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Operations Research Games: A Survey

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Abstract

This paper surveys the research area of cooperative games associated with several types of operations research problems in which various decision makers (players) are involved. Cooperating players not only face a joint optimisation problem in trying, e.g., to minimise total joint costs, but also face an additional allocation problem in how to distribute these joint costs back to the individual players. This interplay between optimisation and allocation is the main subject of the area of operations research games. It is surveyed on the basis of a distinction between the nature of the underlying optimisation problem: connection, routing, scheduling, production and inventory.

1 Introduction

Typically, operations research analyses situations in which one decision maker, guided by an objective function, faces an optimisation problem. The theory concentrates on the question of how to act in an optimal way and, in particular, on the issues of computational complexity and the design of efficient algorithms. Game theory on the other hand analyses situations involving at least two interacting decision makers (called players), with possibly diverging interests. Roughly speaking, it deals with mathematical models of competition and cooperation.

Competitive or noncooperative game theory studies situations in which the players can negotiate about what to do (i.e., pre-play communication is allowed), but enforceable binding agreements are assumed not to be possible. Therefore strategic analysis and individual incentives play a prominent role here. In cooperative game theory enforceable binding agreements are possible and also side payments may be allowed. Now the main issue is fair allocation, either of joint costs or joint revenues.

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Since the early developments of operations research and game theory there has been a strong interplay between the two disciplines. Especially the interrelation between operations research and noncooperative game theory is well-known: between duality results in mathematical programming and minimax results for zero-sum games, between linear complementarity and bimatrix games, between Markov decision processes and stochastic games, and between optimal control theory and differential games.

The interrelation between operations research and cooperative game theory is of a more recent date and is summarised under the heading of operations research games. One can say that an important part of the interplay between cooperative games and operations research stems from the basic (discrete) structure of a graph, network or system that underlies various types of combinatorial optimisation problems. If one assumes that at least two players are located at or control parts (e.g., vertices, edges, resource bundles, jobs) of the underlying system, then a cooperative game can be associated with this type of optimisation problem. In working together, the players can possibly create extra gains or save costs compared to the situation in which everybody optimises individually. Hence the question arises how to share the extra revenues or cost savings.

One way to analyse this question is to study the general properties (e.g., balancedness or convexity) of all games arising from that specific type of operations research problem and to apply a suitable existing game theoretic solution concept (e.g., core or Shapley value) to this class. Another way is to create a context specific allocation rule. Such a rule can be based either on desirable properties in this specific context or on a kind of decentralised mechanism that prescribes an allocation on the basis of the algorithmic process along which a jointly optimal combinatorial structure is established (e.g., following an algorithm to create a minimal cost spanning tree). A general reference on cooperative games is Driessen (1988) and various specific operations research games are treated in Curiel (1997).

The original request of TOP to the authors of this paper was to write a *complete* survey on operations research games. Given the abundance of papers on this topic, starting more or less from the beginning of the seventies, and the vast increase during the nineties, this constitutes a "mission impossible". Moreover, the definition of operations research games is not so strict that a unique selection of research streams is prescribed. Consequently, the choice of topics treated in this survey is somewhat biased towards our own expertise, knowledge and interests. A first general aim of this survey is to give an unacquainted reader a flavour of the things that are going on in this interesting field of research. Our second aim is to provide a rather up to date state-of-the-art. Last but not least we hope it stimulates researchers to enter this field. There are still many questions to be asked, gaps to be filled and extensions to be investigated. We have included a brief final section with our ideas for possible future research lines; not in a very detailed and elaborate way, but mainly by making some hints and stating some catchwords to provoke possibly different individual associations and to enter new research tracks.

To better structure the survey, we have chosen to make a division of operations research games into five categories, primarily based on the nature of the underlying optimisation problem. In our view, this categorisation also allows for a better insight into the various relationships in methodology, techniques and results across the different classes of operations research games. We distinguish between:

(i) *Connection*: fixed tree, spanning tree

(ii) *Routing*: Chinese postman, travelling salesman

(iii) *Scheduling*: sequencing, permutation, assignment

(iv) *Production*: linear production, flow

(v) *Inventory*

Each category will be treated in a separate section. Within each category we have chosen one representative class which is discussed in some detail, starting from the level of a person who is not familiar with the topic. For this reason, relatively much attention is paid to the modelling phase, i.e., how to go from operations research to game theory. After discussing the main results from the literature for the representative class, the other classes within the same category are treated in a more compact way.

To conclude the introduction we mention some topics which could be considered as being inside the theory of operations research games, or at least closely related, but which will not be discussed in this survey. For those interested we have added a selected list of references. Within a financial context we mention bankruptcy games, cf. O'Neill (1982), Aumann and Maschler (1985), Curiel et al. (1987), Young (1988), Kaminski (2000) and Calleja et al. (2001), deposit games, cf. Izquierdo and Rafels (1996) and Borm et al. (2001) and shortest path games, cf. Fragnelli et al. (2000), Voorneveld and Grahn (2000) and Grahn (2001). An interesting class with clear optimisation features deals with cost sharing issues, cf. Moulin and Shenker (1992a), Moulin and Shenker (1992b) and Sprumont (1998). A nice survey of this literature can be found in Koster (1999). Another type of games which directly involves combinatorial structures is the class of communication games and all its variants. Surveys can be found in Slikker and Van den Nouweland (2001) and Bilbao (2000). An interesting recent contribution which uses techniques from linear production is Suijs et al. (2001).

2 Preliminaries

In this section we introduce some basic notation and define a number of elementary concepts in cooperative game theory, which we use throughout the paper.

The set of real numbers is denoted by $\mathbb{R}$. For a finite set N we denote by $\mathbb{R}^N$ the set of real vectors of length $|N|$, where the coordinates correspond to the elements of N. $\mathbb{R}^N_+$ is the set of all elements of $\mathbb{R}^N$ in which all coordinates are nonnegative and $\mathbb{R}^N_{++}$ denotes the set of all vectors in which all coordinates are positive. The set of subsets of N is denoted by 2^N. For $S \subset N$, e^S denotes the vector in $\mathbb{R}^N$ with $e^S_i = 1$ for all $i \in S$ and $e^S_i = 0$ for all $i \in N \backslash S$.

A *cooperative game with transferable utility*, or *TU game*, is described by a pair (N, v), where $N = \{1, \ldots, n\}$ denotes the set of players and $v : 2^N \rightarrow \mathbb{R}$ is the *characteristic function*, assigning to every coalition $S \subset N$ of players a value $v(S)$, representing the maximal total monetary reward the members of this group can obtain when they cooperate. By convention, $v(\emptyset) = 0$.

The *imputation set* $I(v)$ of a game (N, v) is defined as the set of individually rational allocations of $v(N)$:

$$I(v) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N), \forall_{i \in S} : x_i \geqslant v(\{i\})\}$$

and the *core* is defined as

$$C(v) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N), \forall_{S \subset N} : \sum_{i \in S} x_i \geqslant v(S)\}.$$

So the core consists of all allocations of $v(N)$ such that no coalition S has an incentive to part company with $N \backslash S$ and establish cooperation on its own. A TU game (N, v) is called *balanced* if it has a nonempty core and *totally balanced* if the core of every subgame is nonempty, where the subgame corresponding to some coalition $T \subset N, T \neq \emptyset$ is the game (T, v^T) with $v^T(S) = v(S)$ for all $S \subset T$.

Every game can be uniquely decomposed as a linear combination of unanimity games. For $T \subset N, T \neq \emptyset$, the *unanimity game* (N, u_T) is defined by

$$u_T(S) = \begin{cases} 1 & \text{if } T \subset S, \\ 0 & \text{otherwise} \end{cases}$$

for all $S \subset N$.

An *order* on the players is a bijection $\sigma : N \to \{1, \ldots, n\}$, where $\sigma(i) = j$ means that player i is at position j. The set of all orders on N is denoted by Π_N. For every order $\sigma \in \Pi_N$, we define the *marginal vector* $m^\sigma(v)$ recursively by

$$m^\sigma_{\sigma^{-1}(k)}(v) = v(\{\sigma^{-1}(1), \ldots, \sigma^{-1}(k)\}) - v(\{\sigma^{-1}(1), \ldots, \sigma^{-1}(k-1)\})$$

for all $k \in \{1, \ldots, n\}$. The *Shapley value* of (N, v) is defined as (cf. Shapley (1953))

$$\Phi(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi_N} m^\sigma(v).$$

The Shapley value is a linear operator on the class of all TU games and for the unanimity game (N, u_T) it equals

$$\Phi(u_T) = \frac{1}{|T|} e^T.$$

A game (N, v) is called *superadditive* if for all coalitions $S, T \subset N$ with $S \cap T = \emptyset$ we have

$$v(S) + v(T) \leqslant v(S \cup T)$$

and *convex* if for all $i \in N$ and all $S \subset T \subset N \backslash \{i\}$ we have

$$v(S \cup \{i\}) - v(S) \leqslant v(T \cup \{i\}) - v(T).$$

In a superadditive game, it will always be beneficial for two disjoint coalitions to cooperate and form a larger coalition. In a convex game, a player's marginal contribution to a large coalition is larger than his marginal contribution to a smaller coalition, which is stronger than superadditivity. A game is convex if and only if its core is the convex hull of all marginal vectors. Furthermore, every convex game is totally balanced.

The *excess* of coalition $S \subset N$ with respect to an imputation $x \in I(v)$ is defined by

$$E(S,x) = v(S) - \sum_{i \in S} x_i.$$

The *excess vector* with respect to x, denoted by $\theta(x)$, is the vector in $\mathbb{R}^{2^n}$ containing the excesses of all coalitions in (weakly) decreasing order.

For a game (N,v) with $I(v) \neq \emptyset$, the *nucleolus* is defined (cf. Schmeidler (1969)) as the unique imputation $nu(v)$ such that $\theta(nu(v)) \leqslant_L \theta(x)$ for all $x \in I(v)$. A vector $x \in \mathbb{R}^t$ is *lexicographically smaller* than $y \in \mathbb{R}^t$, i.e., $x \leqslant_L y$, if $x = y$ or if there exists an $s \in \{1, \ldots, t\}$ such that $x_k = y_k$ for all $k \in \{1, \ldots, s-1\}$ and $x_s < y_s$.

A *population monotonic allocation scheme* (cf. Sprumont (1990)), or *pmas*, for the game (N,v) is a collection of vectors $y^S \in \mathbb{R}^S$ for all $S \subset N, S \neq \emptyset$ such that

$$\sum_{i \in S} y_i^S = v(S)$$

for all $S \subset N, S \neq \emptyset$ and

$$y_i^S \leqslant y_i^T \tag{2.1}$$

if $S, T \subset N$ and $i \in N$ are such that $S \subset T$ and $i \in S$.

In many operations research settings, one does not consider rewards to coalitions, but costs. A *cost game* is a special kind of TU game, usually denoted by (N,c), in which $c(S)$ is interpreted as the minimal total costs the members of coalition S have to make when they cooperate. Again, by convention, $c(\emptyset) = 0$.

Because of the different interpretation of a cost game, many of the definitions for reward games, as presented above, have to be adjusted to this context. For instance, the core of a cost game (N,c) is defined by

$$C(c) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = c(N), \forall_{S \subset N} : \sum_{i \in S} x_i \leqslant c(S)\}.$$

In a similar way, the definitions of imputation set, nucleolus and pmas are altered. In this cost setting, the natural counterpart of convexity, as defined for reward games, is concavity. A cost game (N,c) is called *concave* if for all $i \in N$ and all $S \subset T \subset N \backslash \{i\}$ we have

$$c(S \cup \{i\}) - c(S) \geqslant c(T \cup \{i\}) - c(T).$$

Similarly, the counterpart of superadditivity is subadditivity.

3 Connection

In this section we consider operations research problems which involve connection networks in an interactive cooperative setting. We look at two such problems in particular: maintenance problems, which involve a fixed tree network, and minimum cost spanning tree problems, in which the connection network is still to be decided upon.

First, we look at maintenance problems, which form a special class of fixed tree problems. Our exposition is mainly based on the overview given in Koster (1999).

The idea behind a maintenance problem is the following. A group of players is connected by some fixed network to a certain service provider, e.g., by a road network to a community centre. This network is a tree in which the service provider is situated at the root. Each road in this network has some maintenance costs associated with it and the question is how the maintenance costs of the entire network should be divided in a fair way among all users.

Formally, a *maintenance problem* is a triple (G, t, N) where

- $G = (V, E)$ is a tree with vertex set V and edge set E; the root r has only one adjacent edge.
- $t : E \to \mathbb{R}_+$ is a nonnegative cost function on the edges of the tree.
- $N = \{1, \ldots, n\}$ is a finite player set; each player $i \in N$ is located at some vertex $v(i) \in V$ and every vertex in V except the root corresponds to exactly one player.

In order to analyse maintenance problems, we introduce some more notation. First note that every vertex $v \in V$ is connected to the root of the tree by a unique path P_v (including v itself). We denote the edge in P_v that is incident on v by e_v. The precedence relation $\preccurlyeq$ on V is defined by

$$v' \preccurlyeq v \Leftrightarrow v' \text{ is on the path } P_v.$$

A *trunk* of G is a set of vertices $R \subset V$ which is closed under the relation $\preccurlyeq$, i.e., if $v \in R$ and $v' \preccurlyeq v$, then $v' \in R$. The set of *followers* of player $i \in N$ is given by $F(i) = \{j \in N \mid v(i) \preccurlyeq v(j)\}$ and the set of *predecessors* by $P(i) = \{j \in N \mid v(j) \preccurlyeq v(i)\}$. The total costs of a trunk R equal

$$T(R) = \sum_{v \in R \setminus \{r\}} t(e_v).$$

With each maintenance problem $\Gamma = (G, t, N)$ we associate a *maintenance game* (N, c_Γ) defined by

$$c_\Gamma(S) = \min\{T(R) \,|\, v(i) \in R \text{ for all } i \in S \text{ and } R \text{ is a trunk}\} \tag{3.1}$$

for all $S \subset N, S \neq \emptyset$ and $c_\Gamma(\emptyset) = 0$. By nonnegativity of the cost function, the trunk R that minimises total costs in (3.1) is the smallest trunk R_S containing all the vertices at which the members of S are located, i.e.,

$$c_\Gamma(S) = \sum_{v \in R_S \setminus \{r\}} t(e_v).$$

The *dual unanimity game* (N, u_T^*) with $T \subset N, T \neq \emptyset$ is defined by

$$u_T^*(S) = \begin{cases} 1 & \text{if } T \cap S \neq \emptyset, \\ 0 & \text{otherwise} \end{cases}$$

for all $S \subset N$. The coalition T in u_T^* can be seen as having some veto control: if no member of T is present in a coalition, this particular coalition has value 0. Note that u_T^* is a concave game.

Proposition 3.1 *Let $\Gamma = (G, t, N)$ be a maintenance problem. Then the associated cost game (N, c_Γ) can be decomposed in the following way:*

$$c_\Gamma = \sum_{i \in N} t(e_{v(i)}) u^*_{F(i)}.$$

The decomposition of c_Γ in terms of dual unanimity games is interpreted as follows. In order to determine the costs of a coalition S, we have to find the smallest trunk containing all members of S. Edge $e_{v(i)}$ is present in this smallest trunk whenever a member of S is a follower of player i, i.e., $S \cap F(i) \neq \emptyset$.

Because all coefficients of the dual unanimity games in Proposition 3.1 are nonnegative, every maintenance game is concave. As a consequence, the core of such a game is nonempty and has a nice structure. The literature offers a large number of characterisations of the core of maintenance games, two of which will be presented here. The first one is in terms of trunks.

Proposition 3.2 *A vector of cost shares $x \in \mathbb{R}^N$ is an element of $C(c_\Gamma)$ if and only if $x \geqslant 0$ and $\sum_{i \in R} x_i \leqslant T(R)$ for each trunk R.*

The second characterisation of the core states that a cost allocation is a core element whenever the costs associated with each edge are divided among those players using that particular edge.

Proposition 3.3 *A vector of cost shares $x \in \mathbb{R}^N$ is an element of $C(c_\Gamma)$ if and only if there exist $y^1, \ldots, y^n$ such that y^j is a vector in the unit simplex in $\mathbb{R}^{F(j)}$ for all $j \in N$ and*

$$x_i = \sum_{j \in P(i)} y_i^j t(e_{v(j)})$$

for all $i \in N$.

Next, we turn our attention to (one point) solutions of maintenance problems. A function Ψ is a *maintenance solution* if it assigns to every maintenance problem $\Gamma = (G, t, N)$ a vector of cost shares $\Psi(\Gamma) \in \mathbb{R}_+^N$.

The first solution is given by a painting story, which is based on Maschler et al. (1995). Suppose the vertices of the tree are homes and the edges are roads connecting these homes to a community centre, which is located at the root of the tree. The costs $t(e)$ of road $e \in E$ are now to be interpreted as the number of days it takes a single worker to paint the stripes on the road. The following rules are used to determine how the road network is to be painted:

(i) Every worker works equally fast with speed 1.

(ii) Every worker keeps working as long as the road from his residence to the community centre has not been completed.

(iii) Every worker does his job on an unfinished road segment between the community centre and his home.

(iv) If the road between a worker's predecessor in the tree and the community centre is not yet fully completed, he has to work on that part of the network.

(v) Every worker is doing his job as close to his residence as conditions (i)-(iv) allow.

Let $P(\Gamma)$ denote the cost allocation for maintenance problem Γ that follows from (i)-(v). The computation of this *home-down painting solution* is illustrated in the following example:

Example 3.1 Consider the maintenance problem with $N = \{1, \ldots, 4\}$ as presented in Figure 3.1, where the numbers on the edges represent the costs.

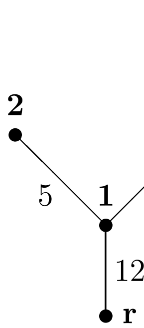

Figure 3.1: A maintenance problem

First we have to determine where each player starts painting. Due to conditions (iv) and (v), players 1, 2 and 3 start on $\{r, 1\}$ and player 4 starts on $\{1, 3\}$. After four time units, $\{r, 1\}$ is completed and player 1 has finished his job. Next, the segment $\{1, 3\}$ is completed by 3 and 4, while player 2 continues with $\{1, 2\}$. Finally, players 2 and 4 finish their "own" segments. The computations are summarised in Figure 3.2. The italic numbers indicate where the players are painting

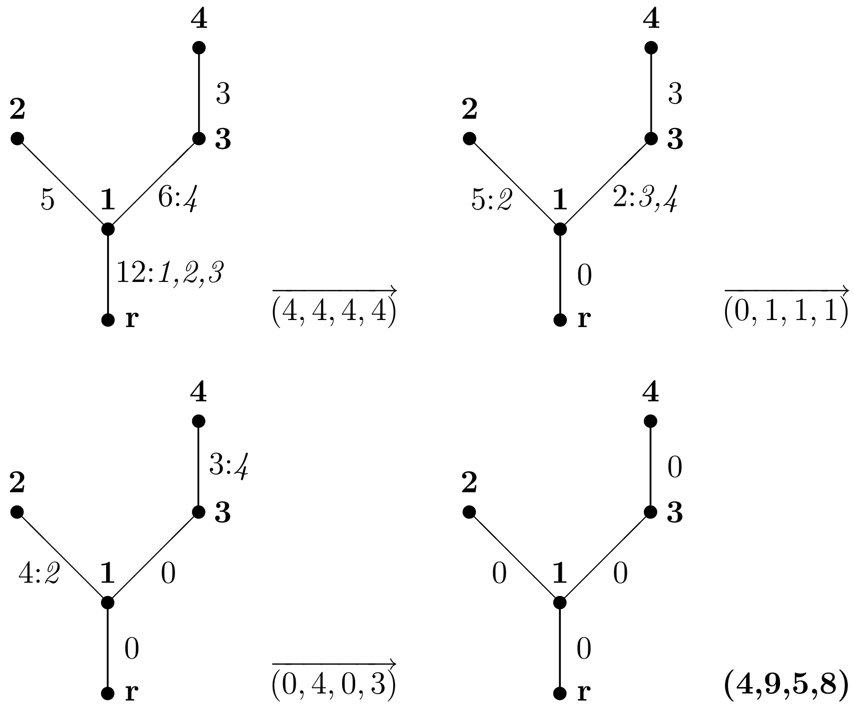

Figure 3.2: Home-down painting solution

at each iteration and the vectors underneath the arrows represent the corresponding marginal costs. The home-down painting solution of this maintenance problem equals $(4,4,4,4)+(0,1,1,1)+(0,4,0,3)=(4,9,5,8)$. $\triangleleft$

The home-down painting solution $P(\Gamma)$ turns out to be the nucleolus of the corresponding maintence game (N, c_Γ) (cf. Maschler et al. (1995)).

Theorem 3.4 *Let $\Gamma=(G,t,N)$ be a maintenance problem. Then $P(\Gamma)=nu(c_\Gamma)$.*

Consistency of the home-down painting solution is studied in Granot et al. (1996), Granot and Maschler (1998) and Van Gellekom and Potters (1999).

An alternative painting solution is given by dropping condition (iv) and replacing condition (v) by

(v') Every worker is doing his job as close to the community centre as conditions (i)-(iii) allow.

Rules (i)-(iii) and (v') determine the *down-home painting solution* for maintenance problems, which we denote by P', and which is given by

$$P'(\Gamma)=\sum_{j\in P(i)}\frac{1}{|F(j)|}t(e_{v(j)}). \tag{3.2}$$

Example 3.2 Consider the maintenance problem in Example 3.1. The down-home painting solution equals

$$P'(\Gamma)=12(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})+5(0,1,0,0)+6(0,0,\frac{1}{2},\frac{1}{2})+3(0,0,0,1)=(3,8,6,9)$$

$\triangleleft$

This down-home painting solution is the Shapley value of the corresponding maintentance game.

Theorem 3.5 *Let $\Gamma=(G,t,N)$ be a maintenance problem. Then $P'(\Gamma)=\Phi(c_\Gamma)$.*

Because c_Γ is concave, the Shapley value lies in the core of the game. This also follows immediately from Proposition 3.3 and equation (3.2).

In order to define a third painting solution, we need to introduce some more concepts. A *pseudo subtree* of a tree $G=(V,E)$ is a connected subgraph $G'=$

(V', E') such that there exists an $r' \in V'$ which is minimal in G' with respect to $\preccurlyeq$ and which has only one adjacent edge in E'. A *weight system* for maintenance problem $\Gamma = (G, t, N)$ is a pair $\beta = (\mathcal{T}, w)$, where $\mathcal{T} = (G^1, \ldots, G^p)$ is a partition of G into pseudo subtrees and $w \in \mathbb{R}_+^N$ is a weight vector such that for all $i \in N$ with $t(e_{v(i)}) > 0$ there is a $j \in F(i)$, who is in the same pseudo subtree as i, with $w_j > 0$. The set of all weight systems for Γ is denoted by $\mathcal{B}(\Gamma)$.

Now we define the *weighted down-home painting solution* P^β corresponding to some weight system $\beta \in \mathcal{B}(\Gamma)$. In this context, every pseudo subtree G^k has its own local community centre, which is situated at the root of G^k. The solution is determined by the following rules:

(i") Every worker works with speed w_i.

(ii") Every worker keeps working as long as the road from his residence to his local community centre has not been completed.

(iii") Every worker does his job on an unfinished road segment between the local community centre and his home.

(iv") If the road between a worker's predecessor in the tree and the local community centre is not yet fully completed, he has to work on that part of the network.

(v") Every worker is doing his job as close to his residence as conditions (i")-(iv") allow.

From Proposition 3.3 it follows that every weighted down-home allocation is a core element. The converse is also true, thus establishing a third characterisation of the core (Bjørndal et al. (1999)).

Theorem 3.6 *Let $\Gamma = (G, t, N)$ be a maintenance problem. Then for every $x \in C(c_\Gamma)$ there exists a weight system $\beta \in \mathcal{B}(\Gamma)$ such that $x = P^\beta(\Gamma)$.*

A similar result in the context of irrigation networks can be found in Koster et al. (1998). Related problems with applications of fixed tree problems are discussed in Megiddo (1978) and Galil (1980). Some computational issues are addressed in Granot and Granot (1992b).

A maintenance problem in which the fixed tree is a line graph is called an *airport problem.* Airport problems were introduced by Littlechild and Owen (1973) and its Shapley value and nucleolus as well as their properties were studied in Littlechild (1974), Littlechild and Owen (1977), Littlechild and Thompson (1977), Dubey (1982) and Potters and Sudhölter (1999).

The description of an airport problem can be shortened to a pair (N, k), where $N = \{1, \ldots, n\}$ is the player set and $k \in \mathbb{R}^N_+$ is a vector of marginal costs, which are interpreted as follows. Every player owns an airplane with certain characteristics, which determine the minimal length of a landing strip this plane can use. Assuming that the players are ordered in increasing length of this strip (i.e., $k_1 \leqslant \ldots \leqslant k_n$) and maintenance costs are linear in strip length, player i's total costs equal $\sum_{j=1}^{i} k_j$, where k_j represents the extra costs of maintaining the longer strip of player j in relation to the shorter strip of player $j-1$. The problem is how to divide the maintenance costs of a strip that accommodates all airplanes, $\sum_{i \in N} k_i$, among the players.

With each airport problem (N, k) we associate an *airport game* (N, c) with cost function $c(S) = \sum_{j=1}^{i} k_j$, where $i = \max\{j \mid j \in S\}$. Since this game is a special case of a maintenance game, it is concave and we have a nice expression for the Shapley value. First, note that one can decompose the cost function c as follows:

$$c = k_1 u^*_N + k_2 u^*_{\{2,\ldots,n\}} + \ldots + k_n u^*_{\{n\}}.$$

Hence, the Shapley value of (N, c) equals

$$\Phi(c) = k_1(\frac{1}{n}, \ldots, \frac{1}{n}) + k_2(0, \frac{1}{n-1}, \ldots, \frac{1}{n-1}) + \ldots + k_n(0, \ldots, 0, 1).$$

Of course, the results for maintenance games w.r.t. the core, the nucleolus and the weighted Shapley values induce easy expressions for airport games in a similar way. A nice application of airport games is provided by Aadland and Kolpin (1998), who look at irrigation networks. In Branzei et al. (2001), airport problems are considered in which there are restrictions on the level of side payments that are feasible.

Next, we consider a class of problems that is closely related to maintenance problems: *minimum cost spanning tree* or *mcst problems.* Contrary to maintenance problems, in mcst problems the connecting network is not fixed, but an integral part of the decision problem. Our discussion is mainly based on Feltkamp (1995).

Consider a group of villages, each of which needs to be connected to some source, either directly or via other villages. Every possible connection has some nonnegative costs associated with it and the problem is how to connect every village to the source such that the total costs of creating the network are minimal. Kruskal (1956) and Prim (1957) provide two greedy algorithms for solving this kind of problem. A historic overview of mcst problems can be found in Graham and Hell (1985).

Constructing an mcst, however, is only part of the problem. In addition to minimising total costs, a cost allocation problem has to be addressed as well. Claus and Kleitman (1973) introduced this cost allocation problem, whereupon Bird (1976) treated this problem with game theoretic methods and proposed a cost allocation rule, known as the Bird rule.

Formally, an msct problem is a triple $\mathcal{T} = (N, *, t)$, where $N = \{1, \dots n\}$ is the player set, $*$ is the source and $t : E_{N^*} \to \mathbb{R}_+$ is the nonnegative cost function. E_S is defined as the set of all edges between pairs of elements of $S \subset N^*$, so that (S, E_S) is the complete graph on S:

$$E_S = \{\{i, j\} \,|\, i, j \in S, i \neq j\}.$$

Because connection costs are nonnegative, it is obvious that a minimal cost graph that connects all players to the source is indeed a tree, which explains the name of the problem.

Given an mcst problem $\mathcal{T} = (N, *, t)$ and an mcst (N^*, R) for the grand coalition, *Bird's tree allocation*, $\beta^R(\mathcal{T})$ is constructed by assigning to each player $i \in N$ the cost of the first edge on the unique path in (N^*, R) from player i to the source $*$. The computation of this allocation can be integrated into the Prim algorithm, which, starting from a fixed root, constructs an mcst by consecutively adding edges with the lowest cost, without introducing cycles.

Algorithm 3.7 (Bird's rule)
input: an mcst problem $(N, *, t)$
output: an edge set $R \subset E_{N^*}$ *of an mcst and corresponding Bird allocation* $\beta^R(\mathcal{T})$

1. *Choose the source* $*$ *as root.*

2. *Initialise* $R = \emptyset$.

3. *Find a minimal cost edge* $e = \{i, j\} \in E_{N^*} \backslash R$ *incident on* $*$ *or any of the vertices present in one of the edges in* R *in such a way that joining* e *to* R *does not introduce a cycle.*

4. *One of i and j, say j, was previously connected to the source and the other vertex i is a player who was not yet connected to the source. Assign the cost $\beta_i^R(\mathcal{T}) = t(e)$ to agent i.*

5. *Join e to R.*

6. *If not all vertices are connected to the root in the graph (N^*, R), go back to step 3.*

Note that the Bird allocation depends on the actual mcst the algorithm arrives at, which is determined by the choices made in step 3 of the algorithm.

The following example illustrates the algorithm.

Example 3.3 Consider the mcst problem $\mathcal{T}$ with $N = \{1, 2, 3\}$ as presented in Figure 3.3, where the numbers on the edges represent the costs.

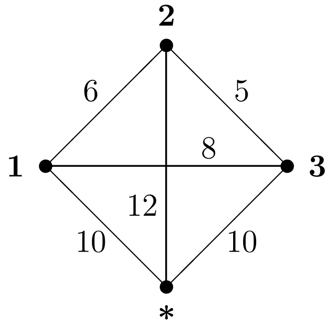

Figure 3.3: A minimum cost spanning tree problem $\mathcal{T}$

When we apply Algorithm 3.7 to this problem, the first edge we join to R is either $\{*, 1\}$ or $\{*, 3\}$. Suppose we choose the first one, then we set $\beta_1^R(\mathcal{T}) = 10$. Subsequently, we add $\{1, 2\}$ to R, set $\beta_2^R(\mathcal{T}) = 6$, add $\{2, 3\}$ and set $\beta_3^R(\mathcal{T}) = 5$. This gives us a cost allocation of $(10, 6, 5)$. On the other hand, suppose we start with $\{*, 3\}$. Then we end up with cost allocation $\beta^R(\mathcal{T}) = (6, 5, 10)$.

The two minimum cost spanning trees are drawn in Figure 3.4. $\triangleleft$

With each mcst problem $\mathcal{T} = (N, *, t)$ we associate a *mcst game* $(N, c^{\mathcal{T}})$, where $c^{\mathcal{T}}(S)$ represents the minimal costs of a tree on $S^* = S \cup \{*\}$:

$$c^{\mathcal{T}}(S) = \min\{\sum_{e \in R} t(e) \,|\, R \subset E_{S^*} \text{ and } (S^*, R) \text{ is a tree}\}$$

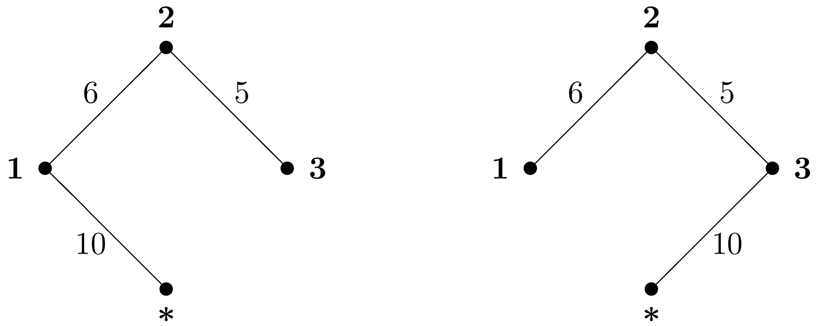

Figure 3.4: Two minimum cost spanning trees

for all $S \subset N, S \neq \emptyset$ and $c^{\mathcal{T}}(\emptyset) = 0$. The following theorem comes from Granot and Huberman (1981).

Theorem 3.8 *Let $\mathcal{T} = (N, *, t)$ be a minimum cost spanning tree problem. Then for every minimum cost spanning tree (N^*, R), Bird's allocation rule $\beta^R(\mathcal{T})$ is an extreme point of the core of the corresponding minimum cost spanning tree game $(N, c^{\mathcal{T}})$.*

It immediately follows that every mcst is balanced. An alternative proof for nonemptiness of the core is given in Granot and Huberman (1982).

A further overview of mcst problems is given in Aarts (1994) and the core and nucleolus are studied in Granot and Huberman (1984) and Solymosi et al. (1998). Aarts and Driessen (1993) study the irreducible core of mcst games, which is a subset of the core, and present two algorithms to determine this set. In Moretti, Norde, Pham Do, and Tijs (2001) and Norde, Moretti, and Tijs (2001), existence of population monotonic allocation schemes for mcst games is investigated. In Van den Nouweland et al. (1993) it is shown that every nonnegative monotonic game arises from an mcst problem in which there are costs associated with the vertices as well as with the edges.

There are a large number of variations on the mcst problem as presented above. In Feltkamp et al. (1994), minimum cost spanning extension problems are introduced, in which there is a fixed tree, which has to be extended in such a way that total extension costs are minimal. In this framework, two allocation rules are presented that are inspired by Kruskal's algorithm for finding minimum cost spanning trees. In Suijs (2001), mcst problems are studied in which the connection costs consist of two parts: construction costs and maintenance costs. Since the latter costs are unknown ex ante, connection costs are represented by random variables. An algorithm to determine an "optimal" network is presented and a two stage Bird allocation is defined and shown to be a core allocation of the corresponding cooperative stochastic minimum spanning tree game (cf. Suijs (2000)).

4 Routing

In this section we discuss classes of operations research problems in which the objective is to find a route of minimal costs within a graph. First, we discuss the class of Chinese postman games as introduced in Hamers, Borm, Van de Leensel, and Tijs (1999). Second, we discuss travelling salesman games as introduced in Potters et al. (1992). We discuss two variants of the travelling salesman problem: the fixed routing problem and the Steiner travelling salesman problem.

In the Chinese postman problem, which is introduced in Mei-Ko Kwan (1962), one considers a situation in which a postman has to deliver mail to each street of a certain city. He has to start and finish at the post office. For each street costs are involved each time the postman visits this street. The postman should choose a route to visit all streets in such a way that costs are minimised. The main difference between several classes of Chinese postman problems can be found in the underlying graph that describes the street plan of the city. For the classical problem, in which the underlying graph is undirected, Edmonds and Johnson (1973) present a polynomially bounded matching algorithm that provides a route with minimal costs.

A cost allocation problem arises if in the underlying graph each edge corresponds to a different player. Because all players need the mail delivery service and the nature of this service requires the server to travel from the post office and visit all edges (players) before returning to the post office, the cost allocation problem is concerned with a fair allocation of the cost of a cheapest Chinese postman tour in the graph. That is, the cost of a cheapest tour, which starts at the post office, visits each edge at least once and returns to the post office.

Formally, a *Chinese postman* or *CP problem* is a tuple $\Gamma = (N, G, v_0, g, t)$, where $N = \{1, \ldots, n\}$ is the set of players, $G = (V, E)$ is a connected undirected graph with vertex set V and edge set E, $v_0 \in V$ represents the post office, $g : E \to N$ is a bijection relating the players to the edges and $t : E \to \mathbb{R}_+$ is a nonnegative cost function assigning costs to the edges. An S-tour with respect to v_0 associated with coalition $S \subset N$ is a closed walk $(v_0, e_1, \ldots, e_k, v_0)$ that starts at the post office v_0, visits each player in S at least once and returns to v_0, i.e., $S \subset \{g(e_j) \mid j \in \{1, \ldots, k\}\}$. Note that an S-tour may also use edges corresponding to players outside S. The set of all S-tours is denoted by $D(S)$.

Suppose a coalition S is served according to the S-tour $(v_0, e_1, \ldots, e_k, v_0) \in D(S)$, then the total costs of this tour are $\sum_{j=1}^{k} t(e_j)$. We will assume that each player $i \in S$ pays the costs $t(g^{-1}(i))$ himself. In this way we already allocate the *separable costs* $\sum_{i \in S} t(g^{-1}(i))$ of an S-tour. Note that these separable costs are independent of the chosen S-tour. The remaining *nonseparable costs* for coalition S, $\sum_{j=1}^{k} t(e_j) - \sum_{i \in S} t(g^{-1}(i))$, have to be allocated to its members in some way. This gives rise to the *Chinese postman* or *CP game* (N, c) corresponding to $\Gamma = (N, G, v_0, g, t)$, defined by

$$c(S) = \min_{(v_0, e_1, \ldots, e_k, v_0) \in D(S)} [\sum_{j=1}^{k} t(e_j) - \sum_{i \in S} t(g^{-1}(i))].$$

for all $S \subset N$. In the following example, we show that a CP game need not be balanced.

Example 4.1 Consider the CP problem (N, G, v_0, g, t) with $N = \{1, \ldots, 5\}$, graph $G = (V, E)$ as depicted in Figure 4.1, $t(e_j) = 1$ and $g(e_j) = j$ for all $j \in \{1, ..., 5\}$.

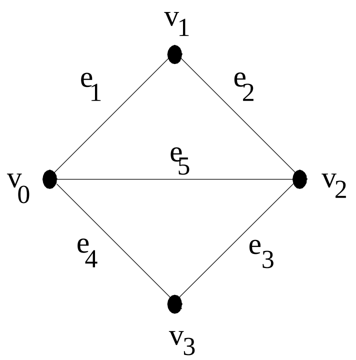

Figure 4.1: A Chinese postman problem

Denote the corresponding CP game by (N, c). Then $c(N) = 1$ and $c(S) = 0$ for $S \in \mathcal{A} = \{\{1, 2, 5\}, \{3, 4, 5\}, \{1, 2, 3, 4\}\}$. Let $x \in \mathbb{R}^N$ and suppose $x \in C(c)$. Then

$$2 = 2c(N) = \sum_{S \in \mathcal{A}} \sum_{i \in S} x_i \leqslant \sum_{S \in \mathcal{A}} c(S) = 0.$$

Contradiction, so (N, c) is not balanced. $\triangleleft$

In spite of this result, balancedness, total balancedness and concavity have been established for CP games that arise from some specific classes of graphs. A graph $G = (V, E)$ is said to be *globally CP balanced (totally balanced, concave)* if the induced CP game is balanced (totally balanced, concave) for all possible $v_0 \in V$ and all nonnegative cost functions on the edges. G is called *locally CP balanced (totally balanced, concave)* if the induced CP game is balanced (totally balanced, concave) for some $v_0 \in V$ and all cost functions.

In Theorems 4.1 - 4.4 some results are stated from Hamers (1997), Granot et al. (1999) and Granot and Hamers (2000).

Theorem 4.1 *Let G be a connected undirected graph. Then the following three assertions are equivalent:*

(i) G is weakly Euler.

(ii) G is globally CP balanced.

(iii) G is locally CP balanced.

A graph is called weakly Euler if each biconnected component[1] in G is Eulerian (i.e., the degree of every vertex is even).

Theorem 4.2 *Let G be a connected undirected graph. Then the following five assertions are equivalent:*

(i) G is weakly cyclic.

(ii) G is globally CP concave.

(iii) G is globally CP totally balanced.

(iv) G is locally CP concave.

[1]A biconnected component of a graph G is a maximal subgraph of G in which each pair of vertices is connected by at least two edge disjoint paths.

(v) G is locally CP totally balanced.

A graph G is called weakly cyclic if each biconnected component is a circuit.

The Chinese postman problem in which the underlying graph is directed has also been studied in the literature. All definitions for the undirected case as presented above can be extended to the directed case in a straightforward way.

Theorem 4.3 *Let G be a strongly connected directed graph. Then G is globally CP balanced.*

The proof of Theorem 4.3 translates the problem to a linear programming problem and applies a balancedness result established in Owen (1975).

Theorem 4.4 *Let G be a strongly connected directed graph. Then G is directed weakly cyclic if and only if G is globally CP concave.*

A directed weakly cyclic graph is a 1-sum[2] of directed circuits.

We conclude the discussion on CP games by considering an allocation rule for the class of problems in which the underlying graph is an undirected weakly Euler graph. This class of CP problems with player set N is denoted by WE^N.

In order to introduce a rule that divides for each $\Gamma = (N, G, v_0, g, t) \in WE^N$ the costs of a minimal N-tour among the players, we need the notion of followers of a bridge with respect to v_0. An edge of G is called a *bridge* if removal of this edge leads to a disconnected graph. We denote the set of bridges in G by $B(G)$. Edge $e \in E$ is called a *follower* of b with respect to v_0 if each path that contains both v_0 and e also contains b. The set of followers of b will be denoted by $F_b(G, v_0)$. Note that $b \in F_b(G, v_0)$ and that the set of followers depends on the location of v_0 in the graph.

Let $b \in B(G)$. Then the postman needs to cross this bridge twice if he is to make a tour containing some edge in $F_b(G, v_0)$. It seems reasonable that each player in $F_b(G, v_0)$ will pay an equal share of the costs of crossing b for the second time. So, if a tour that visits a certain player contains bridges, he has to contribute a

[2]The 1-sum of graphs G and H is defined as the graph derived from G and H by coalescing one vertex in G with another vertex in H.

fair share in the nonseparable costs of all these bridges. Formally, the division rule $\gamma : WE^N \to \mathbb{R}^N$ is defined for all $\Gamma = (N, G, v_0, g, t) \in WE^N$ by

$$\gamma_{g(e)}(\Gamma) = \sum_{b \in B(G): e \in F_b(G,v_0)} \frac{t(b)}{|F_b(G, v_0)|}$$

for all $e \in E$.

The following example illustrates the γ rule.

Example 4.2 Consider the CP problem (N, G, v_0, g, t), where the graph G is depicted in Figure 4.2 (left), $t(b_1) = 52$, $t(b_2) = 44$ and $t(b_3) = 33$.

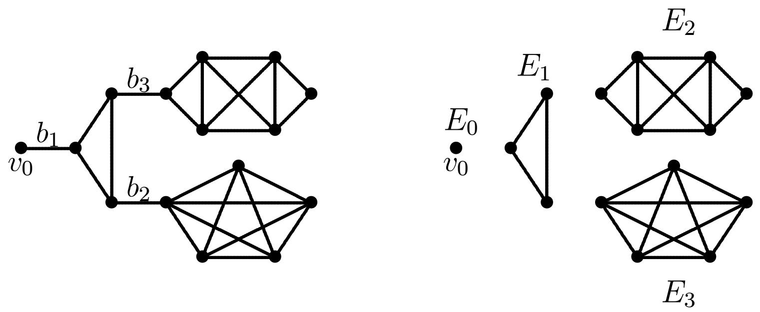

Figure 4.2: Weakly Euler graph G (left); the components of G after removal of bridges (right)

Observe that G is indeed a weakly Euler graph, because the removal of the bridges b_1, b_2 and b_3 leads to the components E_0, E_1, E_2 and E_3, which are all Eulerian. Because $|F(G, b_1)| = 26$, $|F(G, b_2)| = 11$ and $|F(G, b_3)| = 11$, according to the γ rule each player in $F(G, b_1)\backslash(F(G, b_2) \cup F(G, b_3))$ pays $\frac{52}{26} = 2$, each player in $F(G, b_2)$ pays $\frac{52}{26} + \frac{44}{11} = 6$ and each player in $F(G, b_3)$ pays $\frac{52}{26} + \frac{33}{11} = 5$. $\triangleleft$

The γ rule can be characterised by two different sets of properties. The first characterisation uses five properties called efficiency, standard, null, symmetry and additivity. This characterisation is based on decomposing a CP problem into a number of simple subproblems and uses the additive structure of the γ rule.

The second characterisation uses three properties that are explained below. Before we can formulate these properties, we need the notions of bridge cluster of a weakly Euler graph and the condensation of a graph with respect to an extreme bridge. A *bridge cluster* is a maximal set of edges that need the same set of bridges to be connected to the post office. So for $\Gamma = (N, G, v_0, g, t) \in WE^N$ and $B(G) = \{b_1, \ldots, b_q\}$ we have the bridge clusters $\{C_j(G, v_0)\}_{j \in \{0,\ldots,q\}}$, where

$$C_0(G, v_0) = E \backslash \cup_{b \in B(G)} F_b(G, v_0)$$

is the set of edges that do not need any bridge to be connected to v_0 and for all $j \in \{1, \ldots, q\}$

$$C_j(G, v_0) = F_{b_j}(G, v_0) \backslash \cup_{b \in B(G) \cap F_{b_j}(G, v_0), b \neq b_j} F_b(G, v_0)$$

is the cluster of edges that need the bridges $\{b \in B(G) \,|\, b_j \in F_b(G, v_0)\}$ to be connected to v_0. A bridge $b_j \in B(G)$ is called an *extreme bridge* of G if it has no other bridge as a follower, or equivalently, if $C_j(G, v_0) = F_{b_j}(G, v_0)$. The following example illustrates the notions of bridge cluster and extreme bridge.

Example 4.3 Consider the graph G in Example 4.2. Then $C_0(G, v_0) = E_0$ and $C_j(G, v_0) = E_j \cup \{b_j\}$ for $j \in \{1, 2, 3\}$. The extreme bridges are b_2 and b_3. $\triangleleft$

Next, we describe a procedure to construct the condensed graph of a weakly Euler graph G with respect to an extreme bridge. Let $v_0 \in V$ and let $b \in B(G)$ be an extreme bridge of G. Let v_1^* be incident on b such that there exists a path between v_0 and v_1^* in the graph $(V, E \backslash \{b\})$. Let $V(F_b(G, v_0))$ be the set of vertices incident on the edges in $F_b(G, v_0)$. The graph $\underline{G}$ arises from G by removing all edges $F_b(G, v_0)$ and vertices $V(F_b(G, v_0)) \backslash \{v_1^*\}$. Let $|F_b(G, v_0)| = m$, then the graph G^* arises from $\underline{G}$ by connecting a circuit of length m to the vertex v_1^*. The graph G^* is called the *condensed graph* of G with respect to the extreme bridge b. Note that G^* is also a weakly Euler graph. Moreover, the number of edges in G and G^* coincide.

Example 4.4 Consider the graph G in Example 4.2. Figure 4.3 shows the graph G^* that arises from G by condensation with respect to the extreme bridge b_3. $\triangleleft$

The condensed CP problem of $\Gamma = (N, G, v_0, t, g) \in WE^N$ with respect to the extreme bridge $b \in B(G)$ is $\Gamma_b = (N, G^*, v_0, t^*, g^*)$, where $G^* = (V^*, E^*)$ is the condensed graph of G with respect to b, $g^* : E^* \to N$ is a bijection such that $g^*(e) = g(e)$ for all $e \in E \backslash F_b(G, v_0)$ and the cost function $t^* : E^* \to \mathbb{R}_+$ is defined by

$$t^*(e) = \begin{cases} t(e) & \text{if } e \in E \backslash F_b(G, v_0), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\Gamma = (N, G, v_0, g, t) \in WE^N$. Consider the following three properties for a division rule $f : WE^N \to \mathbb{R}^N$:

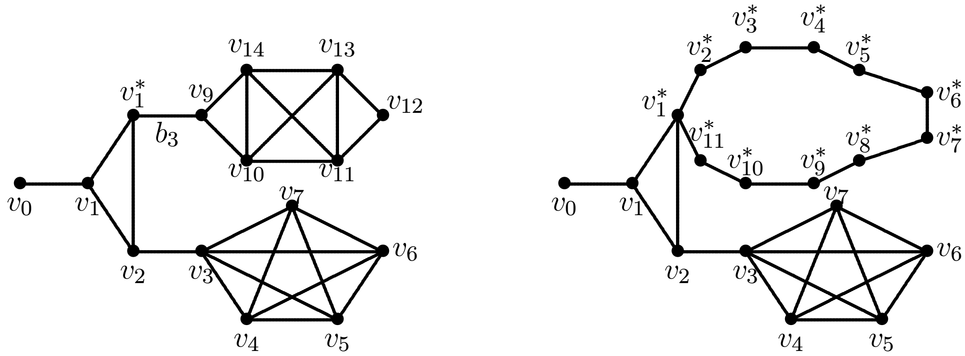

Figure 4.3: The condensed graph G^* from G with respect to b_3

- *Efficiency:* $\sum_{i \in N} f_i(\Gamma) = \sum_{b \in B(G)} t(b)$.
- *Bridge cluster symmetry:* Let $B(G) = \{b_1, \ldots, b_q\}$, then $f_{g(e_1)}(\Gamma) = f_{g(e_2)}(\Gamma)$ for all $e_1, e_2 \in C_j(G, v_0)$, $j \in \{0, \ldots, q\}$.
- *Condensation property:* Let b be an extreme bridge of G and let $\Gamma_b = (N, G^*, v_0, g^*, t^*)$ be the condensed problem with respect to b, then $f_{g(e)}(\Gamma) = f_{g(e)}(\Gamma_b)$ for all $e \in E \backslash F_b(G, v_0)$.

Bridge cluster symmetry states that each group of players that need the same set of bridges to be connected to the post office will contribute the same share in the nonseparable costs. The condensation property is a kind of consistency property. All players who are not in the bridge cluster corresponding to the removed bridge face the same problem in this reduced graph as in the original graph. Now, a rule is called consistent if in both situations this rule assigns to each player in this group the same costs.

Theorem 4.5 *The allocation rule $\gamma : WE^N \to \mathbb{R}^N$ is the unique rule that satisfies efficiency, bridge cluster symmetry and the condensation property.*

Whereas in the Chinese postman problem each edge in the graph has to be visited at least once, in the travelling salesman problem one aims to find a tour that visits all the vertices in the graph exactly once. For example, a professor has to make a trip visiting several universities. He has to start at his own university, visit all other universities exactly once and then return to his home university. The problem is to select a route in which total travel costs are minimised. It is well known that finding such a route is an NP-hard problem. Nevertheless, many real life problems are related to the travelling salesman problem. This has resulted in many heuristic approaches to find good solutions to several variants of this problem. For a review on the travelling salesman problem we refer to Lawler et al. (1985).

Fishburn and Pollack (1983) introduce the cost allocation problem that arises if in the underlying graph each vertex, except the one that corresponds to the home location, corresponds to a different player. The cost allocation is concerned with a fair allocation of the cost of a cheapest Hamiltonian circuit in the graph. That is the cheapest tour that starts in the vertex that corresponds to the home location, visits all other vertices precisely once and returns home.

Formally, a *travelling salesman* or *TS problem* is a tuple $(N, *, t)$, where $N = \{1, \ldots, n\}$ is the set of players, $*$ represents the home location and $t : E_{N^*} \to \mathbb{R}_+$ is the cost function assigning costs to the edges connecting the vertices in $N^* = N \cup \{*\}$. We assume that t satisfies the triangle inequality. E_S is defined as the set of all edges between pairs of elements of S, so that (S, E_S) is the complete graph on S:

$$E_S = \{\{i, j\} \,|\, i, j \in S, i \neq j\}.$$

By defining the worth of a coalition S as the minimal costs of a Hamiltonian circuit in the graph $(S \cup \{*\}, E_{S \cup \{*\}})$, we obtain the corresponding *travelling salesman* or *TS game*.

The following example, due to Tamir (1989), illustrates that TS games need not be balanced.

Example 4.5 Consider the TS problem $(N, *, t)$ with player set $N = \{1, \ldots, 6\}$, $t(\{i, j\}) = 1$ for all edges $\{i, j\}$ depicted in Figure 4.4 and for all other edges $\{i, j\}$, $t(\{i, j\})$ equals the minimal costs of a path connecting i to j using the depicted edges.

We denote the corresponding TS game by (N, c). Then $c(N) = 8$ (with optimal tour $(*, 4, 5, 6, 1, 2, 3, *)$), $c(\{1, 2, 4, 5\}) = 5$, $c(\{3, 4, 5, 6\}) = 5$ and $c(\{1, 2, 3, 6\}) = 5$. Let $x \in \mathbb{R}^N$ and suppose $x \in C(c)$, then

$$16 = 2c(N) = \sum_{i \in \{1,2,4,5\}} x_i + \sum_{i \in \{3,4,5,6\}} x_i + \sum_{i \in \{1,2,3,6\}} x_i \leqslant 5 + 5 + 5 = 15.$$

Contradiction, so (N, c) is not balanced. $\triangleleft$

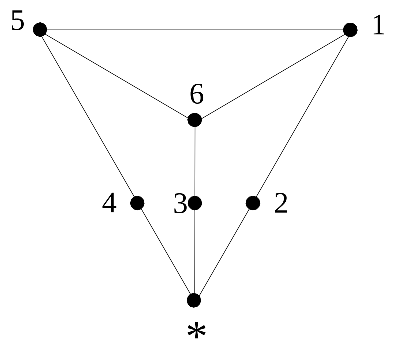

Figure 4.4: A travelling salesman problem

In case there are less than six players some results with respect to balancedness are established. Potters et al. (1992) show that 3-person TS games have a nonempty core. Tamir (1989) shows that each 4-person TS game has a nonempty core and provides Example 4.5 showing that a 6-person TS game can have an empty core. Finally, Kuipers (1993) proves that 5-person TS games are balanced.

The travelling salesman model can be extended to the case in which the costs depend on the direction in which the salesman travels through the edges. In this context, Potters et al. (1992) provide a 4-person TS game with an empty core.

Potters et al. (1992) also introduce the class of fixed routing games. The idea of a fixed routing game is that the salesman decides about the Hamiltonian circuit he will use to visit all the players. Then the value of a coalition S in a fixed routing game is defined as the costs of the restricted tour that visits the players in S in the same order as prescribed by the original Hamiltonian circuit and skips all other players. Potters et al. (1992) show that fixed routing games have a nonempty core if the chosen Hamiltonian circuit is an optimal route for the related TS problem. Derks and Kuipers (1997) give a time efficient algorithm that calculates core elements of fixed routing games. In Kuipers et al. (2000) and Solymosi et al. (1998) $\mathcal{O}(n^4)$ algorithms are provided that calculate the nucleolus of fixed routing games.

Finally, we mention Steiner TS games. These games arise from situations in which some of the edges between pairs of players may be absent. The value of a coalition in a Steiner TS game corresponds to the costs of the cheapest Steiner tour. A Steiner tour is a closed trail that starts in the home location and visits each vertex of S at least once. For these games Herer and Penn (1995), Granot et al. (2000)

and Granot and Hamers (2000) have characterised concavity by the structure of the available edges.

5 Scheduling

In this section we discuss classes of operations research games that are related to scheduling problems. First, we discuss various classes of sequencing games as initiated by Curiel et al. (1989). We focus on balancedness and convexity and discuss two context specific solution concepts: the equal gain splitting rule and the split core. Second, we consider permutation games, introduced in Tijs et al. (1984), where we focus on total balancedness. Finally, we discuss assignment games, introduced in Shapley and Shubik (1971), which form a special class of permutation games and have some appealing properties with respect to the structure of the core.

The main characteristic of a sequencing situation is that a number of jobs (tasks, operations) have to be processed in some order on a (number of) machine(s) in such a way that some cost criterion is minimised. In spite of this common characteristic, sequencing situations can be classified on the basis of many features. We mention the number of machines, the specific properties of machines (e.g., parallel, serial), the chosen cost criterion (e.g., maximum completion time, weighted completion time), restrictions on the jobs (e.g., ready times, due dates) and possibly the specific order in which the jobs have to be processed on the machines (e.g., job-shop, flow-job). Obviously, sequencing situations arise in many applications: the process of manufacturing cars, allocating patients to surgery rooms, maintenance of airplanes, etc. For a review of scheduling theory we recommend Lawler et al. (1993).

As a specific example we describe the class of one-machine sequencing situations as introduced in Curiel et al. (1989). In a one-machine sequencing situation there is a queue of players, each with one job, in front of a machine. Each player must have his job processed on this machine. The finite set of players is denoted by $N = \{1, ..., n\}$. The positions of the players in the queue are described by a bijection $\sigma \in \Pi_N$. We assume that there is an initial order $\sigma_0 \in \Pi_N$ on the jobs before the processing of the machine starts. The processing time p_i of the job of player i is the time the machine takes to handle this job. For each player $i \in N$ the costs of spending time in the system can be described by a linear cost function $c_i : \mathbb{R}_+ \to \mathbb{R}$ defined by $c_i(t) = \alpha_i t$ with $\alpha_i > 0$. A *sequencing situation* as described above is denoted by

(N, σ_0, p, α) with $p, \alpha \in \mathbb{R}^N_{++}$.

The completion time $C(\sigma, i)$ of the job of player i if the jobs are processed (in a semi-active way) according to the order $\sigma \in \Pi_N$ is given by

$$C(\sigma, i) = \sum_{\{j \in N | \sigma(j) \leqslant \sigma(i)\}} p_j.$$

A processing order is called semi-active if there does not exist a job which could be processed earlier without altering the processing order, i.e., if there are no unnecessary delays. The total costs of all players if the jobs are processed according to the order σ equal $\sum_{i \in N} \alpha_i C(\sigma, i)$. Clearly, because Π_N is finite, there exists an order for which total costs are minimised. A processing order that minimises total costs and thus maximises total cost savings is an order in which the jobs are processed in decreasing order with respect to the urgency index u_i defined by $u_i = \frac{\alpha_i}{p_i}$ (cf. Smith (1956)).

Example 5.1 Consider a one-machine sequencing situation (N, σ_0, p, α), where $N = \{1, 2, 3\}$, $\sigma_0 = (1, 2, 3)$, $p = (2, 2, 1)$ and $\alpha = (4, 6, 5)$. Then the urgencies for the players are $u_1 = 2$, $u_2 = 3$ and $u_3 = 5$, respectively. Hence, the optimal processing order is $(3, 2, 1)$ with total costs $5 \cdot 1 + 6 \cdot 3 + 4 \cdot 5 = 43$. $\lhd$

Note that an optimal order can be obtained from the initial order by consecutive switches of neighbours i, j with i directly in front of j and $u_i < u_j$. This process will be referred to as the Smith algorithm.

By rearranging from the initial order to an optimal order, an allocation problem arises: how should the maximal total cost savings the players can obtain be divided among the players? Again, this problem is tackled using cooperative game theory by analysing corresponding sequencing games.

For a sequencing situation (N, σ_0, p, α) the costs $C_S(\sigma)$ of coalition S with respect to a processing order σ equal $C_S(\sigma) = \sum_{i \in S} \alpha_i C(\sigma, i)$. We want to determine the maximal cost savings of a coalition S when its members decide to cooperate. For this, we have to define which rearrangements of the coalition S are admissible with respect to the initial order. A bijection $\sigma \in \Pi_N$ is called *admissible* for S if it satisfies the following condition:

$$P(\sigma, j) = P(\sigma_0, j)$$

for all $j \in N \backslash S$, where for any $\tau \in \Pi_N$ the set of predecessors of a player $j \in N$ with respect to τ is defined as $P(\tau, j) = \{k \in N \mid \tau(k) < \tau(j)\}$.

This condition implies, in particular, that the starting time of each player outside the coalition S is equal to his starting time in the initial order and the players of S are not allowed to "jump" over players outside S. The set of admissible orders for a coalition S is denoted by $\mathcal{A}(S)$.

By defining the value of a coalition S as the maximum cost savings coalition S can achieve by means of an admissible rearrangement we obtain the corresponding *sequencing game* (N, v), which is defined by

$$v(S) = \max_{\sigma \in \mathcal{A}(S)} \{\sum_{i \in S} \alpha_i [C(\sigma_0, i) - C(\sigma, i)]\} \tag{5.1}$$

for all $S \subset N$.

Expression (5.1) can be rewritten in terms of $g_{ij} = \max\{0, \alpha_j p_i - \alpha_i p_j\}$, which equals the cost savings attainable by player i and j when i is directly in front of j, regardless of the exact position in the order. For this we need the notion of connected coalition. A coalition S is called *connected* with respect to σ if for all $i, j \in S$ and $k \in N$ such that $\sigma(i) < \sigma(k) < \sigma(j)$ it holds that $k \in S$. The Smith algorithm and (5.1) imply the following proposition.

Proposition 5.1 *Let (N, σ_0, p, α) be a sequencing situation and let (N, v) be the corresponding sequencing game. Then for any coalition S that is connected with respect to σ_0 we have*

$$v(S) = \sum_{i,j \in S : \sigma_0(i) < \sigma_0(j)} g_{ij}.$$

For a coalition T that is not connected with respect to σ_0 the definition of admissible orders implies that

$$v(T) = \sum_{S \in T \backslash \sigma_0} v(S),$$

where $T \backslash \sigma_0$ is the set of components of T, a component of T being a maximally connected subset of T.

Example 5.2 Let $N = \{1, 2, 3\}$, $\sigma_0 = (1, 2, 3)$, $p = (2, 2, 1)$ and $\alpha = (4, 6, 5)$. It is readily verified that $g_{12} = g_{23} = 4$ and $g_{13} = 6$. Then $v(\{i\}) = 0$ for all $i \in N$, $v(\{1, 2\}) = v(\{2, 3\}) = 4$, $v(\{1, 3\}) = v(\{1\}) + v(\{3\}) = 0$ and $v(N) = 14$. $\triangleleft$

The following theorem, due to Curiel et al. (1989), shows that sequencing games are convex games.

Theorem 5.2 *Let (N, σ_0, α, p) be a sequencing situation. Then the corresponding sequencing game (N, v) is convex.*

In particular, Theorem 5.2 implies that sequencing games are (totally) balanced.

Another way of proving balancedness of sequencing games is by explicitly constructing core allocations. We will show that the the equal gain splitting rule, introduced in Curiel et al. (1989), and the split core, introduced in Hamers et al. (1996), are rules that yield allocations that are in the core of the corresponding sequencing games.

Recall that the set of predecessors of player $i \in N$ with respect to the processing order σ is given by $P(\sigma, i) = \{j \in N \mid \sigma(j) < \sigma(i)\}$. We define the set of followers of $i \in N$ with respect to σ to be $F(\sigma, i) = \{j \in N \mid \sigma(j) > \sigma(i)\}$. The *equal gain splitting* or *EGS rule* is a map that assigns to each sequencing situation (N, σ_0, p, α) a vector in $\mathbb{R}^N$, which is defined by

$$EGS_i(N, \sigma_0, p, \alpha) = \frac{1}{2} \sum_{j \in F(\sigma_0, i)} g_{ij} + \frac{1}{2} \sum_{k \in P(\sigma_0, i)} g_{ki} \tag{5.2}$$

for all $i \in N$. Equation (5.2) means that the EGS rule assigns to each player half of the gains of all neighbour switches he is actually involved in when reaching an optimal order from the initial order.

From (5.2) it readily follows that the EGS rule is efficient, i.e.,

$$\sum_{i \in N} EGS_i(N, \sigma_0, p, \alpha) = \sum_{i,j \in N : \sigma_0(i) < \sigma_0(j)} g_{ij} = v(N).$$

Example 5.3 Let $N = \{1, 2, 3\}$, $\sigma_0 = (1, 2, 3)$, $p = (2, 2, 1)$ and $\alpha = (4, 6, 5)$. Because $g_{12} = g_{23} = 4$ and $g_{13} = 6$ we have $EGS_1(N, \sigma_0, p, \alpha) = \frac{1}{2}(4 + 6) = 5$, $EGS_2(N, \sigma_0, p, \alpha) = \frac{1}{2}(4 + 4) = 4$ and $EGS_3(N, \sigma_0, p, \alpha) = \frac{1}{2}(6 + 4) = 5$. Moreover, we have $\sum_{i \in N} EGS_i(N, \sigma_0, p, \alpha) = 4 + 4 + 6 = 14 = v(N)$. $\triangleleft$

A nice feature of the EGS rule is that it can be characterised using three appealing properties. Let SEQ^N denote the class of one-machine sequencing situations with player set N. Consider the following properties for a rule $f : SEQ^N \to \mathbb{R}^N_+$ with $(N, \sigma_0, p, \alpha) \in SEQ^N$:

- *Efficiency:* Let π be an optimal processing order for N. Then f is called efficient if $\sum_{i \in N} f_i(N, \sigma_0, p, \alpha) = C_N(\sigma_0) - C_N(\pi)$.

- *Equivalence property:* Let $i \in N$ and $(N, \sigma_1, p, \alpha) \in SEQ^N$ be such that $P(\sigma_0, i) = P(\sigma_1, i)$. Then f satisfies the equivalence property if $f_i(N, \sigma_0, p, \alpha) = f_i(N, \sigma_1, p, \alpha)$.

- *Switch property:* Let $i, j \in N$ be such that $|\sigma_0(i) - \sigma_0(j)| = 1$. Let $(N, \sigma_1, p, \alpha) \in SEQ^N$ be such that $\sigma_1(i) = \sigma_0(j)$, $\sigma_1(j) = \sigma_0(i)$ and $\sigma_1(k) = \sigma_0(k)$ for all $k \in N \backslash \{i, j\}$. Then f satisfies the switch property if $f_i(N, \sigma_0, p, \alpha) - f_i(N, \sigma_1, p, \alpha) = f_j(N, \sigma_0, p, \alpha) - f_j(N, \sigma_1, p, \alpha)$.

The equivalence property states that the order of a player's predecessors does not affect his allocation. For explaining the switch property, let two players be neighbours in a sequencing situation. If these players switch positions, then the switch property states that in this new situation the allocation is increased (or decreased) equally for both players. These three properties characterise the EGS rule.

Theorem 5.3 *The EGS rule is the unique rule on SEQ^N that satisfies efficiency, the equivalence property and the switch property.*

The proof of Theorem 5.3 is one by induction on the number of misplacements. A pair $\{i, j\}$ is called a *misplacement* in an order σ if they are neighbours in σ and the urgency of the player in front is smaller than the urgency of its neighbour.

Generalising the EGS rule, we consider gain splitting (GS) rules in which each player obtains a nonnegative part of the gain of all neighbour switches he is involved in to reach the optimal order. Again, the total gain of a neighbour switch is only divided among the two players that are involved. Formally, we define for all $i \in N$ and all $\lambda \in \Lambda$

$$GS_i^{\lambda}(N, \sigma_0, p, \alpha) = \sum_{j \in F(\sigma_0, i)} \lambda_{ij} g_{ij} + \sum_{k \in P(\sigma_0, i)} (1 - \lambda_{ki}) g_{ki},$$

where $\Lambda = \{\{\lambda_{ij}\}_{i,j \in N, \sigma_0(i) < \sigma_0(j)} \,|\, \forall_{i,j \in N, \sigma_0(i) < \sigma_0(j)} : 0 \leqslant \lambda_{ij} \leqslant 1\}$. Note that $GS^{\lambda}(N, \sigma_0, p, \alpha) = EGS(N, \sigma_0, p, \alpha)$ in case every λ_{ij} equals $\frac{1}{2}$.

Example 5.4 If we take $\lambda_{12} = \frac{3}{4}$, $\lambda_{13} = \frac{1}{3}$ and $\lambda_{23} = 1$ in the sequencing situation of Example 5.3, then $GS^{\lambda}(N, \sigma_0, p, \alpha) = (5, 5, 4)$. $\triangleleft$

The *split core* of a sequencing situation (N, σ_0, p, α) is defined by

$$SPC(N, \sigma_0, p, \alpha) = \{GS^\lambda(N, \sigma_0, p, \alpha) \mid \lambda \in \Lambda\}.$$

The split core can be characterised using similar properties as in the characterisation of the EGS rule. Finally, we state that the EGS rule and the split core generate core allocations for sequencing games.

Theorem 5.4 *Let $(N, \sigma_0, p, \alpha) \in SEQ^N$ and let (N, v) be the corresponding sequencing game. Then $SPC(N, \sigma_0, p, \alpha) \subset C(v)$.*

Yet another proof for balancedness is provided in Curiel et al. (1995). They introduce the class of component additive games, which contains the class of sequencing games, and prove that the average of two specific marginal vectors, the β rule, lies in the core of such a game. In fact, it turns out that the β rule coincides with the EGS rule within the class of sequencing games.

In the literature many other classes of sequencing games are studied. Hamers et al. (1995) extend the class of one-machine sequencing situations considered by Curiel et al. (1989) by imposing ready times on the jobs. In this case the corresponding sequencing games are balanced, but not necessarily convex. For a special subclass of sequencing games with ready times, however, convexity can be established. Borm et al. (1999) consider some classes of sequencing situations in which due dates are imposed on the jobs and different cost criteria are used: weighted completion time, weighted tardiness and weighted penalty. Several convexity results are established.

Instead of imposing restrictions on the jobs, Hamers, Klijn, and Suijs (1999), Calleja et al. (2001) and Van den Nouweland et al. (1992) extend the number of machines. Hamers, Klijn, and Suijs (1999) consider sequencing situations with m parallel and identical machines in which no restrictions on the jobs are imposed. Again, the weighted completion time criterion is used. Balancedness is established for two-machine situations by showing that these games are component additive games. In case there are more than two machines, balancedness is shown for two special classes. Calleja et al. (2001) establish balancedness for a special class of sequencing games that arise from two-machine sequencing situations in which a maximal weighted cost criterion is considered. Van den Nouweland et al. (1992) consider multiple machine flow-shop sequencing situation with a dominant machine.

Convexity is established in case the first machine is the dominant machine by showing that this class of games coincides with the class of sequencing games discussed in Curiel et al. (1989). In case another machine is the dominant machine, the corresponding game need not be balanced.

Van Velzen and Hamers (2001) consider some classes of sequencing games that arise from relaxations of classical sequencing situations. By allowing more admissible rearrangements, coalitions have more possibilities to maximise their profit. Balancedness is shown for some of these classes. Other related papers in the field of sequencing games are Curiel et al. (1994), Hamers (1995), Suijs et al. (1997) and Curiel et al. (1997).

Permutation games, introduced by Tijs et al. (1984), arise from situations in which every player has one job and one machine. Every job has to be processed on a machine and each machine can process every job, but no machine is allowed to process more than one job. If player i processes his job on the machine of player j, the processing costs are a_{ij}. Let $N = \{1, \dots, n\}$ be the set of players. The corresponding *permutation game* (N, v) is the cooperative game defined by

$$v(S) = \sum_{i \in S} a_{ii} - \min_{\pi \in \Pi_S} \sum_{i \in S} a_{i\pi(i)}$$

for all $S \subset N, S \neq \emptyset$ and $v(\emptyset) = 0$. The number $v(S)$ denotes the maximal cost savings a coalition S can obtain by processing their jobs according to an optimal schedule compared to the situation in which every player processes his job on his own machine. The following example illustrates that a permutation game need not be convex.

Example 5.5 Let $N = \{1, 2, 3\}$ be the player set and let

$$A = \begin{pmatrix} 8 & 4 & 2 \\ 2 & 4 & 10 \\ 5 & 6 & 10 \end{pmatrix}$$

be the cost matrix. Then the corresponding permutation game (N, v) is given by:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	6	11	0	12

E.g., the optimal schedule for the grand coalition is to process player 1's job on machine 3, player 2's job on machine 1 and player 3's job on machine 2, giving total cost savings of 8+4+10-(2+2+6)=12.

For this game we have

$$v(\{1,2,3\}) - v(\{1,3\}) = 1 < 6 = v(\{1,2\}) - v(\{1\}),$$

which implies that (N, v) is not convex. $\triangleleft$

It can be shown that the core of a permutation game is nonempty. Since every subgame of a permutation game is again a permutation game, we have the following result.

Theorem 5.5 *Permutation games are totally balanced.*

For Theorem 5.5 several proofs are presented in the literature. We mention Tijs et al. (1984), using the Birkhoff-Von Neumann theorem on doubly stochastic matrices. Curiel and Tijs (1986) use an equilibrium existence theorem of Gale (1984) for a discrete exchange economy with money. Klijn et al. (2000) use the existence of envy-free allocations in specific economies with indivisible objects and money to prove balancedness of permutation games.

An interesting subclass of permutation games is the class of assignment games, introduced in Shapley and Shubik (1971). These games are inspired on two-sided markets in which indivisible objects are exchanged for money. Applications that can be analysed using assignment games are, e.g., private markets in used cars, real estate markets and auctions.

Formally, assignment games arise from bipartite matching situations. Let M and N be two finite, disjoint sets. For each $i \in M$ and $j \in N$ the monetary value of a matching between i and j is given by $a_{ij} \geqslant 0$. Corresponding to this situation an assignment game is defined in the following way. On the player set $M \cup N$, the value of the coalition $S \cup T$, $S \subset M, T \subset N$ is defined to be the maximum that $S \cup T$ can obtain by making matchings between players in S and T. If $S = \emptyset$ or $T = \emptyset$ no suitable pairs can be made and therefore the value of such a coalition equals 0.

The following example illustrates that an assignment game need not be convex.

Example 5.6 Let $M = \{1, 2\}$ and $N = \{3, 4\}$. Let $a_{13} = 3$, $a_{14} = 5$, $a_{23} = 1$ and $a_{24} = 4$. The coalitions with nonzero value in the corresponding assignment game (N, v) are presented in the following table:

S	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$	$\{2,4\}$
$v(S)$	3	5	1	4

S	$\{1,2,3\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	$\{1,2,3,4\}$
$v(S)$	3	5	5	4	7

E.g., the optimal assignment for the grand coalition consists of a matching between players 1 and 3 and a matching between players 2 and 4.

For this game we have

$$v(\{1,2,3,4\}) - v(\{1,2,4\}) = 2 < 3 = v(\{1,3\}) - v(\{1\}),$$

which implies that (N, v) is not convex. $\triangleleft$

Every assignment game is a permutation game. Let $A = (a_{ij})_{i\in M, j\in N}$ denote a matrix with corresponding assignment game $(M \cup N, v)$, then the $|M \cup N| \times |M \cup N|$ matrix

$$\begin{bmatrix} 0 & -A \\ 0 & 0 \end{bmatrix}$$

gives rise to a permutation game, which equals v. Hence, it follows from Theorem 5.5 that assignment games are totally balanced.

In contrast to permutation games, the structure of the core of assignment games has been studied extensively. Shapley and Shubik (1971) show that the set of core allocations coincides with the set of solutions of the linear programming problem that is the dual of the optimal assignment problem. Moreover, they observe that the core corresponds to the set of competitive price equilibria of an economy associated with the assignment problem (cf. Debreu and Scarf (1963)). Shapley and Shubik (1971) also prove that the core expressed as a set of utility vectors for the players in M (or N) is a lattice. It is easy to see that the lattice is of a special type called the "45°-lattice". Quint (1991a) shows that also the converse is true, i.e., that every 45°-lattice can be associated with the core of an appropriately defined assignment game. Balinsky and Gale (1990) show that the number of extreme points of the core cannot exceed $\binom{2k}{k}$, where $k = \min\{|M|, |N|\}$. More recently, Hamers, Klijn, Solymosi, Tijs, and Villar (1999) have shown that the core of an assignment game satisfies the CoMa property, i.e., the core is the convex hull of some marginal vectors. Nuñez and Rafels (2000) relate the extreme points of the core to reduced marginal vectors. Solymosi and Raghavan (1994) present an $\mathcal{O}(n^4)$ algorithm to find the nucleolus of assignment games. In Solymosi and Raghavan (2000) the stability of the core of assignment games is investigated. For neighbour games, i.e., the class of games that equals the intersection of the classes of assignment games and component additive games, Hamers, Klijn, Solymosi, Tijs, and Vermeulen (1999) provide an $\mathcal{O}(n^2)$ algorithm to find the nucleolus. The relation between the core of assignment games and permutation games has been studied in Curiel and Tijs (1986) and Quint (1996).

Further papers dealing with assignment problems or closely related games are Kaneko (1982), Owen (1992), Sasaki (1995), Llorca et al. (1999), Sánchez-Soriano et al. (2000), Sánchez-Soriano et al. (2001), Quint (1991b) and Roth and Sotomayor (1989). The latter provides an overview of stable matchings, a concept formalised and analysed first in Gale and Shapley (1962).

6 Production

This section first surveys the results in the model of production economics as initiated by Owen (1975). The prime focus is on the Owen set and, in particular, on its characterisation as provided in Van Gellekom et al. (2000).

In Owen's production economy the situation is as follows. The production process is linear and freely accessible for every group of agents (players). There is a finite set N of players, a finite set R of resources and these resources can be used to produce consumption goods (products). The finite set of products is denoted by P. The production technologies are described by a production matrix A, where A_{rp} represents the number of units of resource $r \in R$ necessary to produce one unit of product $p \in P$. The products can be sold at a fixed market price (independent of the quantities produced), given by a vector $c \in \mathbb{R}^P$.

The maximal profit that can be made from a resource bundle $b \in \mathbb{R}^R$ is then equal to the maximum of the linear program

$$\max_{x \in \mathbb{R}^P_+} \{c^\top x \mid Ax \leqslant b\},$$

where the coordinate x_p denotes the amount of product p that is produced. Further, each player owns a bundle of resources. These resource bundles are summarised in a matrix B of size $|R| \times |N|$: the column of B corresponding to player i denotes player i's initial resource bundle.

The players try to maximise their profits. They can work on their own, but they are allowed to cooperate by pooling their resources. Pooling is favourable, because the maximal (joint) profit after pooling is always at least as high as the sum of the profits of the players separately. For, when cooperating, they could still make the same products they make on their own. Therefore, it is assumed that all players cooperate, yielding a maximal (total) profit. The question arises how to divide this profit among the agents in a fair way. So, again, this type of situation features not only an optimisation aspect in finding an optimal production plan, but also an allocation aspect in how to divide the corresponding profits.

A situation as described above is called an *LP process* and it is summarised by $L = (N, R, P, A, B, c)$. We make the following natural assumptions:

(i) $A \geqslant 0, B \geqslant 0$,

(ii) $Be^N > 0$,

(iii) $\exists_{p \in P} : c_p \geqslant 0$,

(iv) $c_p > 0 \Rightarrow \exists_{r \in R} : A_{rp} > 0$ (no gains without input).

The class of LP processes with arbitrary but finite player set satisfying (i)-(iv) is denoted by $\mathcal{L}$.

To analyse the allocation problem due to cooperation, we consider associated TU games. For $L = (N, R, P, A, B, c) \in \mathcal{L}$, the corresponding *LP game* v_L is defined by

$$v_L(S) = \max_{x \in F(S)} c^\top x$$

for every $S \subset N$, where $F(S) = \{x \in \mathbb{R}_+^P \,|\, Ax \leqslant Be^S\}$. Note that Be^S represents the total resource bundle available to coalition S and $v_L(S)$ is the maximal profit the players in S can jointly generate by pooling their resources.

From duality theory we know that

$$v_L(S) = \min_{y \in F^*} y^\top Be^S$$

with $F^* = \{y \in \mathbb{R}_+^R \,|\, y^\top A \geqslant c^\top\}$, since it is readily checked that the feasible regions $F(S)$ and F^* are both nonempty.

It is important to note that the feasible region F^* of the dual program does not depend on the coalition S one is considering and hence can be readily used to determine v_L just by changing the objective function.

Example 6.1 Consider the LP process L with $N = \{1,2,3\}$, two resources and two products:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}, B = \begin{bmatrix} 28 & 42 & 0 \\ 28 & 0 & 35 \end{bmatrix} \text{ and } c^\top = \begin{bmatrix} 6 & 8 \end{bmatrix}.$$

The dual feasible region F^* (for any coalition) is given by

$$2y_1 + y_2 \geqslant 6, y_1 + 4y_2 \geqslant 8, y_1 \geqslant 0, y_2 \geqslant 0.$$

From this we readily derive (e.g., by comparing the value of the objective function in the corner points) that the corresponding LP game v_L is given by

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v_L(S)$	0	104	0	0	168	154	146	250

Note that $(Be^N)^\top = \begin{bmatrix} 70 & 63 \end{bmatrix}$ and $\min_{y \in F^*} y^\top Be^N = 250$ is (uniquely) attained in $(\frac{16}{7}, \frac{10}{7})$, while for $S = \{1,2\}$, $(Be^S)^\top = \begin{bmatrix} 70 & 28 \end{bmatrix}$ and $\min_{y \in F^*} y^\top Be^S = 168$ is attained in $(0,6)$. $\triangleleft$

For $L = (N, R, P, A, B, c) \in \mathcal{L}$ we define the *Owen set* by

$$Owen(L) = \{y^\top B \in \mathbb{R}^N \mid y \in F^*, v_L(N) = y^\top Be^N\}.$$

So to determine an element of the Owen set, an *Owen vector*, we first have to determine an optimal solution $y \in \mathbb{R}^R$ of the dual program for the grand coalition N. For each $r \in R$, y_r is interpreted as the shadow price for resource r. Then, for each $i \in N$, $(y^\top B)_i$ represents the shadow value of the initial resource bundle for player i.

Example 6.2 Consider the LP process L of Example 6.1. As we have seen, the unique optimal solution of the dual program for N equals $y = (\frac{16}{7}, \frac{10}{7})$.

Hence, the Owen set of this LP process consists of one point z with

$$z = y^\top B = \begin{bmatrix} 104 \\ 96 \\ 50 \end{bmatrix},$$

where for example $96 = \frac{16}{7} \cdot 42 + \frac{10}{7} \cdot 0$ reflects the shadow value of the initial resource bundle $(42, 0)$ of player 2. Note that $Owen(L) \subsetneqq C(v_L)$. $\triangleleft$

Each Owen vector belongs to the core of the corresponding LP game.

Theorem 6.1 *Let $L = (N, R, P, A, B, c) \in \mathcal{L}$. Then $Owen(L) \subset C(v_L)$.*

Proof: Take $z \in Owen(L)$ and let $y \in F^*$ with $y^\top Be^N = v_L(N)$ be such that $z = y^\top B$. Then

$$\sum_{i \in N} z_i = \sum_{i \in N} (y^\top B)_i = y^\top Be^N = v_L(N)$$

and, for all $S \subset N$,

$$\sum_{i \in S} z_i = y^\top Be^S \geqslant v_L(S),$$

since $y \in F^*$ and thus y is feasible for the dual program corresponding to S. $\square$

In particular, Theorem 6.1 implies that every LP game is balanced. In fact, since each subgame of an LP game is also an LP game itself (corresponding to the natural "sub"-LP process), LP games are totally balanced and nonnegative (by definition). LP games even fill up the class of all nonnegative totally balanced games.

Theorem 6.2 *Every LP game is nonnegative and totally balanced and, conversely, every nonnegative and totally balanced TU game is an LP game.*

Proof: It suffices to show the "converse" part. Let $v \geqslant 0$ be a totally balanced TU game. Define $D(v) = (N, R, P, A, B, c)$ by $R = N$, $P = 2^N \backslash \{\emptyset\}$, $A = \left[\ldots e^S \ldots\right]$, $B = I_N$ and $c^\top = [\ldots v(S) \ldots]$. It is easy to check that $D(v) \in \mathcal{L}$ and $v_{D(v)} = v$. $\square$

The LP process $D(v)$ in the proof of Theorem 6.2 is called the *direct LP process* corresponding to the TU game v. Here, players are the resources (think of labourers), coalitions can be produced, each player has only himself to offer on the labour market and, finally, the price of each coalition (a product) is determined by the underlying game v. For direct LP processes, the Owen set exhausts the core.

Proposition 6.3 *Let (N, v) be a TU game. Then $Owen(D(v)) = C(v)$.*

Taking into account Example 6.2 and Proposition 6.3, one can conclude that the Owen set is not a game theoretic solution concept: it does not depend on the data of the game v only, but it needs more. Put differently, two different LP processes which both lead to the same LP game may have different Owen sets.

Owen (1975) has shown that the core of the r-fold replication of an LP process converges to the Owen set when r tends to infinity. Samet and Zemel (1984) give a necessary and sufficient condition for finite convergence.

In the literature many generalisations of LP processes can be found. Granot (1986) and Curiel et al. (1989) consider LP processes where (simple) control games on (bundles of) resources determine the resource bundle available to each coalition. If the underlying control games are balanced, then, following the Owen approach, core elements of the corresponding LP games can be constructed.

Feltkamp et al. (1993) analyse production economies with a finite number of (linear) production sites at different locations. At each location there are fixed prices for the products (in an insatiable market) and there is a finite amount of resources available, which is controlled by the players. If the production sites were isolated, nothing new would be obtained. However, transport of products, resources and technology between the sites is allowed along exogenously given routes. The possible transport routes are modelled by directed graphs. It is assumed that there are linear losses during transport and linear transportation costs. Conditions are provided such that the corresponding LP game is balanced, and that a core element can be found by solving only the dual of the linear program of the grand coalition.

Another extension is provided by Timmer et al. (2000), where situations are considered involving the linear transformation of products (LTP situations). A typical feature of LTP situations is the fact that resources themselves have economic value, since they can be sold directly next to being used in several transformation techniques. Moreover, transformation techniques can have more than one output good, so the model allows for byproducts. Again, (total) balancedness of the corresponding LTP games can be derived.

Extensions of the results on both LP and LTP situations to a context of a countable, infinite number of production/tansformation techniques can be found in Fragnelli et al. (1999), Timmer et al. (2000) and Tijs et al. (2001). Multiobjective LP games are considered in Nishizaki and Sakawa (2001). Existence of stable outcomes is shown and, using a duality theorem from multiobjective programming, the concept of Owen set is generalised to this framework. Linear production in a monotonic setting is studied in Bird (1981). A nice survey can be found in Timmer (2001).

Now we return to our original setting of LP processes and provide a characterisation of the Owen set.

An LP rule F is a set valued function on $\mathcal{L}$ such that

$$F(N, R, P, A, B, c) \subset \mathbb{R}^N$$

for each $(N, R, P, A, B, c) \in \mathcal{L}$.

An LP rule F satisfies *one-person efficiency* if for all $L = (N, R, P, A, B, c) \in \mathcal{L}$ with $|N| = 1$ and $B = e^R$ we have that

$$F(L) = \{v_L(N)\},$$

i.e., if there is only one agent owning one unit of all resources, then F assigns to him the maximal profit that can be made from his resource bundle.

The property of rescaling means that an LP rule should be independent of the units in which the resources are measured. Formally, an LP rule F satisfies *rescaling* if for all $(N, R, P, A, B, c) \in \mathcal{L}$ and all $R \times R$ diagonal matrices D with positive diagonal elements, it holds that

$$F(N, R, P, DA, DB, c) = F(N, R, P, A, B, c).$$

The property of shuffling considers the influence of (combinations of) the splitting and merging of the resource bundles of the various players (in the process possibly changing the number of players). An LP rule F satisfies *shuffling* if for all $(N, R, P, A, B, c) \in \mathcal{L}$ and all nonnegative $N \times M$ matrices X with M finite and $Xe^M = e^N$ it holds that

$$F(N, R, P, A, B, c)X = F(M, R, P, A, BX, c).$$

Here, with $L \in \mathcal{L}$, $F(L)X = \{z^\top x \mid z \in F(L)\}$.

The property of consistency has to do with the special case that every player owns exactly one unit of exactly one resource and different players own different resources. Suppose now that the agents agree on an element y prescribed by the solution rule F and suppose player i takes y_i and leaves. In the reduced LP process without player i we now impose that the resources of i can still be used but at a price of y_i per unit, which is equivalent to saying that the price of a product decreases with y_i for every unit needed of this resource. The rule F is said to satisfy consistency if the restriction of y to the remaining agents is a solution prescribed by F in the above defined reduced LP process. An LP rule F satisfies *consistency* if for all $L = (N, R, P, A, B, c) \in \mathcal{L}$ with $N = R$, $B = I_N$ and $|N| \geqslant 2$ and for all $y \in F(L)$ and $i \in N$ we have

$$y_{-i} \in F(L_{-i}),$$

where

$$L_{-i} = (N\backslash\{i\}, R\backslash\{i\}, P, A_{-i*}, I_{N\backslash\{i\}}, \tilde{c})$$

with A_{-i*} denoting the submatrix of A obtained by deleting the ith row and

$$\tilde{c}_p = c_p - y_i A_{ip}$$

for all $p \in P$.

The final property we introduce is deletion. It says that if a production technology is not needed to make the maximal profit for the grand coalition of all players, this technology can be deleted without deleting solutions prescribed by the LP solution rule. As is the case for consistency, deletion is only required for special LP processes. An LP rule F satisfies *deletion* if for all $L = (N, R, P, A, B, c) \in \mathcal{L}$ with $N = R$ and $B = I_N$ and for all $Q \subset P$ for which

$$L^{-Q} = (N, R, P\backslash Q, A_{-*Q}, B, c_{-Q})$$

is such that

$$v_L(N) = v_{L^{-Q}}(N)$$

it holds that

$$F(L) \subset F(L^{-Q}).$$

Here, A_{-*Q} denotes the submatrix of A obtained by deleting all columns corresponding to elements in Q.

It is interesting to note that the five properties above imply nonemptiness and (general) efficiency of an LP rule.

Theorem 6.4 *The Owen set is the unique LP rule satisfying one-person efficiency, rescaling, shuffling, consistency and deletion. Moreover, these five properties are logically independent.*

Another type of production economy is represented by a *flow situation.* Flow situations were first investigated from an interactive cooperative point of view by Kalai and Zemel (1982a) and Kalai and Zemel (1982b). To let the reader get acquainted with the subject, we have chosen to follow the lines set out by Curiel et al. (1988).

Without giving precise definitions, a flow situation is modelled as a directed graph with two distinct nodes: a source and a sink. On each of the arcs there is a (nonnegative) capacity restriction and an associated simple control game which describes which coalitions of players are allowed to use the arc. A game (N, v) is called *simple* if $v(S) \in \{0, 1\}$ for all $S \subset N$ and $v(N) = 1$. A coalition is allowed to use a particular arc if its value equals 1 in the associated control game. In the corresponding *flow game* the value of a coalition S is the maximal flow through the network from source to sink where only arcs are used which are controlled by S.

Example 6.3 Consider the network of Figure 6.1 with one source, one sink, one intermediate node and three arcs a_1, a_2, a_3 with capacities $c_1 = 10$, $c_2 = 3$ and $c_3 = 6$, respectively. The corresponding control games, with player set $N = \{1, 2, 3\}$, are $w_1 = u_{\{1\}}$, $w_2 = u_{\{1,2\}}$ and $w_3 = u_{\{1,3\}}$.

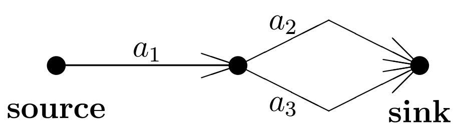

Figure 6.1: A flow network

The coalition $\{1, 3\}$ controls the arcs a_1 and a_3, so the maximal flow for $\{1, 3\}$ equals 6, resulting in $v(\{1, 3\}) = 6$ in the corresponding flow game v. This flow game is given by $v(\{i\}) = 0$ for all $i \in N$, $v(\{1, 2\}) = 3$, $v(\{1, 3\}) = 6$, $v(\{2, 3\}) = 0$ and $v(N) = 9$. The unique minimum cut corresponding to the coalition N is $\{a_2, a_3\}$. By the max-flow min-cut theorem of Ford and Fulkerson (1962), the sum $c_2 + c_3$ equals $v(N)$.

To define a *minimum cut solution* (*MC solution*), take arbitrary core elements of the control games w_2 and w_3 corresponding to the arcs a_2 and a_3 in the minimum cut and divide the corresponding capacities proportional to these core elements. Taking, e.g., $(\frac{1}{3}, \frac{2}{3}, 0) \in C(w_2)$ and $(\frac{1}{2}, 0, \frac{1}{2}) \in C(w_3)$, one obtains the MC solution $(1, 2, 0) + (3, 0, 3) = (4, 2, 3)$, which belongs to the core of the flow game v. $\triangleleft$

Note that an MC solution can only be defined if all control games (in a minimum cut) have a nonempty core.

Theorem 6.5

(i) If all control games are balanced, then MC solutions belong to the core of the flow game and hence, the flow game is balanced.

(ii) Every nonnegative balanced game arises from a flow situation with balanced control games.

(iii) If all control games are dictatorial (i.e., for every arc a_k, there is a player $i \in N$ such that the control game w_k equals $u_{\{i\}}$), then the corresponding flow game is totally balanced.

(iv) Every nonnegative totally balanced game arises from a flow situation with dictatorial control.

Related results on flow situations can be found in Granot and Granot (1992a). Extensions to multicommodity flow situations (cf. Assad (1978)) can be found in Derks and Tijs (1985) and Derks and Tijs (1986).

An interesting recent contribution to the theory of flow situations is the characterisation of the MC solution (as a set valued solution) for so-called simple flow situations, i.e., situations where each player dictatorially controls exactly one arc, other arcs are publicly available (with control games w with $w(S) = 1$ for all coalitions S) and all arcs have a capacity of 1. The MC solution in this context has to be understood as the set of all the vectors $e^S \in \mathbb{R}^N$ for coalitions S which fully control a minimum cut (i.e., without public arcs). This characterisation can be found in Reijnierse et al. (1996) and uses the properties of one-person efficiency, consistency and converse consistency. Moreover, it is shown that the extreme points of the core of a simple flow game coincide with the MC solution. So, in particular, for simple flow situations the core of the related flow game is nonempty if and only if there is a minimum cut which does not contain a public arc.

Various instances of LP games and flow games can be seen as special cases of *linear programming games* (cf. Dubey and Shapley (1984)). An interesting paper which aims at a unification of techniques within combinatorial game theory, providing a unified proof of balancedness, is Potters (1987).

To conclude this section on production, we want to mention Shapley and Shubik (1967), where more general types of production functions are considered, and Sandsmark (1999), where uncertainty is taken into account. An interesting recent application of flow techniques is found in Koster et al. (1999).

7 Inventory

In this section we consider a recent application of game theory within models of inventory control. Inventory management itself is a relatively old branch within operations research and many books have been written on mathematical inventory models, e.g. Hax and Candea (1984) and Tersine (1994). The main objective of inventory management is to minimise average (long term) costs per time unit, while guaranteeing a prespecified minimal level of service.

Firms can save on inventory costs by cooperating. For instance, if there is a fixed cost per order, firms will have to pay less ordering costs if they order simultaneously as a group, rather than separately. This again raises an allocation problem: how should the total minimal inventory costs of the grand coalition be divided among the firms? This problem has been analysed in Meca et al. (1999), on which this section is based.

To fix ideas, we first look at an extremely basic one-firm inventory problem. The firm faces a (deterministic) demand of d units of a specific good per time unit. It is not allowed to run out of stock and the lead time, the time between placement of the order and arrival of the goods, is assumed to be zero. The firm faces two kinds of costs. First, there are ordering costs. For each order the firm places it has to pay a fixed cost a, independent of the quantity ordered. Second, there are holding costs: the costs of carrying one good in stock for one time unit are assumed to be constant and are denoted by h.

Denote by Q the quantity ordered each time the firm places an order. The time between two successive orders then equals Q/d time units. A cycle is defined as a time interval of length Q/d starting at a point in time when an order is placed. By m we denote the number of orders placed per time unit: $m = d/Q$. Because there are on average d/Q orders per time unit, the average ordering costs equal ad/Q. The average inventory level equals $Q/2$, so the average holding costs per time unit equal $hQ/2$. In total, the average inventory costs are $AC(Q)$ when ordering the quantity Q per order:

$$AC(Q) = a\frac{d}{Q} + h\frac{Q}{2}.$$

Minimising average costs over all $Q \geqslant 0$, we obtain an optimal ordering size of $Q^* = \sqrt{2ad/h}$, giving an optimal number of orders per time unit of $m^* = d/Q^* = \sqrt{dh/(2a)}$ and a minimal average costs of $AC(Q^*) = 2am^*$.

An n-firm *inventory situation*, denoted by (N, d, h, a), consists of a finite set $N = \{1, \ldots, n\}$ of firms, a vector $d \in \mathbb{R}^N_{++}$ of demand levels, a vector $h \in \mathbb{R}^N_{++}$ of holding cost parameters and an ordering costs parameter $a > 0$. By Q_i we denote the ordering quantity (per order) of firm $i \in N$.

We claim that in the optimum, firms always synchronise their cycle lengths and place their orders simultaneously. To see this, suppose firm 2 has a longer cycle than firm 1. Then total cost will decrease if firm 2 shortens its cycle length to player 1's length. Ordering costs will decrease, because the number of orders goes down, and holding costs will decrease, because firm 2's average inventory level goes down.

The cycle length of firm $i \in N$ equals Q_i/d_i, so in optimum we have

$$Q_i = \frac{d_i}{d_1} Q_1$$

for all $i \in N$. Using this, the average costs for the firms in N,

$$AC(Q_1, \ldots, Q_n) = a\frac{d_1}{Q_1} + \sum_{i \in N} h_i \frac{Q_i}{2},$$

reduce to

$$AC(Q_1) = a\frac{d_1}{Q_1} + \frac{Q_1}{2d_1} \sum_{i \in N} h_i d_i.$$

Minimising this with respect to Q_1 yields and optimal ordering level

$$Q_i^* = \sqrt{\frac{2ad_i^2}{\sum_{j \in N} d_j h_j}}$$

and an optimal number of orders per time unit of

$$m_N = \frac{d_i}{Q_i^*} = \sqrt{\frac{\sum_{j \in N} d_j h_j}{2a}} = \sqrt{\sum_{j \in N} m_j^2}$$

and minimal average costs $AC(Q_1^*) = 2am_N$. In fact, both ordering costs and holding costs equal am_N in the optimum. Note that the minimal costs only depend on a, which is public information, and the m_i. So in order to calculate the minimal costs, it suffices for each firm $i \in N$ only to reveal their private optimum m_i and not the actual d_i and h_i.

In view of these last remarks, in the remainder of this section we only look at ordering costs and suppress the private parameters d and h. An *ordering cost* *situation* can then be described by a 3-tuple (N, a, m) with $m \in \mathbb{R}^N_{++}$. If a coalition S of firms cooperates, then their optimal ordering costs are

$$a\sqrt{\sum_{i \in S} m_i^2}. \tag{7.1}$$

Consequently, we define a corresponding *ordering cost game* (N, c_o), where $c_o(S)$ equals the expression in (7.1) for all $S \subset N$.

Proposition 7.1 *Let (N, a, m) be an ordering cost situation and let (N, c_o) be the corresponding ordering cost game. Then (N, c_o) is concave and monotonic.*

Another property of the class of ordering cost games is that it is closed with respect to nonnegative scalar multiplication, but not with respect to addition.

Ordering cost games are a special kind of production games, as introduced by Shapley and Shubik (1967). An interesting solution concept for this general class of games is the proportional rule, which for ordering cost games boils down to

$$\pi_i(c_o) = \frac{b(\{i\})}{b(N)} = \frac{am_i^2}{\sqrt{\sum_{j \in N} m_j^2}}$$

for all $i \in N$.

This proportional rule has some nice properties. First of all it provides a core element which is pmas extendable. Note that since (N, c_o) is a cost game, the reverse inequality in monotonicity condition (2.1) should hold.

Theorem 7.2 *Let (N, c_o) be an ordering cost game. Then there exists a population monotonic allocation scheme $y = \{y_{iS}\}, i \in S, S \subset N, S \neq \emptyset$ of (N, c_o) such that $y_{iN} = \pi_i(c_o)$ for all $i \in N$.*

Proof: Define for all $i \in S, S \subset N, S \neq \emptyset$

$$y_{iS} = \frac{am_i^2}{\sqrt{\sum_{j \in S} m_j^2}}.$$

Then, for all $S \subset N, S \neq \emptyset$

$$\sum_{i \in S} y_{iS} = \sum_{i \in S} \frac{am_i^2}{\sqrt{\sum_{j \in S} m_j^2}} = a\sqrt{\sum_{j \in S} m_j^2} = c_o(S)$$

and for all $S, T \subset N$ such that $S \neq \emptyset$ and $S \subset T$ and for all $i \in S$

$$y_{iS} = \frac{am_i^2}{\sqrt{\sum_{j\in S} m_j^2}} \geqslant \frac{am_i^2}{\sqrt{\sum_{j\in T} m_j^2}} = y_{iT}.$$

In particular, we have $y_{iN} = \pi_i(c_o)$ for all $i \in N$, so the proportional rule provides a core element. □

The proportional rule can be characterised by means of a monotonicity property. An ordering cost rule f is called *monotonic* if for all ordering cost games (N, c_o) and $(N, \bar{c}_o)$ we have that $c_o(N)f_i(c_o) \geqslant \bar{c}_o(N)f_i(\bar{c}_o)$ whenever $c_o(\{i\}) \geqslant \bar{c}_o(\{i\})$. Basically, if we have two inventory situations with the same total costs to share and a player generates more costs on his own in one situation than in the other, then he should pay more in the former situation than in the latter.

Theorem 7.3 *The proportional rule is the unique rule on the class of ordering cost games satisfying efficiency and monotonicity.*

An alternative characterisation of the proportional rule using a kind of transfer property and null player property instead of monotonicity is provided in Meca et al. (2001). In this paper also equilibria outcomes of a "constructive" noncooperative approach are analysed. In the same spirit, but focusing on sharing the benefits from joint storage, is Tijs et al. (2000).

A model of inventory games within a context of stochastic uncertainty is given in Hartman et al. (2000). Slikker et al. (2001) provides a nice application in terms of a multiple news vendor problem.

8 Future

Notwithstanding the huge literature on operations research games, our general impression is that the theory is still only in a rather initial phase. This has to do with two related aspects: simplicity and (restricted) applicability of the current models.

Issues to be considered in the future involve:

- dynamics: changes in the player set and other time-related aspects,
- strategic incentives (coopetition),
- minimising private information exchange,

- consistency, monotonicity and continuity arguments for allocation rules,
- stochastic uncertainty,
- asymmetric information between the players with respect to the data of the underlying operations research problem.

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