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OPERATOR NORMS OF POWERS OF THE VOLTERRA OPERATOR

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1. Introduction. The Volterra operator $V : L^2[0,1] \to L^2[0,1]$ will be defined by

$$Vf(x) = \int_0^x f(t)\,dt, \tag{1.1}$$

where f is real valued function.

Definition 1.1. The *operator norm*, $\|.\|$, is defined by

$$\|T\| = \sup_{\|f\|_2=1} \|Tf\|_2, \tag{1.2}$$

where

$$\|f\|_2 = \left[\int_0^1 |f(t)|^2\ dt\right]^{1/2}. \tag{1.3}$$

It is not difficult to show that the operator norm of V is $2/\pi$. In [**5**] N. Lao and R. Whitley give the numerical evidence which led them to the conjecture that

$$\lim_{m\to\infty} \|m!V^m\| = 1/2. \tag{1.4}$$

The purpose of this article is to verify that this is indeed the case. The analysis will be presented for a more general operator defined as follows.

Definition 1.2. The linear operator $A : L^2[0,1] \to L^2[0,1]$ is given by

$$Af(x) = \int_0^x a(x-t)f(t)\,dt, \tag{1.5}$$

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where a is a nonnegative, nondecreasing L^2-integrable function on $[0, 1]$.

A is a Hilbert-Schmidt operator. It will be convenient to state some definitions and results concerning cones and u_0-positive operators. The general theory will be found in [**4**] from which the following are taken.

Definition 1.3. Let E be a real Banach space. A set $K \subset E$ is called a *cone* if the following conditions are satisfied:

(a) the set is closed,

(b) if $u, v \in K$ then $\alpha u + \beta v \in K$ for all nonnegative real numbers α, β,

(c) of each pair of vectors $f, -f$ at least one does not belong to K provided that $f \neq 0$.

We write $f \geq 0$ if $f \in K$, and $f \geq g$ if $f - g \in K$.

Definition 1.4. A cone is called *reproducing* if every element $f \in E$ can be represented in the form

$$f = u - v, \quad u, v \in K.$$

Example 1.1. The collection of nonnegative functions in C, the space of functions which are continuous on a bounded closed set, is a reproducing cone. In fact it is *solid*, that is to say, it contains interior points.

Example 1.2. Although $L^2[0, 1]$ does not contain a solid cone it does in fact contain the reproducing cone of functions which are positive almost everywhere, since every function $f \in L^2[0, 1]$ can be represented, as

$$f = f_+ - f_-,$$

where f_+ and f_- are nonnegative and belong to $L^2[0, 1]$.

Definition 1.5. The operator A defined on E is u_0-*positive* if there exists $u_0 \in K$ and a fixed positive integer p such that for each element

$f \in K$ there are positive numbers α and β, which depend on f, so that

$$\alpha\ u_0 \leq A^p f \leq \beta\ u_0.$$

Example 1.3. The Volterra operator V is not u_0-positive. For simplicity we take $p = 1$; the proof for a general value is similar. Suppose there did exist a nonnegative function u_0, and positive scalars α, β, such that

$$\alpha(f)\ u_0(x) \leq \int_0^x f(t)\,dt \leq \beta(f)\ u_0(x), \tag{1.6}$$

for all nonnegative functions f. Set $f_1(t) = 1$ on the right and $f_2(t) = t$ on the left to give, for almost all $x \in [0,1]$,

$$\frac{x}{\beta(f_1)} \leq u_0(x) \leq \frac{x^2}{2\alpha(f_2)},$$

which is clearly not true for all x.

Example 1.4. The operator G defined on $L^2[0,1]$ by

$$Gf(x) = (1-x)\int_0^x tf(t)\,dt + x\int_x^1 (1-t)f(t)\,dt, \tag{1.7}$$

is u_0-positive, with $p = 1$ and $u_0(x) = x(1-x)$.

Let $f \in L^2[0,1]$ and be positive almost everywhere; then

$$\begin{aligned} \frac{Gf(x)}{x(1-x)} &= \frac{1}{x}\int_0^x tf(t)\,dt + \frac{1}{(1-x)}\int_x^1 (1-t)f(t)\,dt \\ &\geq \min_{0\leq x\leq 1}\left[\frac{1}{x}\int_0^x tf(t)\,dt + \frac{1}{(1-x)}\int_x^1 (1-t)f(t)\,dt\right] \\ &= \alpha. \end{aligned} \tag{1.8}$$

It follows that

$$Gf(x) \geq \alpha\ x(1-x). \tag{1.9}$$

Clearly $\alpha \geq 0$ since f is positive almost everywhere and not identically zero; in fact, it must be positive. For, suppose that $\alpha = 0$, in which case the lefthand side of (1.8) would vanish for some value of x, call this value x_0. If $0 < x_0 < 1$ this would imply that $Gf(x_0) = 0$. However, f is not identically zero and the integrands in (1.7) are positive; consequently, this cannot occur. On the other hand, if $x_0 = 0$ then

$$\lim_{x\to 0} \frac{Gf(x)}{x(1-x)} = \int_0^1 (1-t)f(t)\,dt,$$

which is not zero unless f is zero. The case of $x_0 = 1$ is treated in a similar fashion. We can take

$$\beta = \max_{0\leq x\leq 1} \left[\frac{1}{x}\int_0^x tf(t)\,dt + \frac{1}{(1-x)}\int_x^1 (1-t)f(t)\,dt\right]. \tag{1.10}$$

Finally we quote from [**4**] the following results. These will be found in the summary on pages 329–330.

Theorem 1.6 (Krasnosel'skiĭ). *Let K be a reproducing cone and T a u_0-positive linear operator. Then*

(a) *T has a unique eigenfunction which is in K,*

(b) *the corresponding eigenvalue, λ_0, is simple,*

(c) *if λ is any other eigenvalue, then*

$$|\lambda| < \lambda_0.$$

2. Equivalent formulation. The problem of finding the norm of A, defined by (1.5), is equivalent to that of finding the square root of the norm of A^*A, where A^* is the adjoint of A, given by

$$A^*f(x) = \int_x^1 a(t-x)f(t)\,dt. \tag{2.1}$$

We have to estimate the largest eigenvalue of A^*A.

Theorem 2.1. *Let the operator $B : L^2[0,1] \to L^2[0,1]$ be defined by*

$$Bf(x) = \int_0^{1-x} a(1-x-t)f(t)\,dt; \tag{2.2}$$

then

$$A^*A = B^2. \tag{2.3}$$

Proof. Let $f \in L^2[0,1]$; then

$$A^*Af(x) = \int_x^1 a(t-x)\int_0^t a(t-s)f(s)\,ds\,dt,$$

replace $t \mapsto 1-t$ to give

$$\begin{aligned} A^*Af(x) &= \int_0^{1-x} a(1-x-t)\int_0^{1-t} a(1-t-s)f(s)\,ds\,dt \\ &= B^2f(x). \qquad \square \end{aligned}$$

We note in passing the more usual Fredholm form of the operators:

$$\begin{aligned} A^*Af(x) = B^2f(x) = &\int_0^x f(s)\int_x^1 a(t-x)a(t-s)\,dt\,ds \\ &+ \int_x^1 f(s)\int_s^1 a(t-x)a(t-s)\,dt\,ds. \end{aligned} \tag{2.4}$$

Thus the problem of finding the spectral radius of A^*A can be replaced by that of finding the spectral radius of B^2. Denote it by λ_0^2 and the corresponding eigenfunction by ϕ_0. We shall show that ϕ_0 is of constant sign. This will enable us to estimate bounds for λ_0.

3. The u_0-positivity of B. We now show that the operator B defined by (2.2) is u_0-positive, with $p = 2$.

Lemma 3.1. *Let* $g(0) \neq 0$, $g(1) = 0$, $g'(t) \leq 0$, $a(t) \geq 0$, $a'(t) \geq 0$, $0 \leq t \leq 1$. *Then*

$$\max_{0\leq x_0\leq 1} g(x_0)\frac{\int_0^{x_0} a(t)\,dt}{\int_0^1 a(t)\,dt} \leq \frac{\int_0^x a(x-t)g(t)\,dt}{\int_0^x a(t)\,dt} \leq g(0). \tag{3.1}$$

Proof. We note first that g decreases to zero.

Let x_0 satisfy $0 \leq x_0 \leq 1$; then

(a) $0 \leq x \leq x_0$, since g is a decreasing function, $g(t) \geq g(x_0)$, for $0 \leq t \leq x_0$, and so

$$\begin{aligned}\frac{\int_0^x a(x-t)g(t)\,dt}{\int_0^x a(t)\,dt} &\geq g(x_0)\frac{\int_0^x a(x-t)\,dt}{\int_0^x a(t)\,dt}\\ &= g(x_0).\end{aligned} \tag{3.2}$$

(b) $x_0 \leq x \leq 1$, the integrand is positive and so

$$\begin{aligned}\int_0^x a(x-t)g(t)\,dt &\geq \int_0^{x_0} a(x-t)g(t)\,dt\\ &= \int_0^{x_0} [a(x-t) - a(x_0-t)]g(t)\,dt\\ &\quad + \int_0^{x_0} a(x_0-t)g(t)\,dt,\end{aligned}$$

which, since a' is nonnegative, gives

$$\int_0^x a(x-t)g(t)\,dt \geq \int_0^{x_0} a(x_0-t)g(t)\,dt.$$

It follows that

$$\int_0^x a(x-t)g(t)\,dt \geq \int_0^{x_0} a(x_0-t)g(t)\,dt \geq g(x_0)\int_0^{x_0} a(t)\,dt.$$

Now $\int_0^x a(t)\,dt \leq \int_0^1 a(t)\,dt$ and so

$$\frac{\int_0^x a(x-t)g(t)\,dt}{\int_0^x a(t)\,dt} \geq g(x_0)\frac{\int_0^{x_0} a(t)\,dt}{\int_0^1 a(t)\,dt}. \tag{3.3}$$

The combination of (3.2) and (3.3) gives, for any x_0 in $[0,1]$,

$$\begin{aligned}\frac{\int_0^x a(x-t)g(t)\,dt}{\int_0^x a(t)\,dt} &\geq \min\left[g(x_0), g(x_0)\frac{\int_0^{x_0} a(t)\,dt}{\int_0^1 a(t)\;dt}\right] \\ &= g(x_0)\frac{\int_0^{x_0} a(t)\,dt}{\int_0^1 a(t)\,dt}.\end{aligned} \tag{3.4}$$

Since the lefthand side is independent of x_0, we have

$$\frac{\int_0^x a(x-t)g(t)\,dt}{\int_0^x a(t)\,dt} \geq \max_{0\leq x_0\leq 1} g(x_0)\frac{\int_0^{x_0} a(t)\,dt}{\int_0^1 a(t)\,dt}. \tag{3.5}$$

The upper bound in (3.1) follows from the fact that $g(t) \leq g(0)$, $0 \leq t \leq 1$. $\square$

Theorem 3.2. *Let* $f(t) \geq 0$, $0 \leq t \leq 1$. *Then*

$$\alpha u_0 \leq B^2 f \leq \beta u_0, \tag{3.6}$$

where

$$u_0(x) = \int_0^{1-x} a(t)\,dt \tag{3.7}$$

$$\alpha = \max_{0\leq x_0\leq 1}\left[\frac{\int_0^{x_0} a(t)\,dt}{\int_0^1 a(t)\,dt}\int_0^{1-x_0} a(1-x_0-u)f(u)\,du\right] \tag{3.8}$$

$$\beta = \int_0^1 a(1-u)f(u)\,du. \tag{3.9}$$

Proof. In (3.1) replace $x \mapsto 1-x$ and set

$$g(t) = \int_0^{1-t} a(1-t-u)f(u)\,du. \tag{3.10}$$

This function will satisfy the conditions of Lemma 3.1, and the result follows. $\square$

Hence, B is u_0-positive, and so by Theorem 1.6 the eigenvalue which gives the spectral radius of B is positive and the corresponding eigenfunction is nonnegative.

4. Mean value theorem. The proof of the next theorem is a generalization of one given by Collatz [**2**] for a finite dimensional operator, see also [**1**] and [**3**].

Theorem 4.1. *For any positive function* $f \in C[0,1]$ *the eigenvalue* λ_0 *which corresponds to an eigenfunction of constant sign satisfies*

$$(4.1) \qquad \inf_{0<\tau<1} \left\{ \frac{\int_0^{1-\tau} a(1-\tau-x)u_0(x)f(x)\,dx}{u_0(\tau)f(\tau)} \right\} \le \lambda_0 \le \sup_{0<\tau<1} \left\{ \frac{\int_0^{1-\tau} a(1-\tau-x)u_0(x)f(x)\,dx}{u_0(\tau)f(\tau)} \right\},$$

where

$$(4.2) \qquad u_0(x) = \int_0^{1-x} a(t)\,dt, \quad 0<x<1.$$

Proof. Let ϕ_0 be the eigenvector which corresponds to λ_0, and as we have seen, $\phi_0 \in K$. Multiply (2.2) by $u_0 f$ and integrate to give

$$\lambda_0 \int_0^1 u_0(x)f(x)\phi_0(x)\,dx = \int_0^1 u_0(x)f(x)\int_0^{1-x} a(1-x-t)\phi_0(t)\,dt\,dx.$$

Interchange the order of integration, then

$$\lambda_0 \int_0^1 u_0(x)f(x)\phi_0(x)\,dx = \int_0^1 \phi_0(t)\int_0^{1-t} a(1-x-t)u_0(x)f(x)\,dx\,dt.$$

Hence

$$\lambda_0 \int_0^1 u_0(x) f(x)\phi_0(x)\,dx$$
$$= \int_0^1 u_0(t)\phi_0(t) f(t)\left[\frac{1}{u_0(t)f(t)}\int_0^{1-t} a(1-t-x)u_0(x)f(x)\,dx\right]dt.$$

The expression inside the square brackets is nonnegative, and so we can use the integral mean value theorem to give

(4.3)
$$\lambda_0 = \frac{\int_0^{1-\tau} a(1-\tau-x)u_0(x)f(x)\,dx}{u_0(\tau)f(\tau)}, \quad \text{for some } \tau, \quad 0<\tau<1,$$

from which the desired result follows. □

5. The Volterra operator. We now apply the results of the previous section to the problem of finding upper and lower bounds for $\|m!V^m\|$ where

$$(m-1)!V^m f(x) = \int_0^x (x-t)^{m-1} f(t)\,dt. \tag{5.1}$$

Theorem 5.1. *Let λ_0 be the largest eigenvalue of $(m-1)!V^m$. Then*

$$\frac{1}{2m} < \lambda_0 < \frac{1}{m}. \tag{5.2}$$

Proof. In this case $a(x) = x^{m-1}$ and the corresponding operator is u_0-positive, with

$$u_0(x) = \frac{(1-x)^m}{m}.$$

Hence the eigenfunction which corresponds to λ_0 is of constant sign, and so (4.3) becomes

$$\begin{aligned}\lambda_0 &= \frac{\int_0^{1-\tau}(1-\tau-x)^{m-1}(1-x)^m\,dx}{(1-\tau)^m}\\ &= \int_0^1 (1-x)^{m-1}(1-x+\tau x)^m\,dx.\end{aligned} \tag{5.3}$$

It follows that

$$\int_0^1 (1-x)^{2m-1}\,dx < \lambda_0 < \int_0^1 (1-x)^{m-1}\,dx,$$

which gives (5.2). □

The next corollary follows from the definition.

Corollary 5.2.

$$\frac{1}{2} < \|m!V^m\| < 1. \tag{5.4}$$

The upper bound in (5.2) can be improved by the use of the next result.

Theorem 5.3. *Let*

$$Af(x) = \int_0^x a(x-t)f(t)\,dt, \tag{5.5}$$

where $f \in L^2[0,1]$, *then*

$$\lambda_0^2 = \|A\|^2 \le \int_0^1 \int_0^t a^2(x)\,dx\,dt. \tag{5.6}$$

Proof. As we have seen

$$A^*A = B^2,$$

where

$$Bf(x) = \int_0^{1-x} a(1-x-t)f(t)\,dt.$$

Hence,

$$
\begin{aligned}
|\lambda_0 f(x)|^2 &= |Bf(x)|^2 \\
&\le \int_0^{1-x} a^2(1-x-t)\,dt \int_0^{1-x} f^2(t)\,dt \\
&\le \int_0^{1-x} a^2(1-x-t)\,dt \int_0^{1} f^2(t)\,dt.
\end{aligned} \tag{5.7}
$$

Integrate this from 0 to 1 to give

$$
\begin{aligned}
\lambda_0^2 &\le \int_0^1 \int_0^{1-x} a^2(1-x-t)\,dt\,dx \\
&= \int_0^1 \int_x^1 a^2(t-x)\,dt\,dx \\
&= \int_0^1 \int_0^t a^2(x)\,dx\,dt. \qquad \square
\end{aligned}
$$

In the present case

$$
Af(x) = (m-1)!V^m f(x) = \int_0^x (x-t)^{m-1} f(t)\,dt, \tag{5.8}
$$

and an easy calculation gives

$$
\|m!V^m\| \le \left[\frac{m^2}{2m(2m-1)}\right]^{1/2} = \frac{1}{2}\left(1 - \frac{1}{2m}\right)^{-1/2}. \tag{5.9}
$$

This together with the lower bound in Corollary 5.2 gives the result stated in the introduction.

Theorem 5.4.

$$
\lim_{m\to\infty} \|m!V^m\| = \frac{1}{2}. \tag{5.10}
$$

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