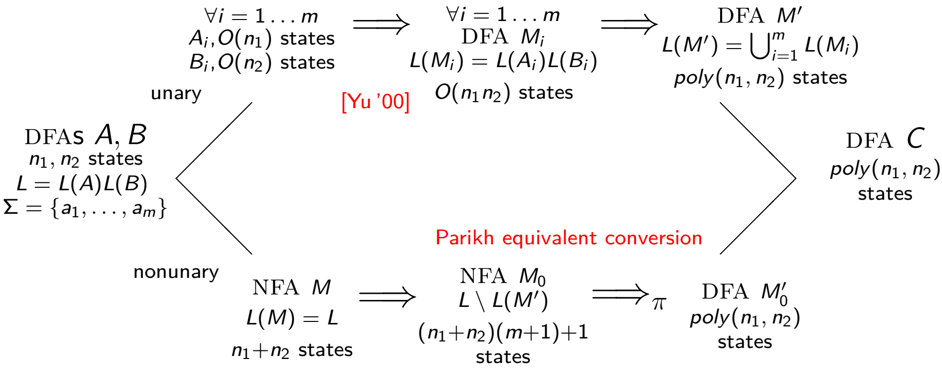
$\Longrightarrow_{\pi}$ Parikh equivalent conversion

DFA M'_0
$poly(n_1, n_2)$
states

DFA C
$poly(n_1, n_2)$
states

Concatenation under Parikh equivalence: proof idea



Theorem

Given two DFAs A and B of n_1 and n_2 states, respectively, there exists a DFA of polynomial number of states in n_1 and n_2 that is Parikh equivalent to $L(A)L(B)$. Moeover, this cost is tight.

Projection under Parikh equivalence

Given a word $w \in \Sigma^*$, the *projection* of w over an alphabet $\Sigma' \subseteq \Sigma$, is the word $P_{\Sigma'}(w)$ obtained by removing from w all the symbols which are not in Σ'. (see, e.g., [Jirásková & Masopust 12]).

Example:

$$P_{\{a,b\}}(a^n b^n c^n) = a^n b^n$$

Projection under Parikh equivalence

Under Parikh equivalence, $e^{O(\sqrt{n \cdot \ln n})}$ is enough and this is tight.

DFA A		NFA A'		DFA M
$L(A)$	$\Longrightarrow$	$L(A') = P_{\Sigma'}(L(A))$	$\Longrightarrow_\pi$	$L(M) =_\pi L(A')$
n states		n states		$e^{O(\sqrt{n \cdot \ln n})}$ states

Operation	Standard equivalence	Parikh equivalence
$L_1 \cup L_2$	$n_1 n_2$	$n_1 n_2$
$L_1 \cap L_2$	$n_1 n_2$	$n_1 n_2$
L_1^c	n_1	n_1
$L_1 L_2$	$(2n_1 - 1)2^{n_2 - 1}$	$poly(n_1, n_2)$
L_1^*	$2^{n_1 - 1} + 2^{n_1 - 2}$	$poly(n_1)$
$L_1 \sqcup\!\sqcup L_2$	$2^{n_1 n_2} - 1$	$poly(n_1, n_2)$
L_1^R	2^{n_1}	n_1
$P_{\Sigma_0}(L_1)$	$3 \cdot 2^{n_1 - 2} - 1$	$e^{O(\sqrt{n_1 \cdot \ln n_1})}$

[Yu '00, Campeanu&Salomaa&Yu '02, Yu&Zhuang&Salomaa '94, Jiraskova&Masopust '12]

Intersection does not commute with Parikh mapping

$\psi(a^+b^+ \cap b^+a^+) \neq \psi(a^+b^+) \cap \psi(b^+a^+)$ holds; in fact,

$$\begin{aligned} \psi(a^+b^+ \cap b^+a^+) &= \emptyset \\ \psi(a^+b^+) \cap \psi(b^+a^+) &= \{(i,j) \mid i,j \geq 1\}. \end{aligned}$$

Complement does not commute with Parikh mapping

$\psi((a^*b^*)^c) \neq (\psi(a^*b^*))^c$ holds; in fact,

$$\begin{aligned} \psi((a^*b^*)^c) &= \{(i,j) \mid i,j \geq 1\} \\ (\psi(a^*b^*))^c &= \emptyset. \end{aligned}$$

Problem: intersection

A, B DFAs
n_1, n_2 states

$\Longrightarrow$

M DFA
$\psi(L(M)) = \psi(L(A)) \cap \psi(L(B))$
How many states needed?

Problem: complement (left open!)

A DFA
n states

$\Longrightarrow$

M DFA
$\psi(L(M)) = (\psi(L(A)))^c$
How many states needed?

Intersection: revisited

We use a modification of the following result:

Theorem ([Kopczyński&To '10])

There is a polynomial p such that for each n-state NFA A *over* $\Sigma = \{a_1, \ldots, a_m\}$,

$$\psi(L(A)) = \bigcup_{i \in I} Z_i$$

where:

- I *is a set of at most* $p(n)$ *indices*
- *for* $i \in I$, $Z_i \subseteq \mathbb{N}^m$ *is a linear set of the form:*

  $$Z_i = \{\alpha_0 + n_1\alpha_1 + \cdots + n_k\alpha_k \mid n_1, \ldots, n_k \in \mathbb{N}\}$$

  with

  - $0 \leq k \leq m$
  - *the components of* α_0 *are bounded by* $p(n)$
  - $\alpha_1, \ldots, \alpha_k$ *are linearly independent vectors from* $\{0, 1, \ldots, n\}^m$

Intersection: revisited

Theorem

Let A, B be DFAs with respectively n_1, n_2 states over $\Sigma = \{a_1, \ldots, a_m\}$. There exists a DFA M whose Parikh map is equal to $\psi(L(A)) \cap \psi(L(B))$ and which contains

$$O(n^{(2m-1)(3m^3+6m^2)+2} p(n)^{2(3m^3+6m^2)+m})$$

states, where:

- $n = \max\{n_1, n_2\}(m+1) + 1$
- $p(n) = O(n^{3m^2} m^{m^2/2+2})$

Proof.

Revisiting the Ginsburg and Spanier's proof [Ginsburg&Spanier '64] of the closure property of semilinear sets under intersection. □

Conclusion

Under Parikh equivalence:

- For $\cup$, $\cdot$, *, ^{c}, $\cap$, $\sqcup\!\sqcup$, and ^{R}, we obtain a polynomial state complexity, in contrast to the intrinsic exponential state complexity in the classical equivalence.
- For P_{Σ_0} we prove a superpolynomial state complexity, which is lower than the exponential one of the corresponding classical operation.
- For each two deterministic automata A and B, it is possible to obtain a deterministic automaton with a polynomial number of states, whose accepted language has as Parikh image $\psi(L(A)) \cap \psi(L(B))$.

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Thank you for your attention