OPERATOR THEORY AND HARMONIC ANALYSIS

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CONTENTS

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It is assumed that the reader has a basic knowledge of metric spaces, topology, measure theory, and the theory of bounded linear operators on Banach and Hilbert spaces. Suitable references for this material are the books "Real and Complex Analysis" by W. Rudin, "Real Analysis" by H.L. Royden, "Introduction to Topology and Modern Analysis" by G.F. Simmons, "Functional Analysis" by F. Riesz and B. Sz.-Nagy, and "Linear Operators, Part I, General Theory" by N. Dunford and J.T. Schwartz. Later, we shall also expect some knowledge of Fourier theory and partial differential equations.

Lecture 1. Spectral Theory of Bounded Operators

Much of the material in the first two lectures is presented in greater detail in the books "Perturbation Theory for Linear Operators" by T. Kato and "Spectral Theory" by E.R. Lorch, as well the book by Dunford and Schwartz just mentioned.

In these two lectures, $\mathcal{X}$ denotes a non-trivial complex Banach space, while $\mathcal{H}$ denotes a non-trivial complex Hilbert space.

(A) Spectra and resolvents of bounded operators on Banach spaces.

Let T be a bounded operator on $\mathcal{X}$. The *norm* of T is

$$\|T\| = \sup\{\, \|Tu\| : u \in \mathcal{X},\ \|u\| = 1 \,\}.$$

The Banach algebra of all bounded operators on $\mathcal{X}$ is denoted by $\mathcal{L}(\mathcal{X})$. As with matrices, $\lambda \in \mathbb{C}$ is called an *eigenvalue* of T if there exists a non-zero vector $u \in \mathcal{X}$ such that $Tu = \lambda u$. The *resolvent set* $\rho(T)$ of T is the set of all $\lambda \in \mathbb{C}$ for which $(T - \lambda I)$ is a one-one mapping and $(T - \lambda I)^{-1} \in \mathcal{L}(\mathcal{X})$. The *spectrum* $\sigma(T)$ of T is the complement of $\rho(T)$. Clearly every eigenvalue of T lies in $\sigma(T)$.

In a finite dimensional space, every one-one operator is an isomorphism, and $\sigma(T)$ is precisely the finite set of eigenvalues of T.

For $\lambda \in \rho(T)$, the *resolvent operator* $R_T(\lambda) \in \mathcal{L}(\mathcal{X})$ is defined by

$$R_T(\lambda) = (\lambda I - T)^{-1}.$$

These operators satisfy the *resolvent equation*

$$R_T(\lambda) - R_T(\mu) = (\mu - \lambda)\, R_T(\lambda)\, R_T(\mu)$$

for all λ, $\mu \in \rho(T)$. Thus, in particular, $R_T(\lambda)$ and $R_T(\mu)$ commute.

Example. Let l^p denote the Banach space of all sequences

$$x = (x_1, x_2, \dots, x_n, \dots)$$

of complex numbers, with finite norm

$$\|x\| = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} .$$

Given a sequence $d = (d_1, d_2, \dots)$ such that $\sup_n |d_n| < \infty$, define the operator $D = \operatorname{diag}(d)$ on l^p by $(Dx)_n = d_n x_n$ for every n. Then $D \in \mathcal{L}(l^p)$ and $\sigma(D)$ is the closure of the set $\{d_1, d_2, \dots\}$ in $\mathbb{C}$. Moreover

$$R_D(\lambda) = \operatorname{diag}\left(\frac{1}{\lambda - d_1}, \frac{1}{\lambda - d_2}, \dots\right)$$

for all $\lambda \in \rho(D)$.

We can treat infinite series of vectors (or operators) in the same way that we treat infinite series of complex numbers. For example, we say that an infinite series of operators $\sum T_n$ is absolutely convergent if the series $\sum \| T_n \|$ is convergent, in which case $\sum T_n$ is necessarily convergent in $\mathcal{L}(\mathcal{X})$ and $\| \sum T_n \| \le \sum \| T_n \|$.

Also several of the definitions for real and complex functions can be extended to vector valued (or operator valued) functions $u(t)$ defined for a real or complex variable t and taking values in $\mathcal{X}$ (or $\mathcal{L}(\mathcal{X})$). For instance $\lim_{t \to a} u(t) = v$ means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\| u(t) - v \| < \varepsilon$ whenever $0 < |t - a| < \delta$. A function u is continuous in a region E if $\lim_{t \to a} u(t) = u(a)$ for every $a \in E$. We define $u'(t)$ to be the derivative of $u(t)$ whenever the limit

$$u'(t) = \frac{du}{dt}(t) = \lim_{h \to 0} \frac{u(t+h) - u(t)}{h}$$

exists. For a piecewise continuous function u defined on a rectifiable curve γ, we define the Riemann integral $\int_\gamma u(t)\, dt$ to be the limit in $\mathcal{X}$ (or in $\mathcal{L}(\mathcal{X})$) of the appropriate sums $\sum_j (\gamma(t_j) - \gamma(t_{j-1})) u(t_j)$.

Moreover, several of the formulae such as

$$\begin{aligned}
\frac{d}{dt}(\alpha u(t) + \beta v(t)) &= \alpha u'(t) + \beta v'(t) \\
\int (\alpha u(t) + \beta v(t))\, dt &= \alpha \int u(t)\, dt + \beta \int v(t)\, dt \\
\left\| \int u(t)\, dt \right\| &\le \int \| u(t) \|\, |dt|
\end{aligned}$$

which are valid for real valued functions, also hold for vector valued and operator valued functions. Also, for holomorphic functions (i.e. functions which are defined and differentiable everywhere on an open subset of the complex plane), Cauchy's integral theorem holds and there is a Taylor expansion about every point in that set. If a function is bounded and holomorphic on the entire complex plane, then Liouville's theorem holds, meaning that such a function is constant.

Whenever $\|T\| < 1$ then (on defining $T^0 = I$)

$$(I-T)^{-1} = \sum_{n=0}^{\infty} T^n$$

and the series is absolutely summable. A consequence of this is the fact that, if $\lambda \in \rho(T)$ and

$$|\zeta - \lambda| < \|R_T(\lambda)\|^{-1} ,$$

then

$$R_T(\zeta) = (I-(\lambda-\zeta)R_T(\lambda))^{-1}R_T(\lambda) = \sum_{n=0}^{\infty}(\lambda-\zeta)^n R_T(\lambda)^{n+1} .$$

This shows that $\rho(T)$ is an open subset of $\mathbb{C}$, and that $R_T(\zeta)$ is holomorphic in ζ with the above Taylor expansion about each point $\lambda \in \rho(T)$. Hence

$$\frac{d^n}{d\zeta^n}R_T(\zeta) = (-1)^n n!\, R_T(\zeta)^{n+1} .$$

Theorem A. *Suppose that T is a bounded operator on $\mathcal{X}$. Then $\sigma(T)$ is a non–empty compact subset of $\mathbb{C}$.*

Proof. In view of the above remarks, it suffices to show that $\sigma(T)$ is bounded and non–empty. Take any $|\zeta| > \|T\|$. Then

$$R_T(\zeta) = \zeta^{-1}(I-\zeta^{-1}T)^{-1} = \sum_{n=0}^{\infty}\zeta^{-n-1}T^n$$

and

$$\|R_T(\zeta)\| \le \frac{1}{|\zeta| - \|T\|} .$$

Therefore $\sigma(T)$ is bounded and $R_T(\zeta) \to 0$ as $\zeta \to \infty$. If $\sigma(T)$ were empty, then $R_T(\zeta)$ would be a bounded entire function on $\mathbb{C}$ which tends to zero at infinity, and therefore, by Liouville's theorem, $R_T(\zeta) = 0$ for every $\zeta \in \mathbb{C}$. However, this is impossible as the inverse of an operator on a non–trivial Banach space cannot be zero. We conclude that $\sigma(T)$ is a non–empty subset of $\mathbb{C}$. $\square$

(B) Holomorphic functional calculi of bounded operators.

Let T be a bounded operator on $\mathcal{X}$. Let Ω be an open subset of $\mathbb{C}$ which contains $\sigma(T)$, and let $H(\Omega)$ denote the space of all complex valued holomorphic functions defined on Ω.

Let us use the word *contour* to mean a finite collection of oriented smooth closed curves, and let us say that a contour γ *envelopes* $\sigma(T)$ in Ω if γ is contained in $\Omega \setminus \sigma(T)$ and

$$\frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta - \alpha} = \begin{cases} 1, & \text{if } \alpha \in \sigma(T) \\ 0, & \text{if } \alpha \notin \Omega . \end{cases}$$

For each function $f \in H(\Omega)$, define the operator $f(T) \in \mathcal{L}(\mathcal{X})$ by

$$f(T) = \frac{1}{2\pi i}\int_\gamma f(\zeta)(\zeta I - T)^{-1}\, d\zeta = \frac{1}{2\pi i}\int_\gamma f(\zeta) R_T(\zeta)\, d\zeta$$

where γ is a contour which envelopes $\sigma(T)$ in Ω.

Since γ envelopes $\sigma(T)$ in Ω, and the integrand is a holomorphic operator valued function on $\Omega \setminus \sigma(T)$, it follows that $f(T) \in \mathcal{L}(\mathcal{X})$, and by Cauchy's theorem, that the definition of $f(T)$ is independent of the particular choice of γ.

The terminology $f(T)$ is natural, in view of the following result, which can be found in Chapter VII of the book already referred to by Dunford and Schwartz. In the Notes and Remarks at the end of that Chapter, there is an account of the historical development of this theory. People involved include E.H. Moore, D. Hilbert, F. Riesz, C. Neumann, I.M. Gelfand, N. Dunford and A.E. Taylor.

Theorem B. *Let $T \in \mathcal{L}(\mathcal{X})$, and suppose that Ω is an open set which contains $\sigma(T)$. Then the mapping from $H(\Omega)$ to $\mathcal{L}(\mathcal{X})$ which maps f to $f(T)$ satisfies the following properties:*

(1) *if f, $g \in H(\Omega)$ and $\alpha \in \mathbb{C}$ then $f(T) + \alpha g(T) = (f + \alpha g)(T)$;*

(2) *if f, $g \in H(\Omega)$ then $f(T)g(T) = (fg)(T)$;*

(3) *if $p(\zeta) = \sum_{k=0}^n c_k \zeta^k$ then $p(T) = \sum_{k=0}^n c_k T^k$;*

(4) *if $\alpha \in \rho(T)$ and $R_\alpha(\zeta) = (\zeta - \alpha)^{-1}$ then*

$$R_\alpha(T) = (T - \alpha I)^{-1} = -R_T(\alpha);$$

(5) *if $\{f_\alpha\}$ is a net in $H(\Omega)$ which converges uniformly on compact subsets of Ω to $f \in H(\Omega)$ then $f(T) = \lim_\alpha f_\alpha(T)$;*

(6) *if $f \in H(\Omega)$ then $f(\sigma(T)) = \sigma(f(T))$.*

Proof.

(1) This follows from the linearity of the integral.

(2) Since the definitions of $f(T)$ and $g(T)$ are independent of the contours which envelope $\sigma(T)$ in Ω, we are free to chose suitable ones for our purposes. Let U be an open subset of Ω which contains $\sigma(T)$, and whose closure $\overline{U}$ is a compact subset of Ω. Let Γ and γ be contours such that Γ envelopes $\sigma(T)$ in U, and γ envelopes $\overline{U}$ in Ω. Then, by the resolvent equation, we have

$$\begin{aligned}
f(T)g(T) &= \frac{1}{2\pi i}\int_\gamma f(\zeta)R_T(\zeta)\,d\zeta\,\frac{1}{2\pi i}\int_\Gamma g(w)R_T(w)\,dw \\
&= \frac{1}{(2\pi i)^2}\int_\gamma\int_\Gamma f(\zeta)g(w)R_T(\zeta)R_T(w)\,d\zeta\,dw \\
&= \frac{1}{(2\pi i)^2}\int_\gamma f(\zeta)R_T(\zeta)\int_\Gamma \frac{g(w)}{w-\zeta}\,dw\,d\zeta \\
&\quad - \frac{1}{(2\pi i)^2}\int_\Gamma g(w)R_T(w)\int_\gamma \frac{f(\zeta)}{w-\zeta}\,d\zeta\,dw \\
&= 0 + \frac{1}{2\pi i}\int_\Gamma f(w)g(w)R_T(w)\,dw \\
&= (fg)(T)\,.
\end{aligned}$$

(We can interchange the order of integration because the integrals are absolutely convergent.)

(3) It follows from (1) and (2) that it suffices to prove this result for the functions 1 and Id defined by $1(\zeta)=1$ and $Id(\zeta)=\zeta$ for all $\zeta\in\mathbb{C}$. Let γ be a contour which envelopes the closed disc of radius $\|T\|$, and hence $\sigma(T)$, in $\mathbb{C}$. Then

$$\begin{aligned}
1(T) &= \frac{1}{2\pi i}\int_\gamma (\zeta I - T)^{-1}\,d\zeta \\
&= \frac{1}{2\pi i}\int_\gamma \sum_{n=0}^{\infty}\frac{T^n}{\zeta^{n+1}}\,d\zeta \\
&= \sum_{n=0}^{\infty} T^n\frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta^{n+1}} \\
&= I
\end{aligned}$$

since

$$\int_\gamma \frac{d\zeta}{\zeta^{n+1}} = \begin{cases} 2\pi i & \text{if } n=0 \\ 0 & \text{otherwise.}\end{cases}$$

Using the same argument, we see that

$$Id(T) = \frac{1}{2\pi i}\int_\gamma \zeta(\zeta I - T)^{-1}\,d\zeta = \sum_{n=0}^{\infty} T^n\frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta^n} = T\,.$$

(4) Let $\alpha \in \rho(T)$. Then, by the above properties, we obtain:

$$\begin{aligned}(T-\alpha I)R_\alpha(T) &= 1(T) = I \\ R_\alpha(T)(T-\alpha I) &= 1(T) = I.\end{aligned}$$

Therefore $R_\alpha(T) = (T-\alpha I)^{-1}$.

(5) Since γ has finite length and

$$\begin{aligned}&\left\| \int_\gamma (f_\alpha(\zeta) - f(\zeta)) R_T(\zeta)\, d\zeta \right\| \\ &\qquad \le \max_{\zeta\in\gamma} \{ \, |f_\alpha(\zeta) - f(\zeta)| \, \| R_T(\zeta) \| \, \} \ \text{length}(\gamma) ,\end{aligned}$$

it follows that $f(T) = \lim_\alpha f_\alpha(T)$.

(6) Suppose $f \in H(\Omega)$. Let $\lambda \in \sigma(T)$. For $\zeta \in \Omega$, let

$$g(\zeta) \;=\; \begin{cases} \dfrac{f(\zeta)-f(\lambda)}{\zeta-\lambda} & \text{when } \zeta \neq \lambda \\ f'(\lambda) & \text{when } \zeta = \lambda\ . \end{cases}$$

Then $g \in H(\Omega)$ and $f(T) - f(\lambda)I = g(T)(T-\lambda I)$. If $f(\lambda) \in \rho(f(T))$ then $f(T) - f(\lambda)I$ has a bounded inverse, and hence so does $(T-\lambda I)$, which contradicts the assumption that $\lambda \in \sigma(T)$. Therefore $f(\lambda) \in \sigma(f(T))$.

Now let $\lambda \in \sigma(f(T))$. If $\lambda \notin f(\sigma(T))$ then $h(\zeta) = (f(\zeta)-\lambda)^{-1}$ is holomorphic on a neighbourhood of $\sigma(T)$, say Ω' . Applying the above results to $H(\Omega')$, we get $h(T)(f(T)-\lambda I) = I$, which contradicts the assumption that $\lambda \in \sigma(f(T))$. Therefore $\lambda \in f(\sigma(T))$. □

We say that T has a *bounded* $H(\Omega)$ *functional calculus* ("bounded" because $\| f(T) \| \le c \| f \|_\infty$ for all $f \in H(\Omega)$). There is a straightforward consequence concerning spectral decompositions.

Suppose that T is an operator whose spectrum $\sigma(T)$ is a pairwise disjoint union of non–empty compact sets

$$\sigma(T) \;=\; \sigma_1 \cup \sigma_2 \cup \cdots \cup \sigma_N\ .$$

For each k , let Ω_k be an open subset which contains σ_k , chosen so that the Ω_k 's are pairwise disjoint, let $\Omega = \cup_k \Omega_k$, and define the holomorphic functions $\chi_k \in H(\Omega)$ by

$$\chi_k(\zeta) \;=\; \begin{cases} 1 & \text{if } \zeta \in \Omega_k \\ 0 & \text{otherwise.} \end{cases}$$

Then ${\chi_k}^2 = \chi_k$, $\chi_k \chi_j = 0$ if $k \neq j$, and $\sum_{k=1}^N \chi_k = 1$.

Thus the operators P_k defined by $P_k = \chi_k(T)$ form a *family of spectral projections* associated with T, meaning that they satisfy

$$P_k{}^2 = P_k \ , \ P_k P_j \ = \ 0 \ \text{if} \ k \neq j \ , \ \text{and} \ \sum_{k=1}^{N} P_k = I$$

as well as $P_k T = T P_k$.

It is easy to infer from this, that the range $\mathcal{X}_k$ of P_k is a closed subspace of $\mathcal{X}$ which is invariant under T, that $\mathcal{X} = \oplus_k \mathcal{X}_k$, and that, if T_k denotes the restriction of T to $\mathcal{X}_k$, then $T_k \in \mathcal{L}(\mathcal{X}_k)$ and $\sigma(T_k) = \sigma_k$. Thus we have a *spectral decomposition* of $\mathcal{X}$ associated with T.

In the case when $\mathcal{X}$ is finite dimensional and the sets σ_k consist of distinct eigenvalues, then this decomposition of $\mathcal{X}$ into $\mathcal{X}_k$ can be used to obtain the Jordan canonical form of T, as is shown, for example, in Kato's book.

Exercise. For the operator D in the Example in Section A, show that $f(D) = \operatorname{diag}(f(d_1), f(d_2), \dots)$. In this case, $\| f(D) \| = \sup_{\lambda \in \sigma(D)} |f(\lambda)| \leq \| f \|_\infty$. Actually this operator D has a bounded $\mathcal{F}$ functional calculus, where $\mathcal{F}$ is the space of all bounded complex valued functions defined on $\{d_n\}$.

Exercise. Prove that the functional calculus defined above is unique in the following sense. There is no other mapping from $H(\Omega)$ to $\mathcal{L}(\mathcal{X})$ which satisfies properties (1), (2), (5), and either (3) or (4) of Theorem B.

Exercise. For each $t \in \mathbb{R}$, let g_t denote the holomorphic function $g_t(\zeta) = e^{-t\zeta}$. Define $e^{-tT} \in \mathcal{L}(\mathcal{X})$ by $e^{-tT} = g_t(T)$. Use the holomorphic functional calculus of T and the properties of the exponential functions to obtain whatever properties you can concerning the operators e^{-tT}. Show that the unique C^1 function $u : \mathbb{R} \to \mathcal{X}$ which satisfies

$$\frac{du}{dt}(t) \ = \ -Tu(t) \ , \ u(0) \ = \ v$$

is given by $u(t) = e^{-tT} v$.

Lecture 2. Spectral Theory of Unbounded Operators

Material in Section D and subsequent sections is based on my paper [M^c] on H_∞ functional calculi, and on subsequent papers with A. Yagi [M^{c}Y], T. Qian [M^{c}Q], M. Cowling, I. Doust and A. Yagi [CDM^{c}Y], and P. Auscher and A. Nahmod [AM^{c}N], all of whom I take this opportunity to thank.

(C) Spectra and resolvents of closed operators in Banach spaces.

Recall that $\mathcal{X}$ denotes a non–trivial complex Banach space. By an *operator* in $\mathcal{X}$ we mean a linear mapping $T : \mathcal{D}(T) \to \mathcal{X}$, where the *domain* $\mathcal{D}(T)$ is a linear subspace of $\mathcal{X}$. (Note that we do not require that subspaces are closed.) The *range* of T is denoted by $\mathcal{R}(T)$ and the *nullspace* by $\mathcal{N}(T)$. The *norm* of T is

$$\|T\| \;=\; \sup\{\,\|Tu\| : u \in \mathcal{D}(T), \|u\| = 1\,\}$$

so that $0 \le \|T\| \le \infty$. We say that T is *bounded* if $\|T\| < \infty$, and that T is *bounded on* $\mathcal{X}$ if it is bounded and $\mathcal{D}(T) = \mathcal{X}$. The algebra of all bounded operators on $\mathcal{X}$ is denoted by $\mathcal{L}(\mathcal{X})$. We call T *densely–defined* if $\mathcal{D}(T)$ is dense in $\mathcal{X}$, and *closed* if its graph, $\mathcal{G}(T) = \{(u, Tu) : u \in \mathcal{D}(T)\}$ is a closed subspace of $\mathcal{X} \times \mathcal{X}$. The space of all closed operators is denoted $\mathcal{C}(\mathcal{X})$.

A complex valued function f defined on $\mathcal{X}$ is called a *conjugate linear functional* on $\mathcal{X}$ if $f(\alpha u + \beta v) = \bar{\alpha} f(u) + \bar{\beta} f(v)$, for every α, $\beta \in \mathbb{C}$ and u, $v \in \mathcal{X}$. The complex Banach space $\mathcal{X}^*$ of all conjugate linear functionals on $\mathcal{X}$, under the norm

$$\|f\| \;=\; \sup\{\,|f(u)| : \|u\| = 1\,\}\ ,$$

is called the *adjoint space* of $\mathcal{X}$.

If T is a densely–defined operator in $\mathcal{C}(\mathcal{X})$, then its *adjoint* operator $T^* \in \mathcal{C}(\mathcal{X})$ is defined as follows. The domain of T^* consists of all $g \in \mathcal{X}^*$ such that $g(Tu) = f(u)$ for every $u \in \mathcal{D}(T)$ and some $f \in \mathcal{X}^*$. Since T is assumed to be densely–defined, the $f \in \mathcal{X}^*$ is uniquely defined by g, and so we define $T^* g = f$.

In the case that $\mathcal{X}$ is a Hilbert space, we can identify $\mathcal{X}^*$ with $\mathcal{X}$ by taking $f(x) = \langle f, x\rangle$. If T is a densely–defined operator in $\mathcal{X}$ such that $T = T^*$, then T is called a *self–adjoint* operator in $\mathcal{X}$.

When new operators are constructed from old, the domains are taken to be the largest for which the constructions make sense. For example, if S and T are linear operators, then $S + T$ and ST are the linear operators defined by

$$\begin{aligned}(S+T)u &= Su + Tu\ , \qquad & u \in \mathcal{D}(S+T) &= \mathcal{D}(S) \cap \mathcal{D}(T)\ ,\\ (ST)u &= S(Tu)\ , \qquad & u \in \mathcal{D}(ST) &= \{u \in \mathcal{D}(T) : Tu \in \mathcal{D}(S)\}\ ,\end{aligned}$$

and, if T is one–one, then

$$\mathcal{D}(T^{-1}) \;=\; \mathcal{R}(T)\;.$$

Care needs to be taken with the domains of operators. We write $S \subset T$ if $\mathcal{D}(S) \subset \mathcal{D}(T)$ and $Su = Tu$ for all $u \in \mathcal{D}(S)$. So $S = T$ if and only if $S \subset T$ and $T \subset S$.

Exercise. Show that

$$\begin{aligned}
S+T &= T+S\,,\\
(S+T)+U &= S+(T+U)\,,\\
S+0 &= S\,,\\
0S &\subset S0 \;=\; 0\,,\\
S-S &\subset 0\,,\\
(ST)U &= S(TU)\,,\\
S(T+U) &\supset ST+SU\,,\\
(S+T)U &= SU+TU\,,
\end{aligned}$$

and, if S is one–one,

$$S^{-1}S \subset I \text{ and } SS^{-1} \subset I\,.$$

Sometimes $S(T+U) \neq ST+SU$, as the example given by $T = -U = I$ and $\mathcal{D}(S) \neq \mathcal{X}$ shows. If S and T are one–one, then so is ST, and $(ST)^{-1} = T^{-1}S^{-1}$.

Exercise. If $B \in \mathcal{L}(\mathcal{X})$ and $T \in \mathcal{C}(\mathcal{X})$, then the following operators are closed: B, TB, $B^{-1}T$ (if B is one–one) and T^{-1} (if T is one–one).

The *resolvent set* $\rho(T)$ of T is the set of all $\lambda \in \mathbb{C}$ for which $(T-\lambda I)$ is one–one and $(T-\lambda I)^{-1} \in \mathcal{L}(\mathcal{X})$. The *spectrum* $\sigma(T)$ of T is the complement of $\rho(T)$, together with ∞ if $T \notin \mathcal{L}(\mathcal{X})$.

For $T \in \mathcal{C}(\mathcal{X})$ and $\lambda \in \rho(T)$, define the *resolvent operator* $R_T(\lambda) \in \mathcal{L}(\mathcal{X})$ by

$$R_T(\lambda) \;=\; (\lambda I - T)^{-1}\;.$$

Lemma C. *Suppose T is a closed operator in $\mathcal{X}$ and $\lambda \in \rho(T)$. Let $R_\lambda(\zeta) = (\zeta-\lambda)^{-1}$ for every $\zeta \in \mathbb{C}$, and let $R_\lambda(\infty) = 0$. Then $-R_\lambda(\sigma(T)) = \sigma(R_T(\lambda))$.*

Proof. Suppose $\infty \in \sigma(T)$. Then $T \notin \mathcal{L}(\mathcal{X})$ and hence $T - \lambda I \notin \mathcal{L}(\mathcal{X})$. Therefore

$$R_\lambda(\infty) \;=\; 0 \;\in\; \sigma(R_T(\lambda))\;.$$

Next, let $\mu \in \sigma(T) \cap \mathbb{C}$ and assume that $-(\mu-\lambda)^{-1} = (\lambda-\mu)^{-1} \in \rho(R_T(\lambda))$. Then

$$P \;=\; \frac{1}{\lambda-\mu}\,R_T(\lambda)\left(R_T(\lambda) - \frac{1}{\lambda-\mu}\,I\right)^{-1} \;\in\; \mathcal{L}(\mathcal{X})\;.$$

Therefore $P(T-\mu I) \subset I$ and $(T-\mu I)P = I$, so that $\mu \in \rho(T)$, which contradicts the assumption that $\mu \in \sigma(T)$. Therefore $-R_\lambda(\sigma(T)) \subset \sigma(R_T(\lambda))$.

Now, suppose $\mu \in \sigma(R_T(\lambda))$ and assume that $-\mu \notin R_\lambda(\sigma(T))$. Then $\mu \in \mathbb{C}$ and $\lambda - 1/\mu \in \rho(T)$. Let

$$Q \;=\; -\frac{1}{\mu}(I + \frac{1}{\mu}R_T(\lambda - 1/\mu)) \;\in\; \mathcal{L}(X) .$$

Then $Q(R_T(\lambda) - \mu I) = (R_T(\lambda) - \mu I)Q = I$. So $\mu \in \rho(R_T(\lambda))$. However, this is a contradiction, and hence $\sigma(R_T(\lambda)) \subset -R_\lambda(\sigma(T))$. □

In the above, we have extended the complex plane $\mathbb{C}$ to the Riemann sphere $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. The topology on the Riemann sphere consists of the usual neighbourhoods in $\mathbb{C}$ together with the *neighbourhoods of* ∞ which are the sets of the form $U \cup \{\infty\}$, where $\mathbb{C} \setminus U$ is a compact subset of $\mathbb{C}$. A function f defined in a neighbourhood of ∞ is *holomorphic at* ∞ if $f(1/\zeta)$ is holomorphic at $\zeta = 0$.

Let T be a closed operator in $\mathcal{X}$ and let $\lambda \in \rho(T)$. Then the function R_λ defined in the last lemma, is a homeomorphism on $\mathbb{C}_\infty$ and is holomorphic on $\mathbb{C}_\infty \setminus \{\lambda\}$. It therefore follows from the last lemma that $\sigma(T)$ is a non–empty compact subset of $\mathbb{C}_\infty$.

We now develop the holomorphic functional calculus for closed operators with non–empty resolvent set, following the treatment of Dunford and Schwartz.

Let Ω be an open subset of $\mathbb{C}_\infty$ which contains $\sigma(T)$, but does not contain $\lambda \in \mathbb{C}$, and let $H(\Omega)$ be the space of holomorphic functions defined on Ω. Then R_λ induces a bijection between $H(\Omega)$ and $H(R_\lambda(\Omega))$, where $f \in H(\Omega)$ is mapped to $f \circ R_\lambda^{-1}$. Using this bijection we define

$$f(T) \;=\; (f \circ R_\lambda^{-1})(-R_T(\lambda)) ,$$

for every $f \in H(\Omega)$. The following result is then a consequence of Theorem B.

Theorem C. *Let T be a closed operator in $\mathcal{X}$ with non–empty resolvent set, and suppose that Ω is a proper open subset of $\mathbb{C}_\infty$ which contains $\sigma(T)$. Then the mapping from $H(\Omega)$ to $\mathcal{L}(\mathcal{X})$ which maps f to $f(T)$ satisfies the following properties:*

(1) *if f, $g \in H(\Omega)$ and $\alpha \in \mathbb{C}$ then $f(T) + \alpha g(T) = (f + \alpha g)(T)$;*
(2) *if f, $g \in H(\Omega)$ then $f(T)g(T) = (fg)(T)$;*
(3) *$1(T) = I$;*
(4) *if $\alpha \in \rho(T)$ and $R_\alpha(\zeta) = (\zeta - \alpha)^{-1}$ then*

$$R_\alpha(T) \;=\; (T - \alpha I)^{-1} \;=\; -R_T(\alpha) ;$$

(5) *if $\{f_\alpha\}$ is a net in $H(\Omega)$ which converges uniformly on compact subsets of Ω to $f \in H(\Omega)$ then $f(T) = \lim_\alpha f_\alpha(T)$;*
(6) *if $f \in H(\Omega)$ then $f(\sigma(T)) = \sigma(f(T))$.*

Exercise. State and prove a theorem concerning the uniqueness of this holomorphic functional calculus.

Exercise. Prove A.E. Taylor's formula

$$f(T) \;=\; f(\infty)I + \frac{1}{2\pi i}\int_\delta f(\zeta)(\zeta I - T)^{-1}\,d\zeta$$

in the case when $\infty \in \sigma(T)$ and δ is a contour which envelopes $\sigma(T)$ in Ω in some appropriate sense.

(D) Holomorphic functional calculi of operators of type $S_{\omega+}$.

For $0 \le \omega < \mu < \pi$, define the closed and open sectors in the complex plane $\mathbb{C}$:

$$\begin{aligned} S_{\omega+} &= \{\zeta \in \mathbb{C} : |\arg\zeta| \le \omega\} \cup \{0\} \\ S^0_{\mu+} &= \{\zeta \in \mathbb{C} : \zeta \ne 0, |\arg\zeta| < \mu\}. \end{aligned}$$

We employ the following subspaces of the space $H(S^0_{\mu+})$ of all holomorphic functions on $S^0_{\mu+}$.

$$H_\infty(S^0_{\mu+}) \;=\; \{\, f \in H(S^0_{\mu+}) : \|f\|_\infty < \infty \,\}\,,$$

where $\|f\|_\infty = \sup\{\, |f(\zeta)| : \zeta \in S^0_{\mu+} \,\}$,

$$\Psi(S^0_{\mu+}) \;=\; \{\, \psi \in H(S^0_{\mu+}) : \exists s > 0, |\psi(\zeta)| \le C|\zeta|^s(1+|\zeta|^{2s})^{-1} \,\}$$

and

$$F(S^0_{\mu+}) \;=\; \{\, f \in H(S^0_{\mu+}) : \exists s > 0, |f(\zeta)| \le C(|\zeta|^{-s} + |\zeta|^s) \,\}\,,$$

so that

$$\Psi(S^0_{\mu+}) \;\subset\; H_\infty(S^0_{\mu+}) \;\subset\; F(S^0_{\mu+}) \;\subset\; H(S^0_{\mu+})\,.$$

Let $0 \le \omega < \pi$. An operator $T \in \mathcal{C}(\mathcal{X})$ is said to be of *type* $S_{\omega+}$ if $\sigma(T) \subset S_{\omega+}$ and, for each $\mu > \omega$, there exists C_μ such that

$$\|(T - \zeta I)^{-1}\| \;\le\; C_\mu|\zeta|^{-1}\,, \quad \zeta \notin S_{\mu+}\,.$$

Example 1. Suppose that T is a self–adjoint operator in a Hilbert space $\mathcal{H}$ and that $\langle Tu\,,\,u\rangle \ge 0$ for every $u \in \mathcal{X}$. Then T is an operator of type S_{0+}.

Example 2. More generally, for $0 \le \omega \le \frac{\pi}{2}$, suppose that T is an ω*-accretive* operator in a Hilbert space $\mathcal{H}$, by which we mean that $\sigma(T) \subset S_{\omega+}$ and $\langle Tu\,,\,u\rangle \in$

$S_{\omega+}$ for all $u \in \mathcal{D}(T)$. Then it is easy to show that $\| R_T(\zeta) \| \leq (\operatorname{dist}(\zeta, S_{\omega+}))^{-1}$ for all $\zeta \notin S_{\omega+}$, so that T is an operator of type $S_{\omega+}$. We remark that ω-accretive operators necessarily have dense domain, and are self–adjoint if and only if $\omega = 0$.

Example 3. Let $0 < \omega < \frac{\pi}{2}$. A family of bounded operators $\{S(z) : z \in S^0_{\omega+}\}$ on $\mathcal{X}$ is called a *holomorphic semigroup* if

(1) $S(z)S(w) = S(z+w)$ for all z, $w \in S^0_{\omega+}$;

(2) $S(z)$ is holomorphic on $S^0_{\omega+}$.

Let T be the *generator of* $\{S(z)\}$, namely the operator in $\mathcal{X}$ such that

$$Tu = \lim_{t \downarrow 0} \frac{S(t)u - u}{t}$$

with domain consisting of all $u \in \mathcal{X}$ for which the limit exists. Then $-T$ is an operator of type $S_{(\frac{\pi}{2}-\omega)+}$ in $\mathcal{X}$.

Example 4. Define

$$T = \bigoplus_{k \in \mathbb{N}} \begin{bmatrix} 2^{-k} & 1 \\ 0 & 2^{-k} \end{bmatrix}$$

in the Hilbert space $\mathcal{H} = \oplus \mathbb{C}^2 = \{(u_1, u_2, \ldots) : u_k \in \mathbb{C}^2\}$ with inner product $((u_1, u_2, \ldots), (v_1, v_2, \ldots)) = \sum (u_k, v_k)$. Then, for $\xi < 0$,

$$(T - \xi I)^{-1} = \bigoplus_{k \in \mathbb{N}} \begin{bmatrix} (2^{-k} - \xi)^{-1} & -(2^{-k} - \xi)^{-2} \\ 0 & (2^{-k} - \xi)^{-1} \end{bmatrix},$$

so $\| (T - \xi I)^{-1} \| \geq \sup_k |2^{-k} - \xi|^{-2} \geq |\xi|^{-2}$. Thus T is *not* of type $S_{\omega+}$ for any ω.

Suppose that T is a one–one operator of type $S_{\omega+}$ with dense domain and dense range in $\mathcal{X}$. Every such operator has a holomorphic functional calculus which is consistent with the usual definition of polynomials of an operator. By this we mean that for each $\mu > \omega$ and for each $f \in F(S^0_{\mu+})$, there corresponds a densely–defined operator $f(T) \in \mathcal{C}(\mathcal{X})$. Further, if $f, g \in F(S^0_{\mu+})$ and $\alpha \in \mathbb{C}$, then

$$\begin{aligned} \alpha f(T)u + g(T)u &= (\alpha f + g)(T)u \\ g(T)f(T)u &= (gf)(T)u \end{aligned}$$

for every $u \in \mathcal{D}(f(T))$ for which one side (and hence the other side) of each equation is defined.

In particular, if $\psi \in \Psi(S^0_{\mu+})$, then

$$\psi(T) = \frac{1}{2\pi i} \int_\gamma (T - \zeta I)^{-1} \psi(\zeta) \, d\zeta$$

where γ is the unbounded contour $\{\zeta = re^{\pm i\theta} : r \geq 0\}$ parametrised clockwise around $S_{\omega+}$, and $\omega < \theta < \mu$. Clearly, this integral is absolutely convergent in $\mathcal{L}(\mathcal{X})$, and it is straightforward to show, using Cauchy's theorem, that the definition is independent of the choice of $\theta \in (\omega, \mu)$, and that if ψ is holomorphic on a neighbourhood of $S_{\omega+}$, then this definition is consistent with the previous one. Moreover $\sigma(\psi(T)) = \psi(\sigma(T))$.

One way to construct this functional calculus is to first define these operators $\psi(T)$ by this integral when $\psi \in \Psi(S^0_{\mu+})$, and to then define $f(T)$ when $f \in F(S^0_{\mu+})$ as follows.

Suppose $f \in F(S^0_{\mu+})$. Then for some c and k, $|f(\zeta)| \leq c(|\zeta|^k + |\zeta|^{-k})$ for every $\zeta \in S^0_{\mu+}$. Let

$$\psi(\zeta) = \left(\frac{\zeta}{(1+\zeta)^2}\right)^{k+1} .$$

Then ψ, $f\psi \in \Psi(S^0_{\mu+})$ and $\psi(T)$ is one–one. So $(f\psi)(T)$ is a bounded operator on $\mathcal{X}$, and $\psi(T)^{-1}$ is a closed operator in $\mathcal{X}$. Define $f(T) \in \mathcal{C}(\mathcal{X})$ by

$$f(T) = (\psi(T))^{-1}(f\psi)(T) .$$

One must then check that the properties of the functional calculus are satisfied.

This functional calculus is the unique one in which the following Convergence Lemma holds. It is one of the most important properties which is satisfied by this functional calculus.

Theorem D (The Convergence Lemma). *Let $0 \leq \omega < \mu \leq \pi$. Let T be an operator of type $S_{\omega+}$ which is one–one with dense domain and range. Let $\{f_\alpha\}$ be a uniformly bounded net in $H_\infty(S^0_{\mu+})$, which converges to $f \in H_\infty(S^0_{\mu+})$ uniformly on compact subsets of $S^0_{\mu+}$, such that $\{f_\alpha(T)\}$ is a uniformly bounded net in $\mathcal{L}(\mathcal{X})$. Then $f(T) \in \mathcal{L}(\mathcal{X})$, $f_\alpha(T)u \to f(T)u$ for all $u \in \mathcal{X}$, and $\| f(T) \| \leq \sup_\alpha \| f_\alpha(T) \|$.*

Proof. Let $\psi(\zeta) = \zeta(1+\zeta)^{-2}$ and $\psi_\alpha(\zeta) = \psi(\zeta)(f_\alpha(\zeta) - f(\zeta))$. Then, for each α, $\psi_\alpha \in \Psi(S^0_{\mu+})$ and

$$\begin{aligned}
\| \psi_\alpha(T) \| &\leq \text{const.} \int_\gamma \frac{|f_\alpha(\zeta) - f(\zeta)|}{1+|\zeta|^2} |d\zeta| \\
&\leq \frac{\text{const.}}{r}(\| f \|_\infty + \| f_\alpha \|_\infty) + \text{const.} \sup_{r^{-1} \leq |\zeta| \leq r} \{|f_\alpha(\zeta) - f(\zeta)|\}
\end{aligned}$$

for all $r > 1$. Since the net $\{f_\alpha\}$ is uniformly bounded and converges on compact subsets of $S^0_{\omega+}$, it follows that $\| \psi_\alpha(T) \| \to 0$. Now take any $u \in \mathcal{X}$. Then

$$\| f_\alpha(T)\psi(T)u - f(T)\psi(T)u \| = \| \psi_\alpha(T)u \| \leq \| \psi_\alpha(T) \| \| u \| \to 0 .$$

Therefore

$$\| f(T)\psi(T)u \| \;\leq\; (\sup_{\alpha} \| f_\alpha(T) \|) \, \| \psi(T)u \| \; .$$

Since $\psi(T)$ has dense range it follows that $\| f(T) \| \leq \sup_\alpha \| f_\alpha(T) \|$ and that $f_\alpha(T)u \to f(T)u$ for every $u \in \mathcal{X}$. $\square$

The Convergence Lemma allows us to give immediate proofs of many of the formulae used in studying such entities as powers of T and semigroups associated with T.

Let us consider the powers of a one–one operator T of type $S_{\omega+}$ with dense domain and dense range. (The density assumptions can often be removed.) For $\alpha \in \mathbb{C}$, let $f_\alpha \in F(S^0_{\mu+})$ be given by $f_\alpha(\zeta) = \zeta^\alpha$, where $\mu > \omega$. Then define $T^\alpha \in \mathcal{C}(\mathcal{X})$ by

$$T^\alpha = f_\alpha(T) \; .$$

From the functional calculus of T, formulae such as

$$T^\alpha T^\beta u \;=\; T^{\alpha+\beta} u \; , \qquad u \in \mathcal{D}(T^\beta) \cap \mathcal{D}(T^{\alpha+\beta}) \; ,$$

follow immediately. From the Convergence Lemma, it is also easy to prove standard formulae for these operators such as the following. If $0 < s < 1$, then

$$T^s u \;=\; \frac{1}{\beta} \lim_{\substack{R\to\infty \\ \epsilon\to 0}} \int_\epsilon^R t^{-s}(I + tT)^{-1} Tu \, dt \; , \qquad u \in D(T) \; ,$$

where

$$\beta \;=\; \int_0^\infty t^{-s}(1+t)^{-1} dt \; .$$

We can develop the theory of the associated holomorphic semigroups in a similar manner. Suppose T is a one–one operator with dense domain and dense range which is of type $S_{\omega+}$, where $0 \leq \omega < \frac{\pi}{2}$. For each $z \in S^0_{(\frac{\pi}{2}-\omega)+}$ let

$$g_z(\zeta) = e^{-z\zeta}.$$

Then $g_z \in F(S^0_{\frac{\pi}{2}+})$ and the operators

$$e^{-zT} = g_z(T)$$

form a holomorphic semigroup.

Exercise. State and prove a theorem concerning the uniqueness of this holomorphic functional calculus.

Exercise. Show that, whenever $f(T) \in \mathcal{L}(\mathcal{X})$, then $\overline{f}(T^*) \in \mathcal{L}(\mathcal{X}^*)$ and $f(T)^* = \overline{f}(T^*)$, where $\overline{f}(\zeta) = \overline{f(\overline{\zeta})}$.

Lecture 3. Quadratic Estimates

In this lecture and the next, $\mathcal{H}$ denotes a non–trivial complex Hilbert space.

We are primarily interested in characterising and studying those one–one operators T of type $S_{\omega+}$ in $\mathcal{H}$ which have a *bounded* H_∞ functional calculus, meaning that for all $f \in H_\infty(S^0_{\mu+})$ (where $\mu > \omega$), $f(T) \in \mathcal{L}(\mathcal{H})$ and

$$\| f(T) \| \leq c_\mu \| f \|_\infty$$

for some constant c_μ. In particular, we show that this is the case if and only if the quadratic norm $\| u \|_T$ is equivalent to the original norm $\| u \|$ on $\mathcal{H}$.

In the Hilbert space case, every one–one operator T of type $S_{\omega+}$ necessarily has dense domain and dense range, so that the Convergence Lemma is always valid.

Further, $\mathcal{D}(T) \cap \mathcal{R}(T)$ is dense in $\mathcal{H}$. If T is not one–one, then $\mathcal{H} = \mathcal{N}(T) \oplus \mathcal{H}_0$ where $\mathcal{N}(T) = \{u \in \mathcal{H} : Tu = 0\}$, and $\mathcal{H}_0$ is the closure of $\mathcal{R}(T)$, so that we do not lose generality by restricting our attention to one–one operators. (No orthogonality is implied by the symbol $\oplus$.) See [CDM^cY] for a treatment of these results, and for the situation in Banach spaces. References for the results which follow also include [Y], [M^c], [M^cY], [M^cQ], and [AM^cN]. There is related material in [deL].

(E) Quadratic norms of operators of type $S_{\omega+}$ in Hilbert spaces.

Suppose, for $0 \leq \omega < \pi$, that T is a one–one operator of type $S_{\omega+}$ in $\mathcal{H}$. Given $\psi \in \Psi(S^0_{\mu+})$ and $t > 0$, define $\psi_t \in \Psi(S^0_{\mu+})$ by $\psi_t(\zeta) = \psi(t\zeta)$. The operators $\psi_t(T) \in \mathcal{L}(\mathcal{H})$, defined in the previous section, depend continuously on t, so the integrals in the following results make sense.

Lemma E. *Let T be a one–one operator of type $S_{\omega+}$ in $\mathcal{H}$. Let ψ, $\underline{\psi} \in \Psi(S^0_{\mu+})$ where $\mu > \omega$. Then there exists a constant c such that*

(i)
$$\| (f\psi_t)(T) \| \leq c \| f \|_\infty$$

for all $f \in H_\infty(S^0_{\mu+})$ and all $t > 0$, and

(ii)
$$\left\{ \int_0^\infty \left\| \int_\alpha^\beta \psi_\tau(T)\underline{\psi}_t(T)g(\tau)\,\frac{d\tau}{\tau} \right\|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \leq c \left\{ \int_\alpha^\beta \| g(\tau) \|^2 \,\frac{d\tau}{\tau} \right\}^{\frac{1}{2}}$$

for all continuous functions g *from* $[\alpha,\beta]$ *to* $\mathcal{H}$, *and all* $0<\alpha<\beta<\infty$.

Proof. If $\gamma=\{\zeta=re^{\pm i\theta}: r>0\}$, $\omega<\theta<\mu$, is an unbounded contour parametrised clockwise around $S_{\omega+}$, then

$$(f\psi_t)(T) = \frac{1}{2\pi i}\int_\gamma f(\zeta)\psi(t\zeta)(T-\zeta I)^{-1}\,d\zeta\,.$$

So, for some $s>0$,

$$\begin{aligned}\|(f\psi_t)(T)\| &\le \text{const.}\ \|f\|_\infty \int_\gamma \frac{|t\zeta|^s}{1+|t\zeta|^{2s}}\frac{|d\zeta|}{|\zeta|}\\ &\le c\|f\|_\infty\end{aligned}$$

which proves (i). To obtain (ii), first verify that

$$\begin{aligned}\left\|(\psi_\tau\underline{\psi}_t)(T)\right\| &\le \text{const.}\int_\gamma \frac{|\tau\zeta|^s}{1+|\tau\zeta|^{2s}}\frac{|t\zeta|^s}{1+|t\zeta|^{2s}}\frac{|d\zeta|}{|\zeta|}\\ &\le \begin{cases}\text{const.}\ (t/\tau)^s(1+\log(\tau/t)) & \text{if } 0<t\le\tau<\infty\\ \text{const.}\ (\tau/t)^s(1+\log(t/\tau)) & \text{if } 0<\tau\le t<\infty\,.\end{cases}\end{aligned}$$

Therefore a variant of Schur's estimate becomes

$$\begin{aligned}&\int_0^\infty\left\|\int_\alpha^\beta \psi_\tau(T)\underline{\psi}_t(T)g(\tau)\frac{d\tau}{\tau}\right\|^2\frac{dt}{t}\\ &\quad\le \int_0^\infty\left\{\int_\alpha^\beta\left\|\psi_\tau(T)\underline{\psi}_t(T)\right\|^{\frac12}\left\|\psi_\tau(T)\underline{\psi}_t(T)\right\|^{\frac12}\|g(\tau)\|\frac{d\tau}{\tau}\right\}^2\frac{dt}{t}\\ &\quad\le \int_0^\infty\left\{\int_\alpha^\beta\left\|\psi_\tau(T)\underline{\psi}_t(T)\right\|\frac{d\tau}{\tau}\int_\alpha^\beta\left\|\psi_\tau(T)\underline{\psi}_t(T)\right\|\|g(\tau)\|^2\frac{d\tau}{\tau}\right\}\frac{dt}{t}\\ &\quad\le \sup_t\left\{\int_\alpha^\beta\left\|(\psi_\tau\underline{\psi}_t)(T)\right\|\frac{d\tau}{\tau}\right\}\sup_\tau\left\{\int_0^\infty\left\|(\psi_\tau\underline{\psi}_t)(T)\right\|\frac{dt}{t}\right\}\left\{\int_\alpha^\beta\|g(\tau)\|^2\frac{d\tau}{\tau}\right\}\\ &\quad\le c^2\int_\alpha^\beta\|g(\tau)\|^2\frac{d\tau}{\tau}\end{aligned}$$

as required. $\square$

We are now in a position to define the quadratic norm $\|u\|_{T,\psi}$ associated with a one–one operator T of type $S_{\omega+}$ in $\mathcal{H}$ and a non–zero function $\psi\in\Psi(S^0_{\mu+})$. It is defined by

$$\|u\|_{T,\psi} = \left\{\int_0^\infty\|\psi_t(T)u\|^2\frac{dt}{t}\right\}^{\frac12}$$

on the space $\mathcal{H}^0_{T,\psi}$ of all $u \in \mathcal{H}$ for which the integral is finite.

Exercise. Show that there exist two more functions θ, $\underline{\theta} \in \Psi(S^0_{\mu+})$ such that

$$\int_0^\infty \theta(\tau)\underline{\theta}(\tau)\psi(\tau)\,\frac{d\tau}{\tau} = 1\,.$$

Prove, using the Convergence Lemma, that

$$\lim_{\substack{\alpha\to 0\\ \beta\to\infty}} \psi_{\alpha,\beta}(T)u = \lim_{\substack{\alpha\to 0\\ \beta\to\infty}} \int_\alpha^\beta \theta_\tau(T)\underline{\theta}_\tau(T)\psi_\tau(T)u\,\frac{d\tau}{\tau} = u$$

for all $u \in \mathcal{H}$, where $\psi_{\alpha,\beta} \in \Psi(S^0_{\mu+})$ is defined by

$$\psi_{\alpha,\beta}(\zeta) = \int_\alpha^\beta \theta_\tau(\zeta)\underline{\theta}_\tau(\zeta)\psi_\tau(\zeta)\,\frac{d\tau}{\tau}\,.$$

Exercise. Show that $\| u \|_{T,\psi}$ is indeed a norm on $\mathcal{H}^0_{T,\psi}$.

Exercise. Suppose that $\{v_\alpha\}$ is a Cauchy net in $\mathcal{H}^0_{T,\psi}$ in the topology associated with $\|\cdot\|_{T,\psi}$, and that $v_\alpha \to v \in \mathcal{H}$ in the original norm topology. Prove that $v \in \mathcal{H}^0_{T,\psi}$ and that $\| v_\alpha - v \|_{T,\psi} \to 0$.

Proposition E. *Suppose that T is a one–one operator of type $S_{\omega+}$ in $\mathcal{H}$, and that ψ, $\underline{\psi}$ are non–zero functions in $\Psi(S^0_{\mu+})$ where $\mu > \omega$. Then there exists a constant c such that, for every $f \in H_\infty(S^0_{\mu+})$ and every $u \in \mathcal{H}^0_{T,\psi} \cap \mathcal{D}(f(T))$,*

$$\| f(T)u \|_{T,\underline{\psi}} \le c \| f \|_\infty \, \| u \|_{T,\psi}\,.$$

Proof. Let $u \in \mathcal{H}^0_{T,\psi} \cap \mathcal{D}(f(T))$. Choose θ, $\underline{\theta}$ and $\psi_{\alpha,\beta}$ as in the exercise above. By using the holomorphic functional calculus of T, and applying Lemma E, we have

$$\begin{aligned}
\| \psi_{\alpha,\beta}(T)f(T)u \|_{T,\underline{\psi}} &= \left\{ \int_0^\infty \left\| \underline{\psi}_t(T)\psi_{\alpha,\beta}(T)f(T)u \right\|^2 \frac{dt}{t} \right\}^{\frac12} \\
&= \left\{ \int_0^\infty \left\| \int_\alpha^\beta \underline{\psi}_t(T)\theta_\tau(T)\underline{\theta}_\tau(T)\psi_\tau(T)f(T)u \frac{d\tau}{\tau} \right\|^2 \frac{dt}{t} \right\}^{\frac12} \\
&= \left\{ \int_0^\infty \left\| \int_\alpha^\beta \theta_\tau(T)\underline{\psi}_t(T)(f\underline{\theta}_\tau)(T)\psi_\tau(T)u \frac{d\tau}{\tau} \right\|^2 \frac{dt}{t} \right\}^{\frac12}
\end{aligned}$$

$$\leq \text{ const. } \left\{ \int_\alpha^\beta \| (f\underline{\theta}_\tau)(T)\psi_\tau(T)u \|^2 \frac{d\tau}{\tau} \right\}^{\frac{1}{2}} \qquad \text{by (ii)}$$

$$\leq c \| f \|_\infty \left\{ \int_\alpha^\beta \| \psi_\tau(T)u \|^2 \frac{d\tau}{\tau} \right\}^{\frac{1}{2}} \qquad \text{by (i)}$$

$$\leq c \| f \|_\infty \| u \|_{T,\psi} .$$

Using this we see that $\{\psi_{\alpha,\beta}(T)f(T)u\}$ is a Cauchy net in $\mathcal{H}^0_{T,\psi}$ as $(\alpha,\beta) \to (0,\infty)$. Moreover, by the first of the above exercises, we see that it converges in $\mathcal{H}$ to $f(T)u$. So, by the third exercise, $f(T)u \in \mathcal{H}^0_{T,\psi}$, and satisfies the required estimate. □

On considering the case when $f \equiv 1$, in which case $f(T) = I$, we see that the spaces $\mathcal{H}^0_{T,\psi}$ are independent of ψ, and indeed of μ, so can be denoted by $\mathcal{H}^0_T$, and that the norms $\| u \|_{T,\psi}$ are equivalent, meaning that, for each pair of non–zero functions ψ and $\underline{\psi}$, there exists a positive constant c such that $c^{-1} \| u \|_{T,\psi} \leq \| u \|_{T,\underline{\psi}} \leq c \| u \|_{T,\psi}$ for all $u \in \mathcal{H}^0_T$. We write this as $\| u \|_{T,\psi} \approx \| u \|_{T,\underline{\psi}}$. Henceforth we write $\| u \|_T$ in place of any one of these equivalent norms.

Define the Banach space $\mathcal{H}_T$ to be the completion of $\mathcal{H}^0_T$ under the norm $\| u \|_T$. Actually $\mathcal{H}_T$ is a Hilbert space, because $\mathcal{H}^0_T$ is an inner product space under

$$(u\ , v)_T = \int_0^\infty (\psi_t(T)u\, , \psi_t(T)v) \frac{dt}{t} .$$

Exercise. Prove that $\mathcal{R}(T(I+T)^{-2}) = \mathcal{D}(T) \cap \mathcal{R}(T) \subset \mathcal{D}(T^\alpha) \cap \mathcal{R}(T^\alpha) \subset \mathcal{H}^0_T \subset \mathcal{H}$ for all $\alpha \in (0,1)$, and hence that $\mathcal{H}^0_T$ is dense in $\mathcal{H}$. Also prove that $\mathcal{R}(T(I+T)^{-2})$ is dense in $\mathcal{H}_T$.

We say that T satisfies a *quadratic estimate* if $\mathcal{H} \subset \mathcal{H}_T$ and $\| u \|_T \leq c \| u \|$ for all $u \in \mathcal{H}$, and that T satisfies a *reverse quadratic estimate* if $\mathcal{H}_T \subset \mathcal{H}$ and $\| u \| \leq c \| u \|_T$ for all $u \in \mathcal{H}_T$. Our next aim is to show that T satisfies a quadratic estimate if and only if its adjoint T^* satisfies a reverse quadratic estimate, or indeed, if and only if its dual with respect to any pairing does.

Two complex Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ form a *dual pair* $\langle \mathcal{X}, \mathcal{Y} \rangle$ if there is a bilinear or a sesquilinear form $\langle \cdot, \cdot \rangle$ on $\mathcal{X} \times \mathcal{Y}$ which satisfies the following properties

$$|\langle u\, , v \rangle| \leq c \| u \|_\mathcal{X} \| v \|_\mathcal{Y} \qquad \text{for all } u \in \mathcal{X} \text{ and all } v \in \mathcal{Y}$$

$$\| u \|_\mathcal{X} \leq c_1 \sup_{v \in \mathcal{Y}} \frac{|\langle u\, , v \rangle|}{\| v \|_\mathcal{Y}} \qquad \text{for all } u \in \mathcal{X}$$

$$\| v \|_\mathcal{Y} \leq c_2 \sup_{u \in \mathcal{X}} \frac{|\langle u\, , v \rangle|}{\| u \|_\mathcal{X}} \qquad \text{for all } v \in \mathcal{Y}$$

for some constants c, c_1 and c_2. (A sesquilinear form $\langle u\,,\,v\rangle$ is a complex valued function on $\mathcal{X}\times\mathcal{Y}$ which is linear in u and conjugate linear in v.)

Suppose $\langle\mathcal{X}\,,\,\mathcal{Y}\rangle$ is a dual pair. A *dual pair* $\langle T\,,\,T'\rangle$ *of operators of type* $S_{\omega+}$ consists of operators T of type $S_{\omega+}$ in $\mathcal{X}$ and T' of type $S_{\omega+}$ in $\mathcal{Y}$ which satisfy

$$\langle Tu\,,\,v\rangle \;=\; \langle u\,,\,T'v\rangle$$

for all $u\in\mathcal{D}(T)$ and all $v\in\mathcal{D}(T')$.

Exercise. Let T^* be the adjoint of an operator T of type $S_{\omega+}$ in a Hilbert space $\mathcal{H}$ with inner product $(\cdot\,,\cdot)$. Show that $\langle T\,,\,T^*\rangle$ is a dual pair of operators of type $S_{\omega+}$ in $\langle\mathcal{H}\,,\,\mathcal{H}\rangle$ under the pairing $\langle u\,,\,v\rangle=(u\,,\,v)$.

Exercise. If $\langle T\,,\,T'\rangle$ is a dual pair of operators of type $S_{\omega+}$ in $\langle\mathcal{X}\,,\,\mathcal{Y}\rangle$ and if $\psi\in\Psi(S^0_{\mu+})$ for some $\mu>\omega$, then

$$\langle\psi(T)u\,,\,v\rangle \;=\; \langle u\,,\,\overline{\psi}(T')v\rangle$$

for all $u\in\mathcal{X}$ and $v\in\mathcal{Y}$, where $\overline{\psi}(\zeta)=\overline{\psi(\overline{\zeta})}$ when the pairing is sesquilinear, and $\overline{\psi}=\psi$ when the pairing is linear. Moreover there exists a constant c such that $\|\,\overline{\psi}(T')\,\|\le c\,\|\,\psi(T)\,\|$ for all $\psi\in\Psi(S^0_{\mu+})$.

Example. If γ is a contour in $\mathbb{C}$ and $1\le p\le\infty$, let $L_p(\gamma)$ be the Banach space of equivalence classes of measurable complex valued functions u on γ for which

$$\|\,u\,\|_p \;=\; \left\{\int_\gamma |u(z)|^p\;|dz|\right\}^{1/p} <\infty \quad 1\le p<\infty\,,$$

or

$$\|\,u\,\|_\infty \;=\; \operatorname*{ess\,sup}_{z\in\gamma}|u(z)|<\infty\,.$$

For $1\le p\le\infty$ and $1/p+1/q=1$, $\langle L_p(\gamma)\,,\,L_q(\gamma)\rangle$ is a dual pair of Banach spaces under the bilinear pairing

$$\langle u\,,\,v\rangle \;=\; \int_\gamma u(z)v(z)\,dz\,.$$

To be specific,

$$\|\,u\,\|_p \;=\; \sup\Big\{\,|\langle u\,,\,v\rangle| : v\in L_q(\gamma)\,,\,\|\,v\,\|_q=1\,\Big\}\,.$$

Suppose $Tu=wu$ and $T'v=wv$ for some measurable function w with essential range in $S_{\omega+}$. Then $\langle T\,,\,T'\rangle$ forms a dual pair of operators of type $S_{\omega+}$ in $\langle L_p(\gamma)\,,\,L_q(\gamma)\rangle$.

Exercise. Suppose $\langle\mathcal{X}\,,\,\mathcal{Y}\rangle$ is a dual pair of Banach spaces, $\mathcal{Z}$ is a dense linear subspace of $\mathcal{Y}$, and f is a continuous function from a compact interval $[\alpha,\beta]$ to $\mathcal{X}$. Then there exists a Borel function g from $[\alpha,\beta]$ to $\mathcal{Z}$ such that $\|\,g(t)\,\|_{\mathcal{Y}}=1$ for all t and $\|\,f(t)\,\|_{\mathcal{X}}\le 2c_1\langle f(t)\,,\,g(t)\rangle$ for all t.

Theorem E. *Suppose* $\langle T\,, T'\rangle$ *is a dual pair of one–one operators of type* $S_{\omega+}$ *in a dual pair of Hilbert spaces* $\langle \mathcal{H}\,, \mathcal{K}\rangle$ *. Then* $\langle \mathcal{H}_T\,, \mathcal{K}_{T'}\rangle$ *is a dual pair of Hilbert spaces under the induced pairing.*

Proof. Choose $\psi \in \Psi(S^0_{\mu+})$ such that $\int_0^\infty \psi^2(t)t^{-1}\,dt = 1$, and let $\|u\|_T = \|u\|_{T,\psi}$ and $\|v\|_{T'} = \|v\|_{T',\psi}$. It suffices to prove the following bounds.

$$\begin{aligned}
|\langle u, v\rangle| \;&\le\; c\|u\|_T\,\|v\|_{T'} && \text{for all } u\in\mathcal{H}^0_T \text{ and all } v\in\mathcal{K}^0_{T'}\\
\|u\|_T \;&\le C_1 \sup_{v\in\mathcal{K}^0_{T'}} \frac{|\langle u, v\rangle|}{\|v\|_{T'}} && \text{for all } u\in\mathcal{H}^0_T\\
\|v\|_{T'} \;&\le C_2 \sup_{u\in\mathcal{H}^0_T} \frac{|\langle u, v\rangle|}{\|u\|_{T}} && \text{for all } v\in\mathcal{K}^0_{T'}\ .
\end{aligned}$$

On applying the Convergence Lemma, we see that if $u \in \mathcal{H}^0_T$ and $v \in \mathcal{K}^0_{T'}$, then

$$\begin{aligned}
\langle u\,,\, v\rangle \;&=\; \langle \int_0^\infty \psi_t^2(T)u\frac{dt}{t}\,,\, v\rangle\\
&=\; \int_0^\infty \langle \psi_t(T)u\,,\, \overline{\psi}_t(T')v\rangle\frac{dt}{t}\ ,
\end{aligned}$$

so that $|\langle u\,,\, v\rangle| \le c\|u\|_T\,\|v\|_{T'}$.

Now take any $u \in \mathcal{H}^0_T$. Choose $0 < \alpha < \beta < \infty$ such that

$$\frac{9}{10}\|u\|_T^2 \;\le\; \int_\alpha^\beta \|\psi_t(T)u\|^2\,\frac{dt}{t}\ .$$

By the previous exercise, there exists a Borel function g from $[\alpha,\beta]$ to $\mathcal{K}^0_{T'}$ such that

$$\|\psi_t(T)u\|^2 \;\le\; 2c_1\langle\psi_t(T)u\,,\, g(t)\rangle$$

and

$$\|g(t)\| \;=\; \|\psi_t(T)u\|$$

for all t . Let $v = \int_\alpha^\beta \overline{\psi}_t(T')g(t)t^{-1}\,dt$. Then

$$\frac{9}{10}\|u\|_T^2 \;\le\; 2c_1\int_\alpha^\beta \langle\psi_t(T)u\,,\, g(t)\rangle\frac{dt}{t} \;=\; 2c_1\langle u\,,\, v\rangle$$

and

$$\begin{aligned}
\|v\|_{T'} \;&=\; \left\{\int_0^\infty \left\|\int_\alpha^\beta \psi_\tau(T')\overline{\psi}_t(T')g(t)\frac{dt}{t}\right\|^2\frac{d\tau}{\tau}\right\}^{\frac12}\\
&\le\; \text{const.}\left\{\int_\alpha^\beta \|g\|^2\,\frac{dt}{t}\right\}^{\frac12} \qquad \text{(by Lemma E)}\\
&\le\; \text{const.}\ \|u\|_T\ .
\end{aligned}$$

Hence

$$\| u \|_T \leq C_1 \frac{|\langle u , v \rangle|}{\| v \|_{T'}}$$

for some constant C_1.

Similarly, we can show that for any $v \in \mathcal{K}^0_{T'}$, there exists $u \in \mathcal{H}^0_T$ such that

$$\| v \|_{T'} \leq C_2 \frac{|\langle u , v \rangle|}{\| u \|_T} .$$

□

Corollary E. *Suppose $\langle T , T' \rangle$ is a dual pair of one–one operators of type $S_{\omega+}$ in a dual pair of Hilbert spaces $\langle \mathcal{H} , \mathcal{K} \rangle$. Then $\mathcal{H}_T \subset \mathcal{H}$ with $\| u \| \leq c \| u \|_T$ if and only if $\mathcal{K} \subset \mathcal{K}_{T'}$ with $\| v \|_{T'} \leq c' \| v \|$.*

Proof. This follows immediately from the previous theorem. □

(F) Boundedness of holomorphic functional calculi.

We are now in a position to relate square function estimates to bounded H_∞ functional calculi.

If T is a one–one operator of type $S_{\omega+}$ in $\mathcal{H}$, and $0 \leq \omega < \mu < \pi$, then we say that T has a *bounded $H_\infty(S^0_{\mu+})$ functional calculus* if, for all $f \in H_\infty(S^0_{\mu+})$, $f(T) \in \mathcal{L}(\mathcal{H})$ and

$$\| f(T) \| \leq c_\mu \| f \|_\infty .$$

Exercise. If $\langle T , T' \rangle$ is a dual pair of operators of type $S_{\omega+}$ in a dual pair of Hilbert spaces $\langle \mathcal{H} , \mathcal{K} \rangle$, and $\mu > \omega$, show that T has a bounded $H_\infty(S^0_{\mu+})$ functional calculus if and only if T' does, and that

$$\langle f(T)u , v \rangle = \langle u , \overline{f}(T')v \rangle$$

for all $u \in \mathcal{H}$ and $v \in \mathcal{K}$, where $\overline{f}$ was defined in a previous exercise.

Theorem F. *Suppose T is a one–one operator of type $S_{\omega+}$ in $\mathcal{H}$. Then the following statements are equivalent:*

(a) T has a bounded $H_\infty(S^0_{\mu+})$ functional calculus for all $\mu > \omega$;

(b) T has a bounded $H_\infty(S^0_{\mu+})$ functional calculus for some $\mu > \omega$;

(c) $\|f(T)\| \leq c_\mu \|f\|_\infty$ *for all* $f \in \Psi(S^0_{\mu+})$ *and some* $\mu > \omega$;

(d) $T^{is} \in \mathcal{L}(\mathcal{H})$ *for all* $s \in \mathbb{R}$, *and* $\|T^{is}\| \leq c_\mu e^{\mu|s|}$ *for some* $\mu > \omega$;

(e) $\mathcal{D}(T^\alpha) = \mathcal{D}(A^\alpha)$ *and* $\mathcal{D}(T^{*\alpha}) = \mathcal{D}(B^\alpha)$ *for some* $\alpha \in (0,1)$, *where* $A = (T^*T)^{\frac{1}{2}}$ *and* $B = (TT^*)^{\frac{1}{2}}$, *with* $\|A^\alpha u\| \approx \|T^\alpha u\|$ *and* $\|B^\alpha u\| \approx \|T^{*\alpha} u\|$;

(f) $H_T = \mathcal{H}$, *and there exists* $c > 0$ *such that, for every* $u \in \mathcal{H}$,

$$c^{-1}\|u\|_T \leq \|u\| \leq c\|u\|_T .$$

Proof. Suppose that (f) holds. Then, for all $\mu > \omega$, it is a consequence of Proposition E that $\|f(T)u\| \leq c_\mu \|f\|_\infty \|u\|$ for all $u \in \mathcal{H}$ and all $f \in \Psi(S^0_{\mu+})$. Thus $\|f(T)\| \leq c_\mu \|f\|_\infty$ for all $f \in \Psi(S^0_{\mu+})$.

To obtain (a), apply this estimate, the Convergence Lemma, and the fact that every function $f \in H_\infty(S^0_{\mu+})$ is the limit of a uniformly bounded sequence of functions $f_n \in \Psi(S^0_{\mu+})$ in the sense of uniform convergence on compact subsets of $S^0_{\mu+}$. Statements (b) and (c) follow.

So also does (d), because $T^{is} = f_s(T)$ where $f_s \in H_\infty(S^0_{\mu+})$ is given by $f_s(\zeta) = \zeta^{is}$, and $\|f_s\|_\infty = e^{\mu|s|}$.

We shall prove that (d) implies (e) in the next section. We refer the reader to the literature, say [Y] or [M^c], for a proof that (e) implies (f).

A proof that (b) implies (f) which is more in the spirit of harmonic analysis goes as follows. Let ψ_k be a sequence of functions in $\Psi(S^0_{\mu+})$ such that $|\psi_k(\zeta)| \leq 2^{-k}|\zeta|(1 + 2^{-k}|\zeta|)^{-2}$ and $\sum_{k=-\infty}^{k=\infty} \psi_k^2(\zeta) = 1$. Then, for all $u \in \mathcal{H}$,

$$\begin{aligned}
\|u\|_T &\leq c_1 \left\{ \sum_{k=-\infty}^{k=\infty} \|\psi_k(T)u\|^2 \right\}^{\frac{1}{2}} && \text{(proved similarly to Proposition E)} \\
&\leq c_1 \sup_{|\alpha_k|\leq 1} \left\| \sum \alpha_k \psi_k(T)u \right\| && \text{(proved below)} \\
&\leq c_2 \sup_{|\alpha_k|\leq 1} \sup_{\zeta} \left| \sum \alpha_k \psi_k(\zeta) \right| \|u\| && \text{(by (b))} \\
&\leq c_2 \sup_{\zeta} \sum |\psi_k(\zeta)| \, \|u\| \\
&\leq c_3 \|u\| .
\end{aligned}$$

Also, T^* has a bounded $H_\infty(S^0_{\mu+})$ functional calculus, so $\|u\|_{T^*} \leq c_4 \|u\|$, and hence, by Corollary E, $\|u\| \leq c_5 \|u\|_T$ as well, thus proving (f). $\square$

The second inequality above is well known. It is proved in the following way. Let $u_k = \psi_k(T)u$. Let r_k denote the Rademacher functions on $[0,1]$. These are step functions such that $r_k(x) = \pm 1$ for all $x \in [0,1]$, and $\int_0^1 r_j(x) r_k(x) dx = \delta_{jk}$. Then

$$\begin{aligned}
\sum_k \| u_k \|^2 &= \sum_{j,k} \Big(\int_0^1 r_j(x) r_k(x)\, dx\, u_k\, ,\, u_j \Big) \\
&= \int_0^1 \Big\| \sum_k r_k(x) u_k \Big\|^2 dx \\
&\le \Big\{ \sup_{|\alpha_k| \le 1} \Big\| \sum \alpha_k u_k \Big\| \Big\}^2
\end{aligned}$$

as required.

A. Yagi first proved the equivalence of (d), (e) and (f), (with $\| u \|_T$ defined using a particular choice of ψ) [Y]. Subsequently a full statement of this theorem was given in [M^c], though various parts of it have been known for some time. Of course, quadratic estimates have a long history in harmonic analysis, some aspects of which we shall touch on later.

We conclude with the remark that, by virtue of the equivalence of (a) and (b), the definition which we made loosely at the very beginning of this lecture is perfectly alright for the case of one–one operators of type $S_{\omega+}$ in a Hilbert space. That is, we say that T has a *bounded* H_∞ *functional calculus* provided T has a bounded $H_\infty(S^0_{\mu+})$ functional calculus for some, and hence all, $\mu > \omega$.

LECTURE 4. OPERATORS WITH BOUNDED HOLOMORPHIC FUNCTIONAL CALCULI

In this lecture we investigate some classes of operators T of type $S_{\omega+}$ in a non-trivial Hilbert space $\mathcal{H}$ with inner product $(\cdot\,,\cdot)$, and also consider operators of type S_ω, where $S_\omega = S_{\omega+} \cup S_{\omega-}$.

(G) Accretive operators.

Let us first consider self-adjoint operators. A self-adjoint operator S in $\mathcal{H}$ is said to be *positive* if $(Su\,,u) > 0$ for all non-zero $u \in \mathcal{D}(S)$. Every positive self-adjoint operator is a one-one operator of type S_{0+}, and has a bounded Borel functional calculus, so it certainly has a bounded $H_\infty(S^0_{\mu+})$ functional calculus for all $\mu > 0$. Actually, the constant c_μ is 1, meaning that $\| f(S)u \| \leq \| f \|_\infty$ for all $f \in H_\infty(S^0_{\mu+})$.

Let us verify that statement (f) of Theorem F holds in this case. Let $u \in \mathcal{H}$. Then

$$\begin{aligned}
\| u \|_S &= \left\{ \int_0^\infty \| \psi_t(S)\,u \|^2 \,\frac{dt}{t} \right\}^{\frac{1}{2}} \\
&= \left\{ \int_0^\infty (\overline{\psi}_t(S)\psi_t(S)\,u\,,\,u)\,\frac{dt}{t} \right\}^{\frac{1}{2}} \\
&= \left\{ (\int_0^\infty \overline{\psi}_t\psi_t(S)\,u\,\frac{dt}{t}\,,\,u) \right\}^{\frac{1}{2}} \\
&= \kappa \| u \|
\end{aligned}$$

where $\kappa = \left\{ \int_0^\infty |\psi(t)|^2 \,\frac{dt}{t} \right\}^{\frac{1}{2}}$. Therefore, in this special case, the quadratic norm is a positive multiple of the original norm, and so $\mathcal{H}_S = \mathcal{H}$.

An operator T in $\mathcal{H}$ is *accretive* if $\operatorname{Re}(Tu\,,u) \geq 0$ for every $u \in \mathcal{D}(T)$, and T is *maximal accretive* if T is accretive and $\zeta \in \rho(T)$ whenever $\operatorname{Re}\zeta < 0$. Every bounded accretive operator on $\mathcal{H}$ is maximal accretive.

If T is maximal accretive and $\operatorname{Re}\zeta < 0$, then $\| R_T(\zeta) \| \leq |\operatorname{Re}\zeta|^{-1}$, from which it follows that T is of type $\frac{\pi}{2}$.

Every one-one maximal accretive operator T has a bounded $H_\infty(S^0_{\mu+})$ functional calculus for all $\mu > \frac{\pi}{2}$ with constant $c_\mu = 1$, meaning that $\| f(T)u \| \leq \| f \|_\infty$ for all $f \in H_\infty(S^0_{\mu+})$. This fact is essentially equivalent to von Neumann's inequality, namely that $\| g(W) \| \leq 1$ for all operators $W \in \mathcal{L}(H)$ such that $\| W \| \leq 1$, and all functions g which are holomorphic on a neighbourhood of the closed unit disc, and bounded by 1 on the disc. See Chapter XI of [RN] for a discussion of this inequality.

We present, however, a direct proof due to Edwin Franks.

Theorem G. *Suppose that T is a one–one maximal accretive operator in $\mathcal{H}$. Then T has a bounded H_∞ functional calculus, and*

$$\| f(T) \| \leq \| f \|_\infty ,$$

for every $f \in H_\infty(S^0_{\mu+})$ and all $\mu > \frac{\pi}{2}$.

Proof. Let us first suppose that T is bounded and strictly accretive, meaning that

$$\mathrm{Re}\,(Tu\,,\,u) \geq c \| u \|^2$$

for all $u \in \mathcal{H}$ and some $c > 0$.

Choose $\mu > \frac{\pi}{2}$ and $f \in \Psi(S^0_{\mu+})$. Then, since $(T^* + \zeta I)^{-1}$ is holomorphic on a neighbourhood of $\{\, \zeta \in \mathbb{C} : \mathrm{Re}\,\zeta \geq 0 \,\}$, we have

$$\begin{aligned} f(T) &= \frac{1}{2\pi i} \int_{i\mathbb{R}} f(\zeta)\,(T - \zeta I)^{-1}\, d\zeta \\ &= \frac{1}{2\pi i} \int_{i\mathbb{R}} f(\zeta)\,((T - \zeta I)^{-1} + (T^* + \zeta I)^{-1})\, d\zeta\ . \end{aligned}$$

Therefore

$$f(T)u \;=\; \frac{1}{2\pi i} \int_{i\mathbb{R}} f(\zeta)\,(T^* + \zeta I)^{-1}(T^* + T)(T - \zeta I)^{-1} u\, d\zeta$$

for all $u \in \mathcal{H}$. Note that, because of our assumptions on T, the last integral is absolutely convergent. On taking limits as usual, and applying the Convergence Lemma, we find that this formula holds for all $f \in H_\infty(S^0_{\mu+})$ and all $u \in \mathcal{H}$. In particular, setting $f = 1$,

$$u \;=\; \frac{1}{2\pi i} \int_{i\mathbb{R}} (T^* + \zeta I)^{-1}(T^* + T)(T - \zeta I)^{-1} u\, d\zeta$$

and, since the integrand is positive self–adjoint,

$$\| u \| \;=\; \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\| (T^* + T)^{\frac12} (T - i\eta I)^{-1} u \right\|^2 d\eta \right\}^{\frac12} .$$

Moreover, for all $u, v \in \mathcal{H}$,

$$\begin{aligned} |(f(T)u\,,\,v)| &= \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} f(i\eta)\,((T^* + i\eta I)^{-1}(T^* + T)(T - i\eta I)^{-1} u\,,\,v)\, d\eta \right| \\ &\leq \| f \|_\infty \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\| (T^* + T)^{\frac12} (T - i\eta I)^{-1} u \right\|^2 d\eta \right\}^{\frac12} \\ &\qquad \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\| (T^* + T)^{\frac12} (T - i\eta I)^{-1} v \right\|^2 d\eta \right\}^{\frac12} \\ &= \| f \|_\infty \| u \| \| v \| \end{aligned}$$

so $\| f(T) \| \le \| f \|_\infty$ as required.

Now let us consider the general case. Take any $\epsilon > 0$ and let $T_\epsilon = ((T + \epsilon I)^{-1} + \epsilon I)^{-1}$. Then T_ϵ is a bounded strictly accretive operator on $\mathcal{H}$. Moreover, as $\epsilon \to 0$, $(T_\epsilon - \zeta I)^{-1}$ converges in the norm topology of $\mathcal{L}(\mathcal{H})$, to $(T - \zeta I)^{-1}$. This convergence is uniform on sets of the form $\{\zeta \in S^0_{\mu+} : r \le |\zeta| \le R\}$ when $0 < r < R < \infty$. So, for every $\psi \in \Psi(S^0_{\mu+})$, $\psi(T_\epsilon)$ converges in the norm topology to $\psi(T)$ as $\epsilon \to 0$. It therefore follows from the above result for bounded strictly accretive operators, that

$$\| \psi(T) \| \le \| \psi \|_\infty$$

for every $\psi \in \Psi(S^0_{\mu+})$.

We have thus proved statement (c) of Theorem F with $c_\mu = 1$ for all $\mu > \frac{\pi}{2}$. The result follows by applying the Convergence Lemma as in the proof of that theorem. □

Exercise. Extend this theorem to prove that $\| f(T) \| \le \| f \|_\infty$ for functions such as $f_t(\zeta) = e^{-t\zeta}$, $t > 0$, which are bounded on the right half plane, but not on any bigger sector. Hence prove that e^{-tT} is a C_0 semigroup with $\| e^{-tT} \| \le 1$ for all $t > 0$.

We remark that the formula for $f(T)$ used in proving Theorem G involves a kind of Poisson integral, as opposed to the Cauchy integral we mostly use.

Exercise. Prove that every one–one maximal accretive operator T in $\mathcal{H}$ has a bounded harmonic functional calculus (where this needs to be defined). Hence prove that $\| \log T - \log T^* \| \le \pi$.

Exercise. Let W and T be two bounded operators on $\mathcal{H}$ related by the formula $T = (I - W)(I + W)^{-1}$ and hence by $W = (I - T)(I + T)^{-1}$. (Assume that -1 is not in the spectrum of either operator.) Prove that $\| W \| < 1$ if and only if T is strictly accretive.

Here is a proof of one implication. For all $u \in \mathcal{H}$,

$$\begin{aligned} \| u \|^2 - \| Wu \|^2 &= \operatorname{Re}((I - W)u\,,\,(I + W)u) \\ &= \operatorname{Re}(T(I + W)u\,,\,(I + W)u) \\ &> 0\,. \end{aligned}$$

Exercise. Derive von Neumann's inequality as a corollary of Theorem G.

In section D, we defined, for $0 \le \omega \le \frac{\pi}{2}$, T to be an ω-accretive operator in $\mathcal{H}$ if $\sigma(T) \subset S_{\omega+}$ and $(Tu\,,\,u) \in S_{\omega+}$ for all $u \in \mathcal{D}(T)$. Such an operator T is maximal accretive, and so, if it is one–one, has an H_∞ functional calculus. It is also of type $S^0_{\omega+}$, and thus, by Theorem F, for all $\mu > \omega$ there exists c_μ such that $\| f(T) \| \le c_\mu \| f \|_\infty$ for all $f \in H_\infty(S^0_{\mu+})$. We note however, that if $\mu < \frac{\pi}{2}$, then the constant c_μ may be larger than 1.

A third class of operators which occurs in applications are those given in the following exercise. These operators do not necessarily have bounded H_∞ functional calculi.

Exercise. Let $T = VS$ where S is a positive self–adjoint operator in $\mathcal{H}$, and V is a bounded invertible ω-accretive operator. Prove that T is a one–one operator of type $S_{\omega+}$. An example of such an operator is given by $Tu(x) = -b(x)\sum_{j=1}^n \frac{\partial^2 u}{\partial x_j{}^2}(x)$ in $L_2(\mathbb{R}^n)$, where $b \in L_\infty(\mathbb{R}^n, \mathbb{C})$ with $\operatorname{Re} b(x) \ge \kappa > 0$ for almost all $x \in \mathbb{R}^n$.

The theory of accretive operators was developed by T. Kato and others to study operators such as elliptic operators in divergence form with bounded measurable complex coefficients. These are defined as follows.

Consider an elliptic sesquilinear form J defined on $\mathcal{V} \times \mathcal{V}$ by

$$J[u,v] = \int_\Omega \{ \ \Sigma a_{jk} \frac{\partial u}{\partial x_k} \overline{\frac{\partial v}{\partial x_j}} + \Sigma a_k \frac{\partial u}{\partial x_k} \overline{v} + \Sigma b_j u \overline{\frac{\partial v}{\partial x_j}} + a u \overline{v} \ \} \, dx$$

where Ω is an open subset of $\mathbb{R}^n$, $\mathcal{V}$ is a closed linear subspace of the Sobolev space $H^1(\Omega)$ which contains $C_c^\infty(\Omega)$, $a_{jk}, a_k, b_j, a \in L_\infty(\Omega)$ and $\operatorname{Re} \Sigma a_{jk}(x) \zeta_k \overline{\zeta_j} \ge \kappa |\zeta|^2$ for all $\zeta = (\zeta_j) \in \mathbb{C}^n$ and some $\kappa > 0$. Let L be the operator in $L_2(\Omega)$ with largest domain $\mathcal{D}(L) \subset \mathcal{V}$ such that $J[u,v] = (Lu, v)$ for all $u \in \mathcal{D}(L)$ and all $v \in \mathcal{V}$. Then $L + \lambda I$ is a maximal accretive operator in $L_2(\Omega)$ for some positive number λ. Therefore $L + \lambda I$ has a bounded H_∞ functional calculus, and satisfies quadratic estimates.

The part of Theorem F which was actually the first to be used in studying elliptic operators, was the fact that certain operators satisfy statement (d), and hence (e), which was used to determine that the domains of their fractional powers are Sobolev spaces.

The proof that (d) implies (e) is a consequence of the following result with $B = I$, by first taking $S = A$, then interchanging the roles of S and T, and finally repeating the process for the adjoint operators.

Proposition G. *Let S and T be one–one operators of type S_{ω_S+} and S_{ω_T+} in Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively, each having bounded imaginary powers (as in (d) of Theorem F.). Let B be a bounded linear mapping from $\mathcal{H}$ to $\mathcal{K}$ such that $B(\mathcal{D}(T)) \subset \mathcal{D}(S)$ and $\| SBu \| \le c_1 \| Tu \|$. Then, for every $\alpha \in (0,1)$, $B(\mathcal{D}(T^\alpha)) \subset \mathcal{D}(S^\alpha)$ and $\| S^\alpha Bu \| \le c_\alpha \| T^\alpha u \|$ where $c_\alpha = {c_1}^\alpha \| B \|^{1-\alpha}$.*

Proof. Suppose that $c_1 > 0$. It suffices to show that, for any $u \in \mathcal{D}(SBT^{-1})$, the function

$$\phi(z) = \left\| S^z B T^{-z} u \right\| \| B \|^{z-1} c_1^{-z}$$

satisfies $|\phi(z)| \leq 1$ on the strip $\{\zeta \in \mathbb{C} : 0 \leq \mathrm{Re}\zeta \leq 1\}$. From the assumptions, we have the bounds on the lines $Re\,\zeta = 0$ and $Re\,\zeta = 1$. If $\phi(\zeta) \to 0$ at ∞ in the strip, then $|\phi(z)| \leq 1$ by the maximum modulus principle. Otherwise, consider $\phi_n(\zeta) = \phi(\zeta)e^{\zeta^2/n}e^{-1/n}$. Since $\phi(\zeta)$ is bounded, $\phi_n(\zeta) \to 0$ as $\zeta \to \infty$ in the strip. Thus, $|\phi_n(\zeta)| \leq 1$ everywhere in the strip since $|\phi_n| \leq 1$ on the boundary. This is true for all n, so $|\phi(\zeta)| \leq 1$ since $e^{\zeta^2/n}e^{-1/n} \to 1$ as $n \to \infty$. Therefore, as $\mathcal{D}(SBT^{-1})$ is dense in $\mathcal{D}(S^\alpha BT^{-\alpha})$ the result follows. □

(H) Operators of type S_ω and spectral projections.

For $0 \leq \omega < \mu < \frac{\pi}{2}$, define $S_{\omega-} = -S_{\omega+}$ and $S^0_{\mu-} = -S^0_{\mu+}$. Then define the closed and open double sectors $S_\omega = S_{\omega-} \cup S_{\omega+}$ and $S^0_\mu = S^0_{\mu-} \cup S^0_{\mu+}$, and the function spaces

$$\Psi(S^0_\mu) \subset H_\infty(S^0_\mu) \subset F(S^0_\mu) \subset H(S^0_\mu)$$

on them, exactly as before.

Let $0 \leq \omega < \frac{\pi}{2}$. An operator T in $\mathcal{H}$ is said to be of *type* S_ω if $\sigma(T) \subset S_\omega$ and, for each $\mu > \omega$,

$$\|(T - \zeta I)^{-1}\| \leq C_\mu |\zeta|^{-1}, \quad \zeta \notin S_\mu .$$

Most of the results for one–one operators of type $S_{\omega+}$ generalise directly to the case when T is a one–one operator of type S_ω. Indeed the results presented in Lecture 3 remain valid, word for word, subject to the following minor modifications. (i) The contours γ have four rays, so as to enclose both sectors. (ii) The functions $\psi \in \Psi(S^0_\mu)$ used to define $\|u\|_{T,\psi}$ cannot be identically zero on either sector. (iii) At every occurrence, replace $T(I+T)^{-2}$ by $T(I+T^2)^{-1}$, replace T^α by $(T^2)^{\alpha/2}$, and replace $T^{*\alpha}$ by $(T^{*2})^{\alpha/2}$.

The main result is, once again, that T has an H_∞ functional calculus if and only if $\mathcal{H}_T = \mathcal{H}$, with equivalence of norms.

Exercise. Prove that, if T is a one–one operator of type S_ω in $\mathcal{H}$, then T^2 is a one–one operator of type $S_{2\omega+}$. Prove also that $\mathcal{H}_T = \mathcal{H}_{T^2}$. Thus T has a bounded H_∞ functional calculus if and only if T^2 does.

Let us consider the spectral projections associated with the parts of the spectrum in each sector.

For some $\mu > \omega$, define the holomorphic functions χ_+, χ_-, sgn $\in H_\infty(S^0_\mu)$ by $\chi_+(\zeta) = 1$ if $\mathrm{Re}\,\zeta > 0$ and $\chi_+(\zeta) = 0$ if $\mathrm{Re}\,\zeta < 0$, $\chi_-(\zeta) = 1 - \chi_+(\zeta)$, and $\mathrm{sgn}(\zeta) = \chi_+(\zeta) - \chi_-(\zeta)$. Let $E_+ = \chi_+(T)$ and $E_- = \chi_-(T) \in \mathcal{C}(\mathcal{H})$, so that, by the identities of the functional calculus, $\mathcal{D}(E_+) = \mathcal{D}(E_-)$, ${E_+}^2 = E_+$, ${E_-}^2 = E_-$, $E_+E_- = 0|_{\mathcal{D}(E_+)} = E_-E_+$, $E_+ + E_- = I|_{\mathcal{D}(E_+)}$ and $E_+ - E_- = \mathrm{sgn}(T)$.

The operators E_+ and E_- form a pair of closed spectral projections corresponding to the parts of the spectra in $S_{\omega+}$ and $S_{\omega-}$ respectively, though in general they may fail to be bounded operators on $\mathcal{H}$. But, of course, if T has a bounded H_∞ functional calculus, then $E_+, E_- \in \mathcal{L}(H)$ because $\chi_+, \chi_- \in H_\infty(S_\mu^0)$.

We see therefore, that whenever $\mathcal{H}_T = \mathcal{H}$, then E_+ and E_- are bounded spectral projections on $\mathcal{H}$, and that $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, where $\mathcal{H}^+ = \mathcal{R}(E_+)$ and $\mathcal{H}^- = \mathcal{R}(E_-)$ are the corresponding spectral subspaces. (The direct sum $\oplus$ has no connotations of orthogonality.) This is an important reason for obtaining quadratic estimates.

Let us consider some special cases. If T is a one–one self–adjoint operator, then T is of type S_0, and has a bounded Borel functional calculus, so it certainly has a bounded H_∞ functional calculus, again with constants $c_\mu = 1$. In this case $\| E_\pm \| = 1$ so the decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ is an orthogonal one. We remark that there exist one–one operators of type S_0 which do not have a spectral decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ [M^cY].

Another class of operators of type S_ω consists of the ones obtained as follows. Not all such operators have a bounded H_∞ functional calculus.

Theorem H. *Let $T = VS$ where S is a one–one self–adjoint operator in $\mathcal{H}$, and V is a bounded invertible ω-accretive operator. Then T is a one–one operator of type S_ω.*

Proof. There exists a constant c such that, for $\zeta \notin S_\omega$, and $u \in \mathcal{D}(T)$,

$$\begin{aligned} c\|(T-\zeta I)u\|\,\|u\| \;&\geq\; |Im(V^{-1}(T-\zeta I)u,u)| \;=\; |Im\zeta(V^{-1}u,u)| \\ &\geq\; |(V^{-1}u,u)|\ \mathrm{dist}(\zeta,S_\omega) \qquad (\text{because } (V^{-1}u,u)\in S_\omega)\ . \end{aligned}$$

Hence, for some $C > 0$,

$$\|(T-\zeta I)u\| \;\geq\; C\ \mathrm{dist}(\zeta,S_\omega)\ \|u\|\ .$$

The dual of $(T-\zeta I)$ with respect to the pairing $\langle u\,,\,v\rangle = (V^{-1}u,v)$ of $\mathcal{H}$ with itself, is $(T'-\overline{\zeta}I)$ where $T' = V^*S$. Since T' has the same form as T, we also have

$$\|(T'-\zeta I)u\| \;\geq\; C\ \mathrm{dist}(\overline{\zeta},S_\omega)\ \|u\| \;=\; C\ \mathrm{dist}(\zeta,S_\omega)\ \|u\|\ .$$

It follows from these two estimates that $\zeta \notin \sigma(T)$ and

$$\|(T-\zeta I)^{-1}\| \;\leq\; C^{-1}(\ \mathrm{dist}(\zeta,S_\omega))^{-1}\ .$$

We conclude that T is of type S_ω. □

To conclude, let us introduce a specific operator which has this form. In the final lecture, we shall prove that it does indeed have a bounded H_∞ functional calculus, and we shall see the significance of the resulting decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$.

Let γ denote the Lipschitz curve in the complex plane which is parametrised by a Lipschitz function $g : \mathbb{R} \to \mathbb{C}$ such that $g', 1/g' \in L_\infty(\mathbb{R})$ and $g'(x) \in S_{\omega+}$ for all $x \in \mathbb{R}$.

Define the derivative of a Lipschitz function u on γ by

$$u'(z) = \lim_{\substack{h \to 0 \\ z+h \in \gamma}} \frac{u(z+h) - u(z)}{h} .$$

Next use duality to define D_γ to be the closed linear operator in $L_p(\gamma)$ with the largest domain which satisfies

$$\langle D_\gamma u \,,\, v \rangle = \langle u \,,\, iv' \rangle$$

for all Lipschitz functions v on γ with compact support. We are using the pairing

$$\langle u \,,\, v \rangle = \int_\gamma u(z) v(z)\, dz$$

defined in Section E.

Then $\langle D_\gamma \,,\, -D_\gamma \rangle$ is a dual pair of one–one operators of type S_ω in $\langle L_p(\gamma) \,,\, L_q(\gamma) \rangle$, $1 \le p \le \infty$, $1/p + 1/q = 1$.

If V denotes the isomorphism from $L_2(\gamma)$ to $L_2(\mathbb{R})$ induced by the parametrization, $(Vu)(x) = u(g(x))$, then $(VD_\gamma u)(x) = b(x)(DVu)(x)$, where $b = 1/g'$, and $D = \frac{1}{i}\frac{d}{dx}$ with domain $\mathcal{D}(D) = \{ u \in L_2(\mathbb{R}) : Du \in L_2(\mathbb{R}) \}$. Now D is a one–one self–adjoint operator, and the operator of multiplication by b is a bounded invertible ω–accretive operator in $L_2(\mathbb{R})$, so, by Theorem H, the operator $T = bD$ is a one–one operator of type S_ω in $L_2(\mathbb{R})$. It follows that the operator $D_\gamma = V^{-1}TV$ is a one–one operator of type S_ω in $L_2(\gamma)$.

Exercise. Show that, once we have proved that the operator T has a bounded H_∞ functional calculus in $L_2(\mathbb{R})$, then we can conclude that the operator D_γ has a bounded H_∞ functional calculus in $L_2(\gamma)$.

Lecture 5. Singular Integrals

Let us turn our attention to real harmonic analysis. Good references include the books [St1] and [St2] of E.M. Stein.

(I) Convolutions and the functional calculus of $-i\frac{d}{dx}$.

Let us briefly consider the L_2 theory of the gradient operator

$$D = (D_1, D_2, \ldots, D_n) = (\tfrac{1}{i}\frac{\partial}{\partial x_1}, \tfrac{1}{i}\frac{\partial}{\partial x_2}, \ldots, \tfrac{1}{i}\frac{\partial}{\partial x_n}).$$

For $1 \le p < \infty$, let $L_p(\mathbb{R}^n)$ denote the Banach space of (equivalence classes of) complex valued measurable functions u on $\mathbb{R}^n$ for which the norm

$$\| u \|_p = \left\{ \int_{\mathbb{R}^n} |u(x)|^p \, dx \right\}^{\frac{1}{p}} < \infty ,$$

and let $L_\infty(\mathbb{R}^n)$ be the Banach space of (equivalence classes of) measurable functions u on $\mathbb{R}^n$ for which the norm

$$\| u \|_\infty = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} |u(x)| < \infty .$$

(Functions which are equal almost everywhere are identified in the usual way.)

For $1 < j < n$, let D_j denote the operator $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$ in $L_p(\mathbb{R}^n)$ with domain

$$\mathcal{D}(D_j) = \left\{ u \in L_p(\mathbb{R}^n) : \frac{\partial u}{\partial x_j} \in L_p(\mathbb{R}^n) \right\}$$

where the derivative is taken in the sense of distributions.

It is well known that D_j is a closed operator. In the case when $p = 2$, Fourier theory can be used to verify this, and to construct a functional calculus of $D = (D_1, D_2, \ldots, D_n)$. Here is a brief survey of the results from Fourier theory that are needed.

The Fourier transform $\hat{u} = \mathcal{F}(u)$ of a function $u \in L_1(\mathbb{R}^n)$ is defined by

$$\hat{u}(\xi) = \mathcal{F}(u)(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x, \xi\rangle} u(x)\, dx$$

for all $\xi \in \mathbb{R}^n$. The function $\hat{u}$ is a bounded continuous function on $\mathbb{R}^n$ which satisfies $\|\hat{u}\|_\infty \leq \|u\|_1$ for all $u \in L_1(\mathbb{R}^n)$.

If $u \in L_2(\mathbb{R}^n) \cap L_1(\mathbb{R}^n)$, then $\hat{u} \in L_2(\mathbb{R}^n)$, and Parseval's identity

$$\|\hat{u}\|_2 = (2\pi)^{n/2} \|u\|_2$$

holds for all such u, and so the Fourier transform extends to an isomorphism, also called $\mathcal{F}$, from $L_2(\mathbb{R}^n)$ to $L_2(\mathbb{R}^n)$.

If $u \in L_2(\mathbb{R}^n)$, then $D_j u \in L_2(\mathbb{R}^n)$ if and only if $\xi_j \hat{u}(\xi) \in L_2(\mathbb{R}^n, d\xi)$, and

$$(D_j u)^\wedge(\xi) = \mathcal{F}(\tfrac{1}{i}\frac{\partial u}{\partial x_j})(\xi) = \xi_j \int_{\mathbb{R}^n} e^{-i\langle x, \xi\rangle} u(x)\, dx = \xi_j \hat{u}(\xi) .$$

More generally, suppose that p is a polynomial in n variables. Then $p(D)u \in L_2(\mathbb{R}^n)$ if and only if $p(\xi)\hat{u}(\xi) \in L_2(\mathbb{R}^n, d\xi)$, and

$$(p(D)u)^\wedge(\xi) = p(\xi)\hat{u}(\xi) .$$

Exercise. Use Parseval's identity to show that D_j is a closed operator in $L_2(\mathbb{R}^n)$ as claimed above. More generally, show that if p is a polynomial in n variables, then the operator $p(D)$ with domain $\mathcal{D}(p(D)) = \{u \in L_2(\mathbb{R}^n) : p(D)u \in L_2(\mathbb{R}^n)\}$ is a closed operator in $L_2(\mathbb{R}^n)$.

The joint spectrum of $D = (D_1, D_2, \ldots, D_n)$ is $\sigma_n(D) = \mathbb{R}^n$. There are various ways to define this which we need not go into here, but essentially what it means is that $\mathbb{R}^n$ is the support of the functional calculus of D.

For any function $f \in L_\infty(\mathbb{R}^n)$, there is a natural definition of $f(D)$ defined via the Fourier transform, namely

$$(f(D)\, u)^\wedge(\xi) = f(\xi)\, \hat{u}(\xi) .$$

Then $f(D)$ is a bounded operator on $L_2(\mathbb{R}^n)$ with

$$\|f(D)u\|_2 = (2\pi)^{-n/2} \|f\,\hat{u}\|_2 \leq \|f\|_\infty \|u\|_2$$

for all $u \in L_2(\mathbb{R}^n)$. This, together with the facts (i) that the mapping from functions f to operators $f(D)$ is an algebra homomorphism, and (ii) that there is agreement with the natural definition for polynomials of several variables, means that we have a bounded $L_\infty(\mathbb{R}^n)$ functional calculus of D in $L_2(\mathbb{R}^n)$.

The agreement with the natural definition of $p(D)$ for polynomials p of D is in the sense that, if f and $pf \in L_\infty(\mathbb{R}^n)$, then

$$p(D)\, f(D)\, u = (pf)(D)\, u \quad \text{for all } u \in L_2(\mathbb{R}^n) \qquad \text{and}$$
$$f(D)\, p(D)\, u = (fp)(D)\, u \quad \text{for all } u \in \mathcal{D}(p(D)) .$$

It is well known that there is a close connection between the functional calculus of D and convolution operators. For example, if $\phi \in L_1(\mathbb{R}^n)$ and $f = \hat{\phi}$, then, for almost all $x \in \mathbb{R}^n$,

$$f(D)\,u(x) \;=\; \phi * u(x) \;=\; \int_{\mathbb{R}^n} \phi(x-y)\,u(y)\,dy \;.$$

When f is not the Fourier transform of an L_1 function, it may still be possible to represent $f(D)$ as a singular convolution operator. For example, if $r_j(\xi) = -i\xi_j/\,|\xi|$ for $\xi \in \mathbb{R}^n$, then $r_j(D) = R_j$, the j'th Riesz transform. Note that $\|\,R_j u\,\|_2 \le \|\,u\,\|_2$ for all $u \in L_2(\mathbb{R}^n)$. It is well known that

$$R_j u(x) \;=\; r_j(D)u(x) \;=\; \lim_{\varepsilon\to 0} \frac{2}{\sigma_n} \int_{|x-y|>\varepsilon} \frac{x_j - y_j}{|x-y|^{n+1}}\, u(y)\,dy$$

for all $u \in L_2(\mathbb{R}^n)$ and almost all $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Here σ_n is the n dimensional volume of the unit n-sphere $\{x \in \mathbb{R}^{n+1} : |x| = 1\}$. We shall see in Section L that, if $1 < p < \infty$, then $\|\,R_j u\,\|_p \le c_p \,\|\,u\,\|_p$ for some constant c_p.

Exercise. Let

$$\Delta \;=\; \sum_{j=1}^{n} \frac{\partial^2}{\partial {x_j}^2}$$

be the Laplacian on $\mathbb{R}^n$ with domain

$$\mathcal{D}(\Delta) \;=\; W_2^2(\mathbb{R}^n) \;=\; \left\{ u \in L_2(\mathbb{R}^n) \;:\; \frac{\partial^2 u}{\partial x_j \partial x_k} \in L_2(\mathbb{R}^n) \text{ for } 1 \le j \le m,\, 1 \le k \le m \right\} .$$

(i) Prove that $-\Delta$ is a positive self-adjoint operator in $L_2(\mathbb{R}^n)$, and that, letting $|D| = \sqrt{-\Delta}$, then $R_j = -\,|D|^{-1}\,\frac{\partial}{\partial x_j} = -i\,|D|^{-1}\,D_j$.
(ii) Show that the semigroup $S_t = \exp(t\Delta)$, $t > 0$, generated by Δ is represented by

$$S_t u(x) \;=\; e^{t\Delta} u(x) \;=\; \int_{\mathbb{R}^n} k_t(x-y)u(y)\,dy$$

where

$$k_t(x) \;=\; (4\pi t)^{-n/2} e^{-|x|^2/4t}$$

for all $x \in \mathbb{R}^n$. Use the fact that $(S_t u)^\wedge(\xi) = e^{-t|\xi|^2}\hat{u}(\xi)$.

In Section G, we showed that, for any one–one self–adjoint operator T in a Hilbert space, and for any $\psi \in \Psi(S^0_{\mu+})$ where $\mu > 0$, the quadratic norm $\|\,u\,\|_T = \|\,u\,\|_{T,\psi}$ is a multiple of the original norm. That is,

$$\begin{aligned}\|\,u\,\|_T \;&=\; \left\{ \int_0^\infty \|\,\psi_t(T)\,u\,\|^2 \;\frac{dt}{t} \right\}^{\frac12} \\ &=\; \kappa\,\|\,u\,\|\end{aligned}$$

where $\kappa = \left\{\int_0^\infty |\psi(t)|^2 \frac{dt}{t}\right\}^{\frac{1}{2}}$.

Let us now consider the operator $T = -\Delta$ in $\mathcal{H} = L_2(\mathbb{R}^n)$. The quadratic norm using $\psi(\zeta) = \zeta e^{-\zeta}$ is

$$\begin{aligned} \|u\|_{-\Delta} &= \left\{\int_0^\infty \|t\Delta S_t u\|^2 \frac{dt}{t}\right\}^{\frac{1}{2}} \\ &= \left\{\int_0^\infty \int_{\mathbb{R}^n} \left|\frac{\partial}{\partial t} S_t u(x)\right|^2 t\,dxdt\right\}^{\frac{1}{2}} \\ &= \left\{\int_0^\infty \int_{\mathbb{R}^n} \left|\frac{\partial U}{\partial t}(x,t)\right|^2 t\,dxdt\right\}^{\frac{1}{2}} \end{aligned}$$

where $U(x,t) = S_t u(x)$ is the solution of the heat equation $\frac{\partial U}{\partial t}(x,t) = \Delta U(x,t)$ with initial value $U(x,0) = u(x)$, which decays as $t \to \infty$. This is in agreement with the kind of quadratic estimate typically used in harmonic analysis. This norm can also be expressed as

$$\|u\|_{-\Delta} = \left\{\int_0^\infty \int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} \theta_t(x,y)u(y)dy\right|^2 dx\frac{dt}{t}\right\}^{\frac{1}{2}}$$

where $\theta_t(x,y) = t\frac{\partial}{\partial t}k_t(x-y)$.

Exercise. Obtain other equivalent quadratic norms for $-\Delta$ by making other choices of ψ. For example, consider $\psi(\zeta) = \zeta^2 e^{-\zeta}$.

The positive self–adjoint operator $-\Delta$ has a bounded H_∞ functional calculus in $L_2(\mathbb{R}^n)$. In $L_p(\mathbb{R}^n)$, $-\Delta$ is also of type S_{0+}, and has a bounded $H_\infty(S^0_{\mu+})$ functional calculus for all $\mu > 0$. This is a consequence of the Marcinkiewicz multiplier theorem. It also follows from the more general Theorem N to be proved later.

(J) The Hilbert transform and Hardy spaces.

Let us turn our attention to the one dimensional case $n = 1$. A treatment of higher dimensional analogues of this material using Clifford analysis is discussed in [McI].

The operator $D = \frac{1}{i}\frac{d}{dx}$ with domain $W_2^1(\mathbb{R}) = \{u \in L_2(\mathbb{R}) : Du \in L_2(\mathbb{R})\}$ is a one-one self–adjoint operator in $L_2(\mathbb{R})$ with spectrum $\sigma(D) = \mathbb{R}$.

The functional calculus introduced in the previous section is now just the usual functional calculus of the self–adjoint operator D.

In particular, the closed operator $|D|$ defined by

$$(|D|\,u)^\wedge(\xi) = |\xi|\,\hat{u}(\xi)$$

also has domain $W_2^1(\mathbb{R})$ and

$$\begin{aligned} |D| &= \operatorname{sgn}(D)\,D \qquad \text{where} \\ \operatorname{sgn}(D) &= \chi_+(D) - \chi_-(D) = E_+ - E_- \,. \end{aligned}$$

Here $E_+ = \chi_+(D)$ and $E_- = \chi_-(D)$ are the spectral projections corresponding to the positive and negative parts of the spectrum, as defined in Section H. They give us the orthogonal spectral decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$. Since $\mathcal{H} = L_2(\mathbb{R})$, we write $\mathcal{H}^\pm = L_2^\pm(\mathbb{R})$, so that $L_2(\mathbb{R}) = L_2^+(\mathbb{R}) \oplus L_2^-(\mathbb{R})$, which is the *Hardy decomposition* of $L_2(\mathbb{R})$.

Note that $\operatorname{sgn}(D)$ is a bounded operator on $L_2(\mathbb{R})$ satisfying $(\operatorname{sgn}(D))^2 = I$. It is the identity on $L_2^+(\mathbb{R})$, and minus the identity on $L_2^-(\mathbb{R})$. In terms of convolutions, it is represented by

$$\operatorname{sgn}(D)u(x) = \lim_{\varepsilon\to 0} \frac{i}{\pi} \int\limits_{|x-y|>\varepsilon} \frac{1}{x-y}\, u(y)\, dy$$

for all $u \in L_2(\mathbb{R})$ and almost all $x \in \mathbb{R}$, being i times the *Hilbert transform.*

It is interesting to note that, since $E_\pm = \frac{1}{2}(\pm\operatorname{sgn}(D) + I)$ then

$$\begin{aligned} E_\pm u(x) &= \pm \lim_{\varepsilon\to 0} \frac{i}{2\pi} \int\limits_{|x-y|>\varepsilon} \frac{1}{x-y}\, u(y)\, dy + \tfrac{1}{2}\, u(x) \\ &= \pm \lim_{\delta\to 0\pm} \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{1}{x+i\delta-y}\, u(y)\, dy \end{aligned}$$

for all $u \in L_2(\mathbb{R})$ and almost all $x \in \mathbb{R}$.

Let us look at this in a different way using the semigroups generated by $-|D|$.

Given $u \in L_2(\mathbb{R})$ and $t > 0$, define $u_+(t) \in L_2(\mathbb{R})$ by

$$u_+(t) = e^{-tD} E_+ u = e^{-t|D|} E_+ u \,.$$

Then $u_+(t)$ has the properties

$$\left\{ \begin{aligned} \frac{\partial u_+}{\partial t}(t) + Du_+(t) &= 0 \qquad t > 0 \\ \lim_{t\to 0} u_+(t) &= E_+ u \\ \lim_{t\to\infty} u_+(t) &= 0 \qquad . \end{aligned} \right.$$

Also, for $t < 0$, the functions $u_-(t) \in L_2(\mathbb{R})$ defined by

$$u_-(t) = e^{tD} E_- u = e^{-t|D|} E_- u$$

satisfy the properties

$$\left\{\begin{array}{rcll} \dfrac{\partial u_-}{\partial t}(t) + Du_-(t) & = & 0 & t < 0 \\ \lim\limits_{t\to 0} u_-(t) & = & E_-u & \\ \lim\limits_{t\to\infty} u_-(t) & = & 0 & . \end{array}\right.$$

Now these partial differential equations are actually the Cauchy Riemann equations if we identify (x,t) with $x+it \in \mathbb{C}$. That is, the functions U_+ on $\mathbb{C}_+ = \{x+it : t > 0\}$ and U_- on $\mathbb{C}_- = \{x+it : t < 0\}$ defined by

$$\begin{array}{rcll} U_+(x+it) & = & u_+(t)(x)\ , & t > 0\ ,\ x \in \mathbb{R} \\ U_-(x+it) & = & u_-(t)(x)\ , & t < 0\ ,\ x \in \mathbb{R}\ . \end{array}$$

are holomorphic

Exercise. Show that the holomorphic functions U_+ and U_- on the open half spaces $\mathbb{C}_+$ and $\mathbb{C}_-$ can be represented by

$$U_\pm(x+it) = \pm\frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{1}{x+it-y}\, u(y)\, dy$$

for all $u \in L_2(\mathbb{R})$. So, in agreement with the previous expressions for $E_\pm u(x)$,

$$\lim_{t\to 0\pm} U_\pm(x+it) = E_\pm u(x)$$

for almost all $x \in \mathbb{R}$, and

$$\lim_{t\to\pm\infty} U_\pm(x+it) = 0$$

for all $x \in \mathbb{R}$.

Thus the Hardy spaces $L_2^\pm(\mathbb{R})$ consist of those functions $u_\pm \in L_2(\mathbb{R})$ which can be extended holomorphically to functions $U_\pm$ on $\mathbb{C}_\pm$ which decay at infinity.

Exercise. Give a characterisation of the quadratic norm $\| u \|_D$ in terms of these holomorphic extensions of u.

In later sections we shall generalise this material, and consider the Hardy decomposition of $L_2(\gamma)$, where γ is the graph of a Lipschitz function.

LECTURE 6. CALDERÓN–ZYGMUND THEORY

Let us consider some fundamental results concerning real or complex valued functions defined on $\mathbb{R}^n$. We follow the treatment given by E.M. Stein in his excellent books [St1, St2], and recommend that the reader consult these books, as well as those of Y. Meyer [M], M. Christ [Ch] and others, for a fuller treatment of these topics, including the historical background. Similar results also hold for functions defined on a space of homogeneous type. This is a measure space (Ω, μ) which has a metric with the property that $\mu(B(x, 2r)) \leq c\mu(B(x, r))$ for all $r > 0$ and some constant c.

Here, and subsequently, $B(x, r)$ denotes the open ball with centre x and radius r. For a subset B of $\mathbb{R}^n$, the notation $|B|$ denotes its Lebesgue measure, and ${}^cB$ its complement. In all our proofs, the constant c can change from line to line.

(K) Maximal functions and the Calderón–Zygmund decomposition.

(K1) Maximal functions.

We first investigate the maximal functions which are closely related to the theory of singular integrals.

For any locally integrable function f, and any $r > 0$, the Hardy–Littlewood maximal function Mf is defined by

$$(Mf)(x) \;=\; \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y)|dy \;.$$

The main properties of the Hardy–Littlewood maximal functions are as follows.

Theorem K1. *Let f be a complex valued function defined on $\mathbb{R}^n$.*

(a) If $f \in L_p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, then the function Mf is finite almost everywhere.

(b) If $f \in L_1(\mathbb{R}^n)$, then for every $\alpha > 0$

$$|\{x : (Mf)(x) > \alpha\}| \;\leq\; \frac{c}{\alpha} \int_{\mathbb{R}^n} |f(y)|dy$$

where c is a constant which depends only on the dimension n.

(c) If $f \in L_p(\mathbb{R}^n)$, $1 < p < \infty$, then $Mf \in L_p(\mathbb{R}^n)$ and

$$\| Mf \|_p \;\leq\; c_p \| f \|_p$$

where c_p depends on p and the dimension n.

Property (b) is referred to as a weak type $(1,1)$ estimate.

Corollary K1. *If f is a locally integrable function, then*

$$\lim_{r\to 0} |B(x,r)|^{-1} \int_{B(x,r)} f(y)dy \;=\; f(x)$$

almost everywhere.

The proof of the corollary is left as an exercise. See [St1], Chapter 1.

Before proving the theorem, we first state an improved version of the covering lemma of Vitali. We say that a family of sets $\{B_\alpha\}$ covers a set E provided $E \subset \cup_\alpha B_\alpha$.

Lemma K1. *Let E be a measurable subset of $\mathbb{R}^n$ which is covered by a family $\{B_\alpha\}$ of balls of bounded diameter. Then there exists a disjoint sequence (B_k) of these balls so that*

$$\sum_k |B_k| \;\geq\; c|E|$$

where c is a positive constant which depends only on the dimension n ($c = 5^{-n}$ will do).

For a proof of the lemma, see [St1], Chapter 1.

Proof of Theorem K1. We shall prove the theorem by showing that the inequalities hold even for the larger uncentred maximal function $\mathcal{M}f$ which is defined by

$$(\mathcal{M}f)(x) \;=\; \sup_{x\in B} \frac{1}{|B|} \int_B |f(y)|dy \;,$$

where the supremum is taken over all balls B containing x.

For $\alpha > 0$, let $E_\alpha = \{x \in \mathbb{R}^n : (\mathcal{M}f)(x) > \alpha\}$. The definition of $\mathcal{M}f$ implies that for each $x \in E_\alpha$ there exists a ball B_x which contains x so that

$$\int_{B_x} |f(y)|dy \;>\; \alpha|B_x| \,.$$

Thus $|B_x| < \frac{1}{\alpha}\|f\|_1$. The family $\{B_x : x \in E_\alpha\}$ covers E_α. Using the covering lemma above, there exists a mutually disjoint family of balls B_k such that

$$\sum_0^\infty |B_k| \;\geq\; c|E_\alpha| \,.$$

Therefore

$$\| f \|_1 \geq \int_{\cup B_k} |f(y)|dy > \alpha \sum_k |B_k| \geq \alpha c|E|$$

which proves (b).

We now prove (a) and (c) simultaneously.

The case $p = \infty$ is obvious with $c_p = 1$. Hence we suppose that $1 < p < \infty$.

Define $f_1(x) = f(x)$ if $|f(x)| \geq \alpha/2$, and $f_1(x) = 0$ otherwise. Note that $f_1 \in L_1(\mathbb{R}^n)$ if $f \in L_p(\mathbb{R}^n)$. We also have $(\mathcal{M}f)(x) \leq (\mathcal{M}(f_1))(x) + \alpha/2$, so

$$|E_\alpha| \leq |\{x \in \mathbb{R}^n : (\mathcal{M}(f_1))(x) > \alpha/2\}| \leq \frac{2c}{\alpha} \| f_1 \|_1 = \frac{2c}{\alpha} \int_{|f|>\alpha/2} |f| dx .$$

We now set $g = \mathcal{M}f$ and define the distribution function of g as $\lambda(\alpha) = |\{x : |g(x)| > \alpha\}|$. Then

$$\int_{\mathbb{R}^n} (\mathcal{M}f)^p dx = -\int_0^\infty \alpha^p d\lambda(\alpha) = p \int_0^\infty \alpha^{p-1} \lambda(\alpha) d\alpha .$$

Note that the first equality comes from the definition of the distribution function, the second equality from integration by parts.

Substituting the bound of $|E_\alpha|$ into the last integral, we obtain

$$\| \mathcal{M}f \|_p^p \leq p \int_0^\infty \alpha^{p-1} \left[\frac{2c}{\alpha} \int_{|f|>\alpha/2} |f(x)| dx \right] d\alpha .$$

We evaluate the double integral by changing the order of integration, then integrating first with respect to α. Simple calculations show that this gives assertion (c) with the constant $c_p = 2 \left[\frac{5^n p}{p-1} \right]^{1/p}$. $\square$

(K2) The Calderón–Zygmund decomposition.

The Calderón–Zygmund decomposition plays an important role in the real analysis of singular integrals. The idea is to split an integrable function into its "small" and "large" parts, then analyse each part using different techniques.

We first state a lemma using the covering idea of Whitney.

Lemma K2. *Given a closed, nonempty subset E of $\mathbb{R}^n$, then its complement ${}^cE$ is the union of a sequence of cubes Q_k, whose sides are parallel to the axes, whose* *interiors are mutually disjoint, and whose diameters are approximately proportional to their distances from* E *. More explicitly:*

(a) $\bigcup_k Q_k = {}^cE$;

(b) the interiors of Q_k *and* Q_j *are disjoint if* $k \neq j$;

(c) there exist positive constants c_1 *and* c_2 *so that*

$$c_1 d(Q_k) \leq \delta(Q_k, E) \leq c_2 d(Q_k)$$

where $d(Q_k)$ *denotes the diameter of* Q_k *and* $\delta(Q_k, E)$ *is the distance from* Q_k *to* E .

For the proof of this lemma, see [St1], Chapter 1.

The following theorem is referred to as the Calderón–Zygmund decomposition.

Theorem K2. *Suppose* $f \in L_1(\mathbb{R}^n)$ *and* $\alpha > 0$ *. Then there exists a decomposition of* f ,

$$f = g + b = g + \sum_k b_k ,$$

so that

(i) $|g(x)| \leq c\alpha$ *for almost all* $x \in \mathbb{R}^n$;

(ii) there exists a sequence of cubes Q_k *with mutually disjoint interiors so that the support of each* b_k *is contained in* Q_k *and*

$$\int |b_k(x)| dx \leq c\alpha |Q_k| \quad \text{and}$$
$$\int b_k(x) dx = 0 ;$$

(iii) $\sum_k |Q_k| \leq \frac{c}{\alpha} \int |f(x)| dx$

where the constant c *depends only on the dimension* n .

Proof.

We use the uncentred maximal function $\mathcal{M}$ as in the proof of Theorem K1 and define $E_\alpha = \{x \in \mathbb{R}^n : (\mathcal{M}f)(x) > \alpha\}$. Then E_α is an open set and its complement is nonempty. Using the above covering lemma of Whitney, there exists a family of cubes Q_k so that $\bigcup_k Q_k = E_\alpha$ and the interiors of the sets Q_k are mutually disjoint.

Define $g(x) = f(x)$ for $x \in {}^cE_\alpha$ and

$$g(x) = |Q_k|^{-1} \int_{Q_k} f(y)dy$$

if $x \in Q_k$. It follows that $f = g + \sum b_k$, where

$$b_k(x) = \mathcal{X}_{Q_k} \left[f(x) - |Q_k|^{-1} \int_{Q_k} f(y)dy \right]$$

and $\mathcal{X}_{Q_k}$ denotes the characteristic function of Q_k.

By Corollary K1, we have $|f(x)| \leq \alpha$ for almost all $x \in {}^cE_\alpha$. Let B_k be the cube with the same centre as Q_k but whose sides are expanded by the factor c_2 in Lemma K2. Then

$$|B_k|^{-1} \int_{B_k} |f(x)|dx \leq \alpha$$

because the cube B_k intersects ${}^cE_\alpha$.

Since $|B_k| = (c_2)^n |Q_k|$, it follows that $|g(x)| \leq c\alpha$, hence (i) is proved.

It is straightforward from its definition that b_k is supported in Q_k and has the average value 0. Also

$$\int_{\mathbb{R}^n} |b_k|dx \leq 2 \int_{Q_k} |f(x)|dx \leq \alpha |B_k| \leq c\alpha |Q_k| ,$$

thus (ii) is proved.

Property (iii) follows from the weak type $(1,1)$ estimate of the maximal function. □

(L) Singular Integral Operators.

In this section, we investigate singular integral operators T which are expressible in the form

$$(Tf)(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy$$

in some sense, where the kernel may be singular near $x = y$.

Because of this singularity, some care needs to be taken with the above expression. Let us call T a *Calderón–Zygmund singular integral operator* with kernel K if

(1) the operator T is bounded on $L_2(\mathbb{R}^n)$ with norm C_2;

(2) the kernel K is a measurable function with the property that for each continuous function f with compact support, and for almost all x not in the support of f, then $(Tf)(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy$;

(3) there exists a constant $C > 1$ and $0 < \gamma \leq 1$ so that

$$|K(x,y)| \leq \frac{C}{|x-y|^n},$$

$$|K(x,y) - K(x_1,y)| \leq C\frac{|x-x_1|^\gamma}{|x-y|^{n+\gamma}}$$

for $|x - x_1| \leq |x-y|/2$, and

$$|K(x,y) - K(x,y_1)| \leq C\frac{|y-y_1|^\gamma}{|x-y|^{n+\gamma}}$$

for $|y - y_1| \leq |x-y|/2$.

Note that the above condition (3) implies the following condition (3').

(3') There exist constants δ, C so that

$$\int_{{}^cB(y,\delta r)} |K(x,y) - K(x,y_1)|dx \leq C'$$

whenever $y_1 \in B(y,r)$, for all $y \in \mathbb{R}^n$, and

$$\int_{{}^cB(x,\delta r)} |K(x,y) - K(x_1,y)|dy \leq C'$$

whenever $x_1 \in B(x,r)$, for all $x \in \mathbb{R}^n$.

The main result is the following.

Theorem L. *Under the above assumptions, for any p, $1 < p < \infty$, the operator T can be extended to a bounded operator on $L_p(\mathbb{R}^n)$ with the norm bound C_p depending on p, the constants C_2, C and γ in assumptions* (1) *and* (3).

For $p = 1$, the operator T is of weak type $(1,1)$.

Proof. Let us prove that T is of weak type $(1,1)$, using assumption (1) and the first estimate in (3'). The result then follows from the Marcinkiewicz interpolation theorem for $1 < p < 2$, and then the standard duality argument for $2 < p < \infty$.

For $f \in L_1(\mathbb{R}^n) \bigcap L_2(\mathbb{R}^n)$ and $\alpha > 0$, decompose f as $f = g + b = g + \sum_k b_k$ by the Calderón–Zygmund decomposition of Theorem K2. Then

$$|\{x : |Tf(x)| > \alpha\}| \leq |\{x : |Tg(x)| > \alpha/2\}| + |\{x : |Tb(x)| > \alpha/2\}|.$$

It is not difficult to check that $g \in L_2(\mathbb{R}^n)$. Using the facts that T is bounded on $L_2(\mathbb{R}^n)$ and that $|g(x)| \le c\alpha$, we obtain

$$|\{x : |Tg(x)| > \alpha/2\}| \;\le\; c_1\alpha^{-2} \| Tg \|_2^2 \;\le\; c_2\alpha^{-2} \| g \|_2^2 \;\le\; \frac{c_3}{\alpha} \| f \|_1 \;.$$

On the other hand

$$|\{x : |Tb(x)| > \alpha/2\}| \;\le\; \sum_k |B_k| + \sum_k \frac{1}{\alpha} \int_{{}^cB_k} |Tb_k(x)| dx$$

where B_k is the cube with the same centre y_k as that of the cube Q_k in the Calderón–Zygmund decomposition but with sides multiplied by the constant δ in assumption (3'). Because of property (iii) of the decomposition, $\sum_k |B_k| \le c_4\alpha^{-1} \| f \|_1$.

Since the average value of b_k is 0, we have

$$Tb_k(x) \;=\; \int [K(x,y) - K(x,y_k)] b_k(y) dy \;.$$

It then follows from the first estimate in (3') and the estimates (ii), (iii) of the decomposition that

$$\sum_k \int_{{}^cB_k'} |Tb_k(x)| dx \;\le\; c_5 \| f \|_1 \;.$$

On combining these estimates, we find that

$$|\{x : |Tf(x)| > \alpha\}| \;\le\; \frac{c_4}{\alpha} \| f \|_1$$

for all $f \in L_1(\mathbb{R}^n) \bigcap L_2(\mathbb{R}^n)$. We leave it to the reader to complete the proof by proving this weak type $(1,1)$ estimate for all $f \in L_1(\mathbb{R}^n)$. □

Many important operators in harmonic analysis are Calderón–Zygmund operators, the most important being the Hilbert transform and the Riesz transforms. Thus these operators are bounded on $L_p(\mathbb{R})$ and $L_p(\mathbb{R}^n)$ respectively when $1 < p < \infty$. Actually it was these operators that were studied first.

The Riesz transforms play an important role in the analysis of partial differential equations. For example, it follows from the equation

$$\frac{\partial^2 u}{\partial x_j \partial x_k} \;=\; -R_j R_k \Delta u$$

and the boundedness of the Riesz transforms, that the Laplacian controls all second order partial derivatives in the L_p norm. That is, $\left\| \frac{\partial^2 u}{\partial x_j \partial x_k} \right\|_p \le c_p \| \Delta u \|_p$ for all j and k.

More consequences of Theorem L will be given in the forthcoming lectures, arising from functional calculi of elliptic operators in Lecture 7, and from convolution singular integrals on Lipschitz curves γ in Lecture 8.

Exercise L. Generalise Theorem L in the following way.

Let T be a bounded operator on $L_2(\mathbb{R}^n)$. Suppose that T_k is a sequence of bounded operators associated with kernels K_k in the sense of condition (2), such that the kernels K_k satisfy condition (3) with the constants C and γ independent of k. Assume that for each $f \in L_2(\mathbb{R}^n) \bigcap L_1(\mathbb{R}^n)$ there is a subsequence T_{k_j} such that

$$(Tf)(x) = \lim_{j\to\infty} (T_{k_j} f)(x)$$

for almost all $x \in \mathbb{R}^n$.

Prove that T is of weak type $(1,1)$.

Conclude that T can be extended to a bounded operator on $L_p(\mathbb{R}^n)$ with the norm bound C_p depending on p, the constants C and γ, and the L_2 norm of T.

LECTURE 7. FUNCTIONAL CALCULI OF ELLIPTIC OPERATORS

(M) Heat kernel bounds.

Let Ω be a measure space equipped with a distance d and measure ρ. Assume that T is an operator of type $S_{\omega+}$ in $L_2(\Omega)$ with $\omega < \frac{\pi}{2}$. Then $-T$ generates a holomorphic semigroup e^{-tT}. If, for all $t > 0$, these operators can be represented by kernels $k_t(x, y)$, then these kernels are called the heat kernels of T.

We say that the heat kernel of T satisfies a Gaussian upper bound if

$$|k_t(x,y)| \leq \frac{C}{\rho(x,y,t^{1/m})} e^{-c[d(x,y)^m/t]^{1/(m-1)}}$$

where c, C and m are positive constants, and $\rho(x, y, \tau)$ denotes the maximum volume of the two balls with centres x and y and with radius τ.

In many applications, the heat kernel of such an operator also satisfies, for some $0 < \alpha \leq 1$, a Hölder continuity estimate in the variable y:

$$|k_t(x,y) - k_t(x,y_1)| \leq \frac{C[d(y,y_1)]^\alpha}{\rho(x,y,t^{1/m})t^\alpha} e^{-c[d(x,y)^m/t]^{1/(m-1)}}$$

whenever $d(y, y_1) \leq d(x, y)/2$, and a similar estimate in the variable x:

$$|k_t(x,y) - k_t(x_1,y)| \leq \frac{C[d(x,x_1)]^\alpha}{\rho(x,y,t^{1/m})t^\alpha} e^{-c[d(x,y)^m/t]^{1/(m-1)}}$$

whenever $d(x, x_1) \leq d(x, y)/2$.

We remark that there do exist interesting operators which possess Gaussian bounds on their heat kernels, but not the above Hölder continuity estimates.

We now give a list of examples:

(1) In Chapter 5, we saw that the heat kernel of the Laplacian

$$\Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}$$

on $\mathbb{R}^n$ is given explicitly by

$$k_t(x,y) = (4\pi t)^{-n/2} e^{-|x-y|^2/4t}$$

for all $t > 0$. In this case, the above formula for the heat kernel also extends to all complex value t with $\mathrm{Re}\, t > 0$. This formula shows that the heat kernel has the Gaussian bound with $m = 2$ and the corresponding upper bound on its derivatives (with respect to space variables), hence the Hölder continuity estimates hold for both variables x and y.

(2) If Ω is a region in $\mathbb{R}^n$ with smooth boundary and T is the Laplacian on $L_2(\Omega)$ subject to Dirichlet boundary conditions, then it follows from the Feynman–Kac formula that its heat kernel $k_t(x,y)$ is a positive, C^∞ function which satisfies

$$0 \;<\; k_t(x,y) \;\le\; (4\pi t)^{-n/2} e^{-|x-y|^2/4t}$$

for $x \neq y$ and $t > 0$.

The Gaussian bound for complex t with $\mathrm{Re}\, t > 0$ can also be obtained by analytic continuation. See, for example, [D]. The Hölder continuity estimates are also true.

(3) Let V be a nonnegative function on $\mathbb{R}^n$ such that the closed quadratic form

$$Q(f) \;=\; \sum_{j=1}^{n} \| \partial_j f \|^2 + \left\| V^{\frac{1}{2}} \right\|^2$$

is densely defined. There exists a unique positive operator T such that $(Tf\,,\, f) = Q(f)$ and $\mathcal{D}(T^{\frac{1}{2}}) = \mathcal{D}(Q)$. That operator T is called the Schrödinger operator with potential V, usually written as $T = -\Delta + V(x)$. The Trotter formula shows that the heat kernel of T (for $t > 0$) is positive and satisfies the Gaussian upper bound. However, unless V satisfies additional conditions, the heat kernel can be a discontinuous function of the space variables and the Hölder continuity estimates may fail to hold.

(4) Let $A(x,D)$ be an elliptic operator of even order m defined in a bounded domain Ω of $\mathbb{R}^n$ with smooth boundary, and let $B_j(x,D)$, $j = 1,\ldots,m/2$, be operators of order $m_j < m$ defined on the boundary of Ω. We assume

(i) the system $A(x,D), \{B_j(x,D)\}_{j=1}^{m/2}$, as well as its formal adjoint system, are regular on Ω in the sense of Agmon;

(ii) there is an angle $\mu \in (0, \frac{\pi}{2})$ such that the system $e^{i\theta} D_t^m - A(x, D_x)$, $\{B_j(x,D_x)\}_{j=1}^{m/2}$ is an elliptic boundary value problem on $\Omega \times (-\infty < t < \infty))$ which satisfies the coerciveness condition for any $\mu \le \theta \le 2\pi - \mu$.

Let T be the operator with domain

$$\mathcal{D}(T) \;=\; \{ u \in W_2^m(\Omega) : B_j(x,D)u = 0 \text{ on } \partial\Omega, j = 1, \ldots m/2 \}$$

which is defined by $(Tu)(x) = A(x,D)u(x)$ for all $u \in \mathcal{D}(T)$. It is known that the operator defined analogously by the formally constructed adjoint system coincides with the adjoint of T.

Under these assumptions, $T+c_0I$, where c_0 is a sufficiently large positive constant, generates a holomorphic semigroup which possess a heat kernel with Gaussian bound. The Hölder continuity estimates also hold. See [T].

(5) Let $X_1, ..., X_k$ be smooth vector fields on $\mathbb{R}^n$ satisfying the Hörmander subelliptic condition uniformly over $\mathbb{R}^n$, which means the rank of the Lie algebra generated by the vector fields is constant. Let d be the control distance corresponding to the vector fields and ρ the Haar measure. Then the space $(\mathbb{R}^n, d, \rho)$ is of polynomial type with the dimension at infinity n and the local dimension $n' \geq n$. Next let

$$T = -\sum_{i=1}^{k} X_i^* X_i$$

denote the self-adjoint operator associated with the Dirichlet form

$$Q(\phi) = \sum_{i=1}^{k} \| X_i \phi \|_2^2$$

on $L_2(\mathbb{R}^n, d, \rho)$ with domain $\mathcal{D}(Q) = \{u \in L_2(\mathbb{R}^n; d; \rho) : X_i u \in L_2(\mathbb{R}^n; d; \rho) \text{ for all } i\}$.

The operator T then has a heat kernel which satisfies a Gaussian bound with $m = 2$. (The small t bounds were first derived by Sanchez–Calle and the large t bounds were subsequently established by Kusuoka and Stroock). Corresponding bounds on the "first order derivatives" are also true, hence the Hölder continuity estimates hold. See [SC], [R].

(6) Let Ω be a bounded open connected subset of $\mathbb{R}^n$. Let $a = (a_{ij})$ be an $n \times n$ matrix with entries $a_{ij} \in L_\infty(\Omega)$ satisfying $\mathrm{Re} \sum a_{ij}(x)\zeta_i\zeta_j \geq \lambda |\zeta|^2$ for all $x \in \Omega, \zeta \in \mathbb{C}$ and some $\lambda > 0$, and define the quadratic form

$$Q(\phi) = \sum_{i,j=1}^{n} (\partial_i \phi, a_{ij} \partial_j \phi)$$

where the domain $\mathcal{D}(Q)$ is the closure of $C_c^\infty(\Omega)$ with respect to the norm $(Q(\phi) + \| \phi \|^2)^{\frac{1}{2}}$. The associated operator corresponds to the strongly elliptic operator

$$T = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j}$$

with Dirichlet boundary conditions. This operator is ω accretive for some $\omega < \frac{\pi}{2}$.

In the case the coefficients a_{ij} are real, then the heat kernel of T satisfies the Gaussian bound (see [GW] for the case of symmetric coefficients). The Hölder continuity estimates also hold (see [DMc]).

If $\Omega = \mathbb{R}^n$, the Gaussian bound on the heat kernel is still true in the case of real coefficients. See [A], [AMcT]. However, for complex coefficients, there are examples to show that the Gaussian bound may fail to hold in dimensions $n \geq 5$.

(N) Bounded H_∞ functional calculi in L_p spaces.

Let T be a differential operator of type $S_{\omega+}$ in $L_p(\Omega)$. It depends on the operator and the structure of the space that various methods can be used to establish a bounded functional calculus for T. Here we try to give the reader a sketch of these methods.

(1) Fourier multiplier theory: Let T be a (higher order) elliptic partial differential operator with constant coefficients on the Euclidean space $\mathbb{R}^n$, or the half space $\mathbb{R}^n_+$, with Dirichlet boundary conditions. Using the Fourier transform, we can establish the bounds on the symbols of the resolvents. Then a bounded holomorphic functional calculus of T on L_p spaces follows from the Mikhlin's multiplier theorem. This method is restricted to the case of constant coefficients. However, by combining it with a perturbation argument, we can obtain a bounded H_∞ functional calculus when T is a second order elliptic operator on $\mathbb{R}^n$ with Hölder continuous coefficients. See [PS] for the case of $f(T) = T^{is}, s \in \mathbb{R}$.

(2) Pseudo–differential operators: This technique can be used to establish the bounded H_∞ functional calculus of a general elliptic operator with smooth coefficients, acting on q-tuples of functions on a compact manifold with smooth boundary. See [Se], [Du]. In the case of $\mathbb{R}^n$, by using a perturbation argument, the smoothness conditions can be reduced to Hölder continuous coefficients. See [AHS].

(3) Transference method: This technique can be applied to prove the boundedness of $f(T)$ when T is a second order elliptic partial differential operator on an L_p space of a strongly Lipschitz domain. However, this method can only be used to obtain an $H_\infty(S^0_{\mu+})$ functional calculus when $\mu > \frac{\pi}{2}$. See [Du].

(4) Singular integral theory: This is a powerful technique and it is our aim in this section to show how Calderón–Zygmund theory can be used to establish bounded H_∞ functional calculi on L_p spaces.

The main result is the following theorem.

Theorem N. *Assume T is a one–one operator of type $S_{\omega+}$ in $L_2(\mathbb{R}^n)$, $0 \le \omega < \pi/2$, which satisfies the following conditions:*

(1) T has a bounded H_∞ functional calculus in $L_2(\mathbb{R}^n)$ for all $f \in H_\infty(S^0_{\mu+})$.

(2) There exists $\mu > \omega$ so that the kernel $k_z(x,y)$ of the holomorphic semigroup e^{-zT} associated with T satisfies the Hölder continuity estimate:

$$|k_z(x,y) - k_z(x,y_1)| \le c|y_1 - y|^\alpha |z|^{-(n/m+\alpha)} exp(-[\alpha|x-y|^m/|z|]^{1/m-1})$$

for all $z \in S^0_{(\frac{\pi}{2}-\mu)+}$, and some $0 < \alpha \le 1$.

Then if $f \in H_\infty(S^0_\nu)$, for any $\nu > \mu$, the operator $f(T)$ is of weak type $(1,1)$. For any p, $1 < p \le 2$, $f(T)$ can be extended to a bounded operator on $L_p(\mathbb{R}^n)$ with *norm at most* $c\| f \|_\infty$. *A similar result is true for* $2 < p < \infty$ *if the Hölder continuity estimate in condition (2) holds for variable* x *instead of* y.

Proof. First consider functions $f \in \Psi(S^0_{\mu+})$.

To obtain the estimates for $f(T)$, we represent the operator $f(T)$ by using the semigroup e^{-zT}. As in Section D, the operator $f(T)$ on L_2 is given by

$$f(T) \;=\; \frac{1}{2\pi i}\int_\gamma (T-\lambda I)^{-1} f(\lambda)d\lambda\ ,$$

where the contour $\gamma = \gamma_+ \cup \gamma_-$ is given by $\gamma_+(t) = te^{i\nu}$ for $t \geq 0$ and $\gamma_-(t) = -te^{-i\nu}$ for $t \leq 0$.

For $\lambda \in \gamma_+$, substitute

$$(T-\lambda I)^{-1} \;=\; \int_{\Gamma_+} e^{\lambda z} e^{-zT} dz$$

where the curve Γ_+ is defined by $\Gamma_+(t) = te^{(\pi-\nu)/2}$ for $t \geq 0$. We also have similar expression for $\lambda \in \gamma_-$.

We then change the order of integration and obtain

$$f(T) \;=\; \int_{\Gamma_+} e^{-zT} n_+(z)dz + \int_{\Gamma_-} e^{-zT} n_-(z)dz\ ,$$

where

$$n_\pm \;=\; \frac{1}{2\pi i}\int_{\gamma_\pm} e^{\lambda z} f(\lambda)d\lambda\ ,$$

which implies the bound

$$|n_\pm(z)| \;\leq\; c\| f \|_\infty |z|^{-1}$$

Consequently, the kernel $K_f(x,y)$ is given by

$$K_f(x,y) \;=\; \int_{\Gamma_+} k_z(x,y) n_+(z)dz + \int_{\Gamma_-} k_z(x,y) n_-(z)dz\ .$$

Using this representation and the Hölder continuity estimate of the heat kernel, elementary but tedious calculations show that we obtain

$$\int_{{}^cB(y,\delta r)} |K_f(x,y) - K_f(x,y_1)|dx \;\leq\; C$$

and our theorem follows from Theorem L for $f \in \Psi(S^0_{\mu+})$. Note that the constant C in the above estimate does not depend on how fast the function f decays at 0 or ∞.

To extend the result to $f \in H_\infty(S^0_{\mu+})$, take

$$f_k(\lambda) = f(\lambda)\frac{\lambda^{1/k}}{(1+\lambda)^{2/k}} .$$

By the Convergence Lemma (Theorem D), $f_k(T)u$ converges to $f(T)u$ in the L_2 norm for all $u \in L_2(\mathbb{R}^n)$. Hence there is a subsequence k_i so that $f_{k_i}(T)u(x)$ converges to $f(T)u(x)$ almost everywhere. The theorem then follows from the above estimates for $f \in \Psi(S^0_{\mu+})$ and Exercise L.

The case $2 < p < \infty$ follows from duality. □

Exercise. Let $f(T)_p$ denote the extension of $f(T)$ to $L_p(\mathbb{R}^n)$. Show that $f(T)_p = f(T_p)$ for a suitably defined operator T_p of type $S_{\omega+}$ in $L_p(\mathbb{R}^n)$. Thus T_p has a bounded $H_\infty(S^0_{\mu+})$ functional calculus in $L_p(\mathbb{R}^n)$ for all $\mu > \omega$.

Notes.

(1) The theorem is still true when $\mathbb{R}^n$ is replaced by a space of homogeneous type.

(2) The operators in examples (1), (2), (3), (5) and (6) (with real, symmetric coefficients) of Section M are self–adjoint in L_2. The operators in (4) (under suitable conditions) and (6) with complex coefficients are maximal accretive. Hence they all have bounded functional calculi in L_2. Therefore, for all the above operators, the Hölder continuity estimates imply the existence of bounded functional calculi in L_p.

(3) It was proved recently that the above theorem is still true if the condition of Hölder continuity estimates is replaced by an upper bound on the heat kernel which satisfies certain conditions, in which the Gaussian bound is a special case. See [DR].

LECTURE 8. SINGULAR INTEGRALS ON LIPSCHITZ CURVES

In this chapter we consider harmonic analysis on Lipschitz curves. Those who are interested can find a survey of higher dimensional analogues of this material using Clifford analysis in [M^cI].

(O) Convolutions and the functional calculus of $-i\frac{d}{dz}|_\gamma$.

Let us recall the following material from the end of Section H.

Let γ denote the Lipschitz curve in the complex plane which is parametrised by a Lipschitz function $g : \mathbb{R} \to \mathbb{C}$ such that $g', 1/g' \in L_\infty(\mathbb{R})$ and $g'(x) \in S_{\omega+}$ for almost all $x \in \mathbb{R}$.

Define the derivative of a Lipschitz function u on γ by

$$u'(z) = \lim_{\substack{h\to 0\\ z+h\in\gamma}} \frac{u(z+h)-u(z)}{h}$$

for almost all $z \in \gamma$. Next use duality to define D_γ to be the closed linear operator in $L_p(\gamma)$ with the largest domain which satisfies

$$\langle D_\gamma u\,,\, v\rangle = \langle u\,,\, iv'\rangle$$

for all Lipschitz functions v on γ with compact support. We are using the pairing

$$\langle u\,,\, v\rangle = \int_\gamma u(z)v(z)\,dz$$

defined in Section E.

Then $\langle D_\gamma\,, -D_\gamma\rangle$ is a dual pair of one–one operators of type S_ω in $\langle L_p(\gamma)\,, L_q(\gamma)\rangle$, $1 \le p \le \infty$, $1/p + 1/q = 1$. This can be seen by giving an explicit formula for the resolvent. See [M^cQ]. Alternatively, if $p = q = 2$, we can proceed as follows.

If V denotes the isomorphism from $L_2(\gamma)$ to $L_2(\mathbb{R})$ induced by the parametrization, $(Vu)(x) = u(g(x))$, then $(VD_\gamma u)(x) = b(x)(DVu)(x)$, where $b = 1/g'$, and $D = \frac{1}{i}\frac{d}{dx}$ with domain $\mathcal{D}(D) = W_2^1(\mathbb{R}) = \{u \in L_2(\mathbb{R}) : Du \in L_2(\mathbb{R})\}$. Now D is a one–one self–adjoint operator, and the operator of multiplication by b is a bounded invertible ω-accretive operator in $L_2(\mathbb{R})$, so by Theorem H, the operator $T = bD$ is a one–one operator of type S_ω in $L_2(\mathbb{R})$. It follows that the operator $D_\gamma = V^{-1}TV$

is a one–one operator of type S_ω in $L_2(\gamma)$. It is not hard to verify that the dual of D_γ is $-D_\gamma$.

We have seen that every one–one operator of type S_ω has a holomorphic functional calculus, meaning, in this case, that there is a natural way to define the closed operator $f(D_\gamma)$ for each holomorphic function $f \in F(S^0_\mu)$ when $\mu > \omega$. What we would like to know is whether D_γ has a bounded H_∞ functional calculus in $L_p(\gamma)$, especially in $L_2(\gamma)$. The situation is very different from the special case when $\gamma = \mathbb{R}$ and $D_\gamma = D$ which was treated in Section J, because D_γ is typically not self–adjoint, and there is no identity of Parseval for functions in $L_2(\gamma)$.

Of course, if $f \in \Psi(S^0_\mu)$, then $f(D_\gamma) \in \mathcal{L}(L_p(\gamma))$, and the agreement with the natural definition for polynomials p means that if $pf \in \Psi(S^0_\mu)$ also, then

$$\begin{aligned} p(D_\gamma)\,f(D_\gamma)\,u &= (pf)(D_\gamma)\,u \ \text{ for all } \ u \in L_2(\gamma) \quad \text{and} \\ f(D_\gamma)\,p(D_\gamma)\,u &= (pf)(D_\gamma)\,u \ \text{ whenever } \ u \in \mathcal{D}(p(D_\gamma))\,. \end{aligned}$$

Again there is a close connection between the functional calculus of D_γ and convolution operators.

For example, suppose that ϕ is a function in $L_1(\mathbb{R})$ with a holomorphic extension to S^0_ν, $\nu > \omega$, which satisfies $|\phi(z)| \le \frac{c|z|^s}{|z|(1+|z|^{2s})}$ for some c and $s > 0$, and let $f = \hat{\phi}$. Then f has a holomorphic extension, also called f, to S^0_ν, and, whenever $\omega < \mu < \nu$, then $f \in \Psi(S^0_\mu)$. Moreover

$$f(D_\gamma)\,u(z) \;=\; \int_\gamma \phi(z-w)\,u(w)\,dw$$

for all $u \in L_p(\gamma)$ and almost all $z \in \gamma$.

This is not too hard to verify, though not as simple as for the case $\gamma = \mathbb{R}$ treated in Lecture 5, because we can no longer identify the Fourier transform of both sides with $f\hat{u}$. To see that the convolution formula for $f(D_\gamma)$ is reasonable, let us just check the formula $p(D_\gamma)f(D_\gamma) = (pf)(D_\gamma)$ when $p(\xi) = \xi$ and ϕ' satisfies the same bound as ϕ, so that $(\frac{1}{i}\phi')^\wedge = pf \in \Psi(S^0_\mu)$. Then $D_\gamma f(D_\gamma)u(z) = D_\gamma \int_\gamma \phi(z-w)u(w)\,dw = \int_\gamma \frac{1}{i}\phi'(z-w)u(w)\,dw = (pf)(D_\gamma)u(z)$ for all $z \in \gamma$. There is a full treatment of holomorphic extensions of functions and their Fourier transforms in [McQ].

For these functions ϕ, we shall use ϕ_t for the function $\phi_t(z) = \frac{1}{t}\phi(\frac{z}{t})$. This is so that, if $f = \hat{\phi}$, then $f_t(\zeta) = f(t\zeta) = (\phi_t)^\wedge(\zeta)$.

We are now in a position to state our main theorem.

Theorem O. *The operator D_γ has an H_∞ functional calculus in $\mathcal{H} = L_2(\gamma)$.*

Equivalently, $\mathcal{H}_{D_\gamma} = \mathcal{H}$ *and* $\| u \|_{D_\gamma} \approx \| u \|$ *for all* $u \in \mathcal{H} = L_2(\gamma)$.

It suffices to prove that $\| u \|_{D_\gamma} \leq c \| u \|$ for all $u \in \mathcal{H}$, for then, since D_γ is dual to $-D_\gamma$ with respect to the pairing $\langle u, v \rangle = \int_\gamma u(z)v(z)dz$, we can infer, on applying Corollary E, that $\| u \| \leq c \| u \|_{-D_\gamma} = c \| u \|_{D_\gamma}$, and thus get equivalence. This is the reason why we considered pairings on a Hilbert space other than the inner product.

To prove the quadratic estimate $\| u \|_{D_\gamma} \leq c \| u \|$, it suffices to show that for some $\nu > \omega$ and for some sufficiently nice function $\phi \in H(S_\nu^0)$ which satisfies $|\phi(z)| \leq \frac{C|z|^s}{|z|(1+|z|^{2s})}$ for some C and $s > 0$, then there exists a constant c such that

$$\left\{ \int_0^\infty \left\| \int_\gamma \phi_t(\cdot - w) u(w)\, dw \right\|_2^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \leq c \| u \|_2$$

for all $u \in L_2(\gamma)$.

There are by now several ways to prove such an estimate. One way is to apply the following result of S. Semmes [SS]. The proof of this theorem requires more harmonic analysis, such as properties of Carleson measures, than we have developed in these lectures.

Proposition O. *Let* θ_t *be complex valued functions on* $\mathbb{R}^n \times \mathbb{R}^n$, $t > 0$, *such that, for some* $C, \delta > 0$, *and some function* $b \in L_\infty(\mathbb{R}^n)$ *satisfying* $Re\, b \geq \delta$,

$$|\theta_t(x,y)| \leq \frac{Ct^{n+1}}{(t + |x-y|)^{n+2}} \tag{1}$$

$$|\theta_t(x,y) - \theta_t(x,y')| \leq \frac{Ct^n\, |y-y'|}{(t + |x-y|)^{n+2}} \tag{2}$$

$$\int_{\mathbb{R}^n} \theta_t(x,y) b(y)\, dy = 0 \quad \text{for all} \quad x \in \mathbb{R}^n . \tag{3}$$

Then there exists c *such that*

$$\left\{ \int_0^\infty \left\| \int_{\mathbb{R}^n} \theta_t(\cdot, y) u(y)\, dy \right\|_2^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \leq c \| u \|_2$$

for all $u \in L_2(\mathbb{R}^n)$.

This result provides us with a powerful tool for proving quadratic estimates. It is intriguing in that the L_2 estimates follow from the cancellation property of the kernels with respect to any single function $b \in L_\infty(\mathbb{R}^n)$.

Before proceeding, we remark that the quadratic norm for $-\Delta$ was expressed in such a way at the end of Section I, with $\theta_t(x,y) = t\frac{\partial}{\partial t}k_t(x-y)$, a function which clearly satisfies (1), (2) and (3) with $b = 1$.

Proof of Theorem O. Take, for example, $\phi(z) = \frac{z}{(1+z^2)^2}$, and let $\phi_t(z) = \frac{1}{t}\phi(\frac{z}{t})$. The functions defined by $\theta_t(x,y) = \phi_t(g(x) - g(y))$ for almost all $x, y \in \mathbb{R}$ satisfy (1), (2) in Proposition O (with $n = 1$). They also satisfy (3) with $b = g'$. Let us check this when $t = 1$. For all $x \in \mathbb{R}$,

$$\begin{aligned}
\int_0^\infty \theta_1(x,y)b(y)\,dy &= \int_0^\infty \phi(g(x) - g(y))g'(y)\,dy \\
&= \int_\gamma \phi(g(x) - w)\,dw \\
&= \frac{1}{2}\int_\gamma \tfrac{d}{dw}(1 + (g(x) - w)^2)^{-1}\,dw \\
&= 0\,.
\end{aligned}$$

Therefore, by Proposition O,

$$\begin{aligned}
\left\{\int_0^\infty \left\| \int_\gamma \phi_t(\cdot - w)u(w)\,dw \right\|_2^2 \frac{dt}{t}\right\}^{\frac{1}{2}} &\le c\left\{\int_0^\infty \left\| \int_0^\infty \theta_t(\cdot, y)u(g(y))g'(y)\,dy \right\|_2^2 \frac{dt}{t}\right\}^{\frac{1}{2}} \\
&\le c\,\|\, u(g(\cdot))g'(\cdot)\,\|_2 \\
&\le c\,\|\, u\,\|_2
\end{aligned}$$

for all $u \in L_2(\gamma)$ as required. □

Exercise. Suppose $1 < p < \infty$ and $\mu > \omega$. Apply Theorem O and a variant of Theorem N for operators of type S_ω to conclude that the operator D_γ has an $H_\infty(S_\mu^0)$ functional calculus in $L_p(\gamma)$.

(P) The Cauchy integral on a Lipschitz curve γ and Hardy spaces.

When f is not the Fourier transform of an L_1 function, it may still be possible to represent $f(D_\gamma)$ as a singular convolution operator. In particular, we are interested in the holomorphic functions χ_+ , χ_- , $\operatorname{sgn} \in H_\infty(S_\mu^0)$ defined in Section H. Then

$$\operatorname{sgn}(D_\gamma)u(z) = \lim_{\varepsilon\to 0} \frac{i}{\pi} \int_{|z-w|>\varepsilon} \frac{1}{z-w}\,u(w)\,dw$$

for all $u \in L_2(\gamma)$ and almost all $z \in \gamma$, being the *Cauchy singular integral operator on* γ .

Further, $E_{\pm} = \chi_{\pm}(D_\gamma) = \frac{1}{2}(\pm\mathrm{sgn}(D_\gamma) + I)$, so

$$\begin{aligned} E_{\pm}u(z) &= \pm \lim_{\varepsilon\to 0} \frac{i}{2\pi} \int\limits_{|z-w|>\varepsilon} \frac{1}{z-w} u(w)\,dw \;+\; \tfrac{1}{2}\,u(w) \\ &= \pm \lim_{\delta\to 0\pm} \frac{i}{2\pi} \int_\gamma \frac{1}{z+i\delta-w} u(w)\,dw \end{aligned}$$

for all $u \in L_2(\gamma)$ and almost all $z \in \gamma$.

Here $E_+ = \chi_+(D_\gamma)$ and $E_- = \chi_-(D_\gamma)$ are the spectral projections corresponding to the parts of the spectrum $\sigma(D_\gamma)$ in the right and in the left half planes, as defined in Section H. They give us the spectral decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$. Since $\mathcal{H} = L_2(\gamma)$, we write $\mathcal{H}^{\pm} = L_2^{\pm}(\gamma)$, so that $L_2(\gamma) = L_2^+(\gamma) \oplus L_2^-(\gamma)$, which is the *Hardy decomposition* of $L_2(\gamma)$.

Let us also consider the semigroups generated by $-|D_\gamma| = -\mathrm{sgn}(D_\gamma)D_\gamma$.

Given $u \in L_2(\gamma)$ and $t > 0$, define $u_+(t) \in L_2(\gamma)$ by

$$u_+(t) \;=\; e^{-tD_\gamma}E_+u \;=\; e^{-t|D_\gamma|}E_+u\,.$$

Then $u_+(t)$ has the properties

$$\begin{cases} \dfrac{\partial u_+}{\partial t}(t) + D_\gamma u_+(t) = 0 & t > 0 \\ \lim\limits_{t\to 0} u_+(t) = E_+u & \\ \lim\limits_{t\to\infty} u_+(t) = 0 & . \end{cases}$$

Also, for $t < 0$, the functions $u_-(t) \in L_2(\gamma)$ defined by

$$u_-(t) \;=\; e^{tD_\gamma}E_-u \;=\; e^{-t|D_\gamma|}E_-u$$

satisfy the properties

$$\begin{cases} \dfrac{\partial u_-}{\partial t}(t) + D_\gamma u_-(t) = 0 & t < 0 \\ \lim\limits_{t\to 0} u_-(t) = E_-u & \\ \lim\limits_{t\to\infty} u_-(t) = 0 & . \end{cases}$$

Now these partial differential equations are actually the Cauchy Riemann equations if we identify (z,t) with $z+it \in \mathbb{C}$. That is, the functions U_+ on $\Omega_+ = \{z+it : t > 0\}$ and U_- on $\Omega_- = \{z+it : t < 0\}$ defined by

$$\begin{aligned} U_+(z+it) &= u_+(t)(z)\,, & t > 0\,,\ z \in \gamma \\ U_-(z+it) &= u_-(t)(z)\,, & t < 0\,,\ z \in \gamma \end{aligned}$$

are holomorphic.

Exercise. Show that the holomorphic functions U_+ and U_- on the regions Ω_+ and Ω_- can be represented by

$$U_\pm(z+it) = \pm\frac{i}{2\pi}\int_\gamma \frac{1}{z+it-w}\,u(w)\,dw$$

for all $u \in L_2(\gamma)$. So

$$\lim_{t\to 0\pm} U_\pm(z+it) = E_\pm u(z)$$

for almost all $z \in \gamma$, and

$$\lim_{t\to\pm\infty} U_\pm(z+it) = 0$$

for all $z \in \gamma$.

Thus the Hardy spaces $L_2^\pm(\gamma)$ consist of those functions $u_\pm \in L_2(\gamma)$ which can be extended holomorphically to functions $U_\pm$ on $\Omega_\pm$ which decay at infinity.

Exercise. Give a characterisation of the quadratic norm $\| u \|_{D_\gamma}$ in terms of these holomorphic extensions of u.

We saw in the exercise at the end of Section O that D_γ has an $H_\infty(S_\mu^0)$ functional calculus in $L_p(\gamma)$ for $1 < p < \infty$, so the operators considered above are also bounded on $L_p(\gamma)$. Let us summarise these results as follows.

Theorem P. *The Cauchy singular integral operator $C_\gamma = sgn(D_\gamma)$ is a bounded operator on $L_p(\gamma)$, $1 < p < \infty$, as are the spectral projections $E_\pm = \chi_\pm(D_\gamma) = \frac{1}{2}(\pm C_\gamma + I)$ corresponding to the parts of $\sigma(D_\gamma)$ in each half plane. Thus $L_p(\gamma) = L_p^+(\gamma) \oplus L_p^-(\gamma)$ where $L_p^\pm(\gamma) = E_\pm(L_p(\gamma))$ are the Hardy spaces consisting of those functions $u_\pm \in L_p(\gamma)$ which can be extended holomorphically to functions $U_\pm$ on $\Omega_\pm$ which decay at infinity.*

The proof of these theorems builds upon the work of Zygmund, Calderón, Carleson, Stein, Fefferman, Meyer, Coifman and many other people. Calderón first proved Theorem P in 1977, in the case when ω is small [C]. Subsequently, Coifman, M^c^Intosh and Meyer proved the boundedness for all such curves [CM^c^M].

The use of the Calderón rotation method leads to the boundedness of singular double–layer potential operators on $L_p(b\Omega)$, when $b\Omega$ is the boundary of a strongly Lipschitz domain $\Omega \subset \mathbb{R}^{m+1}$. This fact was used soon after by Verchota to solve the Dirichlet and Neumann problems for harmonic functions with L_2 boundary data on such domains by using layer potentials [V]. (These problems had been solved previously by Dahlberg [D1] and by Jerison and Kenig [JK] using other methods.)

The proof in [CM^cM] involved complicated multilinear estimates. Subsequently, other methods were developed which simplified and generalised these results. In particular, there was the $T(b)$ theorem of M^cIntosh and Meyer [M^cM] and of David, Journé and Semmes [DJS], as well as the method of Semmes which we presented [SS]. There were wavelets [M]. And there was the paper by Coifman, Jones and Semmes [CJS] which gave two elementary proofs of the boundedness of C_γ, the first reducing it to quadratic estimates of Kenig in Hardy spaces, and the second using martingales. Actually, no proofs are really elementary, because they all rely on the power of Calderón–Zygmund theory and Carleson measures in some form.

During the same period, the functional calculus aspect of these results was developed, first by Coifman and Meyer [CM1], and more fully by Tao Qian and myself [M^cQ]. Higher dimensional versions of this material were developed in [LM^cS] and [LM^cQ].

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