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OPERATOR MONOTONE FUNCTIONS, POSITIVE DEFINITE KERNELS AND MAJORIZATION

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ABSTRACT. Let $f(t)$ be a real continuous function on an interval, and consider the operator function $f(X)$ defined for Hermitian operators X. We will show that if $f(X)$ is increasing w.r.t. the operator order, then for $F(t) = \int f(t)dt$ the operator function $F(X)$ is convex. Let $h(t)$ and $g(t)$ be C^1 functions defined on an interval I. Suppose $h(t)$ is non-decreasing and $g(t)$ is increasing. Then we will define the continuous kernel function $K_{h,\,g}$ by $K_{h,\,g}(t,s) = (h(t) - h(s))/(g(t) - g(s))$, which is a generalization of the Löwner kernel function. We will see that it is positive definite if and only if $h(A) \leqq h(B)$ whenever $g(A) \leqq g(B)$ for Hermitian operators A, B, and we give a method to construct a large number of infinitely divisible kernel functions.

1. Introduction

Let I be an interval of the real axis and $f(t)$ a real continuous function defined on I. For a bounded Hermitian operator (or matrix) A on a Hilbert space whose spectrum is in I, $f(A)$ stands for the ordinary functional calculus. f is called an *operator monotone function* on I if $f(A) \leqq f(B)$ whenever $A \leqq B$. f is sometimes said to be *operator decreasing* if $-f$ is operator monotone. A continuous function f defined on I is called an *operator convex function* on I if $f(sA + (1-s)B) \leqq sf(A) + (1-s)f(B)$ for every $0 < s < 1$ and for every pair of bounded Hermitian operators A and B whose spectra are both in I. An *operator concave function* is likewise defined. It is known that $g(t)$ is an operator convex function on an open interval I if and only if $g(t)$ is of class $C^2(I)$, and for each $t_0 \in I$, the function $f(t)$ defined by

$$f(t) = \frac{g(t) - g(t_0)}{t - t_0} \quad (t \neq t_0), \qquad f(t_0) = g'(t_0) \tag{1}$$

is operator monotone on I ([14] and [1]).

Let $K(t,s)$ be a real continuous function defined on $I \times I$, and assume that it is symmetric. Then $K(t,s)$ is called a *positive definite kernel* on I if

$$\iint_{I\times I} K(t,s)\phi(t)\phi(s)dt\,ds \geqq 0 \tag{2}$$

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for all real continuous functions ϕ with compact support in I. Suppose $K(t,s) \geqq 0$ for every t, s in I. Then the kernel $K(t,s)$ is said to be *infinitely divisible* on I if $K(t,s)^r$ is a positive definite kernel for every $r > 0$. A kernel $K(t,s)$ is said to be *conditionally positive* on I if (2) holds for every continuous function ϕ on I such that the support of ϕ is compact and the integral of ϕ over I vanishes. A kernel $K(t,s)$ is sometimes said to be *conditionally negative* on I if $-K(t,s)$ is conditionally positive definite on I. In this paper we say that $K(t,s)$ *conditionally vanishes* on I if it is conditionally positive and conditionally negative on I.

The Löwner theorem says that a C^1 function f is operator monotone on an open interval I if and only if the Löwner kernel $K_f(t,s)$ defined by

$$K_f(t,s) = \frac{f(t) - f(s)}{t - s} \quad (t \neq s), \quad K_f(t,t) = f'(t)$$

is positive definite on I and that such a function f possesses a holomorphic extension $f(z)$ onto the open upper half plane Π_+ which maps Π_+ into itself; namely $f(z)$ is a *Pick function*. In this case, by Herglotz's theorem $f(t)$ has an integral representation,

$$f(t) = \alpha + \beta t + \int_{-\infty}^{\infty} \left(-\frac{x}{x^2+1} + \frac{1}{x-t}\right) d\nu(x), \tag{3}$$

where α is real, $\beta \geqq 0$ and ν is a Borel measure so that

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1} d\nu(x) < \infty, \quad \nu(I) = 0.$$

For further details see [2, 3, 6, 13].

Let $f(t)$ be a C^1 function defined on an infinite interval (a, ∞). We will show that the Löwner kernel $K_f(t,s)$ is positive definite on (a,∞) if and only if $K_f(t,s)$ is conditionally positive, $\lim_{t\to\infty} f(t)/t < \infty$ and $f(\infty) := \lim_{t\to\infty} f(t) > -\infty$.

Recall the symbol "$\preceq$" introduced in [16, 17]: let $h(t)$ be a non-decreasing continuous function on I and $g(t)$ an increasing continuous function on I; then h is said to be *majorized* by g and denoted by $h \preceq g$ on I if $h(A) \leqq h(B)$ whenever $g(A) \leqq g(B)$ for A, B whose spectra are both in I. It is clear that $f(t) \preceq t$ on I means that $f(t)$ is operator monotone on I. For a pair of C^1 functions $h(t)$ and $g(t)$, we define a continuous kernel $K_{h,g}(t,s)$ by

$$K_{h,g}(t,s) = \frac{h(t) - h(s)}{g(t) - g(s)} \quad (s \neq t), \quad K_{h,g}(t,t) = \frac{h'(t)}{g'(t)}. \tag{4}$$

The Löwner kernel $K_f(t,s)$ can be written as $K_{f,t}(t,s)$ in this way. We will see that the kernel $K_{h,g}(t,s)$ is positive definite on I if and only if $h \preceq g$ on I. Suppose the range of g is $(0,\infty)$. Then we will show that if the kernel $K_{h,g}$ is positive definite on (a,b), then the kernel $K_{hg,g}$ is conditionally negative and the kernel $K_{g,hg}$ is infinitely divisible. This result must provide a large number of infinitely divisible kernels. We will also give a characterization for $K_{h,g}$ to be conditionally negative.

2. Geometric features

In this section we discuss certain geometric features of operator convex functions. We first make an improvement on the well-known fact about the relation between operator convexity and operator monotonicity which was mentioned in the previous section.

Lemma 2.1. *Let $g(t)$ be a differentiable function on an open interval I. Then $g(t)$ is operator convex if for a point $t_0 \in I$ the function $f(t)$ defined by (1) is operator monotone on I. Conversely, if $g(t)$ is operator convex, then $f(t)$ is operator monotone for each $t_0 \in I$.*

Proof. By representing $f(t)$ as (3) we get

$$g(t) = g(t_0) - \alpha t_0 + (\alpha - \beta t_0)t + \beta t^2 + \int_{-\infty}^{\infty}(-1 + \frac{x - t_0}{x - t} - \frac{x}{x^2+1}(t - t_0))d\nu(x).$$

Therefore $g(t_0) - \alpha t_0 + (\alpha - \beta t_0)t + \beta t^2$ is clearly operator convex. Note that for a given x, $\frac{1}{x-t}$ and $-\frac{1}{x-t}$ are operator convex on $-\infty < t < x$ and $x < t < \infty$, respectively. Thus for each $x \notin I$, $\frac{x-t_0}{x-t}$ is operator convex on I. Thus one can see that $g(t)$ is operator convex. The converse is well-known. □

Proposition 2.2. *Let $f(t)$ be an operator monotone (or decreasing) function on I. Then the indefinite integral $\int f(t)dt$ is an operator convex (or concave) function on I.*

Proof. We show only the case where $f(t)$ is operator monotone. To do so, we prove that $g(t) := \int_c^t f(s)ds$ is operator convex on any compact subinterval $[a, b] \subset I$ for a fixed point c in I. Notice that

$$g(t) = \lim_{n\to\infty} \sum_{k=1}^{n} f(c + \frac{k}{n}(t - c))\frac{1}{n}(t - c) \quad (a \leqq t \leqq b),$$

where the right hand side uniformly converges. Put $h(t) = f(c + \frac{k}{n}(t - c))(t - c)$. Since

$$\frac{h(t) - h(c)}{t - c} = f(c + \frac{k}{n}(t - c))$$

is operator monotone on I, by Lemma 2.1 $h(t)$ is operator convex, and hence so is $g(t)$ on $[a, b]$. □

We note that the above lemma is a slight generalization of Exercise V.3.14 of [2].

Example 2.1. $t \log t$ and $\log \Gamma(t)$ are both operator convex on $(0, \infty)$. $\log \cos t$ is operator concave on $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Proof. Since $\log t$ is operator monotone on $(0, \infty)$, from Proposition 2.2 it follows that $t \log t - t$ is operator convex there, which implies $t \log t$ is operator convex too. Since $\tan t$ and $\frac{\Gamma'(t)}{\Gamma(t)}$ are both operator monotone on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and on $(0, \infty)$, respectively, the required fact follows from Proposition 2.2. □

We remark that the converse of Proposition 2.2 does not hold: for instance, $\frac{1}{t}$ is operator convex on $(0, \infty)$, but $\frac{d}{dt}\frac{1}{t} = -\frac{1}{t^2}$ is not operator monotone there. However we have

Proposition 2.3. *Let $g(t)$ be an operator convex function on $(0, \infty)$. Then $g'(\sqrt{t})$ is operator monotone there.*

Proof. We first assume that $g(t)$ is operator convex on $[0,\infty)$ and $g'(0+)$ is finite. Since $(g(t)-g(0))/t$ is operator monotone on $[0,\infty)$, it admits an expression given by (3) with $\nu([0,\infty))=0$. We then notice that $\frac{1}{x(x^2+1)}$ is integrable with respect to $d\nu(x)$, because $g'(0+)=\lim_{t\to+0}(g(t)-g(0))/t$ is finite. By differentiating

$$g(t)=g(0)+\alpha t+\beta t^2+\int_{-\infty}^{\infty} t(-\frac{x}{x^2+1}+\frac{1}{x-t})d\nu(x)$$

we obtain

$$g'(t)=\alpha+2\beta t+\int_{-\infty}^{\infty}\{-\frac{x}{x^2+1}+\frac{x}{(x-t)^2}\}d\nu(x)\quad (0<t<\infty). \tag{5}$$

Indeed, take any real number $M>0$. Then for $0<t<M$ and $-\infty<x<0$

$$\begin{aligned}|\frac{\partial}{\partial t}t(-\frac{x}{x^2+1}+\frac{1}{x-t})| &= |-\frac{x}{x^2+1}+\frac{x}{(x-t)^2}| \leqq \frac{|x(2tx-t^2+1)|}{(x^2+1)(x-t)^2}\\ &< \frac{2t|x|+t^2+1}{(x^2+1)|x|} \leqq 2M\frac{1}{x^2+1}+(M^2+1)\frac{1}{(x^2+1)|x|}.\end{aligned}$$

The largest side is integrable with respect to $d\nu(x)$. (5) therefore holds for $0<t<M$ and hence for $0<t<\infty$. We therefore get

$$g'(\sqrt{t})=\alpha+2\beta\sqrt{t}+\int_{-\infty}^{\infty}\{-\frac{x}{x^2+1}+\frac{x}{(x-\sqrt{t})^2}\}d\mu(x).$$

Since $(x-\sqrt{t})^2$ is operator monotone on $0<t<\infty$ for $-\infty<x<0$, so is $\frac{x}{(x-\sqrt{t})^2}$. $g'(\sqrt{t})$ is therefore operator monotone on $(0,\infty)$. We next suppose that $g(t)$ is defined on $(0,\infty)$. For sufficiently small $\varepsilon>0$ put $g_\varepsilon(t)=g(t+\varepsilon)$. By the result proved above we get that $g'_\varepsilon(\sqrt{t})=g'(\sqrt{t+\varepsilon})$ is operator monotone on $0<t<\infty$. By letting $\varepsilon\to 0$ we get the operator monotonicity of $g'(\sqrt{t})$. □

It is well-known that a positive continuous function $f(t)$ on $[0,\infty)$ is operator monotone if and only if $f(t)$ is operator concave (cf. Theorem V.2.5 of [2]). Also, there are some extensions of it (for instance, [15]). Now we give an eventual extension with an elementary proof.

Theorem 2.4. *Let $f(t)$ be a continuous function on (a,∞), where $a\geqq-\infty$. Then*

(i) *$f(t)$ is operator decreasing if and only if $f(t)$ is operator convex and $f(\infty)<\infty$;*
(ii) *$f(t)$ is operator monotone if and only if $f(t)$ is operator concave and $f(\infty)>-\infty$.*

Proof. We need to show only statement (i), because (ii) follows from it. Assume $f(t)$ is operator convex and $f(\infty)<\infty$; notice here that $f(\infty):=\lim_{t\to\infty}f(t)$ exists since $f(t)$ is convex. We will prove $f(B) \leqq f(A)$ if $a < A \leqq B$. By considering $B+\delta$ instead of B, we may assume that $B - A \geqq \delta > 0$. Then $\dfrac{\lambda}{1-\lambda}(B-A) > a$ if $0 < \lambda < 1$ and λ is sufficiently close to 1. For such a λ

$$\begin{aligned} f(\lambda B) &= f(\lambda A + (1-\lambda)\frac{\lambda}{1-\lambda}(B-A)) \\ &\leqq \lambda f(A) + (1-\lambda) f(\frac{\lambda}{1-\lambda}(B-A)). \end{aligned}$$

In the case of $f(\infty) = -\infty$, $f(\frac{\lambda}{1-\lambda}(B-A)) \leqq 0$ for λ sufficiently close to 1. We therefore get $f(\lambda B) \leqq \lambda f(A)$ and hence $f(B) \leqq f(A)$. In the case of $f(\infty) > -\infty$, $(1-\lambda)f(\frac{\lambda}{1-\lambda}(B-A)) \to 0$ as $\lambda \to 1-0$. This also yields $f(B) \leqq f(A)$. To see the converse statement we use the well-known technique. Assume $f(t)$ is operator decreasing. Then we obviously get $f(\infty) < \infty$. For $0 < \lambda < 1$ define the unitary operator W on the direct sum of a Hilbert space by

$$W = \begin{pmatrix} \sqrt{\lambda} I & -\sqrt{1-\lambda} I \\ \sqrt{1-\lambda} I & \sqrt{\lambda} I \end{pmatrix}.$$

Since

$$W^* \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} W = \begin{pmatrix} \lambda A + (1-\lambda) B & \sqrt{\lambda(1-\lambda)}(B-A) \\ \sqrt{\lambda(1-\lambda)}(B-A) & \lambda B + (1-\lambda) A \end{pmatrix},$$

for an arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that

$$W^* \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} W \leqq \begin{pmatrix} \lambda A + (1-\lambda) B + \varepsilon & 0 \\ 0 & \lambda B + (1-\lambda) A + \delta \end{pmatrix}.$$

From the assumption we obtain

$$\begin{aligned} W^* \begin{pmatrix} f(A) & 0 \\ 0 & f(B) \end{pmatrix} W &= f\left(W^* \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} W \right) \\ \geqq \begin{pmatrix} f(\lambda A + (1-\lambda) B + \varepsilon) & 0 \\ 0 & f(\lambda B + (1-\lambda) A + \delta) \end{pmatrix}. \end{aligned}$$

By comparing $(1,1)$ elements we have

$$\lambda f(A) + (1-\lambda) f(B) \geqq f(\lambda A + (1-\lambda) B + \varepsilon),$$

which gives the operator convexity of f, because ϵ is arbitrary. $\square$

Recall that $f(t) = t^2$ is operator convex on $(-\infty, \infty)$, but not operator decreasing there: this says we cannot remove the condition $f(\infty) < \infty$ in (i) of the above theorem. The next corollary easily follows from (ii) of Theorem 2.4 and Proposition 2.3.

Corollary 2.5 ([18])**.** *If $f(t)$ is operator monotone on $(0, \infty)$, then so is $-f'(\sqrt{t})$.*

The second statement of the next corollary is well-known (e.g. see [9]).

Corollary 2.6. *If $f(t)$ is operator monotone on (a, ∞), then so is $\dfrac{t-c}{f(t)-f(c)}$ for each $c > a$. Moreover if $a = 0$ and $f(0+) \geqq 0$, then $\dfrac{t}{f(t)}$ is operator monotone on $(0, \infty)$.*

Proof. Since $-f(t)$ is operator convex, for each $c > 0$, $-\dfrac{f(t)-f(c)}{t-c}$ is operator monotone; hence so is $\dfrac{t-c}{f(t)-f(c)}$. In the special case where $a = 0$ and $f(0+) \geqq 0$ we see that $-\dfrac{f(t)-f(0+)}{t}$ is operator monotone on $(0,\infty)$. From $-\dfrac{f(t)}{t} = -\dfrac{f(t)-f(0+)}{t} - \dfrac{f(0+)}{t}$ it follows that $-\dfrac{f(t)}{t}$ is operator monotone, and hence so is $\dfrac{t}{f(t)}$. □

Suppose $f(t)$ is a function defined on a left half line. Then $f(-t)$ is defined on a right half line, $f(t)$ is operator monotone if and only if $f(-t)$ is operator decreasing, and $f(t)$ is operator convex if and only if $f(-t)$ is operator convex. By Theorem 2.4 we therefore get

Corollary 2.7. *Let $f(t)$ be a continuous function on $(-\infty, b)$, where $b \leqq \infty$. Then $f(t)$ is operator monotone (or decreasing) if and only if $f(t)$ is operator convex (or concave) and $f(-\infty) < \infty$ (or $f(-\infty) > -\infty$).*

We have so far dealt with functions on "infinite intervals" except for Proposition 2.2. We now express an operator monotone function on a "finite interval" as a sum of such functions.

Proposition 2.8. *Let $f(t)$ be an operator monotone function on a finite interval (a, b). Then there is a decomposition of $f(t)$ such that*

$$f(t) = f_+(t) + f_-(t) \quad (a < t < b),$$

where $f_+(t)$ and $f_-(t)$ are operator monotone on (a,∞) and $(-\infty, b)$ respectively.

Proof. $f(t)$ is expressed as (3) with $\nu((a, b)) = 0$. It is easy to see that

$$f_+(t) := \alpha + \beta t + \int_{-\infty}^{a} (-\frac{x}{x^2+1} + \frac{1}{x-t}) d\nu(x) \quad (a < t < \infty),$$

$$f_-(t) := \int_{b}^{\infty} (-\frac{x}{x^2+1} + \frac{1}{x-t}) d\nu(x) \quad (-\infty < t < b)$$

satisfy the required fact. □

We remark that by Theorem 2.4 or Corollary 2.7 $f_+(t)$ is operator concave and $f_-(t)$ is operator convex. So we cannot determine whether an operator monotone function on a finite interval is operator concave or whether it is operator convex.

Example 2.2. The formula

$$\tan t = \sum_{n=-\infty}^{\infty} \{ \frac{1}{(n-\frac{1}{2})\pi - t} - \frac{n\pi}{n^2\pi+1} \}$$

says $\tan t$ is operator monotone on $(-\frac{\pi}{2}, \frac{\pi}{2})$. Put

$$f_+(t) = \sum_{n=-\infty}^{0} \{ \frac{1}{(n-\frac{1}{2})\pi - t} - \frac{n\pi}{n^2\pi+1} \}, \quad f_-(t) = \sum_{n=1}^{\infty} \{ \frac{1}{(n-\frac{1}{2})\pi - t} - \frac{n\pi}{n^2\pi+1} \}.$$

Then $f_+(t)$ and $f_-(t)$ are operator monotone on $(-\frac{\pi}{2}, \infty)$ and $(-\infty, \frac{\pi}{2})$, respectively, and $\tan t = f_+(t) + f_-(t)$.

3. Löwner kernels

This section is devoted to the study of the Löwner kernel functions. We first refer to the result shown by Bhatia and Sano in [5]:

Let $f(t)$ be a C^2 function on $[0,\infty)$ such that $f(t) \geqq 0$ and $f(0) = f'(0) = 0$. Then f is operator convex on $[0,\infty)$ if and only if the Löwner kernel $K_f(t,s)$ is conditionally negative definite on $[0,\infty)$.

We extend this as follows:

Theorem 3.1. *Let $f(t)$ be a C^1 function on (a,∞). Then*

(i) *$f(t)$ is operator convex on (a,∞) if and only if the Löwner kernel $K_f(t,s)$ is conditionally negative definite and $\lim_{t\to\infty} \frac{f(t)}{t} > -\infty$;*

(ii) *$f(t)$ is operator concave on (a,∞) if and only if the Löwner kernel $K_f(t,s)$ is conditionally positive definite and $\lim_{t\to\infty} \frac{f(t)}{t} < \infty$.*

To prove this we invoke the following:

Lemma 3.2. *Let $f(t)$ be a C^1 function on (a,b). Then the Löwner kernel $K_f(t,s)$ is conditionally positive definite if and only if $h_c(t) := \frac{f(t)-f(c)}{t-c}$ is operator convex on (a,b) for each c in (a,b).*

Proof. It is known that $K_f(t,s)$ is conditionally positive definite if and only if $\frac{h_c(t)-h_c(c)}{t-c}$ is operator monotone on (a,b) for each c ([6, p. 139]). By Lemma 2.1, this is equivalent with the operator convexity of $h_c(t)$. □

Proof of Theorem 3.1. (i) Assume $f(t)$ is operator convex. Since $f(t)$ is a numerical convex function, we have $\lim_{t\to\infty} \frac{f(t)}{t} > -\infty$. The operator convexity of $f(t)$ guarantees $h(t) := \frac{f(t)-f(c)}{t-c}$ to be operator monotone on (a,∞), and hence operator concave. This implies $\frac{-f(t)-(-f(c))}{t-c}$ is operator convex. By Lemma 3.2, the Löwner kernel $K_{-f}(t,s)$ is conditionally positive definite, that is, $K_f(t,s)$ is conditionally negative definite. We next show the converse statement. By Lemma 3.2 $h(t) := \frac{f(t)-f(c)}{t-c}$ is operator concave on (a,∞). The assumption $\lim_{t\to\infty} \frac{f(t)}{t} > -\infty$ implies $h(\infty) > -\infty$. By Theorem 2.4 $h(t)$ is operator monotone. $f(t)$ is therefore operator convex. Statement (ii) can be shown in the same way as the above or easily follows from (i). □

We give an example which shows the indispensability of the condition $\lim_{t\to\infty} \frac{f(t)}{t} > -\infty$: the Löwner kernel $K_f(t,s) = -(t^2 + st + s^2)$ of $f(t) = -t^3$ is conditionally negative on $(0,\infty)$, but $f(t)$ is not operator convex there.

By combining Theorem 2.4 and Theorem 3.1 we can easily get the next theorem, which clarifies the relation between the positive definiteness and the conditionally positive definiteness of a Löwner kernel.

Theorem 3.3. *Let $f(t)$ be C^1 function on (a,∞). Then*

(i) *the Löwner kernel* $K_f(t,s)$ *is positive definite on* (a,∞) *if and only if* $K_f(t,s)$ *is conditionally positive definite on* (a,∞), $\lim_{t\to\infty}\frac{f(t)}{t}<\infty$ *and* $f(\infty)>-\infty$.

(ii) *the Löwner kernel* $K_f(t,s)$ *is negative definite on* (a,∞) *if and only if* $K_f(t,s)$ *is conditionally negative definite on* (a,∞), $\lim_{t\to\infty}\frac{f(t)}{t}>-\infty$, *and* $f(\infty)<\infty$.

For functions defined on a left half line one can easily obtain the following:

Corollary 3.4. *Let* $f(t)$ *be a* C^1 *function on* $(-\infty,b)$. *Then*

(i) $f(t)$ *is operator convex (or concave) if and only if* $\lim_{t\to-\infty}\frac{f(t)}{t}<\infty$ *(or* $\lim_{t\to-\infty}\frac{f(t)}{t}>-\infty$*) and the Löwner kernel* $K_f(t,s)$ *is conditionally positive (or negative) definite.*

(ii) *The Löwner kernel* $K_f(t,s)$ *is positive (or negative) definite if and only if* $K_f(t,s)$ *is conditionally positive (or negative) definite,* $\lim_{t\to-\infty}\frac{f(t)}{t}<\infty$ *(or* $\lim_{t\to-\infty}\frac{f(t)}{t}>-\infty$*), and* $f(-\infty)<\infty$ *(or* $f(-\infty)>-\infty$*).*

Lastly consider an operator convex function $g(t)$ defined on $(-1,1)$. Since $(g(t)-g(0))/t$ is operator monotone, by using Proposition 2.8 we can write

$$g(t)=g(0)+tf_+(t)+tf_-(t),$$

where $f_+(t)$ and $f_-(t)$ are operator monotone on $(-1,\infty)$ and $(-\infty,1)$, respectively; hence $tf_+(t)$ and $tf_-(t)$ are both operator convex. The Löwner kernels of $tf_+(t)$ and $tf_-(t)$ are therefore conditionally negative definite and conditionally positive definite on $(-1,1)$, respectively.

4. Majorization and kernel functions

In this section we will study a kernel function $K_{h,g}$ defined by (4) and give a method to construct infinitely divisible kernel functions. We remark that this kernel function is a generalization of the Löwner kernel function. By simple calculation of the double integral (2) for this kernel, one can easily see

Lemma 4.1. *Let* $h(t)$ *and* $g(t)$ *be* C^1 *functions on* I, *and suppose that* $g(t)$ *is increasing. Suppose* $t=\tau(x)$ *is a differentiable and increasing function from an interval* J *onto* I. *Then the kernel* $K_{h,\,g}$ *defined by* (4) *is positive definite (or infinitely divisible) on* I *if and only if* $K_{h(\tau),\,g(\tau)}$ *is positive definite (or infinitely divisible) on* J.

Lemma 4.2. *The following statements are equivalent:*

(i) *The kernel* $K_{h,\,g}(t,s)$ *is positive definite on* I.

(ii) *There is an operator monotone function* φ *defined on* $g(I)$ *such that*

$$h(t)=(\varphi\circ g)(t)\quad(t\in I).$$

(iii) $h\preceq g$ *on* I.

Proof. Lemma 4.1 says that the kernel $K_{h,g}$ is positive definite on I if and only if the Löwner kernel $K_{h\circ g^{-1}}$ is positive definite on $g(I)$; this means $h\circ g^{-1}$ is operator monotone on $g(I)$. (i) is hence equivalent with (ii). That (ii) and (iii) are equivalent follows from the definition of "$\preceq$". □

Note that $K_{f,\,g}(t,s)$ is positive definite (or infinitely divisible) if $K_{f,\,h}(t,s)$ and $K_{h,\,g}(t,s)$ are both positive definite (or infinitely divisible): indeed, since the Schur product of kernel functions $K_{f,\,h}$ and $K_{h,\,g}$ is $K_{f,\,g}(t,s)$, it is positive definite too. The following is a part of the Product Lemma given in [16] and [17], but we give a simple proof for the sake of completeness.

Lemma 4.3. *Let $h(t)$ and $g(t)$ be positive C^1 functions on an open interval I. Suppose $h(t)g(t)$ is increasing and its range is $(0,\infty)$. Then the kernel $K_{h,hg}$ is positive definite on I if and only if the kernel $K_{g,hg}$ is as well.*

Proof. Assume the kernel $K_{h,hg}$ is positive definite on I. Then there is an operator monotone function $\varphi(s)$ on $0<s<\infty$ such that $h(t)=\varphi(h(t)g(t))$. Since $\varphi(s)>0$, by Corollary 2.6 $\dfrac{s}{\varphi(s)}$ is operator monotone on $(0,\infty)$. This means

$$\frac{s}{\varphi(s)}\preceq s \quad (0<s<\infty),$$

which is equivalent to

$$g(t)=\frac{h(t)g(t)}{h(t)}\preceq h(t)g(t) \quad (t\in I).$$

By Lemma 4.1 the kernel $K_{g,hg}$ is positive definite on I. This completes the proof. □

Theorem 4.4. *Let $h(t)$ and $g(t)$ be positive C^1 functions defined on I. Suppose g is increasing and its range is $(0,\infty)$. If the kernel $K_{h,\,g}$ is positive definite on I, then for $n\geqq 0,\ m\geqq 1$ the kernels*

$$K_{h^{n+1}g^m,\,h^ng^m},\quad K_{h^ng^{m+1},\,h^ng^m}$$

are conditionally negative, and the kernels

$$K_{h^ig^j,\,h^ng^m}(t,s)=\frac{h^i(t)g^j(t)-h^i(s)g^j(s)}{h^n(t)g^m(t)-h^n(s)g^m(s)}$$

are infinitely divisible for $0\leqq i\leqq n,\ 0\leqq j\leqq m,\ 1\leqq m,\ i+j+1\leqq n+m$.

Moreover, if f is a (not necessarily positive) C^1 function such that the kernel $K_{f,\,g}(t,s)$ is positive definite, then the kernel

$$K_{g,\,e^fg}(t,s)$$

is infinitely divisible.

Proof. Since $K_{h,\,g}$ is positive definite on I, Lemma 4.3 implies that $K_{g/h,\,g}$ is also positive definite on I. The kernel

$$\frac{g(t)h(s)-h(t)g(s)}{g(t)-g(s)}=h(t)\frac{g(t)/h(t)-g(s)/h(s)}{g(t)-g(s)}h(s)$$

is therefore positive definite there. Thus the kernel

$$\begin{aligned}K_{hg,\, g}(t,s) &= \frac{h(t)g(t)-h(s)g(s)}{g(t)-g(s)} \\ &= h(t)+h(s)-\frac{g(t)h(s)-h(t)g(s)}{g(t)-g(s)}\end{aligned}$$

is conditionally negative, because the kernel $h(t)+h(s)$ conditionally vanishes. We note that $h(t)$ is non-decreasing, for $g(t)$ is increasing and $K_{h,\, g}$ is positive definite. This implies that $h(t)g(t)$ is increasing, which gives $K_{hg,\, g}(t,s)>0$ for every t,s. Thus, the reciprocal kernel $K_{g,\, hg}(t,s)$ of $K_{hg,\, g}$ is infinitely divisible (see [12] or p. 458 of [13]), and hence positive definite. Lemma 4.1 deduces positive definiteness of $K_{g,\, g^m}$, for $K_{t^{1/m},\, t}$ is positive definite. By making use of the above fact we see $K_{g^{m+1},\, g^m}$ is conditionally negative. Since $K_{g,\, g^m}$ and $K_{h,\, g}$ are positive definite, so is $K_{h,\, g^m}$, from which it follows that $K_{hg^m,\, g^m}$ is conditionally negative. Suppose $K_{h^{k+1}g^m,\, h^k g^m}$ is conditionally negative for $0 \leqq k \leqq n$. Since $K_{h^k g^m,\, h^{k+1}g^m}$ is positive definite, by the transitive property $K_{g^m,\, h^{n+1}g^m}$ is positive definite. This yields that $K_{g,\, h^{n+1}g^m}$ is positive definite and hence so is $K_{h,\, h^{n+1}g^m}$. By making use of the above fact again we see $K_{h^{n+1}g^{m+1},\, h^{n+1}g^m}$ and $K_{h^{n+2}g^m,\, h^{n+1}g^m}$ are both conditionally negative. Thus we obtain the first requirement. Since $K_{h^n g^m,\, h^{n+1}g^m}$ and $K_{h^n g^m,\, h^n g^{m+1}}$ are both infinitely divisible, by the transitive property one can easily see the second requirement. To see the last requirement we note that if a kernel function is positive definite on I_n such that $I_1 \subset I_2 \subset \cdots$, then so it is on $\bigcup I_n$. We may therefore assume that $f(t)$ is bounded below. Since $1+f(t)/n$ is positive for sufficiently large n and the kernel $K_{(1+f(t)/n),\, g}$ is positive definite on I, $K_{g,\, (1+f(t)/n)^n g}$ is infinitely divisible on I for such an n. Taking the limit shows that $K_{g,\, e^f g}$ is infinitely divisible on I. □

The above result provides a wide variety of infinitely divisible kernels. We give a few examples.

Example 4.1. Put $g(t)=t$ on $(0,\infty)$ and $f(t)=t$ or $f(t)=-1/t$ in Theorem 4.4 to see that

$$\frac{t-s}{te^t-se^s}, \quad \frac{t-s}{te^{-t^{-1}}-se^{-s^{-1}}}$$

are both infinitely divisible on $(0,\infty)$.

We end this paper by giving some necessary and sufficient conditions for a kernel $K_{h,\, g}$ being conditionally negative definite.

Theorem 4.5. *Let $h(t)$ and $g(t)$ be positive C^1 functions defined on an open interval (a,b), where $-\infty \leqq a < b \leqq \infty$. Suppose $g(t)$ is increasing and the range of g is $(0,\infty)$. Then the following are equivalent:*

(i) *the kernel $K_{h,\, g}$ is conditionally negative;*
(ii) *there is an operator convex function φ defined on $(0,\infty)$ such that $\varphi(g(t)) = h(t)$ for $t\in(a,b)$;*
(iii) $\dfrac{h(t)-h(a+0)}{g(t)} \preceq g(t) \quad (a<t<b).$

Proof. (i)⇒(ii). Since $h\circ g^{-1}(t)>0$ and the Löwner kernel $K_{h\circ g^{-1}}$ is conditionally negative definite on $(0,\infty)$, by Theorem 3.1 $\varphi := h\circ g^{-1}$ is an operator convex function on $(0,\infty)$.

(ii)⇒(iii). Since φ is operator convex on $(0,\infty)$, $\dfrac{\varphi(t)-\varphi(0+)}{t}$ is operator monotone on $(0,\infty)$. Hence

$$\frac{\varphi(t)-\varphi(0+)}{t} \preceq t \ (0<t<\infty), \text{ i.e., } \frac{h(t)-h(a+0)}{g(t)} \preceq g(t) \ (a<t<b).$$

This implies (iii).

(iii)⇒(i). Define an operator monotone function f on $(0,\infty)$ by

$$f(g(t)) = \frac{h(t)-h(a+0)}{g(t)}.$$

Since $tf(t)$ is operator convex on $(0,\infty)$, by Theorem 3.1 the Löwner kernel $K_{tf(t)}$ is conditionally negative on $(0,\infty)$, which implies that the kernel $K_{g(t)f(g(t)),\, g(t)}$ is conditionally negative on (a,b). Since $g(t)f(g(t)) = h(t)-h(a+)$, we finally get (i). □

Acknowledgements

Lemma 2.1 has been partly known; for instance, the reader can find it for the special case in Ando's private monograph. Prof. F. Hiai told the author that there is no paper which yet shows it in the general case, and then the referee requested that the author clarify this. Thus Lemma 2.1 was added in the revised version. The author expresses his appreciation for their valuable comments and suggestion.

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