Z. Physik B 27, 85–94 (1977)

Zeitschrift
für Physik B

Operator Orderings and Functional Formulations of Quantum and Stochastic Dynamics

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Received December 28, 1976

We study the connection between operator ordering schemes and the c-number formulations of quantum mechanics, which are based on generating functionals and functional integrals. We show by explicit construction that different operator ordering schemes are related to different functional and functional integral formulations of quantum mechanics. The results of these considerations are applied to classical non-linear stochastic dynamics by using the formal analogy between the Fokker-Planck equation and the Schrödinger equation.

1. Introduction

The c-number formulations of quantum mechanics based on generating functionals and functional integrals often provide convenient compact expressions suitable for formal manipulations, e.g. for the derivation of equations of motion and the generation of perturbation expansions (see e.g. [1–6]).
It is known [7–14] that there exist different functional integral representations of the quantum mechanical propagator; this non-uniqueness reflects the underlying operator ordering problem. In this paper we show by explicit construction that not only the propagator, but the whole apparatus of quantum mechanics may be recast in terms of different functional integral formulations, each of which is related to a particular operator ordering scheme.
In addition, we derive different formulations of quantum mechanics in terms of generating functionals by using the fact that we are free to define the time ordering operation for equal times with the help of an arbitrary operator ordering scheme.
Using these results and the formal analogy between the Fokker-Planck equation and the Schrödinger equation, we establish a class of functional and functional integral formulations of general diffusion processes which have attracted increasing attention recently [15–24]. As a byproduct we get functional integral formulations of the Martin-Rose-Siggia formalism [25] generalized to the case of arbitrary diffusion and drift coefficients.

2. Operator Ordering Schemes and Time Ordering

In this section we establish our notations and discuss various definitions of time ordering for equal times in terms of well known operator ordering schemes [26–29].
We consider a non-relativistic quantum mechanical system consisting of n degrees of freedom with position operators $Q:=(Q_1, \ldots, Q_n)$ and momentum operators $P:=(P_1, \ldots, P_n)$, which obey the canonical commutation relations ($\hbar=1$)

$$[Q_k, P_{k'}]=i\,\delta_{kk'}. \tag{2.1}$$

The dynamics of the system may be derived from the Schrödinger equation

$$\frac{d}{dt}|\psi_t\rangle=-iH|\psi_t\rangle, \tag{2.2}$$

where for simplicity the Hamiltonian $H=H(Q,P)$ is assumed to have no explicit time dependence. For a

Schrödinger operator $A=A(Q,P)$ the corresponding Heisenberg operator

$$A(t):=U_{tt_a}^{-1}AU_{tt_a} \tag{2.3}$$

obeys the Heisenberg equation

$$\frac{dA}{dt}=-i[A(t),H(t)], \tag{2.4}$$

where

$$U_{tt'}:=e^{-i(t-t')H} \tag{2.5}$$

is the time-evolution operator. In the Heisenberg picture the state of the system is described by a time independent state operator W. The expectation value $\langle A\rangle$ of an operator A in the state W is given by

$$\langle A\rangle=\mathrm{Sp}\,WA. \tag{2.6}$$

For notational convenience we combine the Q- and P-operators into a $2n$-component vector operator $\Phi:=(\Phi_+,\Phi_-)=(Q,P)$ with components

$$\Phi_{\alpha k}:=\begin{cases}Q_k & \text{for } \alpha=+\\ P_k & \alpha=-.\end{cases} \tag{2.7}$$

The equal time commutation relations now read

$$[\Phi_1(t),\Phi_2(t)]=-\sigma_{12}^y, \tag{2.8}$$

where the symbol 1 stands for (α_1,k_1) and the matrices σ^x, σ^y and σ^z are related to the Pauli-matrices τ^x, τ^y and τ^z by

$$\sigma_{12}^{x,y,z}:=\delta_{k_1k_2}\tau_{\alpha_1\alpha_2}^{x,y,z}. \tag{2.9}$$

According to (2.4) and (2.8), the equation of motion for $\Phi(t)$ takes the form

$$i\sigma_{12}^y\frac{\partial}{\partial t}\Phi_2(t)+F_1(\Phi(t))=0, \tag{2.10}$$

with

$$F_1(\Phi):=-\sigma_{12}^y[\Phi_2,H(\Phi)]=\frac{\partial H}{\partial\Phi_1}. \tag{2.11}$$

Here and hereafter repeated indices are to be summed over.

At this point we observe that all information on the dynamics of the system is contained in the expectation values of the time ordered products

$$T(\Phi_1(t_1)\dots\Phi_m(t_m)):=\Phi_{\pi(m)}(t_{\pi(m)})\dots\Phi_{\pi(1)}(t_{\pi(1)}). \tag{2.12}$$

Here π represents the permutation such that $t_{\pi(m)}>\dots>t_{\pi(1)}$. Definition (2.12) gives a definite meaning to the T-ordering operation, only in the event when all times involved are different. However, it is convenient to extend the T-ordering to the case when two or more times coincide.

In quantum field theory and in many body theory, the Hamiltonian and all other physical quantities are given in forms which are definitely ordered, e.g. normally or symmetrically ordered. This is the reason why this type of operator ordering is used to define the T-ordering for equal times. Although it is reasonable to proceed in this way, we want to stress that we are free to define the T-ordering for equal times in an arbitrary way. We will find that this freedom of choice is intimately connected with the possibility of writing the theory in terms of different functional formulations. Let us hence study the consequences when we define the T-ordering for equal times as a definitely, but arbitrarily, ordered product.

For simplicity we restrict ourselves to a special class of ordering procedures parametrized by a real ordering parameter u. We define the u-ordered product $\{A(\Phi)\}_u$ of an operator $A(\Phi)$ by $\{\Phi_1\}_u:=\Phi_1$ and by postulating a Wick-theorem with the u-contraction

$$\underbrace{\Phi_1\Phi_2}_{u}:=\Phi_1\Phi_2-\{\Phi_1\Phi_2\}_u$$
$$:=(\tfrac{1}{2}-u)\,i\sigma_{12}^x-\tfrac{1}{2}\sigma_{12}^y. \tag{2.13}$$

When we choose $u=0,\frac{1}{2},1$, we get respectively the well known anti-standard (PQ), Weyl-Wigner and standard (QP) ordering schemes. Every ordering procedure defines a phase space description [26–29]. Hence, via the u-ordering, there corresponds to an operator $A(\Phi)$ in a one-to-one manner, a u-function $a_u(\varphi)$ of the phase space variables $\varphi:=(\varphi_+,\varphi_-)=(q,p)$ which is determined by

$$A(\Phi)=:\{a_u(\Phi)\}_u. \tag{2.14}$$

The functions a_u and $a_{u'}$ are related by

$$a_{u'}=\mathscr{L}_{u'u}a_u. \tag{2.15}$$

Here $\mathscr{L}_{u'u}$ is a linear operator acting on phase space functions $a(\varphi)$ according to

$$\mathscr{L}_{u'u}a(\varphi):=\exp\left(\frac{1}{2}i(u-u')\,\sigma_{12}^x\frac{\partial^2}{\partial\varphi_1\partial\varphi_2}\right)a(\varphi). \tag{2.16}$$

Equivalent to the definition (2.14) is the relation

$$\langle\varphi_+|A(\Phi)|\varphi'_+\rangle=\int\frac{d^n\varphi_-}{(2\pi)^n}\,e^{i\varphi_-(\varphi_++\varphi'_+)}\,a_u(\varphi_u). \tag{2.17}$$

Here $|\varphi_+\rangle=|q\rangle$ is the eigenvector of the position operator $\Phi_+=Q$, $\varphi_+=q$ is the corresponding eigen-

value, and φ_u stands for

$$\varphi_u := (\varphi'_+ + u(\varphi_+ - \varphi'_+), \varphi_-). \tag{2.18}$$

Let us define T_u as the time ordering operation which extends the definition (2.12) to the case of equal times through

$$T_u A(\Phi(t)) := \{A(\Phi(t)\}_u. \tag{2.19}$$

The expression on the r.h.s. should be understood according to the following prescription:

$$\{A(\Phi(t))\}_u = U_{tt_a}^{-1} \{A(\Phi)\}_u U_{tt_a}. \tag{2.20}$$

From Equations (2.14) and (2.19) we obtain

$$A(\Phi(t)) = T_u a_u(\Phi(t)). \tag{2.21}$$

This expression may be written as a limit of a time ordered product taken at different times, since for an arbitrary function $a(\varphi)$ we have

$$\begin{aligned} T_u a(\Phi(t)) &= T_u \mathscr{L}_{0u} a(\Phi_+(t), \Phi_-(t+0)) \\ &= T_u \mathscr{L}_{1u} a(\Phi_+(t+0), \Phi_-(t)). \end{aligned} \tag{2.22}$$

3. Functional Formulation of Quantum Dynamics

In this section we review the functional formulation of quantum mechanics (e.g. [1-4]). We will find that this formulation takes different forms for different definitions of the time ordering for equal times.

The quantity which plays the central role in this formulation is the functional

$$Z_u[\eta] := \langle T_u Y_{t_b t_a}[\eta] \rangle_{ba}. \tag{3.1}$$

Here t_a, t_b are two given times such that $t_b > t_a$, the T_u-ordered product of the operator

$$Y_{t_b t_a}[\eta] := \exp\left(i \int_{t_a}^{t_b} dt\, \eta_3(t)\, \Phi_3(t)\right) \tag{3.2}$$

is the "S-matrix" corresponding to the source-field $\eta(t)$, and the bracket $\langle\ \rangle_{ba}$ is defined as

$$\langle \cdot \rangle_{ba} := \langle \varphi_+^{(b)} |\, U_{t_b t_a}(\cdot)\, | \varphi_+^{(a)} \rangle. \tag{3.3}$$

By functional differentiation of Z_u we have

$$\frac{\delta^m Z_u[\eta]}{i^m \delta\eta_1(t_1) \ldots \delta\eta_m(t_m)} = \langle T_u(\Phi_1(t_1) \ldots \Phi_m(t_m)\, Y_{t_b t_a}[\eta]) \rangle_{ba}. \tag{3.4}$$

From this expression we obtain the m-point functions by setting $\eta = 0$

$$\left.\frac{\delta^m Z_u[\eta]}{i^m \delta\eta_1(t_1) \ldots \delta\eta_m(t_m)}\right|_{\eta=0} = \langle T_u(\Phi_1(t_1) \ldots \Phi_m(t_m)) \rangle_{ba}. \tag{3.5}$$

Such functions characterize the dynamics of the system in the time interval (t_a, t_b). By multiplying them with the density matrix $\langle \varphi_+^{(a)} |\, W U_{t_b t_a}^{-1}\, | \varphi_+^{(b)} \rangle$ and then integrating over $\varphi_+^{(a)}$ and $\varphi_+^{(b)}$, we get the correlation functions of the system in the state W.

As functions of the times $t_1, \ldots, t_m$, the functional derivatives (3.4) and (3.5) of Z_u are discontinuous at those points where at least two times t_i and t_j corresponding to $\delta/\delta\eta_+(t_i)$ and $\delta/\delta\eta_-(t_j)$ coincide; this is due to the fact that the operators Φ_+ and Φ_- do not commute. Nevertheless, via the T_u-ordering, the equal time second and higher order functional derivatives are fixed at these points of discontinuity in a u-dependent way. This may also be considered as a consequence of the fact that the functional $Z_u[\eta]$ itself takes a u-dependent value when the source-field $\eta(t)$ contains δ-functions.

For an arbitrary function $a(\varphi)$ we have according to Equation (3.4)

$$a\left(\frac{\delta}{i\delta\eta(t)}\right) Z_u[\eta] = \langle T_u(a(\Phi(t))\, Y_{t_b t_a}[\eta]) \rangle_{ba}. \tag{3.6}$$

According to relation (2.22) the functional derivatives of Z_u at the points of discontinuity may be written as u-dependent limits

$$\begin{aligned} a\left(\frac{\delta}{i\delta\eta(t)}\right) Z_u &= \mathscr{L}_{0u}\, a\left(\frac{\delta}{i\delta\eta_+(t)}, \frac{\delta}{i\delta\eta_-(t+0)}\right) Z_u \\ &= \mathscr{L}_{1u}\, a\left(\frac{\delta}{i\delta\eta_+(t+0)}, \frac{\delta}{i\delta\eta_-(t)}\right) Z_u. \end{aligned} \tag{3.7}$$

Each of these two relations is equivalent to (3.6).

Specializing (3.6) to $\eta = 0$ and using definition (2.19), we find that a given function $a(\varphi)$ is related to the operator $\{a(\Phi)\}_u$ by

$$a\left(\frac{\delta}{i\delta\eta(t)}\right) Z_u|_{\eta=0} = \langle \{a(\Phi(t))\}_u \rangle_{ba}, \tag{3.8}$$

and conversely according to (2.21), that a given operator $A(\Phi)$ is related to the function $a_u(\varphi)$ by

$$a_u\left(\frac{\delta}{i\delta\eta(t)}\right) Z_u|_{\eta=0} = \langle A(\Phi(t)) \rangle_{ba}. \tag{3.9}$$

From the relations (3.5) and (3.9) and from the definition (3.3) of the bracket $\langle\ \rangle_{ba}$, we see that the operator formalism of quantum mechanics may be expressed in terms of Z_u and its functional derivatives. This functional formulation of quantum mechanics becomes self-contained when we add an equation of motion for Z_u.

From (2.8) and (2.10) we have

$$\sigma_{12}^y \frac{\partial}{\partial t} \frac{\delta Z_u}{\delta\eta_2(t)} + \langle T_u(Y_{t_b t})\, F_1(\Phi(t))\, T_u(Y_{t t_a}) \rangle_{ba} = \eta_1(t)\, Z_u. \tag{3.10}$$

Using (2.21) applied to the operator F_1, we get, according to (3.6), a closed equation for Z_u, namely

$$\sigma_{12}^{y}\frac{\partial}{\partial t}\frac{\delta Z_u}{\delta\eta_2(t)}+f_{1u}\left(\frac{\delta}{i\,\delta\eta(t)}\right)Z_u=\eta_1(t)\,Z_u. \tag{3.11}$$

Here the u-function f_{1u}, which corresponds to F_1, is related to the u-function h_u, which corresponds to the Hamiltonian H, by

$$f_{1u}(\varphi)=\frac{\partial h_u(\varphi)}{\partial\varphi_1}. \tag{3.12}$$

We get the functional Z_u, defined by (3.1), as a solution of (3.11) by imposing the following boundary conditions (see e.g. [4]):

$$Z_u[0]=\langle 1\rangle_{ba}, \tag{3.13}$$

$$\frac{\delta Z_u}{i\,\delta\eta_+(t_a+0)}=\varphi_+^{(a)}\,Z_u, \tag{3.14}$$

$$\frac{\delta Z_u}{i\,\delta\eta_+(t_b-0)}=\varphi_+^{(b)}\,Z_u. \tag{3.15}$$

We remark that for Hamiltonians of the simple type $H=(P_k P_k/2m)+v(Q)$, the associated u-function $h_u=(p_k p_k/2m)+v(q)$ and hence f_{1u} are independent of u. Nevertheless, even in this case an arbitrary operator A corresponds, according to (3.9), to a u-dependent functional derivative of Z_u. The u-dependence of the functional formulation of quantum dynamics appears in perturbation theory also. Let $Z_u^{(0)}$ and Z_u represent the generating functionals of the systems characterized respectively by the Hamiltonians $H^{(0)}$ and $H=H^{(0)}+H^{(1)}$. According to the definition (3.1) and (3.6) Z_u is related to $Z_u^{(0)}$ via

$$Z_u[\eta]=\exp\left(-i\int_{t_a}^{t_b}dt\,h_u^{(1)}\left(\frac{\delta}{i\,\delta\eta(t)}\right)\right)Z_u^{(0)}[\eta]. \tag{3.16}$$

Here $h_u^{(1)}(\varphi)$ is the u-function associated with the operator $H^{(1)}(\Phi)$.

In summary we see that the above equations provide different functional formulations of quantum dynamics, depending on the choice of the underlying ordering scheme. Hence, since there is no preference for a particular ordering scheme, there is no preference for a particular functional formulation.

Conversely, we may consider the above equations as a method of quantization. For a classical system with the Hamiltonian function $h(\varphi)$, we can write down a corresponding quantum mechanical equation of motion which is (3.11) with $f_{1u}(\varphi)$ replaced by $f_1(\varphi):=\partial h/\partial\varphi_1$. Since the second and higher order functional derivatives of the functional Z, which is a solution of this functional differential equation and which obeys the boundary conditions (3.13)–(3.15), are necessarily discontinuous as functions of the occurring times, we have to add a rule for the interpretation of the equal time derivatives of Z. This rule plays the role of a prescription which associates in a definite way a quantum mechanical Hilbert space operator to a classical phase space function $a(\varphi)$. In particular, when we define such a rule by (3.7), denoting this by writing Z_u instead of Z, we see from (3.8) that the associated operator is given as the u-ordered product $\{a(\Phi)\}_u$.

It is the freedom of choice inherent in defining the equal time derivatives that is the functional formulation's way of reflecting the operator ordering problem.

4. Functional Integral Formulation of Quantum Dynamics

In this section we write the generating functional $Z_u[\eta]$ as a functional integral, thereby getting a functional integral formulation of quantum dynamics. As is well known [4–6], such a formulation provides conveniently compact expressions suitable for formal manipulations, e.g. for the generation of perturbation expansions.

In order to guarantee that this c-number method yields correct results we have to define functional integration in a u-dependent way, which reflects the underlying operator ordering problem. We believe that these considerations are a key for a better understanding of the so called ambiguities occurring in functional integral formulations which have been widely discussed in the literature [7–14].

The power of functional integration techniques is fundamentally due to the fact that it is possible to carry out formal manipulations according to a few simple rules without being forced, at each step, to go back to the explicit definition of the functional integration. Specifically, let us assume that functional integration is a linear operation, that it can be interchanged with functional differentiation with respect to the sourcefield, and that it obeys the functional integration by parts lemma (see e.g. [5]). We will show that, in agreement with these rules, the functional $Z_u[\eta]$ defined in (3.1) may be written as

$$Z_u[\eta]=\int_u\delta\varphi\;e^{iS_u[\varphi,\eta]}, \tag{4.1}$$

with the "action" functional

$$S_u[\varphi,\eta]:=\int_{t_a}^{t_b}dt\,L_u(\varphi(t),\dot\varphi(t),\eta(t), \tag{4.2}$$

and the "Lagrangian"

$$L_u(\varphi,\dot\varphi,\eta):=\varphi_{-k}\dot\varphi_{+k}-h_u(\varphi)+\eta_3\varphi_3. \tag{4.3}$$

In (4.1) the u-integral $\int_u \delta\varphi$ means functional integration over all phase space paths $\varphi(t)=(\varphi_+(t), \varphi_-(t))$, subject to $\varphi_+(t_a)=\varphi_+^{(a)}$ and $\varphi_+(t_b)=\varphi_+^{(b)}$. There is no restriction on $\varphi_-(t)$.

According to the rules assumed above we get from (4.1)

$$\frac{\delta^m Z_u[\eta]}{i^m \delta\eta_1(t_1) \dots \delta\eta_m(t_m)} = \int_u \delta\varphi\, \varphi_1(t_1) \dots \varphi_m(t_m)\, e^{iS_u[\varphi,\eta]}, \tag{4.4}$$

and hence for an arbitrary equal time functional derivative of Z_u

$$a\left(\frac{\delta}{i\,\delta\eta(t)}\right) Z_u[\eta] = \int_u \delta\varphi\; a(\varphi(t))\, e^{iS_u[\varphi,\eta]}. \tag{4.5}$$

In order to be consistent with the relation (3.7) for the equal time functional derivatives of Z_u, we must have

$$\begin{aligned}&\int_u \delta\varphi\; a(\varphi(t))\, e^{iS_u[\varphi,\eta]}\\ &= \int_u \delta\varphi\; \mathscr{L}_{0u}\, a(\varphi_+(t), \varphi_-(t+0))\, e^{iS_u[\varphi,\eta]}\\ &= \int_u \delta\varphi\; \mathscr{L}_{1u}\, a(\varphi_+(t+0), \varphi_-(t))\, e^{iS_u[\varphi,\eta]}.\end{aligned} \tag{4.6}$$

These conditions of consistency show that functional integration has to be defined in a u-dependent way. For that reason we have attached the index u to the symbol $\int_u \delta\varphi$, which we call the u-integration. We remark that for Hamiltonians of the simple type $H=(P_k P_k/2m)+v(Q)$, the associated functional S_u is independent of u. Nevertheless, even in this case the conditions of consistency (4.6) require a u-dependent definition of functional integration.

Equation (4.1) may now be formally verified by showing that the functional integral on the r.h.s. satisfies the equation of motion (3.11) and the boundary conditions (3.13)–(3.15). Except for the boundary condition (3.13) this can easily be done by utilizing the rules assumed above. For example, the equation of motion (3.11) follows from (4.5) and the following integration by parts lemma:

$$\begin{aligned}0 &= \int_u \delta\varphi\, \frac{\delta}{\delta\varphi_1(t)}\, e^{iS_u[\varphi,\eta]}\\ &= i\int_u \delta\varphi\; e^{iS_u[\varphi,\eta]} \frac{\delta S_u}{\delta\varphi_1(t)},\end{aligned} \tag{4.7}$$

with

$$\frac{\delta S_u}{\delta\varphi_1(t)} = -i\,\sigma_{12}^y\,\dot{\varphi}_2(t) - f_{1u}(\varphi(t)) + \eta_1(t). \tag{4.8}$$

The boundary condition (3.13) can be shown only when an explicit definition of the u-integration is available. We leave this problem to the next section.

As another example illustrating the usefulness of the functional integration technique, we mention that the perturbative expansion (3.16) follows immediately from (4.1).

The functional integral formulation of quantum mechanics is contained in the relations

$$\begin{aligned}&\int_u \delta\varphi\; \varphi_1(t_1) \dots \varphi_m(t_m)\, e^{iS_u[\varphi,0]}\\ &= \langle T_u(\Phi_1(t_1) \dots \Phi_m(t_m))\rangle_{ba}\end{aligned} \tag{4.9}$$

and

$$\int_u \delta\varphi\; a_u(\varphi(t))\, e^{iS_u[\varphi,0]} = \langle A(\Phi(t))\rangle_{ba}. \tag{4.10}$$

These equations follow when we insert (4.1) into (3.5) and (3.9). In particular, the propagator is given by

$$\int_u \delta\varphi\; e^{iS_u[\varphi,0]} = \langle 1\rangle_{ba} = \langle \varphi_+^{(b)} |\, e^{-i(t_b-t_a)H}\, |\varphi_+^{(a)}\rangle. \tag{4.11}$$

The functional integration technique provides another method of quantization. To a classical system with the Hamiltonian function $h(\varphi)$, we associate a functional $S[\varphi,\eta]$ which is given by (4.2) and (4.3) when h_u is replaced by h. A corresponding quantum theory is then contained in the functional $Z_u[\eta]$ defined by (4.1) with S_u replaced by S. In particular, to a classical phase space function $a(\varphi)$ there then corresponds the quantum mechanical Hilbert space operator $\{a(\Phi)\}_u$ via

$$\int_u \delta\varphi\; a(\varphi(t))\, e^{iS[\varphi,0]} = \langle \{a(\Phi(t))\}_u\rangle_{ba}. \tag{4.12}$$

The choice of the parameter u in relation (4.6), which characterizes the u-integration, thus selects from the possible operator ordering schemes the u-ordering scheme.

5. Functional Integrals as Limits of Lattice Expressions

In the last sections we have demonstrated the usefulness of the functional integration technique without giving an explicit definition of functional integrals and without justifying the rules for manipulating them. In this section we fill this gap by defining the u-integration as a limit of a u-dependent lattice expression. In what follows we assume implicitly that all limits exist. We will not discuss possible relations between this lattice approach to functional integration and the approach based on generalized measure theory (see e.g. the articles and the references given in [30, 31]).

In order to find the lattice definition of the u-integration, we start with a lattice approximation

$$Z_u^{(N)}(\eta) := \langle T_u\, e^{i\sum_{\nu=1}^{N} \Delta_N \eta_3^{(\nu)} \Phi_3^{(\nu)}}\rangle_{ba} \tag{5.1}$$

of the generating functional $Z_u[\eta]$ defined by (3.1). $Z_u^{(N)}$ is a function of the variables $\eta^{(\nu)}:=\eta(t^{(\nu)})$ and depends on the operators $\Phi^{(\nu)}:=\Phi(t^{(\nu)})$. The times

$$t^{(\nu)}:=t_a+\nu\Delta_N, \qquad \nu=0,1,\ldots,N \tag{5.2}$$

split up the interval (t_a,t_b) into N subintervals $(t^{(\nu-1)},t^{(\nu)})$ each of length $\Delta_N:=(t_b-t_a)/N$.
For an arbitrary function $a(\varphi)$ we have according to (5.1)

$$a\left(\frac{\partial}{i\Delta_N\partial\eta^{(\mu)}}\right)Z_u^{(N)}(\eta)=\langle T_u(a(\Phi^{(\mu)})\,e^{i\sum_{\nu=1}^{N}\Delta_N\eta_3^{(\nu)}\Phi_3^{(\nu)}})\rangle_{ba}. \tag{5.3}$$

Carrying out the T_u-ordering and using (2.20) and (3.3), we can write the r.h.s. of (5.3) as

$$\begin{aligned}&\langle\{e^{i\Delta_N\eta_3^{(N)}\Phi_3^{(N)}}\}_u\ldots\{a(\Phi^{(\mu)})\,e^{i\Delta_N\eta_3^{(\mu)}\Phi_3^{(\mu)}}\}_u\\&\ldots\{e^{i\Delta_N\eta_3^{(1)}\Phi_3^{(1)}}\}_u\rangle_{ba}\\&=\langle\varphi_+^{(b)}|\,\{e^{i\Delta_N\eta_3^{(N)}\Phi_3}\}_u\,e^{-i\Delta_N H}\\&\ldots\{a(\Phi)\,e^{i\Delta_N\eta_3^{(\mu)}\Phi_3}\}_u\,e^{-i\Delta_N H}\\&\ldots\{e^{i\Delta_N\eta_3^{(1)}\Phi_3}\}_u\,e^{-i\Delta_N H}\,|\varphi_+^{(a)}\rangle.\end{aligned} \tag{5.4}$$

By employing the completeness relation

$$1=\int d^n\varphi_+^{(\nu)}\,|\varphi_+^{(\nu)}\rangle\,\langle\varphi_+^{(\nu)}| \tag{5.5}$$

$N-1$ times, we get from (5.3) and (5.4)

$$\begin{aligned}a\left(\frac{\partial}{i\Delta_N\partial\eta^{(\mu)}}\right)Z_u^{(N)}(\eta)&=\int d^n\varphi_+^{(1)}\ldots d^n\varphi_+^{(N-1)}\\&\langle\varphi_+^{(\mu)}|\,\{a(\Phi)\,e^{i\Delta_N\eta_3^{(\mu)}\Phi_3}\}_u\,e^{-i\Delta_N H}\,|\varphi_+^{(\mu-1)}\rangle\\&\cdot\prod_{\substack{\nu=1\\ \nu\neq\mu}}^{N}\langle\varphi_+^{(\nu)}|\,\{e^{i\Delta_N\eta_3^{(\nu)}\Phi_3}\}_u\,e^{-i\Delta_N H}\,|\varphi_+^{(\nu-1)}\rangle,\end{aligned} \tag{5.6}$$

where

$$\varphi_+^{(0)}:=\varphi_+^{(a)} \quad\text{and}\quad \varphi_+^{(N)}:=\varphi_+^{(b)}. \tag{5.7}$$

Since we are interested in the limit $N\to\infty$, it is sufficient to retain all terms of order $1/N$ in each of the N factors occurring on the r.h.s. of (5.6). For an arbitrary function $b(\varphi)$ we have according to definition (2.14)

$$\begin{aligned}&\{b(\Phi)\,e^{i\Delta_N\eta_3^{(\nu)}\Phi_3}\}_u\,e^{-i\Delta_N H}\\&=\{b(\Phi)\,e^{i\Delta_N(\eta_3^{(\nu)}\Phi_3-h_u(\Phi))}\}_u+O\left(\frac{1}{N^2}\right).\end{aligned} \tag{5.8}$$

Switching over to matrix elements and using (2.17) in conjunction with the definition (4.3) of L_u, we get

$$\begin{aligned}&\langle\varphi_+^{(\nu)}|\,\{b(\Phi)\,e^{i\Delta_N\eta_3^{(\nu)}\Phi_3}\}_u\,e^{-i\Delta_N H}\,|\varphi_+^{(\nu-1)}\rangle\\&=\int\frac{d^n\varphi_-^{(\nu)}}{(2\pi)^n}\,b(\varphi_u^{(\nu)})\,e^{i\Delta_N L_u(\varphi_u^{(\nu)},\dot\varphi^{(\nu)},\eta^{(\nu)})}+O\left(\frac{1}{N^2}\right),\end{aligned} \tag{5.9}$$

with the abbreviations

$$\dot\varphi^{(\nu)}:=(\varphi^{(\nu)}-\varphi^{(\nu-1)})/\Delta_N, \qquad \varphi_u^{(\nu)}:=(\varphi_{u+}^{(\nu)},\varphi_{u-}^{(\nu)}), \tag{5.10}$$

and

$$\varphi_{u+}^{(\nu)}:=\varphi_+^{(\nu-1)}+u(\varphi_+^{(\nu)}-\varphi_+^{(\nu-1)}), \qquad \varphi_{u-}^{(\nu)}:=\varphi_-^{(\nu)}. \tag{5.11}$$

The idea behind relation (5.9) is a generalization of the idea used in [9, 11, 13, 14] to approximate the short time propagator.
Applying (5.9) to (5.6) with $b(\varphi)=1$ for $\nu\neq\mu$ and $b(\varphi)=a(\varphi)$ for $\nu=\mu$, we arrive at

$$\begin{aligned}&a\left(\frac{\partial}{i\Delta_N\partial\eta^{(\mu)}}\right)Z_u^{(N)}(\eta)\\&=\int d^{(N)}\varphi\;a(\varphi_u^{(\mu)})\,e^{iS_u^{(N)}(\varphi,\eta)}+O\left(\frac{1}{N}\right),\end{aligned} \tag{5.12}$$

where we have introduced the abbreviation

$$d^{(N)}\varphi:=\frac{d^n\varphi_-^{(N)}}{(2\pi)^n}\prod_{\nu=1}^{N-1}\frac{d^n\varphi_+^{(\nu)}d^n\varphi_-^{(\nu)}}{(2\pi)^n}, \tag{5.13}$$

and the function

$$S_u^{(N)}(\varphi,\eta):=\sum_{\nu=1}^{N}\Delta_N L_u(\varphi_u^{(\nu)},\dot\varphi^{(\nu)},\eta^{(\nu)}). \tag{5.14}$$

Proceeding along analogous lines we find for the multitime partial derivative of $Z_u^{(N)}$

$$\begin{aligned}&\frac{\partial^m Z_u^{(N)}(\eta)}{(i\Delta_N)^m\partial\eta_1^{(\nu_1)}\ldots\partial\eta_m^{(\nu_m)}}\\&=\langle T_u(\Phi_1^{(\nu_1)}\ldots\Phi_m^{(\nu_m)}\,e^{i\sum_{\nu=1}^{N}\Delta_N\eta_3^{(\nu)}\Phi_3^{(\nu)}})\rangle_{ba},\end{aligned} \tag{5.15}$$

the relation

$$\begin{aligned}&\frac{\partial^m Z_u^{(N)}(\eta)}{(i\Delta_N)^m\partial\eta_1^{(\nu_1)}\ldots\partial\eta_m^{(\nu_m)}}\\&=\int d^{(N)}\varphi\;\varphi_{u1}^{(\nu_1)}\ldots\varphi_{um}^{(\nu_m)}\,e^{iS_u^{(N)}(\varphi,\eta)}+O\left(\frac{1}{N}\right),\end{aligned} \tag{5.16}$$

which holds independently of whether some of the $\nu_1,\ldots,\nu_m$ coincide or not. Therefore (5.12) may be considered as a special case of (5.16). We now proceed to examine the limit $N\to\infty$. Consider a set of times $t_1,\ldots,t_m$ which fall into the interval (t_a,t_b) and which may or may not be distinct. For a given subdivision (5.2) of this interval we associate with $t_j(j=1,\ldots,m)$ the time $t^{(\nu_j)}$, where ν_j is determined by

$$t^{(\nu_j-1)}<t_j\leqq t^{(\nu_j)}. \tag{5.17}$$

Starting from

$$\begin{aligned}&\langle T_u(\Phi_1(t_1)\ldots\Phi_m(t_m)\,e^{i\int_{t_a}^{t_b}dt\,\eta_3(t)\Phi_3(t)})\rangle_{ba}\\&=\lim_{N\to\infty}\langle T_u(\Phi_1^{(\nu_1)}\ldots\Phi_m^{(\nu_m)}\,e^{i\sum_{\nu=1}^{N}\Delta_N\eta_3^{(\nu)}\Phi_3^{(\nu)}})\rangle_{ba},\end{aligned} \tag{5.18}$$

and using (3.4) and (5.15), we then have

$$\frac{\delta^m Z_u[\eta]}{(i)^m \delta\eta_1(t_1) \dots \delta\eta_m(t_m)} = \lim_{N\to\infty} \frac{\partial^m Z_u^{(N)}(\eta)}{(i\Delta_N)^m \partial\eta_1^{(\nu_1)} \dots \partial\eta_m^{(\nu_m)}}. \quad (5.19)$$

Equation (5.19) relates the functional derivatives of $Z_u[\eta]$ in a natural way to a lattice expression.

Guided by (5.16), we now define the functional u-integration by

$$\int_u \delta\varphi\, \varphi_1(t_1) \dots \varphi_m(t_m)\, e^{iS_u[\varphi,\eta]}$$
$$:= \lim_{N\to\infty} \int d^{(N)}\varphi\, \varphi_{u1}^{(\nu_1)} \dots \varphi_{um}^{(\nu_m)}\, e^{iS_u^{(N)}(\varphi,\eta)}. \quad (5.20)$$

From this definition it is obvious that the u-integration is a linear operation. We have thus for an arbitrary function $a(\varphi)$

$$\int_u \delta\varphi\, a(\varphi(t))\, e^{iS_u[\varphi,\eta]}$$
$$= \lim_{N\to\infty} \int d^{(N)}\varphi\, a(\varphi_u^{(\mu)})\, e^{iS_u^{(N)}(\varphi,\eta)}, \quad (5.21)$$

which is also suggested by (5.12). The u-integral $\int_u \delta\varphi\, G[\varphi]\, e^{iS_u[\varphi,\eta]}$ of an arbitrary functional $G[\varphi]$ may be defined with the help of its Volterra expansion and the definition (5.20).

Because $S_u^{(N)}$ appears, formally at least, as a Riemann approximation of S_u, the lattice definition of the u-integration may be considered as a mnemonicly convenient notation to describe the properties of the limits of lattice expressions by corresponding continuum expressions. The index u attached to the symbol $\int_u \delta\varphi$ reflects the fact that the lattice expression, which is used to define a continuum expression, results from the formal replacement of $\varphi(t)$ by $\varphi_u^{(\nu)}$ instead of $\varphi^{(\nu)}$. This procedure may be called the "discretisation according to the u-point rule". For $u=1/2$ the u-point rule reduces to the midpoint rule (see e.g. [11]).

Equation (5.7) shows that $\int_u \delta\varphi$ should be interpreted as integration over phase space paths $\varphi(t)=(q(t), p(t))$, subject to $\varphi_+(t_a)=\varphi_+^{(a)}=q^{(a)}$ and $\varphi_+(t_b)=\varphi_+^{(b)}=q^{(b)}$.

A direct consequence of the lattice definition is the translational invariance of the u-integration

$$\int_u \delta\varphi\, G[\varphi+\zeta]\, e^{iS_u[\varphi+\zeta,\eta]} = \int_u \delta\varphi\, G[\varphi]\, e^{iS_u[\varphi,\eta]}. \quad (5.22)$$

Here $\zeta(t)$ is an arbitrary phase space path subject to $\zeta_+(t_a)=\zeta_+(t_b)=0$. From relation (5.22) there follows immediately the functional integration by parts lemma

$$\int_u \delta\varphi\, \frac{\delta}{\delta\varphi_1(t)} (G[\varphi]\, e^{iS_u[\varphi,\eta]}) = 0, \quad (5.23)$$

which we have used in (4.7) with $G[\varphi]=1$.

Using the lattice definition (5.20) of the u-integration and combining (5.16) and (5.19), we get (4.4), and from it (4.1) and (4.5) as special cases. Hence, the generating functional $Z_u[\eta]$ and its derivatives may indeed be written in terms of u-integrals. By a comparison of (4.1) and (4.4), we justify that functional integration may be interchanged with functional differentiation with respect to the source field. From this rule and (3.7) the conditions of consistency (4.6) are direct consequences.

In summary, in this section we have given an explicit meaning to all the functional integral expressions and to all the rules for manipulating them which we have used in the last section. As a final remark we mention that the expressions of this and the foregoing sections take a more familiar, but often a less compact form when we rewrite them in terms of the operators $Q(t)=\Phi_+(t)$, $P(t)=\Phi_-(t)$ and the functions $\eta_+(t), \eta_-(t)$, $q(t)=\varphi_+(t)$ and $p(t)=\varphi_-(t)$. For instance the "action" $S_u[\varphi,\eta]$ then reads

$$S_u[q,p,\eta_+,\eta_-] = \int_{t_a}^{t_b} dt (p_k(t)\,\dot{q}_k(t) - h_u(q(t),p(t)) + \eta_{+k}(t)\,q_k(t) + \eta_{-k}(t)\,p_k(t)). \quad (5.24)$$

6. Application to Classical Stochastic Dynamics

Recently there has been an increasing interest in functional integral formulations of diffusion processes [15–21]. They may serve as concise formulations of non-equilibrium thermodynamics and of the Martin-Rose-Siggia-formalism [25] of classical stochastic dynamics. Further, they have been used already [22–24] to extend Wilson's theory [32] from static to dynamic critical phenomena.

Using the results of the last sections and the formal analogy between the Fokker-Planck equation and the Schrödinger equation, we derive in this section a class of functional (integral) formulations of stochastic dynamics.

We consider an n-dimensional diffusion process governed by the Fokker-Planck equation

$$\frac{\partial}{\partial t} w(q,t) = \frac{\partial^2}{\partial q_k \partial q_{k'}} (D_{kk'}(q)\, w(q,t)) - \frac{\partial}{\partial q_k} (K_k(q)\, w(q,t)), \quad (6.1)$$

with the diffusion coefficients $D_{kk'}(q)$ and the drift coefficients $K_k(q)$. Here $w(q,t)$ is the probability density to observe the vector $q:=(q_1,\dots,q_n)$ at time t.

The fundamental solution $g(q,t|q',t')$ of (6.1) represents the conditional probability density to observe the vector q at time t if at time $t'<t$ the vector q' has

been realized. The fundamental solution has the following properties:

$$g(q,t|q',t') \geqq 0 \tag{6.2}$$

$$\int d^n q\, g(q,t|q',t') = 1 \tag{6.3}$$

$$g(q,t|q',t) = \delta(q-q'), \tag{6.4}$$

In addition we have

$$\lim_{t' \to -\infty} g(q,t|q',t') = w_e(q) \tag{6.5}$$

when there exists a unique equilibrium state $w_e(q)$, which is approached in the course of time starting from any initial state.

If the system is at time t_a in the state $w(q,t_a)$, then the common probability density $w(q^{(1)},t_1;\ldots;q^{(m)},t_m)$ to observe the vectors $q^{(1)},\ldots,q^{(m)}$ respectively at times $t_1,\ldots,t_m > t_a$ is given by

$$\begin{aligned} w(q^{(1)},t_1;\ldots;q^{(m)},t_m) &= g(q^{(\pi(m))},t_{\pi(m)}|q^{(\pi(m-1))},t_{\pi(m-1)}) \\ &\ldots g(q^{(\pi(2))},t_{\pi(2)}|q^{(\pi(1))},t_{\pi(1)}) \\ &\cdot \int d^n q\, g(q^{(\pi(1))},t_{\pi(1)}|q,t_a)\, w(q,t_a), \end{aligned} \tag{6.6}$$

where π represents the permutation such that $t_{\pi(m)} \geqq \cdots \geqq t_{\pi(1)} > t_a$.

The Fokker-Planck equation (6.1) is of the form of the Schrödinger equation (2.1). Stochastic dynamics may therefore be formulated in the language of quantum mechanics. The usefulness of such a representation of stochastic dynamics has been demonstrated elsewhere [33, 34]. In the notation of section 2, Equation (6.1) reads

$$\frac{d}{dt}|w_t\rangle = -iH|w_t\rangle, \tag{6.7}$$

with $w(q,t) = \langle q|w_t\rangle$ and

$$H = H(\Phi) = -i\,\Phi_{-k}\,\Phi_{-k'}\,D_{kk'}(\Phi_+) + \Phi_{-k}\,K_k(\Phi_+). \tag{6.8}$$

When we identify the Hamiltonian of the foregoing sections with the non-Hermitian Hamiltonian (6.8), all the quantum mechanical expressions and relations transform into expressions and relations of stochastic dynamics. This is true because in the foregoing sections it was nowhere necessary to assume that the Hamiltonian is Hermitian.

For the state $|w_{t_a}\rangle$ we define the correlation functions for times later than t_a as

$$G_{u|1\ldots m}(t_1,\ldots,t_m|W_{t_a}) := \mathrm{Sp}\, W_{t_a}\, T_u(\Phi_1(t_1)\ldots\Phi_m(t_m)), \tag{6.9}$$

with the state operator

$$W_{t_a} := \int d^n q\, |w_{t_a}\rangle\langle q|. \tag{6.10}$$

For $\alpha_1 = \cdots = \alpha_m = +$ the correlation functions reduce to the moment functions of the diffusion process, namely

$$\begin{aligned} &G_{u|+k_1,\ldots,+k_m}(t_1,\ldots,t_m|W_{t_a}) \\ &= \int d^n q^{(1)}\ldots d^n q^{(m)}\, q_{k_1}^{(1)}\ldots q_{k_m}^{(m)}\, w(q^{(1)},t_1;\ldots;q^{(m)},t_m). \end{aligned} \tag{6.11}$$

The causality of the correlation functions is expressed by the fact that they vanish when the latest time belongs to a momentum operator Φ_-. This follows from their definition (6.9) along with (6.2) and (6.3).

The u-dependence of the correlation functions which results from the u-dependence of the time ordering T_u is very simple. Due to causality and the definition of the T_u-ordering, the 2-point correlation function, for instance, is of the form

$$\begin{aligned} &G_{u|12}(t_1,t_2|W_{t_a}) \\ &= \begin{pmatrix} g_{k_1k_2}(t_1,t_2) & \Theta_u(t_1-t_2)\, f_{k_1k_2}(t_1,t_2) \\ \Theta_u(t_2-t_1)\, f_{k_2k_1}(t_2,t_1) & 0 \end{pmatrix}. \end{aligned} \tag{6.12}$$

Here we have introduced the u-dependent step function

$$= \Theta_u(t) := \begin{cases} 0 & t<0 \\ u & \text{for } t=0. \\ 1 & t>0 \end{cases} \tag{6.13}$$

The extension of the time ordering to equal times, as given by definition (2.19), thus merely fixes the value of the step function at $t=0$. Correspondingly the u-dependence of the multi-point correlation functions is simply due to the u-dependence of the occurring step functions $\Theta_u(t)$. The generating functional for the correlation functions in the time interval (t_a,t_b) is given by

$$Z_u[\eta|W_{t_a}] := \mathrm{Sp}(W_{t_a}\, T_u\, e^{i\int_{t_a}^{t_b} dt\,\eta_3(t)\Phi_3(t)}). \tag{6.14}$$

Using

$$g(q,t|q',t') = \langle q|\, e^{-i(t-t')H}\,|q'\rangle \tag{6.15}$$

and (6.3), we may express the functional $Z_u[\eta|W_{t_a}]$ in terms of the state function $w(q,t_a)$ and the functional $Z_u[\eta]$, defined by (3.1), in the following way

$$Z_u[\eta|W_{t_a}] = \int d^n\varphi_+^{(a)}\, d^n\varphi_+^{(b)}\, Z_u[\eta]\, w(\varphi_+^{(a)},t_a). \tag{6.16}$$

Thus the functionals $Z_u[\eta|W_{t_a}]$ and $Z_u[\eta]$, the latter of which depends on the endpoints a and b, are linearly connected. Therefore, the methods developed in Sections 2–5 directly yield a u-dependent functional formulation and a u-dependent functional integral formulation of diffusion processes, when we identify $h_u(\varphi)$

with the u-function of the Hamiltonian (6.8), explicitly

$$h_u(\varphi) = -i\varphi_{-k}\varphi_{-k'}D_{kk'}(\varphi_+) + \varphi_{-k}\left(K_k(\varphi_+) - 2u\frac{\partial}{\partial\varphi_{+k'}}D_{kk'}(\varphi_+)\right) - iu\frac{\partial}{\partial\varphi_{+k}}K_k(\varphi_+) + iu^2\frac{\partial^2}{\partial\varphi_{+k}\partial\varphi_{+k'}}D_{kk'}(\varphi_+). \quad (6.17)$$

In principle there is no preference for a particular choice of the parameter u which characterizes the underlying operator ordering scheme. Nevertheless, here it is convenient to choose $u=0$, because h_u then takes a simple form which reflects the fact that the Hamiltonian (6.8) presents itself in 0-ordered form.

From the commutation relations (2.8), the equation of motion (2.10), and the Hamiltonian (6.8), it is obvious that the formalism developed so far generalizes the Martin-Rose-Siggia formalism [25] to the case of arbitrary diffusion and drift coefficients and provides, in addition, a class of functional integral formulations of classical stochastic dynamics. (Note that our definitions of Φ_- and H differ from those used in [25] by a factor i.)

Let us consider in some detail the functional (integral) formulations of a diffusion process in the state of equilibrium $|w_e\rangle$. From the assumption (6.5) of an approach to equilibrium, it follows that for fixed times $t_1, \ldots, t_m$ and $t_a \to -\infty$, the correlation functions G_u become independent of the initial state $|w_{t_a}\rangle$ in the infinite past and turn into the equilibrium correlation functions

$$\bar{G}_{u|1\ldots m}(t_1, \ldots, t_m) = \lim_{t_a\to-\infty} G_{u|1\ldots m}(t_1, \ldots, t_m|W_{t_a}). \quad (6.18)$$

Hence, the generating functional for the equilibrium correlation functions taken at arbitrary times is given by

$$\bar{Z}_u[\eta] = \lim_{\substack{t_a\to-\infty \\ t_b\to+\infty}} Z_u[\eta|W_{t_a}], \quad (6.19)$$

where we may use an arbitrary initial state $w(q, t_a)$. Choosing a δ-function for this state, we get from (6.16) and (6.19)

$$\bar{Z}_u[\eta] = \lim_{\substack{t_a\to-\infty \\ t_b\to+\infty}} \int d^n\varphi_+^{(b)}\, Z_u[\eta]. \quad (6.20)$$

The equilibrium generating functional $\bar{Z}_u$ is a solution of the equation of motion (3.11), and according to (3.16) its perturbation expansion is given by

$$\bar{Z}_u[\eta] = e^{-i\int_{-\infty}^{+\infty} dt\, h_u^{(1)}\left(\frac{\delta}{i\delta\eta(t)}\right)}\bar{Z}_u^{(0)}[\eta]. \quad (6.21)$$

The usual choice for the unperturbed Hamiltonian $H^{(0)}$ is a Hamiltonian with constant diffusion and linear drift coefficients. The corresponding equilibrium functional $\bar{Z}_u^{(0)}[\eta]$ is gaussian and is given by

$$\bar{Z}_u^{(0)}[\eta] = \exp\left(-\tfrac{1}{2}\int_{-\infty}^{+\infty} dt_1\, dt_2\, \bar{G}_{u|12}^{(0)}(t_1, t_2)\,\eta_1(t_1)\,\eta_2(t_2)\right). \quad (6.22)$$

Since the unperturbed 2-point equilibrium correlation function $\bar{G}_{u|12}^{(0)}$ is of the form (6.12), we see explicitly from (6.22) that the functional $\bar{Z}_u^{(0)}[\eta]$ takes u-dependent values when the source field $\eta(t)$ contains δ-functions. This u-dependence of $\bar{Z}_u^{(0)}[\eta]$ leads to the u-dependence of the equal time functional derivatives of $\bar{Z}_u^{(0)}[\eta]$, which follow from (6.22) by using the rule

$$\frac{\delta^2}{\delta\eta_1(t_1)\,\delta\eta_2(t_2)}\frac{1}{2}\int_{-\infty}^{+\infty} dt_{1'}\, dt_{2'}\, \bar{G}_{u|1'2'}^{(0)}(t_{1'}, t_{2'}) \cdot \eta_{1'}(t_{1'})\,\eta_{2'}(t_{2'}) = \bar{G}_{u|12}^{(0)}(t_1, t_2) \quad (6.23)$$

for $t_1 = t_2$ also.

The diagrammatic representation of the perturbation expansion (6.21) takes its simplest form for $u=0$. This is due to the simplicity of $h_0^{(1)}(\varphi)$ and the fact that all diagrams vanish which contain self-loops corresponding to $\bar{G}_{0|12}^{(0)}(t, t)$ with $\alpha_1 \neq \alpha_2$.

Equation (6.20) and the functional integral representation (4.1) for $Z_u[\eta]$ suggest the following functional integral representation for the equilibrium generating functional:

$$\bar{Z}_u[\eta] = \int_u \delta\varphi\, e^{i\bar{S}_u[\varphi,\eta]}, \quad (6.24)$$

with

$$\bar{S}_u[\varphi,\eta] := \int_{-\infty}^{+\infty} dt\, L_u(\varphi(t), \dot{\varphi}(t), \eta(t)) = \int_{-\infty}^{+\infty} dt\left(-\frac{i}{2}\varphi_1(t)\,\sigma_{12}^y\,\dot{\varphi}_2(t) - h_u(\varphi(t)) + \eta_3(t)\,\varphi_3(t)\right). \quad (6.25)$$

Equation (6.24) may be justified by arguments analoguous to those used in Sections 4 and 5. In particular, the lattice definition of the u-integration used in (6.24) follows from (6.20) and (5.20) and is given by

$$\int_u \delta\varphi\,\varphi_1(t_1)\ldots\varphi_m(t_m)\,e^{i\bar{S}_u[\varphi,\eta]} = \lim_{\substack{t_a\to-\infty \\ t_b\to+\infty}} \lim_{N\to\infty} \int d^n\varphi_+^{(b)} \int d^{(N)}\varphi\,\varphi_{u1}^{(\nu_1)}\ldots\varphi_{um}^{(\nu_m)}\,e^{iS_u^{(N)}(\varphi,\eta)}. \quad (6.26)$$

Hence, we have explicitly established a class of functional integral formulations of classical stochastic dynamics in the state of equilibrium.

7. Conclusion

We have investigated in some detail the intimate connection between operator ordering schemes and the

functional (integral) formulations of quantum and stochastic dynamics.

We have considered the functional integral formulations in their phase space version only, because we believe that this version has many advantages in comparison to the configuration space version (see e.g. [10, 11, 23, 24]).

We gratefully acknowledge helpful discussions with Professors H. K. Janssen, A. Schmid, and H. Wagner. One of us (H.L.) wishes to thank Professor H. Wagner for the hospitality at the Institut für Festkörperforschung of the KFA Jülich.

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