BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 20, Number 1, January 1989

OPERATOR THEORY AND ALGEBRAIC GEOMETRY

R. G. DOUGLAS, VERN PAULSEN, AND KEREN YAN

Operator theory as the study of bounded linear operators on a complex Hilbert space is nearing the end of its first century. To date most effort has been directed toward the study of a single operator or of a selfadjoint algebra of operators. The work has relied not only on measure theory and functional analysis but on techniques from complex variables, topology, and algebra. But for single operator theory the topology is either planar or general, it is one complex variable, and it is linear algebra or the algebra of polynomials in one variable that is used. For operator algebras the relevant mathematics is more sophisticated and draws on increasing amounts of topology and geometry to the point that one has begun in the last decade to refer to parts of the study of operator algebras as "noncommutative topology and geometry."

In recent years the study of nonselfadjoint operator algebras has also enjoyed considerable success but this development has largely excluded spectral theory. The work of Carey and Pincus **[10]** is an exception. Multivariable spectral theory could be viewed by analogy as "noncommutative algebraic geometry," and such a development was the goal of the module approach to multivariable operator theory presented in **[13]**. The intent was to introduce methods from several variables algebra into operator theory. In this note we announce several results in multivariable operator theory whose proofs rely on techniques which are drawn from algebraic geometry or commutative algebra. Complete details will appear elsewhere.

An operator T on the complex Hilbert space $\mathbf{H}$ is said to be hyponormal if the self-commutator $[T^*, T] = T^*T - TT^*$ is positive definite. Any operator T makes $\mathbf{H}$ into a module over the algebra of polynomials $\mathbf{C}[z]$. A little reflection shows that $\mathbf{H}$ is a module over $\mathrm{Rat}(\sigma(T))$, the algebra of rational functions on the spectrum $\sigma(T)$ of T with poles off $\sigma(T)$. In **[6]** Berger and Shaw showed that if T is hyponormal and $\mathbf{H}$ is a finitely generated $\mathrm{Rat}(\sigma(T))$-module, then $[T^*, T]$ is trace class. Hence T is essentially normal and defines an element $[T]$ in $\mathrm{Ext}(\sigma(T)) = K_1(\sigma(T))$ **[7]**. The trace class commutator enables one to define the Chern character of this class following Helton and Howe **[16]**, Carey and Pincus **[10]**, and Connes **[11]**. Attempts at generalizing the Berger-Shaw result to several variables have failed although the Chern-Weil type construction of Carey-Pincus

Received by the editors June 23, 1988.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 47A25, 47B20, 47A45, 47A60.

Research partially supported by grants from the NSF.

Keren Yan was supported as a CEEC Fellow at Stony Brook during this research.

0273-0979/89 $1.00 + $.25 per page

and Connes would apply if the appropriate hypotheses were fulfilled. Actually, as examination of the coordinate multipliers on the bidisk shows, no complete generalization of the Berger-Shaw result can hold in several variables. We obtain our generalization by restricting the joint spectrum to be contained in an algebraic curve.

An N-tuple $T = (T_1, T_2, \dots, T_N)$ is *joint hyponormal* if $[T_i, T_j] = 0$ and the compound operator $([T_j^*, T_i])$ is positive definite (cf. [**4**]). The joint spectrum we use will always be the Taylor spectrum [**17**].

THEOREM. *Let $T = (T_1, T_2, \dots, T_N)$ be a joint hyponormal N-tuple on* $\mathbf{H}$ *and set*

$$\mathbf{I} = \{p \in \mathbf{C}[z_1, z_2, \dots, z_N]: p(T_1, \dots, T_N) = 0\}.$$

If $\mathbf{H}$ *is a finitely generated* $\mathrm{Rat}(\sigma(T_1, \dots, T_n))$*-module and the Krull dimension of* $\mathbf{C}[z_1, \dots, z_N]/\mathbf{I}$ *is* 1 *(or, equivalently, the zero variety* $\mathbf{Z}(\mathbf{I})$ *is an algebraic curve), then* $([T_j^*, T_i])$ *is trace class.*

Hence, the C^∞-subalgebra of $C^*(T_1, \dots, T_N)$ is what Helton-Howe call a one-dimensional crypto-integral algebra.

COROLLARY. *If* $(S_1, \dots, S_N)$ *are joint subnormal operators on* $\mathbf{H}$ *such that* $\mathbf{H}$ *is a finitely generated* $\mathrm{Rat}(\sigma(S))$*-module and* $\sigma(S_1, \dots, S_N)$ *is contained in an algebraic curve, then* $([S_j^*, S_i])$ *is trace class.*

The proof uses a refined version of Noether's normalization theorem from algebraic geometry (cf. [**8**]). If each of the operators T_i could be expressed as $p_i(X)$ for some polynomial p_i and a fixed hyponormal operator X such that $\mathbf{H}$ were finitely generated as a $\mathrm{Rat}(\sigma(X))$-module, then the result would be immediate. The normalization theorem enables us to essentially reduce to this case.

There are many examples to which our results apply. Let $\mathbf{V}$ be an algebraic curve in $\mathbf{C}^N$ and μ be a finite measure with compact support contained in $\mathbf{V}$. If $\mathbf{H}$ is the closure in $L^2(\mu)$ of $\mathbf{C}[z_1, z_2, \dots, z_N]$, then the restriction T_i of multiplication by z_i provides an N-tuple $(T_1, T_2, \dots, T_N)$ to which the Corollary applies. Although $\mathbf{H}$ could be all of $L^2(\mu)$, that will not be the case if evaluation of the polynomials at some point of $\mathbf{V}$ is continuous in the L^2-norm. Thus if μ restricted to some open set in the relative topology of $\mathbf{V}$ is area measure, the example will be nontrivial.

We now turn to a very different result. The simplest Hilbert module over the disk algebra $A(\mathbf{D})$ is the Hardy module $H^2(\mathbf{D})$. Using von Neumann's characterization of isometries [**18**], the Wold decomposition [**19**], or Beurling's invariant subspace theorem [**6**], one can show that each (closed) submodule of $H^2(\mathbf{D})$ is unitarily equivalent to $H^2(\mathbf{D})$. One can view this as an analytical analogue of the fact that $\mathbf{C}[z]$ is a principal ideal domain.

Attempts at generalizing the Beurling result to the Hardy module $H^2(\mathbf{D}^N)$ for the polydisk algebra $A(\mathbf{D}^N)$ were unsuccessful until it was shown in the middle seventies by Berger, Coburn, and Lebow [**5**] and more generally by Cowen and Douglas [**12**] and Agrawal, Clark, and Douglas [**1**] that not all submodules of $H^2(\mathbf{D}^N)$ are unitarily equivalent. This was extended to more general domains by Agrawal and Salinas **[2]**. Our aim here is a rigidity result which shows just how different submodules are.

If $\mathbf{I}$ is an ideal in $\mathbf{C}[z_1, \ldots, z_N]$, then the closure of $\mathbf{I}$ in $H^2(\mathbf{D}^N)$, denoted $[\mathbf{I}]$, is a submodule of $A(\mathbf{D}^N)$. The results cited above referred to submodules of the form $[\mathbf{I}]$ for which $\mathbf{Z}(\mathbf{I})$ is a finite and discrete subset of $\mathbf{D}^N$. A result of Ahern and Clark **[3]** reveals that all submodules of finite codimension in $H^2(\mathbf{D}^N)$ have this form. Our results cover a much wider class of examples.

THEOREM. *If* $\mathbf{I}_1$ *and* $\mathbf{I}_2$ *are ideals in* $\mathbf{C}[z_1, \ldots, z_N]$ *which satisfy*
(i) *the height of* $\mathbf{I}_i$, *is at least* 2, *and,*
(ii) *each algebraic component of* $\mathbf{Z}(\mathbf{I}_i)$ *intersects* $\mathbf{D}^N$, *then* $[\mathbf{I}_1]$ *and* $[\mathbf{I}_2]$ *are similar if and only if* $\mathbf{I}_1 = \mathbf{I}_2$.

By the height of a general ideal is meant the minimum height of the associated prime ideals.

The proof proceeds using the global isomorphism of $[\mathbf{I}_1]$ and $[\mathbf{I}_2]$ to imply the local isomorphism of $\mathbf{I}_1 \otimes_{\mathbf{C}[\mathbf{z}]} \mathbf{L}$ and $\mathbf{I}_2 \otimes_{\mathbf{C}[\mathbf{z}]} \mathbf{L}$ for every finite dimensional module $\mathbf{L}$ over $\mathbf{C}[\mathbf{z}] = \mathbf{C}[z_1, \ldots, z_N]$. After some additional arguments, a result of Grothendieck **[14]** applies to show that $\mathbf{I}_1 = \mathbf{I}_2$.

Since the proof is completely local, it extends to Hilbert modules over algebras of holomorphic functions on domains such as the unit ball in $\mathbf{C}^N$ or indeed to many convex or pseudoconvex domains of $\mathbf{C}^N$. Moreover, similarity can be replaced by quasisimilarity or something even weaker. If there exist module maps $X\colon [\mathbf{I}_1] \to [\mathbf{I}_2]$ and $Y\colon [\mathbf{I}_2] \to [\mathbf{I}_1]$ with dense range, then $\mathbf{I}_1 = \mathbf{I}_2$. Finally, the Hardy module can be replaced by a Hilbert module formed from the closure of the polynomials or the rational functions in the L^2-space of more general measures on the domain; for example, volume measure. The critical property is that evaluation of the functions and derivatives of the functions at interior points be continuous in the L^2-norm. Hypothesis (i) excludes ideals which are even locally principal, while (ii) ensures that $\mathbf{C}[z_1, \ldots, z_N] \cap [\mathbf{I}] = \mathbf{I}$.

Deciding when the principal ideals

$$\{p\} = p\mathbf{C}[z_1, \ldots, z_N] \quad \text{and} \quad \{q\} = q\mathbf{C}[z_1, \ldots, z_N]$$

give equivalent submodules must involve more then algebra since all principal ideals are isomorphic as modules. However, although $[\{z_1\}], [\{z_2\}]$, and $H^2(\mathbf{D}^N)$ obviously define unitarily equivalent submodules, Hastings showed in **[15]** that the submodule defined by $[\{z_1 - z_2\}]$ is not even quasisimilar to $H^2(\mathbf{D}^N)$. Although the equivalence problem is, in general, quite difficult, we can solve it for homogeneous polynomials.

THEOREM. *If* p_1 *and* p_2 *are homogeneous polynomials in* $\mathbf{C}[z_1, \ldots, z_N]$, *then* $[\{p_1\}]$ *and* $[\{p_2\}]$ *are unitarily equivalent if and only if there exist monomials* $\mathbf{z}$ *and* $\mathbf{z}'$ *such that* $\mathbf{z}p_1 = \mathbf{z}'p_2$; *similar if and only if quasisimilar if and only if the quotient* p_1/p_2 *is bounded above and below on* $\mathbf{T}^N$.

Not all submodules of $H^2(\mathbf{D}^N)$ are of the form $[\mathbf{I}]$ for some ideal $\mathbf{I}$ in $\mathbf{C}[z_1, \ldots, z_N]$. That is not true even for the case $N = 1$. To obtain a rigidity result for general submodules, we must not only restrict the common zero set $\mathbf{Z}(\mathbf{M})$ of the functions in a submodule $\mathbf{M}$ but we must consider the "zero variety" on $\mathbf{T}^N$. For f in $H^2(\mathbf{D}^N)$ let σ_f be the unique singular measure on $\mathbf{T}^N$ for which $P_z(|f|dm - d\sigma_f)$ is the least harmonic majorant of $\log|f|$, where P_z denotes the Poisson kernel.

THEOREM. If $\mathbf{M}_1$ and $\mathbf{M}_2$ are submodules of $H^2(\mathbf{D}^N)$ which satisfy
(i) the Hausdorff dimension of $\mathbf{Z}(\mathbf{M}_i)$ is at most $N-2$, and
(ii) $\inf\{\sigma_f : f \in \mathbf{M}_i\} = 0$,
then $\mathbf{M}_1$ and $\mathbf{M}_2$ are quasisimilar if and only if $\mathbf{M}_1 = \mathbf{M}_2$.

The proof of this last result now involves function theory on the polydisk and a detailed knowledge of the one variable theory. In particular, localization of the interior of $\mathbf{D}^N$ is not enough and one must invoke "localization on the boundary." Hence this proof does not automatically extend to other domains in $\mathbf{C}^N$. We will discuss the details of this at another time.

We have described what we believe are some interesting results in multivariable spectral theory whose proofs depend on nontrivial techniques from algebraic geometry and commutative algebra. We believe that this is only the beginning.

Finally, we want to acknowledge the considerable assistance provided to us by H. Sah in finding the needed techniques from algebra.

REFERENCES

1. O. P. Agrawal, D. N. Clark, and R. G. Douglas, *Invariant subspaces in the polydisk*, Pacific J. Math. **121** (1986), 1–11.

2. O. P. Agrawal and N. Salinas, *Sharp kernels and canonical subspaces*, Amer. J. Math. **109** (1987), 23–40.

3. P. Ahern and D. N. Clark, *Invariant subspaces and analytic continuation in several variables*, J. Math. Mech. **19** (1969/1970), 963–969.

4. A. Athavale, *On joint hyponormality of operators*, Proc. Amer. Math. Soc. **103** (1988), 417–423.

5. C. A. Berger, L. A. Coburn, and A. Lebow, *Representation and index theory for C*-algebras generated by commuting isometries*, J. Funct. Anal. **27** (1978), 51–99.

6. C. A. Berger and B. I. Shaw, *Intertwining, analytic structure, and the trace norm estimate*, Proc. Conf. Operator Theory, Lecture Notes in Math., vol. 345, Springer-Verlag, Berlin and New York, 1973, pp. 1–6.

7. A. Beurling, *On two problems concerning linear transformations in Hilbert space*, Acta Math. **81** (1949), 239–255.

8. N. Bourbaki, *Commutative algebra*, Addison-Wesley, 1972.

9. L. G. Brown, R. G. Douglas, and P. A. Fillmore, *Unitary equivalence modulo the compact operators and extensions of C*-algebras*, Proc. Conf. Operator Theory, Lecture Notes in Math., vol. 345, Springer-Verlag, Berlin and New York, 1973, pp. 58–128.

10. R. W. Carey and J. D. Pincus, *Principal functions, index theory, geometric measure theory and function algebras*, Integral Equations and Operator Theory **2** (1979), 441–483.

11. A. Connes, *Noncommutative differential geometry*, Inst. Hautes Études Sci. Publ. Math. **86** (1986), 41–144.

12. M. J. Cowen and R. G. Douglas, *On moduli for invariant subspaces*, Operator Theory, Birkhauser Verlag, Basel, Adv. Appl. **6** (1982), 65–73.

13. R. G. Douglas and V. I. Paulsen, *Hilbert modules for function algebras*, Longman Research Notes (to appear).

14. A. Grothendieck, *Éléments de géométrie algébrique.* IV, Inst. Hautes Études Sci. Publ. Math. **24** (1965).

15. W. Hastings, *Commuting subnormal operators simultaneously quasisimilar to unilateral shifts*, Illinois J. Math. **22** (1978), 506–519.

16. J. W. Helton and R. Howe, *Traces of commutators of integral operators*, Acta Math. **136** (1976), 271–305.

17. J. Taylor, *A joint spectrum for several commuting operators*, J. Funct. Anal. **6** (1970), 172–191.

18. J. von Neumann, *Allgemeine Eigenwert-theorie Hermitesches Funktional Operatoren*, Math. Ann. **102** (1929), 49–131.

19. H. Wold, *A study in the analysis of stationary time series*, Stockholm, 1938.

Department of Mathematics, SUNY at Stony Brook, Stony Brook, New York 11794-3651 (Current address of R. G. Douglas)

Department of Mathematics, University of Houston, Houston, Texas 77004 (Current address of Vern Paulsen)

Department of Mathematics, University of Iowa, Iowa City, Iowa 52242 (Current address of Keren Yan)