OPERATORS OF PRINCIPAL TYPE WITH INTERIOR BOUNDARY CONDITIONS

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0. Introduction and statement of the main results

In this paper we shall prove results, extending slightly those announced in [16]. The background is some work of Hörmander [9] and Egorov and Kondratev [5], which we shall first describe briefly. We shall always use the same notations for function spaces as Hörmander [7].

Let Ω be a paracompact C^∞ manifold without boundary, $T^*(\Omega)$ the cotangent space, $T^*(\Omega)\diagdown 0$ the space of non zero cotangent vecors and $L^m(\Omega)$ the space of pseudodifferential operators of type 1,0, introduced by Hörmander [8, 10]. In [9] Hörmander studied a pseudodifferential operator $P\in L^m(\Omega)$ with a principal symbol $p\in C^\infty(T^*(\Omega)\diagdown 0)$, positively homogeneous of degree m, such that $C_p \neq 0$ everywhere on the set of zeros of p. Here $C_p\in C^\infty(T^*(\Omega)\diagdown 0)$ is defined by

$$C_p(x,\xi) = 2\operatorname{Im}\sum_{j=1}^{n} p^{(j)}(x,\xi)\overline{p_{(j)}(x,\xi)}. \tag{0.1}$$

where $x=(x_1, x_2, ..., x_n)$ are some local coordinates in Ω and $\xi=(\xi_1, \xi_2, ..., \xi_n)$ are the corresponding dual coordinates in the cotangent space and $p^{(j)}=\partial p/\partial\xi_j$ and $p_{(j)}=\partial p/\partial x_j$. If we fix a strictly positive C^∞ density, then the complex adjoint $P^*\in L^m(\Omega)$ (i.e. the adjoint with respect to the corresponding sesquillinear scalar product) is defined and if we write

$$C_p(x,\xi) = -i\sum_{j=1}^{n}(p^{(j)}(x,\xi)\overline{p_{(j)}(x,\xi)} - \overline{p^{(j)}(x,\xi)}\,p_{(j)}(x,\xi)),$$

we see, using the calculus of pseudodifferential operators, that C_p is the homogeneous principal symbol of $[P, P^*]=PP^*-P^*P$. In particular C_p is independent of the choice of local coordinates. The expression

$$\{p,\bar{p}\} = \sum_{1}^{n}(p^{(j)}\overline{p_{(j)}} - \overline{p^{(j)}}\,p_{(j)})$$

is known as the Poisson bracket of p and $\bar{p}$.

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Hörmander proved in [9] that if $C_p<0$ where $p=0$, then for every compact set $K\subset\Omega$ and $s\in\mathbf{R}$ there is a constant C, such that

$$\|u\|_s\leqslant C(\|Pu\|_{s-m+\frac{1}{2}}+\|u\|_{s-1}),\quad u\in H_s^{\mathrm{loc}}(\Omega)\cap\mathcal{E}'(K).$$

From this estimate it is easy to deduce regularity for the solutions of the equation $Pu=v$ and a local existence theorem for the equation $P^*u=v$. In the case where $C_p>0$ somewhere on the surface $p=0$ he proved non-existence and non-regularity theorems for the equations $P^*u=v$ and $Pu=v$ respectively.

Finally he applied his results to the oblique derivative problem: Let M be an open set in $\mathbf{R}^{n+1}$ with smooth boundary Ω. Let ν be a smooth vector field in $\mathbf{R}^{n+1}$ and consider the following problem: For given functions v defined in M and u_0 defined in Ω find a function u in M, such that

$$\sum_{j=1}^{n+1}\partial^2u/\partial x_j^2=v,\quad \partial u/\partial\nu\Big|_{\Omega}=u_0.\tag{0.2}$$

This problem can be reduced to the study of a certain pseudodifferential operator P in Ω. If ν is nowhere tangential to Ω, we have an elliptic boundary value problem, that is P turns out to be an elliptic operator. In certain cases when ν is not everywhere transversal to Ω, the operator P is of the type above and Hörmander could apply his general results, to prove local existence or local regularity for the problem (0.2), depending on the behaviour of ν near the submanifold of Ω, where ν is tangential.

Egorov and Kondrat'ev [5] have subsequently studied (0.2) using more direct methods. By introducing an extra boundary condition where ν is tangential and adding an error term, to the equation $\partial u/\partial\nu|_\Omega=u_0$, they managed to get a problem for which they could state both existence and uniqueness results.

For the corresponding operator P it should thus be possible to obtain a problem, which is (approximately) uniquely solvable, by adding an error term to the equation $Pu=v$ and adding a suitable boundary condition. Eškin [6] has carried out this program and generalized the results of [5] by studying a larger set of operators P than the set of those resulting from the problem (0.2). Also Višik and Grušin [19], [20] have results in this direction. See also the recent paper [6′] by Eškin.

Here we shall study a class of operators which differ from those of Eškin [6] mainly in that we impose less restrictive geometric conditions on the manifold, where the principal symbol vanishes. On the other hand Eškin allows the principal symbols of his operators to vanish of higher order than we do. We shall also obtain a local result for operators P satisfying only the condition $C_p\neq 0$ when $p=0$. This result is very close to a theorem of Kawai [12].

It should be noted that Egorov [2], [4], Nirenberg and Trèves [15] and Trèves [18] have generalized the results of [9], by studying the equation $Pu=v$ (without extra conditions) for operators P, which degenerate to high order.

We now start to formulate our main results. Let Ω be a paracompact C^∞ manifold and let $P\in L^m(\Omega)$ be properly supported with a principal symbol p positively homogeneous of degree m. (If Ω_1 and Ω_2 are C^∞ manifolds and $A\colon C_0^\infty(\Omega_1)\to \mathcal{D}'(\Omega_2)$ is a continuous operator with distribution kernel K_A, then A is said to be properly supported if $\{(x,y)\in \operatorname{supp} K_A;\ x\in K_2 \text{ or } y\in K_1\}$ is compact for all compact sets $K_j\subset\Omega_j$, $j=1, 2$. This means that Au has compact support for all $u\in C_0^\infty(\Omega_1)$ and that Au can be defined for all $u\in C^\infty(\Omega_1)$. We shall say that a matrix of operators is properly supported (or has C^∞ kernel), if all the entries are properly supported (or have C^∞ kernels).) Let $\Sigma=\Sigma_p\subset T^*(\Omega)\setminus 0$ be the set of zeros of p and let C_p be defined by (0.1) above. We introduce the following two conditions:

(A) C_p never vanishes on Σ_p.

(B) $p'_\xi=(p^{(1)}, ..., p^{(n)})$ is proportional to a real vector on Σ_p and $n=\dim\Omega\geqslant 3$.

Note that p'_ξ is a complex tangent vector to Ω which is independent of the choise of local coordinates. If p satisfies (A), we define Σ_p^+ and Σ_p^- to be the subsets of Σ_p, where $C_p>0$ and $C_p<0$ respectively.

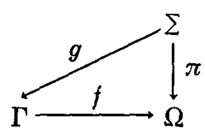
To be able to state the main theorem in the case where p satisfies both (A) and (B), we have to define suitable auxiliary operators and to do so we first have to describe the geometric structure of Σ_p.

If $x=(x_1, ..., x_{n-1}, x_n)$ is any n-tuple, we shall always denote by x' the $(n-1)$-tuple $(x_1, ..., x_{n-1})$. Let $\pi\colon T^*(\Omega)\setminus 0\to\Omega$ be the natural projection. The following proposition will be proved in section 1.

Proposition 0.1. *If p is positively homogeneous and (A) is satisfied, then Σ_p^+ and Σ_p^- are smooth closed conic submanifolds of $T^*(\Omega)\setminus 0$ of codimension 2. If (A) and (B) are satisfied and p is positively homogeneous, then for every $\varrho\in\Sigma_p$ we can find local coordinates $x=(x_1, ..., x_n)$ in a neighbourhood W of $\pi\varrho$, such that the component of ϱ in $T^*(\Omega)|_W\cap\Sigma_p$ is given by the equations $x_n=0$ and $\xi_n=\tau(x',\xi')$. Here $\xi=(\xi_1, ..., \xi_n)$ are the dual coordinates corresponding to x, and $\tau\in C^\infty(\mathbf{R}^{n-1}\times(\mathbf{R}^{n-1}\setminus\{0\}))$ is real valued and positively homogeneous of degree 1 with respect to ξ'.*

Remark. Assume conversely that the surface Σ_p can be given locally by the equations $x_n=0$ and $\xi_n=\tau(x',\xi')$ as in the proposition. Then $\operatorname{grad}_\xi \operatorname{Im} p$ and $\operatorname{grad}_\xi \operatorname{Re} p$ are linearly dependent on Σ_p, because Σ_p has codimension 1 as a submanifold of $\{(x,\xi)\in T^*(\Omega)\setminus 0;\ x_n=0\}$. This means precisely that $\operatorname{grad}_\xi p$ is proportional to a real vector.

We now assume, that (A) and (B) are satisfied and that p is positively homogeneous. It follows from the proposition, that any sufficiently small part of Σ_p is mapped by π into a smooth submanifold of Ω of codimension 1. However $\pi\Sigma_p$ is in general not a submanifold but only the immersion of a certain manifold Γ defined as follows. In Σ we introduce an equivalence relation: If ϱ' and $\varrho'' \in \Sigma$ then $\varrho' \sim \varrho''$ if and only if $\pi\varrho' = \pi\varrho'' = x$ and ϱ' and ϱ'' belong to the same component of $\Sigma \cap \pi^{-1}x$. Let Γ be the corresponding set of equivalence classes and let g: $\Sigma \to \Gamma$ be the natural map. Then we have the commutative diagram:



defining the map f. Now it follows from Proposition 0.1, that there is a unique C^∞ structure on Γ, such that g and f are smooth maps. Moreover f is an immersion. We put $\Gamma^+ = g\Sigma^+$ and $\Gamma^- = g\Sigma^-$. Then Γ is the disjoint union of the submanifolds Γ^+ and Γ^-.

Consider more generally any two C^∞ manifolds X and Y and a given smooth map f: $X \to Y$. The normal bundle $N_f \subset T^*(X) \times T^*(Y)$ of the graph of f is given by

$$N_f = \{((x, {}^t f'(x)\eta), (f(x), -\eta)); \quad x \in X, \ \eta \in T_{f(x)}(Y)\}$$

where ${}^t f'(x)$ is the adjoint of the differential of f at x. With this Lagrangean manifold there is associated for any real m a class $I^m(X \times Y, N_f)$ of distributions in $X \times Y$ with wave front set contained in N_f, and we can regard them as operators from $C_0^\infty(Y)$ to $C^\infty(X)$. (See Hörmander [10].) If $x_1, \ldots, x_\nu$ and $y_1, \ldots, y_n$ are local coordinates near $x^0 \in X$ and $y^0 = f(x^0)$ respectively then N_f is defined by the phase function $\langle f(x) - y, \theta \rangle$, $x \in \mathbf{R}^\nu$, y, $\theta \in \mathbf{R}^n$ and the restriction of any $A \in I^m(X \times Y, N_f)$ to a neighborhood of (x^0, y^0) is of the form

$$Au(x) = (2\pi)^{-(\nu+3n)/4} \iint e^{i\langle f(x) - y, \theta \rangle} a(x, y, \theta)\, u(y)\, dy\, d\theta, \quad u \in C_0^\infty(Y)$$

where $a \in S^{m+(\nu-n)/4}$.

If f is an immersion we can choose a symbol $b(z, y, \theta)$, defined for (z, y) in a neighbourhood of $y^0 \times y^0$ so that $a(x, y, \theta) = b(f(x), y, \theta)$. Thus $Au = (Bu) \circ f$ for a pseudodifferential operator B of order $m + (\nu - n)/4$. Conversely, if $A \in \mathcal{D}'(X \times Y)$, if sing supp $A \subset$ graph f and A is of this form in a neighbourhood of $(x^\circ, f(x^0))$ for any $x^0 \in X$ then $A \in I^m(X \times Y, N_f)$. When f is an immersion we shall write

$$L^m(X, Y, f) = I^{m-(\nu-n)/4}(X \times Y, N_f).$$

We can identify N_f with the pullback $T_X^*(Y)$ of $T^*(Y)$ to X for

$$T^*_X(Y) = \{(x, \eta),\, x \in X,\, \eta \in T^*_{f(x)}(Y)\}.$$

The fact that $N_f \subset T^*(X) \times T^*(Y)$ gives us two maps

$$f_X\colon T^*_X(Y) \ni (x, \eta) \to (x, {}^t f'(x)\eta) \in T^*(X)$$

$$f_Y\colon T^*_X(Y) \ni (x, \eta) \to (f(x), \eta) \in T^*(Y).$$

If f is an immersion it is clear that f_Y is an immersion.

The principal symbol of elements in $I^m(X \times Y, N_f)$ can be considered as defined on $T^*_X(Y) \setminus 0$ and the wavefront set can be considered as a closed conic subset of $T^*_X(Y) \setminus 0$. More precisely the wavefront set $\mathrm{WF}(A) \subset T^*_X(Y)$ of an element $A \in I^m(X \times Y)\, N_f)$ is given by

$$\mathrm{WF}'(A) = \{(f_X(\varrho), f_Y(\varrho));\ \varrho \in \mathrm{WF}(A)\}.$$

See Hörmander [10].

We can apply the preceding discussion to the immersion $f\colon \Gamma^+ \to \Omega$. With the local coordinates in Proposition 0.1 f is locally the map $\mathbf{R}^{n-1} \ni x' \to (x', 0) \in \mathbf{R}^n$. Denote the zerosection in $T^*(\Gamma^+)$ by 0 and put $N^+ = f_{\Gamma^+}^{-1} 0 \subset T^*_{\Gamma^+}(\Omega)$ which is the line bundle of normals of $f(\Gamma^+)$, and put

$$\Sigma_0^+ = \{\varrho \in f_\Omega^{-1} \Sigma^+;\ g f_\Omega(\varrho) = \pi(\varrho)\},$$

where π is the projection in $T^*_{\Gamma^+}(\Omega)$. Then Σ_0^+ is a smooth closed conic submanifold of $T^*_{\Gamma^+}(\Omega)$ of codimension 1 and

$$N^+ \cap \Sigma_0^+ = \emptyset. \tag{0.4}$$

In fact, in the local coordinates of Proposition 0.1 we have

$$N^+ = \{(x', (0, \xi_n)) \in \mathbf{R}^{n-1} \times \mathbf{R}^n\}$$

and

$$\Sigma_0^+ = \{(x', (\xi', \tau(x', \xi'))) \in \mathbf{R}^{n-1} \times \mathbf{R}^n;\quad \xi' \neq 0\}.$$

It is easy to see that the maps $\Sigma_0^+ \to \Sigma^+$ and $\Sigma_0^+ \to T^*(\Gamma^+) \setminus 0$ defined by f_Ω and f_{Γ^+} are diffeomorphisms which together give a diffeomorphism $\mathcal{G}_+\colon \Sigma^+ \to T^*(\Gamma^+) \setminus 0$. We define N^-, Σ_0^- and $\mathcal{G}_-$ analogously.

Since Σ_0^+ and $N^+ \cup ((f_\Omega^{-1}\Sigma) \setminus \Sigma_0^+)$ are disjoint closed conic subsets of $T^*_{\Gamma^+}(\Omega) \setminus 0$ it follows by a partition of unity that there exist operators $R^+ \subset L^0(\Gamma^+, \Omega, f)$ satisfying

(C^+) R^+ has a principal symbol, positively homogeneous of degree 0, which is different from zero on Σ_0^+, but

$$\mathrm{WF}(R^+) \cap (N^+ \cup ((f_\Omega^{-1}\Sigma) \setminus \Sigma_0^+)) = \emptyset.$$

Let (C^-) be the analogous condition, obtained by replacing all $+$ signs by $-$ signs.

Once for all we fix some strictly positive C^∞ densities on Γ^+, Γ^- and Ω. Then the complex adjoint of a continuous operator: $C_0^\infty(\Omega)\to\mathcal{D}'(\Gamma^-)$ is a well defined continuous operator $C_0^\infty(\Gamma^-)\to\mathcal{D}'(\Omega)$. Suppose $R^+\in L^0(\Gamma^+,\Omega,f)$ and $R^{-*}\in L^0(\Gamma^-,\Omega,f)$ are properly supported and satisfy (C^+) and (C^-) respectively and let R^- be the complex adjoint of R^{-*}. Then it follows from [10 section 2.5] that R^+ and R^- can be extended to continuous linear operators $\mathcal{D}'(\Omega)\to\mathcal{D}'(\Gamma^+)$ and $\mathcal{D}'(\Gamma^-)\to\mathcal{D}'(\Omega)$ respectively. Moreover we have.

PROPOSITION 0.2. *R^+ and R^- are continuous*

$$H_s^{\text{loc}}(\Omega)\to H_{s-\frac12}^{\text{loc}}(\Gamma^+) \text{ and } H_s^{\text{loc}}(\Gamma^-)\to H_{s-\frac12}^{\text{loc}}(\Omega)$$

respectively for all $s\in\mathbf{R}$.

This proposition will be proved in section 1. We can now state the main result:

THEOREM 1. *Let Ω be a paracompact C^∞ manifold of dimension $n\geqslant 3$ and assume that $P\in L^m(\Omega)$ is properly supported and has a principal symbol p, positively homogenous of degree m, satisfying (A) and (B). Let*

$$\mathcal{P}=\begin{pmatrix}P & R^-\\ R^+ & 0\end{pmatrix}:\mathcal{D}'(\Omega)\times\mathcal{D}'(\Gamma^-)\to\mathcal{D}'(\Omega)\times\mathcal{D}'(\Gamma^+)$$

be the operator mapping $(u,u^-)\in\mathcal{D}'(\Omega)\times\mathcal{D}'(\Gamma^-)$ to $(Pu+R^-u^-,\ R^+u)\in\mathcal{D}'(\Omega)\times\mathcal{D}'(\Gamma^+)$, where R^+ and R^- satisfy the hypotheses of Proposition 0.2 and

$$(f_\Omega \operatorname{WF}(R^+))\cap(f_\Omega\operatorname{WF}(R^{-*}))=\emptyset. \tag{0.5}$$

Then there exists a properly supported operator

$$\mathcal{E}=\begin{pmatrix}E & E^+\\ E^- & 0\end{pmatrix}:\mathcal{D}'(\Omega)\times\mathcal{D}'(\Gamma^+)\to\mathcal{D}'(\Omega)\times\mathcal{D}'(\Gamma^-),$$

such that:

(i) *$\mathcal{E}\circ\mathcal{P}-I$ and $\mathcal{P}\circ\mathcal{E}-I$ have C^∞ kernels. Here the first I denotes the identity operator in $\mathcal{D}'(\Omega)\times\mathcal{D}'(\Gamma^-)$ and the second the identity operator in $\mathcal{D}'(\Omega)\times\mathcal{D}'(\Gamma^+)$.*

(ii) *E, E^+ and E^- are continuous $H_s^{\text{loc}}(\Omega)\to H_{s+m-\frac12}^{\text{loc}}(\Omega)$, $H_s^{\text{loc}}(\Gamma^+)\to H_{s+\frac12}^{\text{loc}}(\Omega)$ and $H_s^{\text{loc}}(\Omega)\to H_{s+\frac12}^{\text{loc}}(\Gamma^-)$ respectively for all $s\in R$.*

(iii) $\operatorname{WF}'(E^-)\subset\{(\mathcal{G}_-\varrho,\varrho);\quad \varrho\in\Sigma^-\}$

$\operatorname{WF}'(E^+)\subset\{(\varrho,\mathcal{G}_+\varrho);\quad \varrho\in\Sigma^+\}$

$$\begin{aligned}\operatorname{WF}'(E)\subset{}&\{(\varrho,\varrho)\in(T^*(\Omega)\setminus 0)\times(T^*(\Omega)\setminus 0)\}\\ &\cup\{(f_\Omega\varrho,f_\Omega\mu)\in(T^*(\Omega)\setminus 0)\times(T^*(\Omega)\setminus 0);\ \varrho\in\Sigma_0^+,\mu\in\operatorname{WF}(R^+),f_{\Gamma^+}\varrho=f_{\Gamma^+}\mu\}\\ &\cup\{(f_\Omega\mu,f_\Omega\varrho)\in(T^*(\Omega)\setminus 0)\times(T^*(\Omega)\setminus 0);\ \varrho\in\Sigma_0^-,\mu\in\operatorname{WF}(R^{-*}),f_{\Gamma^-}\varrho=f_{\Gamma^-}\mu\}.\end{aligned}$$

From the proof of Theorem 1 it will follow that E^+ and E^- are Fourier integral operators. If the condition (0.5) is not satisfied, we still have an operator $\mathcal{E}$ satisfying (i) and (ii). This can be proved with the methods of section 5 where we study extensions of Theorem 1 when R^+ and R^- are replaced by more general operators.

COROLLARY. *Let P and Ω be as in Theorem 1. Then P induces a bijection:*

$$(\mathcal{D}'(\Omega)/C^\infty(\Omega)) \times (\mathcal{D}'(\Gamma^-)/C^\infty(\Gamma^-)) \to (\mathcal{D}'(\Omega)/C^\infty(\Omega)) \times (\mathcal{D}'(\Gamma^+)/C^\infty(\Gamma^+)).$$

If Ω is compact, then P induces a Fredholm operator

$$C^\infty(\Omega) \times C^\infty(\Gamma^-) \to C^\infty(\Omega) \times C^\infty(\Gamma^+).$$

In the case when only (A) is satisfied, we have a very local result.

THEOREM 2. *Suppose that Ω is a C^∞ manifold and that $P \in L^m(\Omega)$ is properly supported and has a principal symbol p, positively homogeneous of degree m, which satisfies (A). Then for each $\varrho \in \Sigma_p^+$ there exist $\varrho' \in T^*(\mathbf{R}^{n-1}) \setminus 0$ and properly supported operators*

$$R_\varrho^+ : \mathcal{D}'(\Omega) \to \mathcal{D}'(\mathbf{R}^{n-1}), \quad E_\varrho^+ : \mathcal{D}'(\mathbf{R}^{n-1}) \to \mathcal{D}'(\Omega), \quad E_\varrho : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$$

with the following properties, where $A \equiv B$ means that $A - B$ has C^∞ kernel:

(ia) *$R_\varrho^+ E_\varrho^+ T' \equiv T'$ and $PE_\varrho^+ T' \equiv 0$ for all $T' \in L^0(\mathbf{R}^{n-1})$ with $WF(T')$ close to ϱ'.*

(ib) *$PE_\varrho T \equiv T$ and $R_\varrho^+ E_\varrho T \equiv 0$ for all $T \in L^0(\Omega)$ with $WF(T)$ close to ϱ.*

(ii) *$T(E_\varrho P + E_\varrho^+ R_\varrho^+) \equiv T$ for all $T \in L^0(\Omega)$ with $WF(T)$ close to ϱ.*

(ia) and (ib) express that the operator

$$\mathcal{D}'(\Omega) \times \mathcal{D}'(\mathbf{R}^{n-1}) \ni (v, v^+) \to E_\varrho v + E_\varrho^+ v^+ \in \mathcal{D}'(\Omega)$$

is near ϱ a local right inverse modulo C^∞ of the operator

$$\mathcal{D}'(\Omega) \ni u \to (Pu, R_\varrho^+ u) \in \mathcal{D}'(\Omega) \times \mathcal{D}'(\mathbf{R}^{n-1})$$

and (ii) expresses that is as a local left inverse. The proof of Theorem 2 gives additional information on the operators R_ϱ^+, E_ϱ^+ and E_ϱ, in particular on the H_s continuity and the wavefront sets (see also section 5). There is a dual form of Theorem 2, which describes the behavior of P near Σ_p^-. This is based on the observation that the complex adjoint of P has the principal symbol $\bar{p}$ and that $\Sigma_p^- = \Sigma_{\bar{p}}^+$. Recently Kawai ([12] part II th. 2.4) has obtained a result, which is close to our Theorem 2. He assumes that the symbol of P is analytic and uses the theory of hyperfunctions. Our proof will be completely different. A consequence of our results is that if P is as in Theorem 2 and $\Sigma_p^+ \neq \emptyset$ then the operator $\mathcal{D}'(\Omega)/C^\infty(\Omega) \to \mathcal{D}'(\Omega)/C^\infty(\Omega)$ induced by P is not injective and if $\Sigma_p^- \neq 0$ then it is not surjective. (See Hörmander [11] for more general results of this type.)

The plan of the paper is the following: In section 1 we prove Propositions 0.1 and 0.2. In section 2 we state as Theorem 2.1 a local form of a special case and show that it implies Theorems 1 and 2. In section 3 we reduce the proof of Theorem 2.1 to the study of an explicitly given first order pseudodifferential operator. In section 4 we make this study, which completes the proof of Theorems 1 and 2. In section 5 we discuss generalizations of Theorem 1, where R^+ and R^- are replaced by more general operators.

I would finally like to thank Professor Lars Hörmander, who has suggested the subject of this paper and who has given me much help and advise during the work.

§ 1. Proof of Propositions 0.1 and 0.2

Proof of Proposition 0.1. Since p is positively homogeneous it is clear that Σ is conic. In Ω we introduce local coordinates $x=(x_1, ..., x_n)$ and we let $\xi=(\xi_1, ..., \xi_n)$ be the corresponding dual coordinates in the cotangent space. Put $p_1=\operatorname{Re} p$ and $p_2=\operatorname{Im} p$. Then Σ is defined by the two real equations $p_1(x, \xi)=0$ and $p_2(x, \xi)=0$. If we write (0.1) in the form

$$C_p(x, \xi)=2(\langle p'_{2\xi}, p'_{1x}\rangle-\langle p'_{1\xi}, p'_{2x}\rangle), \tag{1.1}$$

we see that $\operatorname{grad} p_1=(p'_{1x}, p'_{1\xi})$ and $\operatorname{grad} p_2=(p'_{2x}, p'_{2\xi})$ are linearly independent on Σ if (A) is satisfied. Here $\langle\ ,\ \rangle$ is the bilinear form on $\mathbf{R}^n$, defined by $\langle x, \xi\rangle=\sum_{j=1}^n x_j\xi_j$, $x, \xi\in\mathbf{R}^n$. Hence Σ is a closed submanifold of codimension 2 of $T^*(\Omega)\setminus 0$, so only the second half of the proposition remains to prove.

Thus we assume that (A) and (B) are satisfied and we shall first study Σ infinitesimally. For $x\in\Omega$, put $\Sigma_x=\{(x, \xi)\in\Sigma\}=\Sigma\cap\pi^{-1}x$. In general, if M is a C^∞ manifold and $m\in M$ we denote by $T_m(M)$ and $T_m^*(M)$ the fibers over m of the tangent space $T(M)$ and the cotangent space $T^*(M)$ respectively.

LEMMA 1.1. *Let $(x, \xi)\in\Sigma$ and let π_* be the natural projection: $T_{(x,\xi)}(\Sigma)\to T_x(\Omega)$. Then π_* has rank $n-1$. Let $N=\operatorname{Im} a\, p'_x$, where the complex number $a\neq 0$ is such that $t=a\, p'_\xi$ is real. Then $\langle t, N\rangle\neq 0$ and N is orthogonal to $\pi_* T_{(x,\xi)}(\Sigma)$ and transversal to $T_{(x,\xi)}(\Sigma_x)$.*

Proof. Let (x, ξ) and a be as in the lemma. Then

$$C_{ap}(x, \xi)=2\operatorname{Im}\sum_{j=1}^n a p^{(j)}(x, \xi)\overline{a p_{(j)}(x, \xi)}=|a|^2 C_p(x, \xi)\neq 0.$$

On the other hand $C_{ap}(x, \xi)=-2\langle t, N\rangle$ so we get

$$0\neq\langle t, N\rangle. \tag{1.2}$$

In particular both t and N are $\neq 0$.

Σ can be defined by the two equations $\operatorname{Re} ap=0$ and $\operatorname{Im} ap=0$. Thus $T_{(x,\xi)}(\Sigma)$ is the set of all $(t_x, t_\xi)\in\mathbf{R}^n\times\mathbf{R}^n$, such that $\langle \operatorname{Im} a\, p'_x, t_x\rangle + \langle \operatorname{Im} a\, p'_\xi, t_\xi\rangle = 0$ and $\langle \operatorname{Re} a\, p'_x, t_x\rangle + \langle \operatorname{Re} a\, p'_\xi, t_\xi\rangle = 0$, or equivalently

$$\langle N, t_x\rangle = 0. \tag{1.3}$$

$$\langle \operatorname{Re} a\, p'_x, t_x\rangle + \langle t, t_\xi\rangle = 0. \tag{1.4}$$

Since $t\neq 0$, we can, for any t_x satisfying (1.3), find t_ξ such that (1.4) is also satisfied. Thus $\pi_* T_{(x,\xi)}(\Sigma)$ is the orthogonal space of N and has dimension $n-1$. Since $T_{(x,\xi)}(\Sigma_x)$ is defined by the equation $\langle t, t_\xi\rangle=0$, it follows from (1.2), that N is transversal to $T_{(x,\xi)}(\Sigma_x)$ and the proof is complete.

Next we study Σ locally. Note that π_* in the preceeding lemma is the differential of the projection $\pi|_\Sigma\colon \Sigma\ni(x,\xi)\to x\in\Omega$.

LEMMA 1.2. *For every point $(x_0,\xi_0)\in\Sigma$ there is an open conic neighbourhood $U\subset T^*(\Omega)\setminus 0$ and two real valued, smooth functions $g(x,\xi)$ and $\gamma(x)$ defined in U, such that:*

(i) $\qquad \langle g'_\xi, \gamma'_x\rangle \neq 0$ *in* U.

(ii) $\qquad \Sigma\cap U = \{(x,\xi)\in U;\ \gamma(x)=g(x,\xi)=0\}$.

Proof. By Lemma 1.1 the differential of $\pi|_\Sigma$ has rank $n-1$. Thus (see [17] pp. 39–41) there is an open neighbourhood U of (x_0,ξ_0), such that $\pi(U\cap\Sigma)$ is a C^∞ manifold of dimension $n-1$, given by an equation $\gamma(x)=0$, where $\gamma\in C^\infty(\pi U)$ and $\gamma'_x\neq 0$ everywhere. Since Σ is conic, we can assume that U is conic. By Lemma 1.1 we have either $\langle \operatorname{grad}_\xi \operatorname{Re} p, \gamma'_x\rangle\neq 0$ or $\langle \operatorname{grad}_\xi \operatorname{Im} p, \gamma'_x\rangle\neq 0$ in U if U is small enough. We put g equal to $\operatorname{Re} p$ or $\operatorname{Im} p$ so that (i) holds. Thus $\{(x,\xi)\in U;\ g(x,\xi)=\gamma(x)=0\}$ is a $(2n-2)$-dimensional manifold, which contains the $(2n-2)$-dimensional manifold $\Sigma\cap U$. If U is small enough they have to be equal and the proof is complete.

It follows from the lemma, that Σ_x is an $(n-1)$-dimensional closed submanifold of $T^*_x(\Omega)\setminus\{0\}$. Let $\varrho=(x_0,\xi_0)\in\Sigma$ and let $\Sigma_{(x_0,\xi_0)}$ be the component of (x_0,ξ_0) in Σ_{x_0}. Since $\Sigma_{(x_0,\xi_0)}$ is conic and $\{(x_0,\xi)\in\Sigma_{(x_0,\xi_0)};\ |\xi|=1\}$ is a compact manifold, we can cover $\Sigma_{(x_0,\xi_0)}$ by a finite number of open conic sets $U_\nu\subset T^*(\Omega)\setminus 0$, $\nu=1,2,\ldots,N$, where we have smooth functions $\gamma_\nu(x)$ and $g_\nu(x,\xi)$ such that (i)–(iii) of Lemma 1.2 are fullfilled.

LEMMA 1.3. *In a neighbourhood of x_0, the equations $\gamma_\nu(x)=0$ define the same hypersurface.*

Proof. Since $\Sigma_{(x_0,\xi_0)}$ is connected and $\{U_\nu\}_{1\leqslant\nu\leqslant N}$ is a finite open covering of $\Sigma_{(x_0,\xi_0)}$, it suffices to prove, that if $\Sigma_{(x_0,\xi_0)}\cap U_\nu\cap U_\mu\neq\emptyset$, then $\gamma_\nu(x)=0$ and $\gamma_\mu(x)=0$ define the same hypersurface in some neighbourhood of x_0. Let $(x_0,\eta)\in\Sigma_{(x_0,\xi_0)}\cap U_\nu\cap U_\mu$. By Lemma 1.2

the projection of a small neighbourhood of (x_0, η) in $\Sigma_{(x_0,\xi_0)}$ is a hypersurface which can either be given by the equation $\gamma_\nu(x)=0$ or $\gamma_\mu(x)=0$. This completes the proof.

We now choose local coordinates with the origin in x_0, such that the equations $\gamma_\nu(x)=0$ are equivalent to the equation $x_n=0$. Put $W'_\delta=\{x'\in\mathbf{R}^{n-1};\ |x'|<\delta\}$ and let $\Sigma_{\varrho,\delta}$ be the component of $\varrho=(x_0,\xi_0)$ in $\Sigma\cap\pi^{-1}\{(x',0)\in\Omega;\ x'\in W'_\delta\}$. Then it follows from the proof of the preceeding lemma, that $\pi\Sigma_{\varrho,\delta}=\{(x',0)\in\Omega;\ x'\in W'_\delta\}$, if $\delta>0$ is small enough. Hence Lemma 1.2 implies:

$$\begin{gathered}\Sigma_{\varrho,\delta} \text{ is locally given by the equations } x_n=0 \text{ and}\\ g(x,\xi)=0, \text{ where } \partial g/\partial\xi_n\neq 0 \text{ if } \delta>0 \text{ is small enough}\end{gathered}\tag{1.5}$$

Thus we have $\Sigma_{\varrho,\delta}\cap\{((x',0),(0,\xi_n))\in T^*(\Omega)\setminus 0;\ x'\in W'_\delta\}=\emptyset$ since $\Sigma_{\varrho,\delta}$ is conic and since Σ is closed we get

$$\begin{gathered}\text{If } \delta>0 \text{ is small enough there is a constant } C>0 \text{ such that}\\ |\xi_n|\leqslant C|\xi'| \text{ for all } ((x',0),\xi)\in\Sigma_{\varrho,\delta}.\end{gathered}\tag{1.6}$$

We now fix $\delta>0$, such that (1.5) and (1.6) hold and let $\mathcal{G}\colon\Sigma_{\varrho,\delta}\to W'_\delta\times(\mathbf{R}^{n-1}\setminus\{0\})$ be the projection $((x',0),\xi)\to(x',\xi')$. It follows from (1.6), that $\mathcal{G}\Sigma_{\varrho,\delta}$ is closed in $W'_\delta\times(\mathbf{R}^{n-1}\setminus\{0\})$ and from (1.5), that it is open. Since $W'_\delta\times(\mathbf{R}^{n-1}\setminus\{0\})$ is connected $(n\geqslant 3)$, we have $\mathcal{G}\Sigma_{\varrho,\delta}=W'_\delta\times(\mathbf{R}^{n-1}\setminus\{0\})$. Now put

$$\tau(x',\xi')=\inf\xi_n;\ \mathcal{G}(x,\xi)=(x',\xi').$$

Then by (1.5) and (1.6) it follows that $\tau\in C^\infty(W'_\delta\times(\mathbf{R}^{n-1}\setminus\{0\}))$ and that τ is positively homogeneous of degree 1 with respect to ξ'. Moreover $\Sigma_{\varrho,\delta}$ is defined by the equations $x_n=0$ and $\xi_n=\tau(x',\xi')$, $(x',\xi')\in W'_\delta\times(\mathbf{R}^{n-1}\setminus\{0\})$. If δ is small enough, τ has an extension to $\mathbf{R}^{n-1}\times(\mathbf{R}^{n-1}\setminus\{0\})$ and Proposition 0.1 follows, if we let $W=\{x\in\mathbf{R}^n;\ |x|<\delta\}$.

Proof of Proposition 0.2. Locally we identify Ω with $\mathbf{R}^n$ and Γ^+ with the hyperplane $x_n=0$. Then R^+ is locally of the form γQ, where γ is the restriction operator $C^\infty(\mathbf{R}^n)\to C^\infty(\mathbf{R}^{n-1})$, defined by $\gamma u(x')=u(x',0)$, $x'\in\mathbf{R}^{n-1}$. Here $Q\in L^0(\mathbf{R}^n)$ is properly supported and satisfies $\mathrm{WF}(Q)\cap\{(x,\xi)\in T^*(\mathbf{R}^n)\setminus 0;\ \xi'=0\}=\emptyset$. Now it is wellknown (see for instance [7]), that γ is continuous $H^{\mathrm{loc}}_{(1,s-1)}(\mathbf{R}^n)\to H^{\mathrm{loc}}_{s-\frac12}(\mathbf{R}^{n-1})$ for all $s\in R$. From Proposition A.2 in the appendix it follows, that Q is continuous $H^{\mathrm{loc}}_s(\mathbf{R}^n)\to H^{\mathrm{loc}}_{(1,s-1)}(\mathbf{R}^n)$ for all s. Thus R^+ is continuous $H^{\mathrm{loc}}_s(\Omega)\to H^{\mathrm{loc}}_{s-\frac12}(\Gamma^+)$ for all s and by the same argument R^{-*} is continuous $H^{\mathrm{loc}}_{-s+\frac12}(\Omega)\to H^{\mathrm{loc}}_{-s}(\Gamma^-)$ for all s. Since R^- is properly supported, we get by duality, that R^- is continuous $H^{\mathrm{loc}}_s(\Gamma^-)\to H^{\mathrm{loc}}_{s-\frac12}(\Omega)$ for all $s\in\mathbf{R}$.

2. Reduction of the proofs of Theorems 1 and 2 to the proof of a certain local theorem

In this section we shall show how Theorems 1 and 2 follow from a local version where the characteristics have a special position. Thus assume that $P \in L^m(\mathbf{R}^n)$ and $\varrho_0 = ((x_0', 0), (\xi_0', 0)) \in T^*(\mathbf{R}^n) \setminus 0$ satisfy:

1°. P is properly supported and has a principal symbol p_m, positively homogeneous of degree m.

2°. There is a neighbourhood of ϱ_0 where $C_{p_m} > 0$ and where p_m vanishes precisely when $x_n = \xi_n = 0$.

Suppose $Q \in L^0(\mathbf{R}^n)$ is properly supported and satisfies:

3°. $\mathrm{WF}(Q) \cap \{(x, \xi) \in T^*(\mathbf{R}^n) \setminus 0;\ \ \xi' = 0\} = \emptyset$.

4°. Q has a principal symbol q, positively homogeneous of degree 0, such that $q(\varrho_0) \neq 0$.

5°. p_m does not vanish anywhere in $\{((x_0', 0), (\xi_0', \xi_n)) \in \mathrm{WF}(Q);\ \ \xi_n \neq 0\}$.

Let γ be the restriction operator $C^\infty(\mathbf{R}^n) \to C^\infty(\mathbf{R}^{n-1})$, defined by $\gamma u(x') = u(x', 0)$, $u \in C^\infty(\mathbf{R}^n)$. By I we shall always denote the identity operator in the appropriate space.

THEOREM 2.1. *If the operator*

$$\mathcal{P} = \begin{pmatrix} P \\ \gamma Q \end{pmatrix} : \mathcal{D}'(\mathbf{R}^n) \to \mathcal{D}'(\mathbf{R}^n) \times \mathcal{D}'(\mathbf{R}^{n-1})$$

is defined by $\mathcal{P}u = (Pu, \gamma Qu)$, $u \in \mathcal{D}'(\mathbf{R}^n)$, *there is a properly supported operator* $\mathcal{E} = (E, E^+) : \mathcal{D}'(\mathbf{R}^n) \times \mathcal{D}'(\mathbf{R}^{n-1}) \to \mathcal{D}'(\mathbf{R}^n)$

$$(u, u^+) \to Eu + E^+u^+,$$

which is a parametrix of $\mathcal{P}$ *near* ϱ_0 *in the following sence:*

(i) *If* $Z \in L^0(\mathbf{R}^n)$ *and* $\mathrm{WF}(Z)$ *is sufficiently close to* ϱ_0, *then*

$$Z(\mathcal{E}\mathcal{P} - I) \equiv 0.$$

(*Here we recall that* $\equiv$ *denotes equality modulo an operator with* C^∞ *kernel.*)

(ii) *If* $Z \in L^0(\mathbf{R}^n)$ *and* $Z' \in L^0(\mathbf{R}^{n-1})$ *and if* $\mathrm{WF}(Z)$ *and* $\mathrm{WF}(Z')$ *are sufficiently close to* ϱ_0 *and* $\varrho_0' = (x_0', \xi_0')$ *respectively, then*

$$(\mathcal{P}\mathcal{E} - I)\begin{pmatrix} Z & 0 \\ 0 & Z' \end{pmatrix} \equiv 0.$$

Moreover $\mathcal{E}$ *has the following properties:*

(iii) E *is continuous* $H_s^{\mathrm{loc}}(\mathbf{R}^n) \to H_{s+m-\frac12}^{\mathrm{loc}}(\mathbf{R}^n)$ *for all* $s \in R$ *and* E^+ *is continuous* $H_s^{\mathrm{loc}}(\mathbf{R}^{n-1}) \to H_{s+\frac12}^{\mathrm{loc}}(\mathbf{R}^n)$ *for all* $s \in \mathbf{R}$.

(iv) $\mathrm{WF}'(E) \subset \{((x, \xi), (x, \xi)) \in (T^*(\mathbf{R}^n)\setminus 0) \times (T^*(\mathbf{R}^n)\setminus 0)\}$

$\cup \{(((x', 0), (\xi', 0)), ((x', 0), \xi)) \in (T^*(\mathbf{R}^n)\setminus 0) \times (T^*(\mathbf{R}^n)\setminus 0);\ ((x', 0), \xi) \in \mathrm{WF}(Q)\}$

and $\mathrm{WF}'(E^+) \subset \{(((x', 0), (\xi', 0)), (x', \xi')) \in (T^*(\mathbf{R}^n)\setminus 0) \times (T^*(\mathbf{R}^{n-1})\setminus 0)\}$

Proof that Theorem 2.1 implies Theorem 2. We shall use the important idea of Egorov [3], [4] and Nirenberg–Treves [15, part II], to simplify the study by using a suitable canonical transformation. For that purpose we shall use the theory of Fourier integral operators, developed by Hörmander [10].

LEMMA 2.2. *Let ϱ, P, and Σ^+ be as in Theorem 2. Then there is an open conic neighbourhood U of ϱ and an injective, homogeneous canonical transformation $\varkappa$: $U \to T^*(\mathbf{R}^n)\setminus 0$ which maps $U \cap \Sigma^+$ into $\{(x, \xi) \in T(\mathbf{R}^n)\setminus 0;\ x_n = \xi_n = 0\}$.*

Proof of Lemma 2.2. Choose local coordinates $x = (x_1, \ldots, x_n)$ with the origin in $\pi\varrho$. Since $C_p(\varrho) \neq 0$, either $\operatorname{grad}_\xi \operatorname{Re} p \neq 0$ or $\operatorname{grad}_\xi \operatorname{Im} p \neq 0$ near ϱ. It is no restriction to assume that $\operatorname{grad}_\xi \operatorname{Re} p \neq 0$ and we can even assume that $(\partial/\partial\xi_n) \operatorname{Re} p \neq 0$ near ϱ. In a conic neighbourhood of ϱ the surface $\operatorname{Re} p = 0$ is then given by an equation $\xi_n = \tau(x, \xi')$, where $\tau \in C^\infty(\mathbf{R}^n \times (\mathbf{R}^{n-1}\setminus\{0\}))$ is real valued and positively homogeneous of degree 1 with respect to ξ'. By the Hamilton–Jacobi theory there is a real valued C^∞ function $\phi = \phi(x, \xi')$, positively homogeneous of degree 1 with respect to ξ', defined for $|x| < \text{const.} > 0$, $\xi' \in \mathbf{R}^{n-1}\setminus\{0\}$, such that

$$\frac{\partial}{\partial x_n} \phi(x, \xi') = \tau(x, \phi'_{x'}(x, \xi')) \quad \phi\,|_{x_n=0} = \langle x', \xi' \rangle \tag{2.1}$$

Put $\phi(x, y, \xi) = x_n \xi_n + \phi(x, \xi') - \langle y, \xi \rangle$. Then it is easy to verify that Φ is a non-degenerate phase function for small x (see [10]). The corresponding canonical relation R_Φ is given by

$$R_\Phi : ((\phi'_{\xi'}(x, \xi'), x_n), \xi) \to (x, (\phi'_{x'}(x, \xi'), \partial\phi(x, \xi')/\partial x_n + \xi_n)).$$

By [10], this relation is locally a canonical transformation $\varkappa_1$, which maps the surface $\xi_n = 0$ into the surface $\xi_n = \tau(x, \xi')$. Since the functional determinant of every canonical transformation is $\neq 0$, we can assume (after having restricted $\varkappa_1$ suitably), that $\varkappa_1$ is a diffeomorphism and that $\varkappa_1^{-1}$ maps the surface $\xi_n = \tau(x, \xi')$ in a homogeneous neighbourhood of ϱ into the surface $\xi_n = 0$. Since canonical transformations leave Poisson brackets invariant it follows from the condition (A), that

$$C_{p_1} \neq 0 \text{ near } \varkappa_1^{-1}\varrho, \text{ where we have put } p_1 = p \circ \varkappa_1. \tag{2.2}$$

Now $\operatorname{Re} p_1 = 0$ for $\xi_n = 0$. Thus (2.2) implies that $(\partial/\partial x_n) \operatorname{Im} p_1 \neq 0$ and therefore the surface $p_1 = 0$ can be given by the equations $\xi_n = 0$ and $x_n = \gamma(x', \xi')$ near $\varkappa_1^{-1}\varrho$, where $\gamma \in C^\infty(\mathbf{R}^{n-1} \times (\mathbf{R}^{n-1}\setminus\{0\}))$ is positively homogeneous of degree 0 with respect to ξ'.

Put $$\Psi(x, y, \xi) = \langle x, \xi\rangle + \gamma(x', \xi')\xi_n - \langle y, \xi\rangle.$$

Then Ψ is a non-degenerate phase function for small $|\xi_n|/|\xi'|$ and the corresponding canonical relation R_Ψ is given by

$$(x + (\xi_n \gamma'_{\xi'}(x', \xi'), \gamma(x', \xi')), \xi) \to (x, (\xi' + \xi_n \gamma'_{x'}(x', \xi'), \xi_n)).$$

Locally, this relation is a homogeneous canonical diffeomorphism $\varkappa_2$, mapping the surface $\xi_n = x_n - \gamma(x', \xi') = 0$ in a neighbourhood of $\varkappa_1^{-1}\varrho$ into the surface $\xi_n = x_n = 0$. It is now clear that $\varkappa = \varkappa_2 \circ \varkappa_1^{-1}$ has the properties required in the lemma.

Remark. Instead of making an explicit construction one can derive Lemma 2.2 from classical theorems on canonical transformations. (See e.g. Duistermaat–Hörmander [1] Proposition 6.1.3.)

From the results in [10] and [1] it follows that there exist properly supported Fourier integral operators G: $\mathcal{D}'(\Omega) \to \mathcal{D}'(\mathbf{R}^n)$ and G': $\mathcal{D}'(\mathbf{R}^n) \to \mathcal{D}'(\Omega)$ with the following properties:

G and G' are continuous from H_s^{loc} to H_s^{loc} for all s (2.3)

$\mathrm{WF}'(G)$ and $\mathrm{WF}'(G')$ are contained in the graphs of $\varkappa$ and $\varkappa^{-1}$ respectively (2.4)

If $A \in L^k(\Omega)$ and $B \in L^k(\mathbf{R}^n)$ have principal symbols a and b respectively then $GAG' \in L^k(\mathbf{R}^n)$ and $G'BG \in L^k(\Omega)$ and they have principal symbols equal to $a \circ \varkappa^{-1}$ and $b \circ k$ near $\varkappa(\varrho)$ and ϱ respectively. Moreover $\mathrm{WF}(GAG') \subset \varkappa(\mathrm{WF}(A))$ and $\mathrm{WF}(G'BG) \subset \varkappa^{-1}\mathrm{WF}(B)$. (2.5)

$$\varkappa(\varrho) \notin \mathrm{WF}(GG' - I) \text{ and } \varrho \notin \mathrm{WF}(G'G - I). \tag{2.6}$$

We put $\varrho_0 = \varkappa(\varrho)$ and $\varrho_0' = (x_0', \xi_0')$, where $((x_0', 0), (\xi_0', 0)) = \varrho_0$. Moreover we put $\tilde{P} = GPG'$, where P is the operator in Theorem 2. Then the pair $(\tilde{P}, \varrho_0)$ satisfies the assumptions 1° and 2° of Theorem 2.1. In fact, by (2.5), $\tilde{P}$ belongs to $L^m(\mathbf{R}^n)$ and has a homogeneous principal symbol p_m which is equal to $p \circ \varkappa^{-1}$ in a neighbourhood of ϱ_0. By the choice of $\varkappa$ the equation $p_m = 0$ is equivalent to $x_n = \xi_n = 0$ in a conic neighbourhood of ϱ_0. Moreover, since canonical transformations preserve Poisson brackets, we have $C_{p_m} = C_p \circ \varkappa^{-1} > 0$ in a neighbourhood of ϱ_0.

With this choice of $(\tilde{P}, \varrho_0)$, let $Q \in L^0(\mathbf{R}^n)$ satisfy the assumtions 3°, 4° and 5° of Theorem 2.1 and also satisfy:

$$\mathrm{WF}(Q) \cap (\mathrm{WF}(I - GG') \cup \varkappa \mathrm{WF}(I - G'G)) = \emptyset. \tag{2.7}$$

Let $\mathcal{E} = (E, E^+)$ be the corresponding local parametrix in Theorem 2.1 and put

$$R_\varrho^+ = \gamma QG, \quad E_\varrho = G'EG, \quad E_\varrho^+ = G'E^+ \text{ and } \varrho' = \varrho_0'.$$

To verify (ia) of Theorem 2, let $T' \in L^0(\mathbf{R}^{n-1})$ with $\mathrm{WF}(T')$ close to ϱ'. Then $(R_\varrho^+ E_\varrho^+ - I)T' = (\gamma QGG'E^+ - I)T' = (\gamma QE^+ - I)T' - \gamma Q(I - GG')E^+T'$. Theorem 2.1 implies that $(\gamma QE^+ - I)T'$ has C^∞ kernel if $\mathrm{WF}(T')$ is sufficiently close to ϱ'. Moreover $\gamma Q(I-GG')E^+T'$ has C^∞ kernel in view of (2.7). (We assume that the reader is familiar with the calculus of wave front sets, developed by Hörmander [10, section 2.5].) This proves the first half of (ia).

Now look at $PE_\varrho^+T' = PG'E^+T' = G'GPG'E^+T' + (I - G'G)PG'E^+T' = G'\tilde{P}E^+T' + (I - G'G)PG'E^+T'$. The term $G'\tilde{P}E^+T'$ has C^∞ kernel by Theorem 2.1. By a simple wavefront calculus we see that this is the case also with the last term. ($\mathrm{WF}'(G')$ is contained in the graph of $\varkappa^{-1}$ by (2.4.)). This proves the second half of (ia).

To prove (ib), we write $(PE_\varrho - I)T = PG'EGT - T$. Looking at the wavefront sets it is easy to see that $(I - G'G)(PG'EGT - T)$ and $(PG'EGT - T)(I - G'G)$ have C^∞ kernels if $\mathrm{WF}(T)$ is sufficiently close to ϱ. Thus, if $\equiv$ denotes equality modulo an operator with C^∞ kernel, we get

$$(PE_\varrho - I)T \equiv G'G(PG'EGT - T)G'G = G'(GPG'EGTG' - GTG')G = G'(\tilde{P}E - I)SG,$$

where $S = GTG'$. By (2.5) we have $S \in L^0(\mathbf{R}^n)$ and $\mathrm{WF}(S)$ is close to ϱ_0 when $\mathrm{WF}(T)$ is close to ϱ. Thus by Theorem 2.1, the operator $(\tilde{P}E - I)S$ has C^∞ kernel and the first part of (ib) follows.

To prove the second part, we write

$$R_\varrho^+ E_\varrho T = \gamma QGG'EGT = \gamma QEGT - \gamma Q(I - GG')EGT.$$

By (2.7) we have $\gamma Q(I - GG')EGT \equiv 0$. Moreover $\gamma QEGT = \gamma QESG + \gamma QEGT(I - G'G)$, where as already observed $S = GTG' \in L^0(\mathbf{R}^n)$ and $\mathrm{WF}(S)$ is close to ϱ_0. Thus $\gamma QESG \equiv 0$ by Theorem 2.1 and $\gamma QEGT(I - G'G) \equiv 0$ since $\mathrm{WF}(T) \cap \mathrm{WF}(I - G'G) = \emptyset$ if $\mathrm{WF}(T)$ is close to ϱ. This proves (ib).

To prove (ii), we write

$$TE_\varrho P = TG'EGP.$$

Looking at the wavefront sets, we see that we get an operator with C^∞ kernel if we multiply $TG'EGP$ from the left or from the right with $(I - G'G)$ when $\mathrm{WF}(T)$ is close to ϱ. In fact, this is obvious in the case of left multiplication, since $\mathrm{WF}(T) \cap \mathrm{WF}(I - G'G) = \emptyset$ when $\mathrm{WF}(T)$ is close to ϱ. From (2.7) and (2.4) we see that $\mathrm{WF}'(Q) \circ \mathrm{WF}'(GP(I - G'G)) = \emptyset$. Thus from (iv) of Theorem 2.1 we see that $TG'EGP(I - G'G)$ has C^∞ kernel, when $\mathrm{WF}(T)$ is close to ϱ, so our statement is true also in the case of right multiplication. Using this result and (2.6) we get

$$T(E_\varrho P+E_\varrho^+R_\varrho^+-I)\equiv G'GTG'EGPG'G+TG'E^+\gamma QG-T$$
$$\equiv G'SE\tilde{P}G+G'GTG'E^+\gamma QG-G'SG\equiv G'S(E\tilde{P}+E^+\gamma Q-I)\,G\equiv 0,$$

where the last equivalence follows from Theorem 2.1. This completes the proof that Theorem 2.1 implies Theorem 2.

We shall next prove that Theorem 2.1 implies Theorem 1. The first step is to prove the following local result:

PROPOSITION 2.3. *Let Ω, P and R^+ be as in Theorem 1. Let $\varrho\in\Sigma_p^+$ and put $\varrho'=\mathcal{G}_+\varrho\in T^*(\Gamma^+)\setminus 0$, where $\mathcal{G}_+$ is the natural diffeomorfism: $\Sigma^+\to T^*(\Gamma^+)\setminus 0$ defined in section 0. Then there exist properly supported operators E_ϱ: $\mathcal{D}'(\Omega)\to\mathcal{D}'(\Omega)$ and E_ϱ^+: $\mathcal{D}'(\Gamma^+)\to\mathcal{D}'(\Omega)$ with the following properties.*

(ia) *If $T'\in L^0(\Gamma^+)$ and $\mathrm{WF}(T')$ is sufficiently close to ϱ' then $(R^+E_\varrho^+-I)\,T'\equiv 0$ and $PE_\varrho^+T'\equiv 0$.*

(ib) *If $T\in L^0(\Omega)$ and $\mathrm{WF}(T)$ is sufficiently close to ϱ, then $(PE_\varrho-I)\,T\equiv 0$ and $R^+E_\varrho T\equiv 0$.*

(ii) *For T as in* (ib): $T(E_\varrho P+E_\varrho^+R^+-I)\equiv 0$.

(iii) *E_ϱ is continuous $H_s^{\mathrm{loc}}(\Omega)\to H_{s+m-\frac12}^{\mathrm{loc}}(\Omega)$ and E_ϱ^+ is continuous $H_s^{\mathrm{loc}}(\Gamma^+)\to H_{s+\frac12}^{\mathrm{loc}}(\Omega)$ for all $s\in\mathbf{R}$. Moreover:*

$\mathrm{WF}'(E_\varrho)\subset\{(\mu,\mu)\in(T^*(\Omega)\setminus 0)\times(T^*(\Omega)\setminus 0)\}\cup\{(f_\Omega\nu, f_\Omega\mu);\ \nu\in\Sigma_0^+,\ \ \mu\in\mathrm{WF}(R^+)$ *and* $f_{\Gamma^+}\nu=f_{\Gamma^+}\mu\}$

$\mathrm{WF}'(E_\varrho^+)\subset\{(\mu,\mathcal{G}_+\mu);\ \ \mu\in\Sigma^+\}$.

Here f_Ω, Σ_0^+, $\mathrm{WF}(R^+)$ and f_{Γ^+} are defined in section 0.

Note that (ia) and (ib) can be expressed more briefly, by stating that

$$\begin{pmatrix}P\\ R^+\end{pmatrix}(E_\varrho\, E_\varrho^+)\begin{pmatrix}T & 0\\ 0 & T'\end{pmatrix}\equiv\begin{pmatrix}T & 0\\ 0 & T'\end{pmatrix}.$$

The proof of this proposition is very similar to the proof of Theorem 2 above, but we have to be more explicit. Note that Eškin [6′] uses a canonical transformation similar to the one in Lemma 2.4 below.

Proof. In view of Proposition 0.1 we can assume that $\Omega=\mathbf{R}^n$ and that the component of ϱ in Σ is given by $x_n=\xi_n-\tau(x',\xi')=0$. In fact, since Proposition 2.3 is a purely local statement and (iii) gives us a good control over the singular supports of the distribution kernels of E_ϱ and E_ϱ^+ it is easy to prove the proposition in the general case, once we have established it in this special one.

We can identify the component of $\pi\varrho'$ in Γ^+ with the hyperplane $x_n=0$ and assume that $\varrho'=(0, \xi_0')$, where $\varrho=(0, (\xi_0', \tau(0, \xi_0')))$. In this component R^+ is of the form γA, where $A \in L^0(\mathbf{R}^n)$ is properly supported and has principal symbol a, positively homogeneous of degree 0 and different from 0 when $x_n=\xi_n-\tau(x', \xi')=0$. Moreover $\mathrm{WF}(A)$ does not intersect the other components of Σ or $\{(x, \xi) \in T^*(\mathbf{R}^n)\setminus 0;\ \xi'=0\}$.

Consider the phase function $\Phi(x, y, \xi)=\phi(x, \xi)-\langle y, \xi\rangle$, $x, y, \xi \in \mathbf{R}^n$, where

$$\phi(x, \xi) = \langle x, \xi\rangle + x_n\chi(\varepsilon\xi_n/|\xi'|)\tau(x', \xi') \tag{2.8}$$

Here $\varepsilon>0$ and $\chi \in C_0^\infty(\mathbf{R})$ is equal to 1 near the origin. Then we have the following explicit analogue of Lemma 2.2.

LEMMA 2.4. *If $\varepsilon>0$ is small enough, Φ is a non-degenerate phase function for small x and induces a homogeneous canonical diffeomorphism $\varkappa^{-1}$ from some neighbourhood of $T_0^*(\mathbf{R}^n)\setminus\{0\}$ (the set of non zero cotangent vectors at the origin) onto some neighbourhood of $T_0^*(\mathbf{R}^n)\setminus\{0\}$, mapping the surface $x_n=\xi_n=0$ into the surface $x_n=\xi_n-\tau(x', \xi')=0$, and such that if $(x, \xi)=\varkappa^{-1}(y, \eta)$ then $x_n=0 \Leftrightarrow y_n=0$ and $x_n=y_n=0 \Rightarrow \xi'=\eta'$, $x'=y'$.*

Proof. Choose $\varepsilon>0$ so small that $|\partial\chi(\varepsilon\xi_n/|\xi'|)\tau(x', \xi)/\partial\xi_n| < \frac{1}{2}$ for all $\xi \neq 0$ when x' is small. Then it is easy to see, that Φ is non-degenerate for small x. The corresponding canonical relation R_Φ is given by

$$\left(x + x_n \operatorname{grad}_\xi \chi\left(\frac{\varepsilon\xi_n}{|\xi'|}\right)\tau(x', \xi'), \xi\right) \to \left(x, \xi + \left(x_n \operatorname{grad}_{x'} \chi\left(\frac{\varepsilon\xi_n}{|\xi'|}\right)\tau(x', \xi'), \chi\left(\frac{\varepsilon\xi_n}{|\xi'|}\right)\tau(x', \xi')\right)\right)$$

Then we have:
$$R_\Phi = R_1 \circ R_2^{-1}, \tag{2.9}$$

where
$$R_1: (x, \xi) \to \left(x, \xi + \left(x_n \operatorname{grad}_{x'} \chi\left(\frac{\varepsilon\xi_n}{|\xi'|}\right)\tau(x', \xi'), \chi\left(\frac{\varepsilon\xi_n}{|\xi'|}\right)\tau(x', \xi')\right)\right) \tag{2.10}$$

and
$$R_2: (x, \xi) \to \left(x + x_n \operatorname{grad}_\xi \chi\left(\frac{\varepsilon\xi_n}{|\xi'|}\right)\tau(x', \xi'), \xi\right). \tag{2.11}$$

Then:

1°. R_1 and R_2 have bijective differentials for small x.

2°. The restrictions of R_1 and R_2 to $T_0^*(\mathbf{R}^n)\setminus\{0\}$ are diffeomorphisms onto $T_0^*(\mathbf{R}^n)\setminus\{0\}$.

Since R_1 and R_2 are homogeneous, we conclude from 1° and 2°, that they are injective for small x and thus by (2.9), that R_Φ near $(T_0^*(\mathbf{R}^n)\setminus\{0\}) \times (T_0^*(\mathbf{R}^n)\setminus\{0\})$ coincides with the graph of a canonical diffeomorphism $\varkappa^{-1}$, mapping some neighbourhood of $T_0^*(\mathbf{R}^n)\setminus\{0\}$

onto some neighbourhood of $T_0^*(\mathbf{R}^n)\setminus\{0\}$. The other properties of $\varkappa^{-1}$ follow from our explicit formulas. The proof is complete.

Choose $\psi\in C_0^\infty(\mathbf{R}^n)$, with $\psi(x)=1$ in a neighbourhood of the origin and let G': $\mathcal{D}'(\mathbf{R}^n)\to\mathcal{D}'(\mathbf{R}^n)$ be the Fourier integral operator given by

$$G'u(x)=\iint\psi(x)\,\psi(y)\,(1-\psi(\xi))\,e^{i(\phi(x,\xi)-\langle y,\xi\rangle)}u(y)\,dy\,d\xi/(2\pi)^n,\quad u\in C^\infty(\mathbf{R}^n),\tag{2.12}$$

where ϕ is given by (2.8). If ψ has its support sufficiently close to the origin, it follows from Lemma 2.4 and the results in [1] and [10] that there exists a properly supported Fourier integral operator G: $\mathcal{D}'(\mathbf{R}^n)\to\mathcal{D}'(\mathbf{R}^n)$ such that

G and G' are continuous $H_s^{\mathrm{loc}}\to H_s^{\mathrm{loc}}$ for all s and $\mathrm{WF}'(G)$ and $\mathrm{WF}'(G')$ are contained in the graphs of $\varkappa$ and $\varkappa^{-1}$ respectively. (2.13)

For every $T\in L^M(\mathbf{R}^n)$ with principal symbol t, the operators GTG' and $G'TG$ belong to $L^M(\mathbf{R}^n)$ and their principal symbols are equal to $t\circ\varkappa^{-1}$ and $t\circ\varkappa$ respectively in a neighbourhood of $T_0^*(\mathbf{R}^n)\setminus\{0\}$. Moreover $\mathrm{WF}(GTG')\subset\varkappa(\mathrm{WF}(T))$ and $\mathrm{WF}(G'TG)\subset\varkappa^{-1}(\mathrm{WF}(T))$. (2.14)

$$(T_0^*(\mathbf{R}^n)\setminus\{0\})\cap(WF(GG'-I)\cup WF(G'G-I))=\emptyset\tag{2.15}$$

If γ is the restriction operator $C^\infty(\mathbf{R}^n)\ni u\to u|_{x_n=0}$, we have:

The distribution kernels of the operators $\gamma-\gamma G'$ and $\gamma-\gamma G$ are smooth near $(0,0)\in\mathbf{R}^{n-1}\times\mathbf{R}^n$. (2.16)

In fact, by (2.8) we have $\phi((x',0),\xi)=\langle(x',0),\xi\rangle$ and using the Fourier inversion formula in (2.12), we get:

$$\begin{aligned}\gamma G'u(x')&=\iint\psi((x',0))\,\psi(y)\,e^{i(\langle(x',0),\xi\rangle-\langle y,\xi\rangle)}u(y)\,dy\,d\xi/(2\pi)^n\\&\quad-\iint\psi((x',0))\,\psi(y)\,\psi(\xi)\,e^{i(\langle(x',0),\xi\rangle-\langle y,\xi\rangle)}u(y)\,dy\,d\xi/(2\pi)^n\\&=(\gamma\psi^2u)(x')-Ku(x'),\quad u\in C^\infty(\mathbf{R}^n),\end{aligned}$$

where K is an operator with C^∞ kernel. Thus $\gamma-\gamma G'$ has smooth kernel near $(0,0)$ and the corresponding statement about $\gamma-\gamma G$ follows if we multiply $\gamma-\gamma G'$ with G and use (2.15).

Now put $\tilde{P}=GPG'$ and $Q=GAG'$ and $\varrho_0=(0,(\xi_0',0))$, where $(0,(\xi_0',\tau(0,\xi_0')))=\varrho$ in Proposition 2.3. Then it follows from (2.14) and Lemma 2.4, that $(\tilde{P},Q,\varrho_0)$ satisfies the assumptions of Theorem 2.1. Let E and E^+ be the corresponding local parametrix operators and put

$$E_\varrho = G'EG\colon \mathcal{D}'(\mathbf{R}^n) \to \mathcal{D}'(\mathbf{R}^n)$$

and

$$E_\varrho^+ = G'E^+\theta\colon \mathcal{D}'(\Gamma^+) \to \mathcal{D}'(\mathbf{R}^n),$$

where θ means multiplication with the characteristic function of the component of Γ^+ which we have identified with the plane $x_n = 0$.

Proof of (iii) *of Proposition 2.3.* The H_s-continuity properties follow at once from the construction. The estimates of the wavefront sets follow from Theorem 2.1 and Lemma 2.4, since $\mathrm{WF}'(G)$ and $\mathrm{WF}'(G')$ are contained in the graphs of $\varkappa$ and $\varkappa^{-1}$ respectively.

Proof of (ia). Let $T' \in L^0(\Gamma^+)$ with $\mathrm{WF}(T')$ close to $(0, \xi_0') = \varrho'$. Then, combining the estimate of $\mathrm{WF}'(R^+)$ given by (C^+) in section 0 with the estimate for $\mathrm{WF}'(E_\varrho^+)$ just proved, we find that $\mathrm{WF}'((I-\theta)R^+E_\varrho^+T') = \emptyset$ and consequently that $(I-\theta)(R^+E_\varrho^+ - I)T' \equiv 0$. To prove the first half of (ia) it therefore suffices to prove that $\theta(R^+E_\varrho^+ - I)T' = \gamma A E_\varrho^+ T' - T' \equiv 0$. We have

$$\gamma A E_\varrho^+ T' = \gamma A G' E^+ \theta T' \equiv \gamma A G' E^+ T' \equiv \gamma G A G' E^+ T' = \gamma Q E^+ T' \equiv T',$$

where the second congruence follows from (2.16) and the third from Theorem 2.1. This proves the first part of (ia). The proof of the second part is exactly the same as in the proof of Theorem 2, so we omit it.

Proof of (ib). The first half is proved exactly as in Theorem 2. To prove the second half, we observe (as in the proof of (ia)), that $(I-\theta)R^+E_\varrho T$ has C^∞ kernel if $\mathrm{WF}(T)$ is sufficiently close to ϱ. Moreover

$$\theta R^+ E_\varrho T = \gamma A G' E G T \equiv \gamma G A G' E G T = \gamma Q E G T \equiv \gamma Q E (G T G') G,$$

where the first congruence follows from (2.16) and the second from (2.15). Now $GTG' \in L^0(\mathbf{R}^n)$ and $\mathrm{WF}(GTG')$ is close to $\varrho_0 = (0, (\xi_0', 0))$ in view of (2.14). Thus by Theorem 2.1, we have $\gamma Q E(GTG') \equiv 0$ and (ib) follows.

The proof of (ii) is almost the same as the proof of the corresponding part of Theorem 2, so we omit it.

The next step in the proof of Theorem 1 will be to construct global left and right parametrices near Σ^+.

Since Σ^+ is closed and conic, we conclude from Proposition 2.3, that for each j in some index set J there exist operators $E_j\colon \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ and $E_j^+\colon \mathcal{D}'(\Gamma^+) \to \mathcal{D}'(\Omega)$, an open conic set $V_j \subset T^*(\Omega)$ and an open set $W_j \subset\subset \Omega$ with the following properties:

The H_s-continuity and the properties of the wave front sets stated in Proposition 2.3 for (E_ϱ, E_ϱ^+) are valid for (E_j, E_j^+). (2.17)

$$T(E_jP+E_j^+R^+-I)\equiv 0 \text{ for every } T\in L^0(\Omega) \text{ with } \mathrm{WF}(T)\subset V_j. \tag{2.18}$$

$$\Sigma^+\subset \bigcup_{j\in J} V_j \quad \text{and } V_j\cap f_\Omega \mathrm{WF}(R^{-*})=\varnothing \text{ for all } j. \tag{2.19}$$

$\{W_j\}_{j\in J}$ is a locally finite covering of Ω and $\overline{\pi V_j}\subset W_j$ for all j. (2.20)

Moreover we can assume that

supp $E_j\subset W_j\times W_j$ and supp $E_j^+\subset W_j\times f^{-1}W_j$, where supp denotes the support of the distribution kernel. (2.21)

In fact, by the estimates (2.17) of $\mathrm{WF}'(E_j)$ and $\mathrm{WF}'(E_j^+)$, we see that we can replace E_j by $\psi_jE_j\psi_j$ and E_j^+ by $\psi_jE_j^+(\psi_j\circ f)$, without changing (2.17) and (2.18) if $\psi_j\in C_0^\infty(W_j)$ and $\psi_j=1$ near $\overline{\pi V_j}$. (Here $f\colon \Gamma\to\Omega$ is defined in section 0.)

Now take functions $0\leqslant t_j\in C^\infty(T^*(\Omega)\setminus 0)$, positively homogeneous of degree 0 with supp $t_j\subset V_j$, such that $\Sigma_{j\in J}\, t_j(x,\xi)>0$ on Σ^+ and take $T_j\in L^0(\Omega)$ with principal symbol t_j, such that $\mathrm{WF}(T_j)\subset$ supp t_j and supp $T_j\subset W_j\times W_j$. Then it follows from (2.20) and (2.21) that the operators

$$F=\Sigma T_jE_j\colon\quad \mathcal{D}'(\Omega)\to\mathcal{D}'(\Omega),$$

$$F^+=\Sigma T_jE_j^+\colon\quad \mathcal{D}'(\Gamma^+)\to\mathcal{D}'(\Omega)$$

and $T=\Sigma T_j\in L^0(\Omega)$ are well defined and properly supported. Moreover

T has a principal symbol, positively homogeneous of degree 0, which is >0 on Σ^+, but $\mathrm{WF}(T)\cap f_\Omega\mathrm{WF}(R^{-*})=\varnothing$. (2.22)

From (2.18) we get

$$FP+F^+R^+\equiv T. \tag{2.23}$$

From (2.17) we get

(F, F^+) has the H_s continuity properties stated for (E, E^+) in Theorem 1. (2.24)

$$\mathrm{WF}'(F)\subset\{(\varrho,\varrho)\in\mathrm{WF}(T)\times\mathrm{WF}(T)\}\cup\{(f_\Omega\varrho, f_\Omega\mu);\ \varrho\in\Sigma_0^+, \mu\in\mathrm{WF}(R^+), f_{\Gamma^+}\varrho=f_{\Gamma^+}\mu\}. \tag{2.25}$$

$$\mathrm{WF}'(F^+)\subset\{(\varrho, \mathcal{G}_+\varrho)\in\Sigma^+\times(T^*(\Gamma^+)\setminus 0)\} \tag{2.26}$$

(2.23) means that we can think of (F, F^+) as the product of a left inverse of $\mathcal{P}$ to the left by T. The construction of a "right inverse" is quite similar, so we only sketch it. As above we cover Σ^+ with small open conic sets V_j, $j\in J$ but this time we also have to cover $T^*(\Gamma^+)\setminus 0$ with small open conic sets V_j', $j\in J$. Let (E_j, E_j^+) be the corresponding local inverse in the sense of Proposition 2.3 and let $S_j \in L^0(\Omega)$ and $S'_j \in L^0(\Gamma^+)$ be such that $\mathrm{WF}(S_j) \subset V_j$ and $\mathrm{WF}(S'_j) \subset V'_j$. With an appropriate choice of E_j, E_j^+, S_j and S'_j the operators

$$L = \sum E_j S_j : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega), \quad L^+ = \sum E_j^+ S'_j : \mathcal{D}'(\Gamma^+) \to \mathcal{D}'(\Omega)$$
$$S = \sum S_j \in L^0(\Omega) \text{ and } S' = \sum S'_j \in L^0(\Gamma^+)$$

are all welldefined and properly supported and we have:

S and S' have principal symbols which are positively homogeneous of degree 0 and strictly positive on Σ^+ and $T^*(\Gamma^+)\setminus 0$ respectively. Moreover $\mathrm{WF}(S) \cap f_\Omega \mathrm{WF}(R^{-*}) = \emptyset$. (2.27)

The operators $PL - S$, PL^+, $R^+L^+ - S'$ and R^+L have C^∞ kernels. (2.28)

(L, L^+) has the same H_s-continuity properties and analogous estimates of the wavefront sets as (F, F^+) in (2.24), (2.25) and (2.26). (2.29)

Now we shall study $\mathcal{D}$ near Σ_p^-. This is easily done by duality. In fact, the complex adjoint P^* of P has the principal symbol $\bar{p}$. Since P satisfies (A) and (B) in section 0 and $C_{\bar{p}} = -C_p$, we see that P^* satisfies (A) and (B) and that $\Sigma_{\bar{p}}^+ = \Sigma_p^-$. For the operator $u \to (P^*u, R^{-*}u)$ we have therefore results analogous to those just obtained for the operator $u \to (Pu, Ru^+)$. Passing to complex adjoints we get the following results for the adjoint operator $(u, u^-) \to Pu + R^- u^-$: There exist properly supported operators

$$F_0,\ L_0 : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega),$$
$$F_0^-,\ L_0^- : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Gamma^-),$$
$$T_0,\ S_0 \in L^0(\Omega) \text{ and } S'_0 \in L^0(\Gamma^-) \text{ such that:}$$

S_0 and T_0 have principal symbols, positively homogeneous of degree 0, which are >0 on Σ^-, but $(\mathrm{WF}(S_0) \cup \mathrm{WF}(T_0)) \cap f_\Omega \mathrm{WF}(R^+) = \emptyset$. (2.30)

S'_0 has a principal symbol, positively homogeneous of degree 0, which is strictly positive. (2.31)

$$PF_0 + R^- F_0^- \equiv T_0. \tag{2.32}$$

(F_0, F_0^-) and (L_0, L_0^-) have the same H_s continuity properties as (E, E^-) in Theorem 1. (2.33)

$$\mathrm{WF}'(F_0) \subset \{(\varrho, \varrho) \in W(T_0) \times \mathrm{WF}(T_0)\} \cup \{(f_\Omega \mu, f_\Omega \nu);\ \nu \in \Sigma_0^-,\ \mu \in \mathrm{WF}(R^{-*}),\ f_{\Gamma^-} \nu = f_{\Gamma^-} \mu\} \tag{2.34}$$

$$\mathrm{WF}'(L_0)\subset\{(\varrho,\varrho)\in \mathrm{WF}(S_0)\times \mathrm{WF}(S_0)\}\cup\{f_\Omega\mu, f_\Omega\nu); \nu\in\Sigma_0^-, \mu\in \mathrm{WF}(R^{-*}), f_{\Gamma^-}\nu=f_{\Gamma^-}\mu\} \tag{2.35}$$

$$WF'(F_0^-)\cup WF'(L_0^-)\subset\{(\mathcal{G}_-\varrho,\varrho)\in(T^*(\Gamma^-)\setminus 0)\times\Sigma^-\} \tag{2.36}$$

The operators L_0P-S_0, L_0^-P, $L_0^-R^- - S_0'$ and L_0R^- have C^∞ kernels. (2.37)

We now construct a right parametrix of $\mathcal{P}$. It follows from (2.27) and (2.30) that the principal symbol of $S+T_0$ is >0 on $\Sigma=\Sigma^+\cup\Sigma^-$. Thus we can find $A\in L^0(\Omega)$, properly supported, such that T_0+A+S is elliptic and

$$\mathrm{WF}(A)\cap\Sigma=\varnothing. \tag{2.38}$$

Since the principal symbol of P is different from zero outside Σ, it follows from (2.38), that there exists $P'\in L^{-m}(\Omega)$, properly supported, such that

$$PP'\equiv A. \tag{2.39}$$

The construction of such a P' is practically the same as the construction of a pseudodifferential parametrix of an elliptic operator and we omit the details. Let $(T_0+A+S)^{-1}\in L^0(\Omega)$ and $S'^{-1}\in L^0(\Gamma^+)$ be properly supported parametrices of the elliptic operators T_0+A+S and S' respectively, so that

$$(T_0+A+S)(T_0+A+S)^{-1}\equiv I \text{ and } S'S'^{-1}\equiv I.$$

Now put $$\mathcal{E}=\begin{pmatrix} E & E^+ \\ E^- & 0\end{pmatrix}: \mathcal{D}'(\Omega)\times\mathcal{D}'(\Gamma^+)\to\mathcal{D}'(\Omega)\times\mathcal{D}'(\Gamma^-),$$

where $E=(L+P'+F_0)(S+T_0+A)^{-1}-L^+S'^{-1}R^+P'(S+T_0+A)^{-1}$

$E^+=L^+S'^{-1}$

$E^-=F_0^-(S+T_0+A)^{-1}$.

Then (ii) and (iii) of Theorem 1 follow from (2.29), (2.33), (2.34), (2.36) and Proposition 0.2.

To prove the second half of (i), means to prove the following equations:

$$PE+R^-E^-\equiv I \tag{2.40}$$

$$PE^+\equiv 0 \tag{2.41}$$

$$R^+E^+\equiv I \tag{2.42}$$

$$R^+E\equiv 0. \tag{2.43}$$

Since $PL^+\equiv 0$ by (2.28), we get

$$PE+R^-E^- \equiv (PL+PP'+PF_0)(S+T_0+A)^{-1}+R^-F_0^-(S+T_0+A)^{-1}$$
$$\equiv (PL+PP'+(PF_0+R^-F_0^-))(S+T_0+A)^{-1}\equiv I.$$

Here the last equivalence follows from (2.28), (2.39), and (2.32). This proves (2.40).

(2.41) and (2.42) follow at once from (2.28).

To prove (2.43), we note that $R^+L\equiv 0$ and $R^+L^+S'^{-1}\equiv I$, by (2.28). Moreover, if we combine (2.30), (2.34) and the condition (0.5) in Theorem 1, we conclude that $R^+F_0\equiv 0$. Thus we get

$$R^+E=(R^+L+R^+P'+R^+F_0)(S+T_0+A)^{-1}-R^+L^+S'^{-1}R^+P'(S+T_0+A)^{-1}\equiv$$
$$\equiv R^+P'(S+T_0+A)^{-1}-R^+P'(S+T_0+A)^{-1}=0.$$

This proves (2.43) and the second half of (i) is now proved.

Applying this result to the complex adjoint

$$\mathcal{P}^*=\begin{pmatrix} P^* & R^{+*} \\ R^{-*} & 0 \end{pmatrix}$$

and then passing to complex adjoints, we see that there exists an operator

$$\mathcal{B}=\begin{pmatrix} B & B^+ \\ B^- & 0 \end{pmatrix}: \mathcal{D}'(\Omega)\times\mathcal{D}'(\Gamma^+)\to\mathcal{D}'(\Omega)\times\mathcal{D}'(\Gamma^-),$$

which is continuous: $H_s^{\text{loc}}(\Omega)\times H^{\text{loc}}_{s+m-\frac{1}{2}}(\Gamma^+)\to H^{\text{loc}}_{s+m-\frac{1}{2}}(\Omega)\times H^{\text{loc}}_{s+\frac{1}{2}}(\Gamma^-)$ for all s and such that $\mathcal{B}\mathcal{P}\equiv I$. It then follows that $\mathcal{B}\equiv\mathcal{E}$ and therefore the first half of (i) in Theorem 1 holds also. In fact, $\mathcal{E}=I\mathcal{E}\equiv\mathcal{B}\mathcal{P}\mathcal{E}\equiv\mathcal{B}I=\mathcal{B}$. This completes the proof that Theorem 2.1 implies Theorem 1.

3. A factorization and further reduction of the proof

In this section we shall reduce the proof of Theorem 2.1 to the study of the system:

$$\Lambda u=v\in C^\infty(\mathbf{R}^n). \quad \gamma u=u_0\in C^\infty(\mathbf{R}^{n-1}). \tag{3.1}$$

Here Λ is the operator given in Lemma 3.1 below and γ is the restriction $C^\infty(\mathbf{R}^n)\to C^\infty(\mathbf{R}^{n-1})$ given by $(\gamma u)(x')=u(x',0)$.

The next lemma will be the essential step in our reduction. Before reading it, the reader should have a look at the appendix, where we define and state some facts about the paces $T^m(\mathbf{R}^n)$. Let P, p_m, Q, $\varrho_0=((x_0',0),(\xi_0',0))$ and $\varrho_0'=(x_0',\xi_0')$ be as in Theorem 2.1. Then there exists an open conic neighbourhood V of $A_Q=\{((x_0',0),(\xi_0',\xi_n))\in \mathrm{WF}(Q)\}$ such that in a neighbourhood of $\bar{V}$ the equation $P_m(x, \xi)=0$ is equivalent to $x_n=\xi_n=0$ and $\bar{V} \cap \{(x), \xi) \in T^*(\mathbf{R}^n)\setminus 0;\ \xi'=0\}=\emptyset$. We can even assume that

$$V=\{(x, \xi) \in T^*(\mathbf{R}^n)\setminus 0;\ (x', \xi') \in V'',\ |x_n|<\delta_V,\ \ \xi_n/|\xi'| \in B_V\} \tag{3.2}$$

where $V'' \in T^*(\mathbf{R}^{n-1})\setminus 0$ is an open conic neighbourhood of ϱ_0' and $B_V \subset\subset \mathbf{R}$ is an open neighbourhood of $\{\xi_n/|\xi_0'|;\ ((x_0', 0), (\xi_0', \xi_n)) \in \mathrm{WF}(Q)\}$ and $\delta_V>0$.

LEMMA 3.1. *There exist properly supported operators $P_0 \in L^{m-1}(\mathbf{R}^n)$ and $\Lambda= D_n - ix_n r(x, D')+s(x, D')$ with the following properties:*

(i) $\mathrm{WF}(P_0) \cap \{(x, \xi) \in T^*(\mathbf{R}^n)\setminus 0;\ \ \xi'=0\}=\emptyset$.

(ii) $r(x, D') \in T^1(\mathbf{R}^n)$ *is properly supported and its symbol is modulo S^∞ equal to $r(x, \xi')$ where r is positively homogeneous of degree* 1 *and* $\operatorname{Re} r>0$.

(iii) $s(x, D') \in T^0(\mathbf{R}^n)$ *is properly supported.*

(iv) $\mathrm{WF}(P-P_0\Lambda) \cap \bar{V}=\emptyset$. (Note that $P_0\Lambda \in L^m(\mathbf{R}^n)$ in view of (i)–(iii) and the appendix.)

(v) P_0 *has a principal symbol $p_{0,m-1}$, which is positively homogeneous of degree $m-1$ and never vanishes in $\bar{V}$.*

Proof. We take a conic neighbourhood W of $\bar{V}$ with the same properties as V. Thus in particular

$$W=\{(x, \xi) \in T^*(\mathbf{R}^n)\setminus 0;\ (x', \xi') \in W'',\ |x_n|<\delta,\ \ \xi_n/|\xi'| \in B_W\}. \tag{3.2'}$$

Put
$$W'=\{(x', \xi) \in \mathbf{R}^{n-1} \times (\mathbf{R}^n\setminus\{0\});\ \ ((x', 0), \xi) \in W\}$$

and
$$(-\delta, \delta) \times W''=\{(x, \xi') \in \mathbf{R}^n \times (\mathbf{R}^{n-1}\setminus\{0\});\ \ |x_n|<\delta,\ \ (x', \xi') \in W''\}.$$

The main step in our proof is the following "preparation theorem":

LEMMA 3.2. *For every $a(x, \xi) \in S^k(W)$ there exist $b(x, \xi) \in S^{k-m}(W)$ and $c(x, \xi') \in S^k((-\delta, \delta) \times W'')$ such that*

$$a(x, \xi)=b(x, \xi)p_m(x, \xi)+c(x, \xi') \tag{3.3}$$

If a is positively homogeneous of degree k then b and c can be chosen positively homogeneous of degree $k-m$ and k respectively.

Proof of Lemma 3.2. By considering Taylor expansions with respect to x_n we shall first find $b' \in S^{k-m}(W)$ and $c \in S^k((-\delta, \delta) \times W'')$ such that (3.3) holds to infinite order at $x_n=0$. By the assumptions in Theorem 2.1 we have $C_{p_m} \neq 0$ when $x_n=\xi_n=0$. In particular

$$\partial p_m/\partial x_n \neq 0 \ \text{ when } \ x_n=\xi_n=0. \tag{3.4}$$

$$\partial p_m/\partial \xi_n \neq 0 \text{ when } x_n = \xi_n = 0. \tag{3.5}$$

Let $p_m \sim \sum_{j=0}^{\infty} d_j(x', \xi) x_n^j$ be the Taylor expansion of p_m. By (3.5) we have $\partial d_0(x', (\xi', 0))/\partial \xi_n \neq 0$. Since p_m vanishes in W precisely when $x_n = \xi_n = 0$, we see that d_0 vanishes precisely when $\xi_n = 0$. Let a have the Taylor expansion

$$a(x, \xi) \sim \sum_{j=0}^{\infty} a_j(x', \xi) x_n^j, \quad a_j \in S^k(W').$$

We look for b' and c with the expansions

$$b'(x, \xi) \sim \sum_{j=0}^{\infty} b_j(x', \xi) x_n^j, \quad c(x, \xi') \sim \sum_{j=0}^{\infty} c_j(x', \xi') x_n^j.$$

That (3.3) holds to infinite order at $x_n = 0$ is equivalent to the system:

$$(j) \qquad a_j(x', \xi) = b_0(x', \xi) d_j(x', \xi) + b_1(x', \xi) d_{j-1}(x', \xi) + \ldots + b_j(x', \xi) d_0(x', \xi) + c_j(x', \xi'),$$
$$j = 0, 1, 2, 3, \ldots .$$

This sytem is solved with respect to b_j and c_j as follows: Put $c_0(x', \xi') = a_0(x', (\xi', 0))$. Then (0) holds for $\xi_n = 0$ and if we then put

$$b_0(x', \xi) = d_0(x', \xi)^{-1} (a_0(x', \xi) - c_0(x', \xi'))$$

it will hold for all ξ_n. c_0 and b_0 will belong to $S^k(W'')$ and $S^{k-m}(W')$ respectively in view of (3.5). Assume inductively that we have already found $b_0, b_1, \ldots, b_{j-1}$ in $S^{k-m}(W')$ and c_0, $c_1, \ldots, c_{j-1}$ in $S^k(W'')$ such that (0), (1), ..., $(j-1)$ hold. Then we can determine $c_j \in S^k(W'')$ such that (j) holds for $\xi_n = 0$ and after that $b_j \in S^{k-m}(W')$ such that it holds for all ξ_n.

We now apply a standard proof of the Borel theorem: Let $\chi(x_n) \in C_0^\infty(\mathbf{R})$ be equal to 1 near $x_n = 0$. If $0 < \lambda_j \to +\infty$ sufficiently fast when $j \to +\infty$, we can put

$$b'(x, \xi) = \sum_{j=0}^{\infty} b_j(x', \xi) \chi(\lambda_j x_n) x_n^j, \quad c(x, \xi') = \sum_{j=0}^{\infty} c_j(x', \xi') \chi(\lambda_j x_n) x_n^j$$

b' and c will then belong to $S^{k-m}(W)$ and $S^k((-\delta, \delta) \times W'')$ respectively and have the desired Taylor expansions. (We omit the details.) We have thus constructed b' and c such that $a - b'p_m - c$ vanishes to infinite order at $x_n = 0$. Put $b = b' + b''$, where $b'' = p_m^{-1}(a - b'p_m - c)$. It follows from (3.4) that $b'' \in S^{k-m}(W)$ and it is trivial to verify that (3.3) holds. This completes the proof of Lemma 3.2.

LEMMA 3.3. *There exist* $r(x, \xi') \in C^\infty((-\delta, \delta) \times W'')$ *and* $p_{0,m-1}(x, \xi) \in C^\infty(W)$ *positively homogeneous of degree* 1 *and* $m-1$ *respectively, such that* $\operatorname{Re} r > 0$ *and* $p_{0,m-1} \neq 0$ *and*

$$p_m(x, \xi) = p_{0,m-1}(x, \xi)(\xi_n - i x_n r(x, \xi')).$$

Proof. Apply Lemma 3.2 with $a(x, \xi) = \xi_n$. Then we get

$$\xi_n = b(x, \xi) p_m(x, \xi) + c(x, \xi').$$

Since p_m vanishes when $x_n = \xi_n = 0$, we have $c((x', 0), \xi') = 0$. Thus $c(x, \xi') = ix_n r(x, \xi')$ for some $r \in C^\infty((-\delta, \delta) \times W'')$ positively homogeneous of degree 1 and we get

$$\xi_n - ix_n r(x, \xi') = b(x, \xi) p_m(x, \xi). \tag{3.6}$$

(3.5) implies that $b((x', 0), (\xi', 0)) \neq 0$. From the condition $C_{p_m}((x', 0), (\xi', 0)) > 0$ it then follows that $\operatorname{Re} r((x', 0), \xi') > 0$. Using this inequality and the fact that $p_m(x, \xi) \neq 0$ when $x_n \neq 0$, we can modify r and b outside $x_n = 0$ so that $\operatorname{Re} r > 0$ in $(-\delta, \delta) \times W''$ and (3.6) still holds. Now put $p_{0,m-1}(x, \xi) = b(x, \xi)^{-1}$ and the lemma follows.

To handle the lower order terms in the factorization (iv) in Lemma 3.1, we need the following easy consequence of Lemmas 3.2 and 3.3:

LEMMA 3.4. *For every* $p'_k \in S^k(W)$ *there exist* $p'_{0,k-1} \in S^{k-1}(W)$ *and* $s_{k-(m-1)}(x, \xi') \in S^{k-(m-1)}((-\delta, \delta) \times W'')$ such that

$$p'_k(x, \xi) = p'_{0,k-1}(x, \xi)(\xi_n - ix_n r(x, \xi')) + p_{0,m-1}(x, \xi) s_{k-(m-1)}(x, \xi'). \tag{3.7}$$

Proof. By Lemma 3.3 (3.7) is equivalent to

$$p_{0,m-1}^{-1} p'_k = p'_{0,k-1} (p_{0,m-1})^{-2} p_m + s_{k-(m-1)}.$$

Thus we can apply Lemma 3.2 with $a = p_{0,m-1}^{-1} p'_k$.

End of proof of Lemma 3.1. We shall construct $p_0 \in S^{m-1}(W)$ with principal part $p_{0,m-1}$ and $s(x, \xi') \in S^0((-\delta, \delta) \times W'')$ such that

$$p(x, \xi) \sim \sum p_0^{(\alpha)}(x, \xi) D_x^\alpha (\xi_n - ix_n r(x, \xi') + s(x, \xi'))/\alpha! \tag{3.8}$$

in W. Here p is the symbol of P, so p has the principal part p_m. We look for p_0 and s with the asymptotic expansions:

$$p_0 \sim \sum_{j=1}^{\infty} p_{0,m-j}, \qquad p_{0,m-j} \in S^{m-j}(W)$$

$$s \sim \sum_{j=2}^{\infty} s_{2-j}, \qquad s_{2-j} \in S^{2-j}((-\delta, \delta) \times W'').$$

By Lemma 3.3 the following statement is satisfied for $N = 1$:

$$(N) \qquad \sum_{\alpha} \Big(\sum_{j=1}^{N} p_{0,m-j}^{(\alpha)}(x,\xi)\Big) D_x^{\alpha}\Big(\xi_n - i x_n r(x,\xi') + \sum_{j=2}^{N} s_{2-j}(x,\xi')\Big)/\alpha!$$
$$\sim p(x,\xi) + p_{m-N}(x,\xi) \text{ for some } p_{m-N} \in S^{m-N}(W).$$

Assume inductively that $p_{0,m-j} \in S^{m-j}$, $1 \leqslant j \leqslant N$ and $s_{2-j} \in S^{2-j}$, $2 \leqslant j \leqslant N$ have already been constructed such that (N) holds. Then by Lemma 3.4 we can find $p_{0,m-(N+1)} \in S^{m-(N+1)}$ and $s_{2-(N+1)} \in S^{2-(N+1)}$ such that $(N+1)$ holds. If we let p_0 and s be the asymptotic sums above, (3.8) follows.

Now extend r, s and p_0 to $C^\infty(\mathbf{R}^n \times (\mathbf{R}^{n-1} \setminus \{0\}))$, $S^0(\mathbf{R}^n \times (\mathbf{R}^{n-1} \setminus \{0\}))$ and $S^{m-1}(\mathbf{R}^n \times (\mathbf{R}^{n-1} \setminus \{0\}))$ respectively so that r satisfies (ii) in Lemma 3.1 and p_0 is of order $-\infty$ in a conic neighbourhood of $\{(x,\xi) \in T^*(R^n) \setminus 0;\ \xi' = 0\}$. This is possible at least if we first shrink W a little. Let $r(x, D')$, $s(x, D')$ and $P_0(x, D)$ be properly supported with symbols modulo $S^{-\infty}$ equal to $r(x,\xi')$, $s(x,\xi')$ and $p_0(x,\xi)$ respectively and put $\Lambda = D_n - i x_n r(x, D') + s(x, D')$. Then it follows from (3.8) and the results in the appendix that (iv) of Lemma 3.1 holds. The properties (i), (iii) and (v) also follow from the construction and Lemma 3.1 is proved.

By condition 4° of Theorem 2.1 we can choose V in Lemma 3.1 so small that $q((x', 0), (\xi', 0)) \neq 0$ in $\bar{V}$. Here q is the principal symbol of Q. The following lemma will help us to pass from the boundary condition $\gamma u = u_0$ in (3.1) to the condition $\gamma Q u = u_0$ in Theorem 2.1.

Lemma 3.5. *There exist properly supported operators $U \in L^0(\mathbf{R}^{n-1})$ and $T \in L^{-1}(\mathbf{R}^n)$, such that U is elliptic,* $\mathrm{WF}(T) \cap \{(x,\xi) \in T^*(\mathbf{R}^n) \setminus 0;\ \xi' = 0\} = \emptyset$

and
$$\gamma Q Z \equiv U \gamma Z + \gamma T \Lambda Z$$
for all $Z \in L^M(\mathbf{R}^n)$ with $\mathrm{WF}(Z) \subset V$, $M \in \mathbf{R}$.

Proof. Let the symbol of Q be $q + q'$, where $q' \in S^{-1}(\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\}))$. Let W be as in the proof above such that $q((x', 0), (\xi', 0)) \neq 0$ in $\bar{W}$. As in the end of the proof of Lemma 3.1, it is easy to construct $u \in S^0(W'')$ and $t \in S^{-1}(W)$ such that

$$(q + q')((x', 0), \xi) \sim u(x', \xi') + \sum t^{(\alpha)}((x', 0), \xi)\,(D_x^\alpha \lambda(x,\xi))\big|_{x_n=0}/\alpha! \tag{3.9}$$

where $\lambda(x,\xi) = \xi_n - i x_n r(x,\xi') + s(x,\xi')$. From the construction it follows that the homogeneous principal part of u is different from 0 in W''. It is now easy to find our operators U and T. (Cf. the end of the proof of Lemma 3.1.)

Let the spaces $H_{(m,s)}(\mathbf{R}^n)$, $H_{(m,s)}^{\mathrm{loc}}(\mathbf{R}^n)$ and $H_{(m,s)}^{\mathrm{comp}}(\mathbf{R}^n)$ be the Sobolev spaces defined in [7] and let $H_{(m,-\infty)}^{\mathrm{loc}}(\mathbf{R}^n)$ be the space of all $u \in \mathcal{D}'(\mathbf{R}^n)$ locally belonging to $H_{(m,s)}(\mathbf{R}^n)$ for some s. The following proposition will be proved in Section 4.

Proposition 3.6. *Let $\Lambda(x, D)=D_n-ix_n r(x, D')+s(x, D')$ be the operator given in Lemma 3.1. Then there exist properly supported operators*

$$F: H^{\text{loc}}_{(0,-\infty)}(\mathbf{R}^n)\to H^{\text{loc}}_{(1,-\infty)}(\mathbf{R}^n),\quad F^+:\mathcal{D}'(\mathbf{R}^{n-1})\to H^{\text{loc}}_{(1,-\infty)}(\mathbf{R}^n)$$

with the following properties:

(i) *F is continuous $H^{\text{loc}}_{(m,s)}(\mathbf{R}^n)\to H^{\text{loc}}_{(m+1,s-\frac{1}{2})}(\mathbf{R}^n)$ for all $s\in\mathbf{R}$ and integers $m\geqslant 0$.*

(ii) *F^+ is continuous $H^{\text{loc}}_s(\mathbf{R}^{n-1})\to H^{\text{loc}}_{(m,s-m+\frac{1}{2})}(\mathbf{R}^n)$ for all $s, m\in\mathbf{R}$.*

(iii) $\text{WF}'(F^+)\subset\{(((x',0),(\xi',0)),(x',\xi'))\in(T^*(\mathbf{R}^n)\setminus 0)\times(T^*(\mathbf{R}^{n-1})\setminus 0)\}$.

(iv) $\text{WF}'(FZ)\subset\{((x,\xi),(x,\xi))\in(T^*(\mathbf{R}^n)\setminus 0)\times(T^*(\mathbf{R}^n)\setminus 0)\}$

$$\cup\,\{(((x',0),(\xi',0)),((x',0),(\xi',\xi_n)))\in T^*(\mathbf{R}^n)\times(T^*(\mathbf{R}^n)\setminus 0)\}$$

for all $Z\in L^M(\mathbf{R}^n)$, $M\in\mathbf{R}$ with
$\text{WF}(Z)\cap\{(x,\xi)\in T^*(\mathbf{R}^n)\setminus 0;\ \ \xi'=0\}=\varnothing$.

(v) *Let $M^{-\infty}$ be the space of operators $H^{\text{loc}}_{(1,-\infty)}(\mathbf{R}^n)\to H^{\text{loc}}_{(2,-\infty)}$ which are continuous $H^{\text{loc}}_{(m,s)}\to H^{\text{loc}}_{(m+1,t)}$ for all $s,t\in\mathbf{R}$ and integers $m>0$. Then $\Lambda F\equiv I$ mod $(M^{-\infty})$, ΛF^+ has C^∞ kernel, $\gamma F=0$ and $\gamma F^+=I$.*

(vi) $F\Lambda+F^+\gamma\equiv I \bmod (M^{-\infty})$.

In the rest of this section we shall prove that Theorem 2.1 follows from Proposition 3.6. Let $U'\in L^0(\mathbf{R}^{n-1})$ be a properly supported parametrix of U in Lemma 3.5. By Lemma 3.1 we can find $P_0'\in L^{-(m-1)}(\mathbf{R}^n)$, properly supported such that

$$(\text{WF}(P_0P_0'-I)\cup\text{WF}(P_0'P_0-I))\cap A_Q=\varnothing \tag{3.10}$$

$$\text{WF}(P_0')\subset V. \tag{3.11}$$

Here we recall that $A_Q=\{((x_0',0),(\xi_0',\xi_n))\in\text{WF}(Q)\}$. With T as in Lemma 3.5 we put

$$E=FP_0'-F^+U'\gamma TP_0'$$

$$E^+=F^+U'.$$

It follows at once from Propositions 3.6 and A.2 that (iii) and the estimate for $\text{WF}'(E^+)$ in (iv) of Theorem 2.1 are valid.

To show the first part of (iv) it is sufficient in view of Proposition 3.6 to show that $EZ\in L^{-m}(\mathbf{R}^n)$ for all $Z\in L^0(\mathbf{R}^n)$ such that

$$\text{WF}(Z)\cap\text{WF}(Q)=\varnothing. \tag{3.12}$$

If Z satisfies (3.12) then $\text{WF}(P_0'Z)$ does not intersect

$$\{(x, \xi) \in T^*(\mathbf{R}^n)\setminus 0;\quad x_n = \xi_n = 0 \text{ or } \xi' = 0\}.$$

Now $\xi_n - ix_n r(x, \xi')$ (the principal symbol of Λ) is $\neq 0$ and belongs to S^1 outside this set. In view of Proposition A.2, we can therefore construct $Z_0 \in L^{-m}(\mathbf{R}^n)$ with $\mathrm{WF}(Z_0) = \mathrm{WF}(P_0'Z)$ such that $P_0'Z \equiv \Lambda Z_0$. (This is the same construction as that of a parametrix of an elliptic operator.) Thus $EZ = FP_0'Z - F^+U'\gamma TP_0'Z \equiv F\Lambda Z_0 - F^+U'\gamma T\Lambda Z_0$. (Here $\equiv$ denotes equality modulo an operator with C^∞ kernel.) Now $U'\gamma T\Lambda Z_0 \equiv U'\gamma QZ_0 - U'U\gamma Z_0 \equiv -\gamma Z_0$ in view of Lemma 3.5 and (3.12). Thus $EZ \equiv (F\Lambda + F^+\gamma)Z_0 \equiv (I+K)Z_0 \equiv Z_0 + KZ_0$ in view of Proposition 3.6. Here $K \in M^{-\infty}$ and it follows from Proposition A.2 that KZ_0 has C^∞ kernel. (Note that $\mathrm{WF}(Z_0) \cap \{(x, \xi) \in T^*(\mathbf{R}^n)\setminus 0;\ \xi' = 0\} = \emptyset$.) Thus EZ is a pseudodifferential operator and (iv) of Theorem 2.1 follows.

To prove (i) of Theorem 2.1, we let $Z \in L^0(\mathbf{R}^n)$ with $\mathrm{WF}(Z)$ close to ϱ_0. Then $Z\mathcal{E}\mathfrak{P} = ZEP + ZE^+\gamma Q$. By (iv) of Theorem 2.1 there exists $Z_0 \in L^0(\mathbf{R}^n)$ properly supported with $\mathrm{WF}(Z_0)$ close to A_Q such that

$$ZEP \equiv ZEPZ_0, \quad ZE^+\gamma Q \equiv ZE^+\gamma QZ_0, \quad ZZ_0 \equiv Z.$$

Then

$$Z\mathcal{E}\mathfrak{P} \equiv ZEPZ_0 + ZE^+\gamma QZ_0 = ZFP_0'PZ_0 - ZF^+U'\gamma TP_0'PZ_0 + ZF^+U'\gamma QZ_0.$$

By Lemma 3.1 we have $P_0'PZ_0 \equiv \Lambda Z_0$. Thus $Z\mathcal{E}\mathfrak{P} \equiv ZF\Lambda Z_0 - ZF^+U'\gamma T\Lambda Z_0 + ZF^+U'\gamma QZ_0$. By Lemma 3.5 and Proposition 3.6 we get $Z\mathcal{E}\mathfrak{P} \equiv ZF\Lambda Z_0 + ZF^+\gamma Z_0 \equiv ZZ_0 \equiv Z$. This proves (i) of Theorem 2.1.

To prove (ii) means to prove the following equations:

$$PEZ \equiv Z \tag{3.13}$$

$$\gamma QEZ \equiv 0 \tag{3.14}$$

$$PE^+Z' \equiv 0 \tag{3.15}$$

$$\gamma QE^+Z' \equiv Z'. \tag{3.16}$$

Here $Z \in L^0(\mathbf{R}^n)$, $Z' \in L^0(\mathbf{R}^{n-1})$ and $\mathrm{WF}(Z)$ and $\mathrm{WF}(Z')$ are close to ϱ_0 and ϱ_0' respectively. By (iv) of Theorem 2.1 and Lemma 3.1 we have $PEZ \equiv P_0\Lambda EZ$. Thus by Proposition 3.6 we get

$$PEZ \equiv P_0\Lambda FP_0'Z - P_0\Lambda F^+U'\gamma TP_0'Z \equiv P_0P_0'Z \equiv Z,$$

which proves (3.13).

Combining (iv) of Theorem 2.1 with Lemma 3.5 we get

$$\gamma QEZ \equiv (U\gamma + \gamma T\Lambda)EZ.$$

Combining this with the definition of E and Proposition 3.6, we see that (3.14) is valid.

(3.15) and (3.16) are easy consequences of Lemma 3.1 and Proposition 3.6. We omit the details. This completes the proof of Theorem 2.1.

4. Proof of Proposition 3.6

Proposition 3.6 states essentially that the system

$$\Lambda u = v \quad \gamma u = u_0 \tag{4.1}$$

is uniquely solvable for given $v \in H^{\text{loc}}_{(0,-\infty)}$ and $u_0 \in \mathcal{D}'(\mathbf{R}^{n-1})$. Stated in this way the result is not new. Eškin [6] has treated much more general problems than (4.1). The new feature here is that we obtain explicit formulas for the solution operators, which enable us to estimate their wavefront sets. This has been essential in the chain of proofs leading from Proposition 3.6 to Theorems 1 and 2.

We begin with an informal discussion. For given functions v in $\mathbf{R}^n$ and u_0 in $\mathbf{R}^{n-1}$ we put

$$u(x) = i\int\left(\int_0^{x_n}\int q(x, y, \xi')\, e^{i\langle x'-y', \xi'\rangle} v(y)\, dy'\, dy_n\right) d\xi' / (2\pi)^{n-1}$$
$$+ \int\left(\int q(x,(y', 0), \xi')\, e^{i\langle x'-y', \xi'\rangle} u_0(y')\, dy'\right) d\xi' / (2\pi)^{n-1}, \tag{4.2}$$

where the symbol q has to be determined in a suitable way. Then we have at least formally:

$$\Lambda u(x) = \int\left(\int q(x,(y', x_n), \xi')\, e^{i\langle x'-y', \xi'\rangle} v(y', x_n)\, dy'\right) d\xi' / (2\pi)^{n-1}$$
$$+ i(2\pi)^{1-n}\int\left(\int_0^{x_n}\int \Lambda(x, D_x)\,(q(x, y, \xi')\, e^{i\langle x', \xi'\rangle})\, e^{-i\langle y', \xi'\rangle} v(y)\, dy'\, dy_n\right) d\xi'$$
$$+ \int\left(\int \Lambda(x, D_x)\,(q(x, (y', 0), \xi')\, e^{i\langle x', \xi'\rangle})\, e^{-i\langle y', \xi'\rangle} u_0(y')\, dy'\right) d\xi' / (2\pi)^{n-1}. \tag{4.3}$$

The first integral here is the boundary term we get when we apply the term D_n in Λ on the first integral in (4.2.)

We shall construct $q \in C^\infty$, such that

$$q(x, (y', x_n), \xi') = \Phi(x' - y'), \tag{4.4}$$

where $\Phi \in C_0^\infty(\mathbf{R}^{n-1})$ and $\Phi = 1$ near the origin, and such that $\Lambda(x, D_x)(q(x, y, \xi')e^{i\langle x', \xi'\rangle})$ and all its derivatives are rapidly decreasing as functions of ξ'. Then if u is given by (4.2) it follows from Fouriers inversion formula that

$$\gamma u(x') = \Phi(x'-x')\, u_0(x') = u_0(x'). \tag{4.5}$$

Using Fouriers inversion formula in the first integral in (4.3) and carrying out the ξ'-integrations in the other two, we get:

$$\Lambda u(x) = v(x) + i\int_0^{x_n}\left(\int k(x,y)\, v(y)\, dy'\right) dy_n + \int k(x,(y',0))\, u_0(y')\, dy', \tag{4.6}$$

where k is smooth. We shall see later that the first integral is in $M^{-\infty}$. Then (4.5) and (4.6) show that u, given by (4.2), is an approximate solution of (4.1).

The program will now be as follows: First we define and investigate certain symbol spaces. After that we define (F, F^+), prove the continuity properties in $H_{(m,s)}$ and that (F, F^+) is a right parametrix of (4.1). By an analogous construction (which we only sketch) there is a left parametrix and this implies that (F, F^+) is also a left parametrix. Finally we estimate $\mathrm{WF}'(F)$ and $\mathrm{WF}'(F^+)$.

We recall the definition of $S^m_{\varrho\delta}(X\times\mathbf{R}^N)$ in [8] and make the following generalization in the case when X is the product of two open sets:

Definition 4.1. Let $X'\subset\mathbf{R}^{n'}$ and $X''\subset\mathbf{R}^{n''}$ be open and ϱ, δ', δ'', m be real numbers. Then we let $S^m_{\varrho\delta'\delta''}(X'\times X''\times\mathbf{R}^N)$ be the set of all $p\in C^\infty(X'\times X''\times\mathbf{R}^N)$ such that for all compact subsets $K\subset X'\times X''$ and multiindices α, β and γ, there is a constant C such that

$$|D^\alpha_{x'} D^\beta_{x''} D^\gamma_\xi p(x',x'',\xi)| \leq C(1+|\xi|)^{m+\delta'|\alpha|+\delta''|\beta|-\varrho|\gamma|}, \quad \text{for all } (x',x'',\xi)\in K\times\mathbf{R}^N.$$

If X' and X'' are as in Definition 4.1 and $\bar X''$ is the closure of X'' in $\mathbf{R}^{n''}$, we let $C^\infty(X'\times\bar X''\times\mathbf{R}^N)$ be the set of all functions $f\in C^\infty(X'\times X''\times\mathbf{R}^N)$, such that f and all its derivatives have continuous extensions to $X'\times\bar X''\times\mathbf{R}^N$. *We now define* $S^m_{\varrho\delta'\delta''}(X'\times\bar X''\times\mathbf{R}^N)$ *by replacing* X'' *by* $\bar X''$ *everywhere in Definition 4.1 (except in the first line).*

Next we extend the notion of asymptotic convergence.

Definition 4.2. Suppose that $p_j\in S^{m_j}_{\varrho\delta'\delta''}(X'\times X''\times\mathbf{R}^N)$, $j=1,2,3,\dots$ and that $m_j\to-\infty$ when $j\to+\infty$. If $p\in C^\infty(X'\times X''\times\mathbf{R}^N)$, and if for every j_0

$$p-\sum_{j=1}^{j_0} p_j \in S^{M(j_0)}_{\varrho\delta'\delta''}(X'\times X''\times\mathbf{R}^N),$$

where $M(j_0)=\max_{j>j_0} m_j$, we write $p\sim\sum_{j=1}^\infty p_j$ and say that p is asymptotically equal to $\sum_{j=1}^\infty p_j$.

We define asymptic convergence in the space $S^m_{\varrho\delta'\delta''}(X'\times\bar X''\times\mathbf{R}^N)$ in exactly the same way.

LEMMA 4.3. *Suppose* $p_j \in S^{m_j}_{\varrho\delta'\delta''}(X' \times \bar{X}'' \times \mathbf{R}^N)$, $j = 1, 2, 3, \ldots$ *where* $m_j \to -\infty$ *when* $j \to +\infty$. *Then there exists* $p \in S^{\max(m_j)}_{\varrho\delta'\delta''}(X' \times \bar{X}'' \times \mathbf{R}^N)$, *such that* $p \sim \sum_{j=1}^{\infty} p_j$. *The corresponding statement holds also in the spaces* $S^m_{\varrho\delta'\delta''}(X' \times X'' \times \mathbf{R}^N)$.

The proof of the lemma is identical with that of the corresponding statement for the spaces $S^m_{\varrho\delta}(X \times \mathbf{R}^N)$ (see [8]): We put $p(x', x'', \xi) = \sum_{j=1}^{\infty} (1 - \chi(\varepsilon_j \xi))\, p_j(x', x'', \xi)$, where $\chi \in C_0^\infty(\mathbf{R}^N)$ is equal to 1 near the origin and $0 < \varepsilon_j \to 0$ sufficiently fast when $j \to +\infty$. We let the reader check the details himself or consult [8].

Definition 4.2 and Lemma 4.3 have immediate extensions to asymptotic sums of the form $p \sim \sum p_\alpha$ where the sum is taken over n-tuples of integers $\geqslant 0$.

Let $U = \{(x_n, y_n) \in \mathbf{R}^2;\ 0 < y_n < x_n \text{ or } x_n < y_n < 0\}$ and let $\bar{U}$ be its closure in $\mathbf{R}^2$.

Definition 4.4. Let $\check{S}^m$ be the space of symbols $p(x, y, \xi')$, $x, y \in \mathbf{R}^n$, $\xi' \in \mathbf{R}^{n-1}$, belonging to $S^m_{101}(\mathbf{R}^{2(n-1)} \times \bar{U} \times \mathbf{R}^{n-1})$, when regarded as functions of $((x', y'), (x_n, y_n), \xi') \in \mathbf{R}^{2(n-1)} \times \bar{U} \times \mathbf{R}^{n-1}$. We let $\check{S}^{-\infty} = \bigcap_{m \in \mathbf{R}} \check{S}^m$.

LEMMA 4.5. *If* $T \in T^m(\mathbf{R}^n)$ *is properly supported with symbol* $t(x, \xi')$ *and if* $q \in \check{S}^k$, *then*

$$e^{-i\langle x', \xi'\rangle} T(x, D'_x)(q(x, y, \xi')\, e^{i\langle x', \xi'\rangle}) \sim \sum t^{(\alpha')}(x, \xi')\, D^{\alpha'}_{x'} q(x, y, \xi') / \alpha'!.$$

Proof. Clearly $t^{(\alpha')} D^{\alpha'}_{x'} q \in \check{S}^{k+m-|\alpha'|}$. Thus by Lemma 4.3 there exists $Q(T, q) \in \check{S}^{k+m}$, such that $Q(T, q) \sim \sum t^{(\alpha')} D^{\alpha'}_{x'} q / \alpha'!$. Put $R(T, q) = e^{-i\langle x', \xi'\rangle} T(q\, e^{i\langle x', \xi'\rangle})$. Then for all T and q as in the lemma, $N > 0$ and multiindices α', β' and γ' we have:

$$D^{\alpha'}_{x'} D^{\beta'}_{y'} D^{\gamma'}_{\xi'} (Q(T, q) - R(T, q)) = O(|\xi'|^{-N}) \text{ when } \xi' \to \infty,$$

uniformly when (x', y', x_n, y_n) belongs to any compact subset of $\mathbf{R}^{2(n-1)} \times \bar{U}$. (4.7)

In fact, this follows if we regard T as a pseudodifferential operator in $\mathbf{R}^{n-1}$, depending on the parameter x_n and regard q as an element of $S^k_{10}(\mathbf{R}^{2(n-1)} \times \mathbf{R}^{n-1})$, depending on the parameters x_n and y_n and apply wellknown results on asymptotic expansions (see Theorem 2.6 in [8]).

Let $T_{x_n} \in T^m(\mathbf{R}^n)$ be the operator with symbol $D_{x_n} t$. Then:

$$D_{x_n} Q(T, q) \equiv Q(T_{x_n}, q) + Q(T, D_{x_n} q) \text{ and } D_{y_n} Q(T, q) \equiv Q(T, D_{y_n} q) \mod (\check{S}^{-\infty}).$$

Similarly: $\quad D_{x_n} R(T, q) = R(T_{x_n}, q) + R(T, D_{x_n} q)$ and $D_{y_n} R(T, q) = R(T, D_{y_n} q)$.

Thus by induction we see that for all α_n and β_n the difference $D^{\alpha_n}_{x_n} D^{\beta_n}_{y_n} (Q(T, q) - R(T, q))$ is asymptotically equal to a finite sum of terms of the type: $Q(T', q') - R(T', q')$, where T' and q' are as in the lemma. Then (4.7) implies that

$$D_x^\alpha D_y^\beta D_{\xi'}^\gamma (Q(T,q) - R(T,q)) = O(|\xi'|^{-N}) \text{ when } \xi' \to \infty,$$

uniformly when $((x', y'), (x_n, y_n))$ belongs to any compact subset of $\mathbf{R}^{2(n-1)} \times \bar{U}$. This is precisely the statement in the lemma.

We can now state how to choose q in (4.2).

LEMMA 4.6. *Let $\Phi \in C_0^\infty(\mathbf{R}^{n-1})$ be equal to 1 near the origin and have support in $\{x' \in \mathbf{R}^{n-1}; |x'| < 1\}$. Then there exists $q \in \dot{S}^0$ with the following properties:*

(i) $$q(x, (y', x_n), \xi') = \Phi(x' - y').$$

(ii) $$e^{-i\langle x', \xi'\rangle} \Lambda(x, D_x)(q(x, y, \xi')\, e^{i\langle x', \xi'\rangle}) \in \dot{S}^{-\infty}.$$

(iii) $\Psi(x, y)\, q(x, y, \xi') \in \dot{S}^{-\infty}$ *for all* $\Psi \in C^\infty(\mathbf{R}^{2n})$, *vanishing near* $\{(x, y) \in \mathbf{R}^{2n};\ x_n = y_n\}$.

(iv) $$q(x, y, \xi') \neq 0 \text{ implies that } |x - y| < 2.$$

(v) *Let* $$R(x, y_n, \xi') = -\int_{y_n}^{x_n} t\, r((x', t), \xi')\, dt,$$

where r is given in Lemma 3.1 and let $\chi \in C_0^\infty(\mathbf{R}^{n-1})$ be equal to 1 near the origin. Then

$$q(x, y, \xi') - \Phi(x' - y')(1 - \chi(\xi'))\, e^{R(x, y_n, \xi')} \in \dot{S}^{-\frac{1}{2}}.$$

We shall first define and investigate a very special class of symbols. After that the proof of Lemma 4.6 will be easy.

Definition 4.7. For $m \in \mathbf{R}$, let $\dot{S}^m$ be the smallest set, closed under addition, that contains all $p \in C^\infty(\mathbf{R}^{2(n-1)} \times \bar{U} \times \mathbf{R}^{n-1})$ for which there are integers $0 \leq k_1 \leq k_2$ and

$$a(x, y, \xi') \in S_{10}^{m+(k_1+k_2)/2}(\mathbf{R}^{2n} \times \mathbf{R}^{n-1}),$$

such that $$p(x, y, \xi') = y_n^{k_1}(x_n - y_n)^{k_2} a(x, y, \xi')\, e^{R(x, y_n, \xi')} \text{ for } |\xi'| > \tfrac{1}{2}.$$

Here $R(x, y_n, \xi') = -\int_{y_n}^{x_n} t\, r((x', t), \xi')\, dt$ as above.

LEMMA 4.8. *Let $c(x, y, \xi') = y_n^{k_1}(x_n - y_n)^{k_2} a(x, y, \xi')$. where k_1 and k_2 are integers ≥ 0 and $a \in S_{10}^m$. Then there is a symbol $b \in S_{10}^m$ such that*

$$\int_{y_n}^{x_n} c((x', t), y, \xi')\, dt = y_n^{k_1}(x_n - y_n)^{k_2+1} b(x, y, \xi'),$$

Proof. Put $x_n - y_n = s$. Then

$$\int_{y_n}^{x_n} c((x', t), y, \xi')\, dt = y_n^{k_1} s^{k_2+1} d(x', s, y, \xi'), \tag{4.8}$$

where $$d(x', s, y, \xi') = \int_0^1 t^{k_2} a((x', y_n + ts), y, \xi')\, dt.$$

Thus $d \in S^m_{10}(\mathbf{R}^{2n} \times \mathbf{R}^{n-1})$ and the lemma follows if we put $b(x, y, \xi') = d(x', x_n - y_n, y, \xi')$.

If we write $x_n r(x, \xi') = y_n r(x, \xi') + (x_n - y_n) r(x, \xi')$, we see from Lemma 4.8 and the definition of R that:

$$R(x, y_n, \xi') = y_n (x_n - y_n) R_1(x, y_n, \xi') + (x_n - y_n)^2 R_2(x, y_n, \xi') \text{ for } |\xi'| > \tfrac{1}{2}, \tag{4.9}$$

where R_1 and $R_2 \in S^1_{10}$. From (4.9) one obtains easily:

$$p \in \dot{S}^m \Rightarrow D^\alpha_x D^\beta_y D^{\gamma'}_{\xi'} p \in \dot{S}^{m+\alpha_n+\beta_n-|\gamma'|} \text{ for all } \alpha, \beta, \gamma'. \tag{4.10}$$

LEMMA 4.9. $\dot{S}^m \subset \tilde{S}^m$.

Proof. By (4.10) it suffices to prove that if $p \in \dot{S}^m$, then

$$p(x, y, \xi') = O(|\xi'|^m), \quad \xi' \to \infty,$$

uniformly when $((x', y'), (x_n, y_n))$ belongs to any compact subset of $\mathbf{R}^{2(n-1)} \times \bar{U}$.

If K is such a compact set, there is a constant $C_K > 0$, such that $\operatorname{Re} R(x, y_n, \xi') < -C_K |x_n| |x_n - y_n| |\xi'|$, when $((x', y'), (x_n, y_n)) \in K$. In fact, by Lemma 3.1 we then have $\operatorname{Re} r(x, \xi') > 2C_K |\xi'|$ for some $C_K > 0$, thus

$$\begin{aligned}\operatorname{Re} R(x, y_n, \xi') &= -\int_{y_n}^{x_n} t \operatorname{Re} r((x', t), \xi')\, dt \leqslant -2C_K |\xi'| \left|\int_{y_n}^{x_n} t\, dt\right| = -C_K |\xi'| (x_n^2 - y_n^2) \\ &= -C_K |\xi'| (x_n + y_n)(x_n - y_n) \leqslant -C_K |x_n| |x_n - y_n| |\xi'|.\end{aligned}$$

Let $p \in \dot{S}^m$. We can assume that

$$p(x, y, \xi') = y_n^{k_1} (x_n - y_n)^{k_2} a(x, y, \xi')\, e^{R(x, y_n, \xi')} \text{ for } |\xi'| > \tfrac{1}{2},$$

where $k_2 \geqslant k_1 \geqslant 0$ are integers and $a \in S^{m+(k_1+k_2)/2}_{10}$. Then for $((x', y'), (x_n, y_n)) \in K$ and $|\xi'| > \frac{1}{2}$, we have $|x_n - y_n| \leqslant |x_n| \geqslant |y_n|$, thus

$$\begin{aligned}|p(x, y, \xi')| &\leqslant C |\xi'|^{m+(k_1+k_2)/2} |y_n|^{k_1} |x_n - y_n|^{k_2} \exp(-C_k |x_n| |x_n - y_n| |\xi'|) \\ &\leqslant C |\xi'|^m (|x_n| |x_n - y_n| |\xi'|)^{(k_1+k_2)/2} \exp(-C_K |x_n| |x_n - y_n| |\xi'|) \\ &\leqslant C |\xi'|^m \sup t^{(k_1+k_2)/2} e^{-C_k t} \leqslant C' |\xi'|^m.\end{aligned}$$

This proves Lemma 4.9.

In the same wav we obtain from (4.9):

LEMMA 4.10. *If* $p \in \dot{S}^k$, *then* $p|_{y_n=0} \in S^k_{10\frac{1}{2}}(\mathbf{R}^{2(n-1)} \times \mathbf{R} \times \mathbf{R}^{n-1})$, *when regarded as a function of* $((x', y'), x_n, \xi') \in \mathbf{R}^{2(n-1)} \times \mathbf{R} \times \mathbf{R}^{n-1}$.

We omit the simple proof. The next lemma is the essential step in our proof of Lemma 4.6.

LEMMA 4.11. *If* $u \in C^\infty(\mathbf{R}^{2(n-1)} \times \bar{U} \times \mathbf{R}^{n-1})$ satisfies $u|_{x_n=y_n} = 0$ *and* $D_{x_n}u(x, y, \xi') - ix_n r(x, \xi')\, u(x, y, \xi') = v(x, y, \xi')$ *for* $|\xi'| > \frac{1}{2}$, *where* $v \in \dot{S}^k$, *then* $u \in \dot{S}^{k-\frac{1}{2}}$.

Proof. We may assume that $v(x, y, \xi') = y_n^{k_1}(x_n - y_n)^{k_2} a(x, y, \xi')\, e^{R(x, y_n, \xi')}$ for $|\xi'| > \frac{1}{2}$, where $k_1 \geqslant k_2 \geqslant 0$ are integers and $a \in S_{10}^{k+(k_1+k_2)/2}$. Then for $|\xi'| > \frac{1}{2}$:

$$u(x, y, \xi') = e^{R(x, y_n, \xi')}\, i \int_{y_n}^{x_n} e^{-R((x', t), y_n, \xi')} v((x', t), y, \xi')\, dt$$

$$= e^{R(x, y_n, \xi')}\, i \int_{y_n}^{x_n} y_n^{k_1} (t - y_n)^{k_2} a((x', t), y, \xi')\, dt.$$

Thus by Lemma 4.8:

$$u(x, y, \xi') = e^{R(x, y_n, \xi')} y_n^{k_1} (x_n - y_n)^{k_2+1} b(x, y, \xi') \quad \text{for } |\xi'| > \tfrac{1}{2},$$

where
$$b \in S_{10}^{k+(k_1+k_2)/2} = S_{10}^{(k-\frac{1}{2})+(k_1+(k_2+1))/2}.$$

Therefore $u \in \dot{S}^{k-\frac{1}{2}}$, as asserted.

Proof of Lemma 4.6. Recursively we shall construct $q_j \in S^{-j/2}$, $j = 0, 1, 2, \ldots$, satisfying:

$$q_j(x, (y', x_n), \xi') = \Phi(x' - y') \text{ if } j = 0 \text{ and } = 0 \text{ if } j > 0, \tag{4.11}$$

such that for each integer $N \geqslant 0$:

$$\text{(N)} \qquad e^{-i\langle x', \xi'\rangle} \Lambda(x, D_x) \Big(\sum_0^N q_j e^{i\langle x', \xi'\rangle}\Big) \sim \sum_{\nu=0}^\infty q_{N\nu}, \quad \text{where } q_{N\nu} \in \dot{S}^{-(N+\nu)/2}.$$

Take $q_0 \in \dot{S}^0$ equal to $\Phi(x' - y')\, e^{R(x, y, \xi')}$ for $|\xi'| > \frac{1}{2}$ and such that $q_0|_{x_n=y_n} = \Phi(x' - y')$. (This is possible because $R(x, x_n, \xi') = 0$.) Then $D_{x_n} q_0 - ix_n r(x, \xi')\, q_0 = 0$ for $|\xi'| > \frac{1}{2}$. Thus by Lemma 4.5:

$$e^{-i\langle x', \xi'\rangle} \Lambda(x, D_x)(q_0 e^{i\langle x', \xi'\rangle}) \sim s(x, \xi')\, q_0 + \sum_{|\alpha'| \geqslant 1} (ix_n r^{(\alpha')}(x, \xi') + s^{(\alpha')}(x, \xi'))\, D_{x'}^{\alpha'} q_0 / \alpha'! \sim \sum_{\nu=0}^\infty q_{0\nu},$$

where
$$q_{0\nu} \in \dot{S}^{-\nu/2}. \text{ (Here } |\alpha'| = \alpha_1 + \ldots + \alpha_{n-1},\ \alpha' = (\alpha_1, \ldots, \alpha_{n-1})).$$

This proves (0). Suppose now that $q_0, \ldots, q_{N-1}$ have already been constructed, such that $(0), \ldots, (N-1)$ hold. Then let $q_N \in C^\infty$ be a solution of the system:

$$D_{x_n} q_N - i x_n r(x, \xi') q_N = -q_{N-1,0}, \quad |\xi'| > \tfrac{1}{2}$$

$$q_N|_{x_n = y_n} = 0.$$

Then $q_N \in \dot{S}^{-N/2}$ by Lemma 4.11 and it follows from $(N-1)$ and Lemma 4.5 that (N) holds with suitable $q_{N\nu} \in \dot{S}^{-(N+\nu)/2}$.

Since $\Phi(x'-y') = 0$ for $|x'-y'| > 1$, it follows from our construction that we can choose our q_j such that

$$q_j(x, y, \xi') = 0 \text{ when } |x'-y'| > 1. \tag{4.12}$$

If $\Psi \in C^\infty(\mathbf{R}^{2n})$ vanishes near $\{(x, y);\ x_n = y_n\}$, we have $\Psi(x, y) p(x, y, \xi') \in \check{S}^{-\infty}$, for every $p \in \dot{S}^m$. In particular:

$$\Psi(x, y)\, q_j(x, y, \xi') \in \check{S}^{-\infty} \tag{4.13}$$

From Lemma 4.3 and its proof it follows, that there exists $q \in \check{S}^0$, satisfying (i) of Lemma 4.6, such that:

$$q \sim \sum_{j=0}^{\infty} q_j \tag{4.14}$$

anu

$$q(x, y, \xi') = 0 \text{ if } |x'-y'| > 1. \tag{4.15}$$

From all the equations (N) it then follows that *(ii)* of Lemma 4.6 is satisfied, and (iii) follows from (4.13) since

$$\Psi(x, y)\, q(x, y, \xi') \sim \sum_{j=0}^{\infty} \Psi(x, y)\, q_j(x, y, \xi').$$

(v) follows directly from the construction.

To make (iv) satisfied, we have to modify q. We replace $q(x, y, \xi')$ by $\chi(x_n - y_n)\, q(x, y, \xi')$, where $\chi \in C_0^\infty(\mathbf{R})$ is $=1$ near the origin and has its support in $\{t \in \mathbf{R};\ |t| < 1\}$. Then (iv) will be satisfied in view of (4.15) and the other properties of q will be preserved, since we have only added the term $(\chi(x_n - y_n) - 1) q(x, y, \xi')$, which belongs to $\check{S}^{-\infty}$ in view of (iii). This completes the proof of Lemma 4.6.

Now take a fixed q as in Lemma 4.6 and define the operators

$$F\colon C^\infty(\mathbf{R}^n) \to C^\infty(\mathbf{R}^n) \text{ and } F^+\colon C^\infty(\mathbf{R}^{n-1}) \to C^\infty(\mathbf{R}^n)$$

by the equations:

$$Fv(x) = i \int \left(\int_0^{x_n} \int q(x, y, \xi')\, e^{i\langle x'-y', \xi' \rangle} v(y)\, dy'\, dy_n \right) d\xi' / (2\pi)^{n-1}$$

$$F^+ u_0(x) = \int \left(\int q(x, (y', 0), \xi')\, e^{i\langle x'-y', \xi' \rangle} u_0(y')\, dy' \right) d\xi' / (2\pi)^{n-1},$$

where $x \in \mathbf{R}^n$, $v \in C^\infty(\mathbf{R}^n)$ and $u_0 \in C^\infty(\mathbf{R}^{n-1})$.

To prove (i) and (ii) of Proposition 3.6, we shall follow Trèves [18]. If H_1 and H_2 are complex Hilbert spaces, let $L(H_1, H_2)$ be the Banach space of bounded linear operators $H_1 \to H_2$; the operator norm will be denoted by $\| \ \|$.

THEOREM 4.12. *Let $s(x', y', \xi')$ be a C^∞ function of $(x', y', \xi') \in \mathbf{R}^{n-1} \times \mathbf{R}^{n-1} \times \mathbf{R}^{n-1}$ with values in $L(H_1, H_2)$ and support in $K \times \mathbf{R}^{n-1}$, where $K \subset\subset \mathbf{R}^{n-1} \times \mathbf{R}^{n-1}$. Suppose that for all multiindices α', β' and γ' there is a constant C, such that*

$$\|D_{x'}^{\alpha'} D_{y'}^{\beta'} D_{\xi'}^{\gamma'} s(x', y', \xi')\| \leq C(1 + |\xi'|)^{k-|\gamma'|} \quad \text{on } K \times \mathbf{R}^{n-1}. \tag{4.16}$$

Then the operator $S: C_0^\infty(\mathbf{R}^{n-1}, H_1) \to C_0^\infty(\mathbf{R}^{n-1}, H_2)$, defined by

$$Sw(x') = \iint s(x', y', \xi')\, e^{i\langle x'-y', \xi'\rangle} w(y')\, dy'\, d\xi', \quad w \in C_0^\infty(\mathbf{R}^{n-1}, H_1), \tag{4.17}$$

can be extended to a bounded linear operator $H_s(\mathbf{R}^{n-1}, H_1) \to H_{s-k}(\mathbf{R}^{n-1}, H_2)$ for all $s \in \mathbf{R}$.

When $H_1 = H_2 = \mathbf{C}$ the theorem is a wellknown result about presudodifferential operators (see [8] p. 154) and the same proof works in the general case. Using this theorem we shall prove:

PROPOSITION 4.13. *If $p \in \dot{S}^k \cup \dot{S}^{k+1/2}$, then the operator*

$$A_p: C_0^\infty(\mathbf{R}^n) \to C^\infty(\mathbf{R}^n),$$

defined by

$$A_p v(x) = \int \left(\int_0^{x_n} \int p(x, y, \xi')\, e^{i\langle x'-y', \xi'\rangle} v(y)\, dy'\, dy_n \right) d\xi', \quad x \in \mathbf{R}^n,\ v \in C_0^\infty(\mathbf{R}^n),$$

can be extended to a continuous linear operator $H_{(0,s)}^{\text{comp}}(\mathbf{R}^n) \to H_{(0,s-k)}^{\text{loc}}(\mathbf{R}^n)$ for all $s \in \mathbf{R}$.

Proof. It suffices to prove that for arbitrary Φ, $\Psi \in C_0^\infty(\mathbf{R}^n)$ the operator $S: C_0^\infty(\mathbf{R}^n) \ni v \to \Phi A_p(\Psi v) \in C_0^\infty(\mathbf{R}^n)$ can be extended to a continuous linear operator $H_{(0,s)} \to H_{(0,s-k)}$ for all s. In fact, A_p can be written as a locally finite sum of operators of this type.

Now the map:

$$H_{(0,s)}(\mathbf{R}^n) \ni u \to (x' \to (x_n \to u(x', x_n))) \in H_s(\mathbf{R}^{n-1}, L^2(\mathbf{R})) \tag{4.18}$$

is a bijective isometry. By the same map we can regard $C_0^\infty(\mathbf{R}^n)$ as a subspace of $C_0^\infty(\mathbf{R}^{n-1}, L^2(\mathbf{R}))$ and we can write S in the form (4.17) with $H_1 = H_2 = L^2(\mathbf{R})$ and $s(x', y', \xi')$ being the operator $L^2(\mathbf{R}) \to L^2(\mathbf{R})$, defined by

$$s(x', y', \xi')\, u(x_n) = \int_0^{x_n} \Phi(x)\, p(x, y, \xi')\, \Psi(y)\, u(y_n)\, dy_n. \tag{4.19}$$

There remains only to prove (4.16) for then we can apply Theorem 4.12. Now $D_{x'}^{\alpha'} D_{y'}^{\beta'} D_{\xi'}^{\gamma'} s(x', y', \xi')$ is a sum of operators of the form (4.19) with some $\Phi, \Psi \in C_0^\infty(\mathbf{R}^n)$ and $p \in \check{S}^{k-|\gamma'|} \cup \dot{S}^{k+1/2-|\gamma'|}$. Thus it suffices to prove (4.16) in the case when $\alpha' = \beta' = \gamma' = 0$. Let us use an elementary lemma (see [18] pp. 93–94 for a proof).

LEMMA 4.14. *Let (X, dx), (Y, dy) be two measure spaces and let $k(x, y)$ be a measurable function on $X \times Y$, such that the functions $\int |k(x, y)| dy$ and $\int |k(x, y)| dx$ belong to $L^\infty(X, dx)$ and $L^\infty(Y, dy)$ respectively and their L^∞ norms are both less than or equal to C. Then*

$$Ku(x) = \int k(x, y)\, u(y)\, dy, \quad u \in L^2(Y, dy) \tag{4.20}$$

defines a bounded linear operator $L^2(Y, dy) \to L^2(X, dx)$ with norm less than or equal to C.

In view of the lemma it suffices to prove

$$\sup_{x_n} \left| \int_0^{x_n} |\Phi(x)\, p(x, y, \xi')\, \Psi(y)|\, dy_n \right| \leqslant \text{const.}\ (1 + |\xi'|)^k \tag{4.21}$$

and

$$\sup_{y_n} \left| \int_{x_n/y_n > 1} |\Phi(x)\, p(x, y, \xi')\, \Psi(y)|\, dx_n \right| \leqslant \text{const.}\ (1 + |\xi'|)^k. \tag{4.22}$$

(4.21) and (4.22) are obvious when $p \in \check{S}^k$ so we assume that $p \in \dot{S}^{k+\frac{1}{2}}$. Then we can even assume that

$$p(x, y, \xi') = y_n^{k_1} (x_n - y_n)^{k_2}\, a(x, y, \xi')\, e^{R(x, y_n, \xi')} \quad \text{for } |\xi'| > \tfrac{1}{2},$$

where $k_2 \geqslant k_1 \geqslant 0$ are integers and $a \in S^{k+(k_1+k_2+1)/2}$. As in the proof of Lemma 4.9 we see that

$$|\Phi(x)\, p(x, y, \xi')\, \Psi(y)| \leqslant \text{const.}\ |\xi'|^{k+(k_1+k_2+1)/2} |x_n|^{k_1} |x_n - y_n|^{k_2} \exp\left(-C |x_n - y_n|\, |x_n|\, |\xi'|\right)$$

for $|\xi'| > \frac{1}{2}$, where $C > 0$. Thus to prove (4.21) and (4.22) it is sufficient to prove

$$\int_0^{x_n} x_n^{k_1} (x_n - y_n)^{k_2} \exp\left(-C(x_n - y_n)\, x_n \lambda\right) dy_n \leqslant \text{const.}\ \lambda^{-(k_1+k_2+1)/2} \tag{4.23}$$

and

$$\int_{y_n}^{+\infty} x_n^{k_1} (x_n - y_n)^{k_2} \exp\left(-C(x_n - y_n)\, x_n \lambda\right) dx_n \leqslant \text{const.}\ \lambda^{-(k_1+k_2+1)/2} \tag{4.24}$$

for all $\lambda > 0$, $x_n \geqslant 0$, $y_n \geqslant 0$. By a change of variables with $T = x_n (C\lambda)^{\frac{1}{2}}$, $t = y_n (C\lambda)^{\frac{1}{2}}$ we get

$$\begin{aligned} &(C\lambda)^{(k_1+k_2+1)/2} \int_0^{x_n} x_n^{k_1} (x_n - y_n)^{k_2} \exp\left(-C(x_n - y_n)\, x_n \lambda\right) dy_n \\ &\quad = \int_0^T T^{k_1} (T - t)^{k_2} \exp\left(-(T - t)\, T\right) dt = T^{k_1 - k_2 - 1} \int_0^{T^2} s^{k_2} e^{-s}\, ds, \end{aligned}$$

which is bounded. This proves (4.23) and (4.24) can be proved similarly. The proof of Proposition 4.13 is complete.

By applying Theorem 4.12 with $H_1 = \mathbf{C}$ and $H_2 = L^2(\mathbf{R})$, one can prove in exactly the same way:

PROPOSITION 4.15. *If $p \in \dot{S}^k \cup \dot{S}^{k+\frac{1}{2}}$, then the operator $B_p: C_0^\infty(\mathbf{R}^{n-1}) \to C^\infty(\mathbf{R}^n)$, defined by*

$$B_p u_0(x) = \int\left(\int p(x, (y', 0), \xi')\, e^{i\langle x'-y', \xi'\rangle} u_0(y')\, dy'\right) d\xi', \quad u_0 \in C_0^\infty(\mathbf{R}^{n-1}),$$

can be extended to a continuous linear operator $H_s^{\text{comp}}(\mathbf{R}^{n-1}) \to H_{(0, s-k)}^{\text{loc}}(\mathbf{R}^n)$ for all $s \in \mathbf{R}$.

Let $\|\ \|_{(m,s)}$ be the norm in $H_{(m,s)}(\mathbf{R}^n)$. Then if m is an integer $\geqslant 0$, the norm $\|v\|_{(m,s)}$ is equivalent to the norm $\Sigma_{k=0}^m \|D_n^k v\|_{(0, s+m-k)}$. To prove that F can be extended to a linear operator $H_{(0,-\infty)}^{\text{loc}} \to H_{(1,-\infty)}^{\text{loc}}$ such that (i) of Proposition 3.6 holds it therefore suffices to prove

(i′) If m and k are integers such that $0 \leqslant k \leqslant m+1 \geqslant 1$ then $D_n^k F$ can be extended to a continuous linear operator $H_{(m,s)}^{\text{loc}} \to H_{(0, m+s-k+\frac{1}{2})}^{\text{loc}}$.

When $k=0$ this follows from Proposition 4.13 since the symbol q, used in the definition of F, can be written $q = q_0 + q'$, where $q_0 \in \dot{S}^0$ and $q' \in \dot{S}^{-\frac{1}{2}}$. By the same argument as used by Hörmander [8] to prove the composition formula for pseudodifferential operators, we are allowed to calculate formally and obtain

$$D_n F v(x) = i\int\left(\int_0^{x_n}\int D_{x_n} q(x, y, \xi')\, e^{i\langle x'-y', \xi'\rangle} v(y)\, dy' dy_n\right) d\xi' / (2\pi)^{n-1}$$
$$+ \int\left(\int q(x, (y', x_n), \xi')\, e^{i\langle x'-y', \xi'\rangle} v(y', x_n)\, dy'\right) d\xi' / (2\pi)^{n-1}, \; v \in C^\infty(\mathbf{R}^n). \quad (4.25)$$

Note that the last integral is equal to $v(x)$ by (i) of Lemma 4.6 and Fourier's inversion formula. By induction we get

$$D_n^k F v(x) = i\int\left(\int_0^{x_n}\int D_{x_n}^k q(x, y, \xi')\, e^{i\langle x'-y', \xi'\rangle} v(y)\, dy'\, dy_n\right) d\xi' / (2\pi)^{n-1}$$
$$+ \sum_{j=0}^{k-1} T_j(x, D_{x'})\, D_n^j v, \quad v \in C^\infty(\mathbf{R}^n),$$

where $T_j \in T^{k-j-1}(\mathbf{R}^n)$ are properly supported. Here we can apply Proposition 4.13 on the first integral and the results in the appendix on the other terms, to see that (i′) holds. This proves (i) of Proposition 3.6 and we omit the proof of (ii) which is quite similar.

Definition 4.16. We let $N^{-\infty}$ be the class of operators $K: C_0^\infty(\mathbf{R}^n) \to C^\infty(\mathbf{R}^n)$ of the form

$$Ku(x) = \int_0^{x_n} \int k(x, y)\, u(y)\, dy'\, dy_n, \quad u \in C_0^\infty(\mathbf{R}^n),$$

where $k \in C^\infty(\mathbf{R}^{2(n-1)} \times \bar{U})$ is a function of $((x', y'), (x_n, y_n))$.

Then we have

Each $K \in N^{-\infty}$ can be extended to a continuous linear operator $H_{(m,s)}^{\text{comp}}(\mathbf{R}^n) \to H_{(m+1,t)}^{\text{loc}}(\mathbf{R}^n)$ for all $s, t \in \mathbf{R}$ and integers $m \geq 0$. (4.26)

Proof of (v) *of Proposition 3.6.* It is evident that $\gamma F = 0$ and by Fourier's inversion formula we see that $\gamma F^+ = I$.

By the usual argument we are allowed to operate under the sign of integration and get:

$$T F v(x) = i \int \left(\int_0^{x_n} \int T(x, D_{x'}) (q(x, y, \xi')\, e^{i\langle x'-y', \xi' \rangle})\, v(y)\, dy'\, dy_n \right) d\xi' / (2\pi)^{n-1}$$

for all $v \in C^\infty(\mathbf{R}^n)$ and $T \in T^k(\mathbf{R}^n)$. Combining this with (4.25) and the immediately following remark, we get:

$$\Lambda F v(x) = v(x) + \int \left(\int_0^{x_n} \int q_{-\infty}(x, y, \xi')\, e^{i\langle x'-y', \xi' \rangle}\, v(y)\, dy'\, dy_n \right) d\xi', \quad v \in C^\infty(\mathbf{R}^n),$$

where $q_{-\infty}(x, y, \xi') = -i(2\pi)^{-n+1} e^{-i\langle x', \xi' \rangle} \Lambda(x, D_x) (q(x, y, \xi')\, e^{i\langle x', \xi' \rangle})$, Thus $q_{-\infty} \in \check{S}^{-\infty}$ by (ii) of Lemma 4.6 and therefore:

$$\Lambda F v(x) = v(x) + \int_0^{x_n} \int k(x, y)\, v(y)\, dy'\, dy_n, \quad v \in C^\infty(\mathbf{R}^n),$$

where $k(x, y) = \int q_{-\infty}(x, y, \xi')\, e^{i\langle x'-y', \xi' \rangle} d\xi'$ belongs to $C^\infty(\mathbf{R}^{2(n-1)} \times \bar{U})$ as a function of $((x', y'), (x_n, y_n))$. This proves that

$$\Lambda F \equiv I \bmod (N^{-\infty}), \tag{4.27}$$

so by (4.26) it follows that $\Lambda F \equiv I \bmod (M^{-\infty})$. That ΛF^+ has C^∞ kernel is proved in the same way and we omit the details. This completes the proof of (v) of Proposition 3.6.

Proof of (vi) *of Proposition 3.6.* We shall first construct operators $G: C^\infty(\mathbf{R}^n) \to C^\infty(\mathbf{R}^n)$ and $G^+: C^\infty(\mathbf{R}^{n-1}) \to C^\infty(\mathbf{R}^n)$, such that:

$$G\Lambda + G^+\gamma \equiv I \bmod (N^{-\infty}). \tag{4.28}$$

LEMMA 4.17. *Let ${}^t\Lambda$ be the real adjoint of Λ. Then there exists $g \in \check{S}^0$, such that:*

(i) ${}^t\Lambda(y, D_y)(g(x, y, \xi')e^{i\langle x'-y', \xi'\rangle}) \in \check{S}^{-\infty}$.

(ii) $g(x, (y', x_n), \xi') = \Phi(x'-y')$, where $\Phi \in C_0^\infty(\mathbf{R}^{n-1})$ is $=1$ near the origin.

(iii) $g(x, y, \xi') = 0$ when $|x-y| > 2$.

The proof of Lemma 4.17 is almost identical with that of Lemma 4.6, so we omit it. Now put:

$$Gv(x) = i\int\left(\int_0^{x_n}\int g(x, y, \xi')\, e^{i\langle x'-y', \xi'\rangle} v(y)\, dy'\, dy_n\right) d\xi'/(2\pi)^{n-1}, \quad v \in C^\infty(\mathbf{R}^n)$$

and

$$G^+u_0(x) = \int\left(\int g(x, (y', 0), \xi')\, e^{i\langle x'-y', \xi'\rangle} u_0(y')\, dy'\right) d\xi'/(2\pi)^{n-1}, \quad u_0 \in C^\infty(\mathbf{R}^{n-1}).$$

These equations define our operators G and G^+. For $u \in C^\infty(\mathbf{R}^n)$ we get after a partial integration:

$$\begin{aligned} G\Lambda u(x) = {} & i\int\left(\int_0^{x_n}\int {}^t\Lambda(y, D_y)\,(g(x, y, \xi')\, e^{i\langle x'-y', \xi'\rangle})\, u(y)\, dy'\, dy_n\right) d\xi'/(2\pi)^{n-1} \\ & + \int\left(\int g(x, (y', x_n), \xi')\, e^{i\langle x'-y', \xi'\rangle} u(y', x_n)\, dy'\right) d\xi'/(2\pi)^{n-1} \\ & - \int\left(\int g(x, (y', 0), \xi')\, e^{i\langle x'-y', \xi'\rangle} u(y', 0)\, dy'\right) d\xi'/(2\pi)^{n-1}. \end{aligned}$$

Here the last two integrals are boundary terms originating from the term D_{y_n} in $\Lambda(y, D_y)$. Lemma 4.17 implies that the first integral is $=\int_0^{x_n}\int k(x, y)u(y)dy'dy_n$, where $k \in C^\infty$ and that the second integral is $=u(x)$. The last integral is $=-G^+\gamma u(x)$. This proves (4.28).

Next we show that (G, G^+) is approximately equal to (F, F^+). From (v) of Proposition 3.6 and (4.27) it follows that

$$(G, G^+)\begin{pmatrix}\Lambda\\ \gamma\end{pmatrix}(F, F^+) = (G, G^+)\begin{pmatrix}I+K_1 & K_1^+\\ 0 & I\end{pmatrix} = (G + GK_1, G^+ + GK_1^+),$$

where $K_1 \in N^{-\infty}$ and K_1^+ has C^∞ kernel. On the other hand (4.28) implies that

$$(G, G^+)\begin{pmatrix}\Lambda\\ \gamma\end{pmatrix}(F, F^+) = (I + K_2)(F, F^+) = (F + K_2F, F^+ + K_2F^+),$$

where $K_2 \in N^{-\infty}$. Thus $(G + GK_1, G^+ + GK_1^+) = (F + K_2F, F^+ + K_2F^+)$ or equivalently:

$$\begin{aligned} F - G &= GK_1 - K_2F \\ F^+ - G^+ &= GK_1^+ - K_2F^+. \end{aligned}$$

By (4.28) we get:

$$F\Lambda + F^+\gamma - I \equiv F\Lambda + F^+\gamma - G\Lambda - G^+\gamma \equiv (F-G)\Lambda + (F^+ - G^+)\gamma$$
$$\equiv (GK_1 - K_2F)\Lambda + (GK_1^+ - K_2F^+)\gamma \bmod (N^{-\infty}).$$

Using Proposition 4.13, we show as in the proof of (i) that G can be extended to a continuous operator $H^{\text{loc}}_{(m,s)} \to H^{\text{loc}}_{(m+1,s-1)}$ for all $s \in \mathbf{R}$ and integers $m \geq 0$. Moreover it is wellknown that γ is continuous $H^{\text{loc}}_{(1,s)}(\mathbf{R}^n) \to H^{\text{loc}}_{s+\frac{1}{2}}(\mathbf{R}^{n-1})$ for all s, so it follows that $(GK_1 - K_2F)\Lambda + (GK_1^+ - K_2F^+)\gamma \in M^{-\infty}$. This proves (vi) of Proposition 3.6.

(iii) of Proposition 3.6 follows from the following two facts:

(a) By Proposition 2.5.7 in [10] we have

$\mathrm{WF}'(F^+) \subset \{((x, (\xi', 0)), (x', \xi')) \in (T^*(\mathbf{R}^n) \setminus 0) \times (T^*(\mathbf{R}^{n-1}) \setminus 0\}$

(b) From (iii) of Lemma 4.6 it follows that the distribution kernel $k(x, y')$ of F^+ is smooth outside the plane $x_n = 0$.

Proof of (iv) of Proposition 3.6. We shall first prove

$$((x, \xi), (y, \eta)) \in \mathrm{WF}'(F) \Rightarrow \xi' = \eta' = 0 \text{ or } \xi' = \eta' \neq 0 \text{ and } x' = y'. \tag{4.29}$$

To do so we note that

$$Fu(x) = \int_0^{x_n} Q_{x_n y_n}(x', D_{x'})\, u(x', y_n)\, dy_n, \; u \in C_0^\infty(\mathbf{R}^n),$$

where $Q_{x_n y_n}$ is given by

$$Q_{x_n y_n}(x', D_{x'})\, w = i \iint q(x, y, \xi')\, e^{i\langle x'-y', \xi'\rangle} w(y')\, dy'\, d\xi' / (2\pi)^{n-1}, \quad w \in C_0^\infty(\mathbf{R}^{n-1}).$$

Clearly $Q_{x_n y_n}$ is a locally bounded function of $(x_n, y_n) \in \bar{U}$ with values in $L^0(\mathbf{R}^{n-1})$. On any compact set where $x' \neq y'$ we therefore have uniform bounds for the derivatives of the kernel $Q_{x_n y_n}(x', y')$ with respect to x' and y' which proves that $\xi' = \eta' = 0$ if $((x, \xi), (y, \eta)) \in \mathrm{WF}'(F)$ and $x' \neq y'$. Let $\Phi \in C_0^\infty(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1})$, $\Psi \in C_0^\infty(\mathbf{R} \times \mathbf{R})$. Since the wave front set of the kernel of a pseudo-differential operator belongs to the normal bundle of the diagonal the Fourier transform of the distribution $\Phi(x', y') Q_{x_n y_n}(x', y')$ with respect to (x', y') is rapidly decreasing when (ξ', η') belongs to any closed cone where $\xi' + \eta' \neq 0$. If the kernel of F is also denoted by F, it follows by integration with respect to x_n and y_n that the Fourier transform of $\Phi(x', y')\Psi(x_n, y_n) F(x, y)$ is rapidly decreasing when (ξ, η) belongs to a closed cone where $\xi' \neq -\eta'$. This proves (4.29).

Since $F(x, y) = 0$ when $|x_n| < |y_n|$ we have

$$((x, \xi), (y, \eta)) \in \mathrm{WF}'(F) \quad \Rightarrow \quad |x_n| \geq |y_n|. \tag{4.30}$$

LEMMA 4.18. *Let $Z \in L^M(\mathbf{R}^n)$ and $\chi \in L^0(\mathbf{R}^n)$ be properly supported and satisfy:*

$$(\mathrm{WF}(\chi) \cup \mathrm{WF}(Z)) \cap \{(x, \xi) \in T^*(\mathbf{R}^n)\setminus 0;\quad \xi' = 0\} = \emptyset \tag{4.31}$$

$$\mathrm{WF}(I-\chi) \cap \{(x, \xi) \in T^*(\mathbf{R}^n)\setminus 0;\quad x_n = \xi_n = 0\} = \emptyset. \tag{4.32}$$

Then $(I-\chi)FZ \in L^{M-1}(\mathbf{R}^n)$, so in particular

$$\mathrm{WF}'((I-\chi)FZ) \subset \{((x, \xi), (x, \xi)) \in (T^*(\mathbf{R}^n)\setminus 0) \times (T^*(\mathbf{R}^n)\setminus 0)\}.$$

Proof. Since the principal symbol of Λ is $\neq 0$ and belongs to S^1 outside $\{(x, \xi) \in T^*(\mathbf{R}^n)\setminus 0; x_n = \xi_n = 0$ or $\xi' = 0\}$ we can find $\Lambda' \in L^{-1}(\mathbf{R}^n)$ with $\mathrm{WF}(\Lambda') \cap \{(x, \xi) \in T^*(\mathbf{R}^n)\setminus 0; \xi' = 0\} = \emptyset$, properly supported and such that $\mathrm{WF}(\Lambda'\Lambda - I) \cup \mathrm{WF}(\Lambda\Lambda' - I)$ is arbitrarily close to $\{(x, \xi) \in T^*(\mathbf{R}^n)\setminus 0; x_n = \xi_n = 0$ or $\xi' = 0\}$. (See Prop. A.1.) Using such a Λ' it is easy to construct $\chi_0 \in L^0(\mathbf{R}^n)$, properly supported such that $\mathrm{WF}(I-\chi_0) \subset \mathrm{WF}(I-\chi)$ and

$$\Lambda(I-\chi) \equiv (I-\chi_0)\Lambda \bmod (L^{-\infty}),$$

where $L^{-\infty}$ is the set of operators with C^∞ kernel. With Λ' as above it suffices to prove that

$$(I-\chi)FZ \equiv (I-\chi)\Lambda' Z.$$

Put $$B = (I-\chi)FZ - (I-\chi)\Lambda' Z.$$

Then $$\Lambda B \equiv (I-\chi_0)\Lambda FZ - (I-\chi_0)\Lambda\Lambda' Z \bmod (L^{-\infty}).$$

By the choice of Λ' we have

$$(I-\chi_0)\Lambda\Lambda' Z \equiv (I-\chi_0)Z \bmod (L^{-\infty}).$$

By (v) of Proposition 3.6 we have

$$(I-\chi_0)\Lambda FZ \equiv (I-\chi_0)Z \bmod (M^{-\infty}Z).$$

In view of Proposition A.2 and (4.31) we have $M^{-\infty}Z \subset L^{-\infty}$. Thus $\Lambda B \equiv 0 \bmod (L^{-\infty})$. By (vi) of Proposition 3.6 we then get

$$B = F\Lambda B + F^+\gamma B + KB \equiv F^+\gamma B + KB \bmod (L^{-\infty}),$$

where $K \in M^{-\infty}$. Using Proposition A.2 we see that $KB \in L^{-\infty}$, thus

$$B \equiv F^+\gamma B \bmod (L^{-\infty}).$$

Take $\chi_1 \in L^0(\mathbf{R}^n)$ such that

$$\mathrm{WF}(I-\chi_1) \cap \{(x, \xi) \in T^*(\mathbf{R}^n)\setminus 0;\quad x_n = \xi_n = 0\} = \emptyset$$

and $$(I-\chi_1)(I-\chi) \equiv (I-\chi) \bmod (L^{-\infty}).$$

Then we get $$B \equiv (I-\chi_1) B \equiv (I-\chi_1) F^{+}\gamma B \equiv 0 \bmod (L^{-\infty}).$$

Here the last equivalence follows from two facts:

1) $(I-\chi_1) F^{+}$ has C^∞ kernel by (iii) of Proposition 3.6.

2) From Propositions A.2 and 3.6 and the definition of B it follows that γB is continuous $H_s^{\text{loc}}(\mathbf{R}^n) \to H_{s-M}^{\text{loc}}(\mathbf{R}^{n-1})$ for all $s \in \mathbf{R}$.

This completes the proof of the lemma.

With χ and Z as in Lemma 4.18 we write

$$FZ = (I-\chi) FZ + \chi FZ.$$

To prove (iv) of Proposition 3.6, it suffices in view of the lemma to estimate $\mathrm{WF}'(\chi FZ)$. Combining (4.29) and (4.30) we get:

$$((x, \xi), (y, \eta)) \in \mathrm{WF}'(\chi FZ) \quad \Rightarrow \quad (x, \xi) \in \mathrm{WF}(\chi),\ (x', \xi') = (y', \eta'),\ |y_n| \leqslant |x_n|.$$

Now the desired estimate follows, since we can choose χ with $\mathrm{WF}(\chi)$ arbitrarily close to

$$\{(x, \xi) \in T^*(\mathbf{R}^n)\setminus 0;\quad \xi_n = x_n = 0\}.$$

This completes the proof of Proposition 3.6 and Theorems 1 and 2 are now completely proved.

Remark 4.19. With the methods of this section one can treat (4.1) in the more general case when $\Lambda(x, D) = D_n - ix_n^k r(x, D') + s(x, D')$. Here k is odd and r and s are the same operators as before. This shows that Theorems 1 and 2 hold with appropriate modifications for more general operators P.

Remark 4.20. At the AMS conference held at Berkeley in August 1971 M Sato announced for the analytic case a stronger result than the conjunction of Lemmas 2.2 and 3.1, which allows one to transform to $\Lambda = D_n - ix_n D_{n-1}$. For this operator the constructions in this section are of course simpler, but we have kept our orginal proofs rather than transferring the burden of proof from section 4 to sections 2 and 3.(1)

§ 5. Extensions of Theorem 1

We let $A \equiv B$ mean that $A - B$ is smooth if A and B are distributions and that the distribution kernel of $A - B$ is smooth if A and B are operators. By Theorem 1 the system

$$Pu \equiv w - R^{-}w^{-}, \quad R^{+}u \equiv u^{+}, \quad u \in \mathcal{D}'(\Omega), \quad w^{-} \in \mathcal{D}'(\Gamma^{-}) \tag{5.1}$$

is equivalent to

(1) (Added in proof.) In a paper to be published jointly with J. J. Duistermaat such transformations will be used to prove a global verison of Theorem 2.

$$w^- \equiv E^- w, \quad u \equiv Ew + E^+ u^+. \tag{5.1'}$$

Here all the operators are given in Theorem 1.

Now let A^+: $\mathcal{D}'(\Omega) \to \mathcal{D}'(\Gamma^+)$ and A^-: $\mathcal{D}'(\Gamma^-) \to \mathcal{D}'(\Omega)$ be continuous linear operators which are also continuous $C^\infty(\Omega) \to C^\infty(\Gamma^+)$ and $C^\infty(\Gamma^-) \to C^\infty(\Omega)$ respectively. Consider the more general system

$$Pu \equiv v - A^- u^-, \quad A^+ u \equiv v^+, \quad u \in \mathcal{D}'(\Omega), \ u^- \in \mathcal{D}'(\Gamma^-). \tag{5.2}$$

If $u^+ = R^+ u$ the equivalence between (5.1) and (5.1′) shows with $w^- = 0$ that the first equation in (5.2) is equivalent to the equations

$$E^-(v - A^- u^-) \equiv 0, \quad u \equiv E(v - A^- u^-) + E^+ u^+.$$

Thus (5.2) is equivalent to

$$E^- A^- u^- \equiv E^- v, \quad A^+ E^+ u^+ \equiv v^+ - A^+ E(v - A^- u^-), \quad u \equiv E(v - A^- u^-) + E^+ u^+ \tag{5.2'}$$

We now assume that there exist continuous linear operators $B^\pm$: $\mathcal{D}'(\Gamma^\pm) \to \mathcal{D}'(\Gamma^\pm)$ which are continuous $C^\infty(\Gamma^\pm) \to C^\infty(\Gamma^\pm)$ and satisfy

$$B^+ A^+ E^+ \equiv A^+ E^+ B^+ \equiv I, \quad B^- E^- A^- \equiv E^- A^- B^- \equiv I.$$

Then we can eliminate u^+ in (5.2′) and a simple calculation shows that (5.2) is equivalent to the system

$$u^- \equiv F^- v, \quad u \equiv Fv + F^+ v^+,$$

where
$$\left.\begin{aligned} &F^- = B^- E^-, \quad F^+ = E^+ B^+ \\ &F = E - EA^- B^- E^- - E^+ B^+ A^+ E + E^+ B^+ A^+ E A^- B^- E^- \end{aligned}\right\} \tag{5.3}$$

Thus we obtain

PROPOSITION 5.1. *Under the assumptions above we have*

$$\tilde{\mathcal{P}} \mathcal{F} \equiv I, \quad \mathcal{F} \tilde{\mathcal{P}} \equiv I.$$

Here
$$\tilde{\mathcal{P}} = \begin{pmatrix} P & A^- \\ A^+ & 0 \end{pmatrix}, \quad \mathcal{F} = \begin{pmatrix} F & F^+ \\ F^- & 0 \end{pmatrix},$$

where F, F^+ and F^- are given by (5.3).

Example 1. Let $A^+ \in L^{m^+}(\Gamma^+, \Omega, f)$ and $A^{-*} \in L^{m^-}(\Gamma^-, \Omega, f)$ be properly supported and have principal symbols positively homogeneous of degree m^+ and m^- respectively. Assume that A^+ and A^{-*} satisfy the obvious analogues of (C^+), (C^-) and (0.5) in section 0. Let

$B^+ \in L^{-m^+}(\Gamma^+)$ and $B^- \in L^{-m^-}(\Gamma^-)$ be elliptic, properly supported with principal symbols positively homogeneous of degree $-m^+$ and $-m^-$ respectively. Put $R^+ = B^+A^+$ and $R^- = A^-B^-$. Then R^+ and R^- satisfy the conditions of Theorem 1. If E, E^+, E^- are the corresponding solution operators, A^+E^+ and E^-A^- have the parametrices B^+ and B^- respectively. Thus we can apply Proposition 5.1. The last equation in (5.3) simplifies to $F \equiv E$. This gives a slight extension of Theorem 1.

Example 2. Let P, R^+, R^-, E, E^+, E^- be as in Theorem 1 and let $A^+ \in L^{m^+}(\Gamma^+, \Omega, f)$ and $A^{-*} \in L^{m^-}(\Gamma^-, \Omega, f)$ be arbitrary. Then in general A^+u is not defined for all $u \in \mathcal{D}'(\Omega)$ and A^- does not map $C^\infty(\Gamma^-)$ into $C^\infty(\Omega)$. Therefore we can not apply Proposition 5.1. However A^+E^+ and E^-A^- still seem to play an essential role for the problem (5.2) so it is interesting to calculate them. Since $E^-A^- = (A^{-*}E^{-*})^*$ and A^{-*} and E^{-*} are the same kind of operators as A^* and E^+, it suffices to calculate A^+E^+.

From Theorem 1 it follows that the distribution kernel of A^+E^+ is smooth outside $\{(x, y) \in \Gamma^+ \times \Gamma^+;\ f(x) = f(y)\}$. Therefore we can localize the study in the following way: By Proposition 0.1 each $x_0 \in \Omega$ has a neighbourhood W, such that $f\Gamma^+ \cap W$ is the union of a finite number of hypersurfaces $f\Gamma_1, f\Gamma_2, \ldots, f\Gamma_N$, where $\Gamma_1, \Gamma_2, \ldots, \Gamma_N$ are the different components of $(f^{-1}W) \cap \Gamma^+$. We can identify $C^\infty((f^{-1}W) \cap \Gamma^+)$ in a natural way with $C^\infty(f\Gamma_1) \times C^\infty(f\Gamma_2) \times \ldots \times C^\infty(f\Gamma_N)$ and A^+ induces a map

$$C^\infty(\Omega) \ni v \to (\gamma_1 A_1 v, \gamma_2 A_2 v, \ldots, \gamma_N A_N v) \in C^\infty(f\Gamma_1) \times \ldots \times C^\infty(f\Gamma_N)$$

where $A_j \in L^{m^+}(\Omega)$ and γ_j is the restriction operator $C^\infty(\Omega) \to C^\infty(f\Gamma_j)$. Similarly E^+ induces a map

$$C_0^\infty(f\Gamma_1) \times \ldots \times C_0^\infty(f\Gamma_N) \ni (u_1, \ldots, u_N) \to E_1 u_1 + E_2 u_2 + \ldots + E_N u_N \in C^\infty(\Omega).$$

Thus A^+E^+ can be locally identified with the matrix:

$$(\gamma_j A_j E_k)_{1 \leq j, k \leq N} \colon C_0^\infty(f\Gamma_1) \times \ldots \times C_0^\infty(f\Gamma_N) \to C^\infty(f\Gamma_1) \times \ldots \times C^\infty(f\Gamma_N).$$

Following the proof of Theorem 1 one can prove (with some work) that in the local coordinates of Proposition 0.1, we have

$$E^+u^+(x) = \iint b(x, y', \xi') \exp\left(i\langle x', \xi'\rangle + ix_n \tau(x', \xi') - i\langle y', \xi'\rangle\right) u^+(y')\, dy'\, d\xi',$$

$$u^+ \in C_0^\infty(\Gamma^+),\ y', \xi' \in \mathbf{R}^{n-1},\ x \in \mathbf{R}^n.$$

Here $b \in S^0_{1,\frac{1}{2}}((\mathbf{R}^n \times \mathbf{R}^{n-1}) \times \mathbf{R}^{n-1})$, τ is given by Proposition 0.1 and we have identified Ω locally with $\mathbf{R}^n$ and Γ^+ with the hyperplane $x_n = 0$. It is possible to calculate the leading term in the asymtotic expansion of b.

Now choose local coordinates with the origin in $x_0 \in W$, such that the x_n-axis is transversal to all the $f\Gamma_j$ at x_0. Near x_0 each $f\Gamma_j$ is then given by an equation $x_n = \lambda_j(x')$, where λ_j is smooth and realvalued. Then for small x

$$E_k u_k(x) = \iint b_k(x, y', \xi') \exp\left(i\langle x', \xi'\rangle + i(x_n - \lambda_k(x'))\,\tau_k(x', \xi') - i\langle y', \xi'\rangle\right) \quad u_k(y')\,dy'\,d\xi$$

for all $u_k \in C_0^\infty(f\Gamma_k)$ with support close to $x'=0$. Here $b_k \in S^0_{1\frac{1}{2}}$ and τ_k is smooth, real valued and positively homogeneous of degree 1 with respect to ξ'. By applying A_j under the sign of integration, we get

$$\gamma_j A_j E_k u_k(x')$$
$$= \iint a_{jk}(x', y', \xi') \exp\left(i\langle x', \xi'\rangle + i(\lambda_j(x') - \lambda_k(x'))\tau_k(x', \xi') - i\langle y', \xi'\rangle\right) u_k(y')\,dy'\,d\xi',$$

where the principal part of $a_{jk} \in S^{m_j^+}_{1\frac{1}{2}}$ can be determined. Thus the study of A^+E^+ is equivalent to the study of a certain system of Fourier integral operators. It seems to be very difficult to find simple nontrivial conditions for such a system to be solvable. However, in the case when all the $f\Gamma_j$ coincide, we have $\lambda_j - \lambda_k = 0$ and A^+E^+ becomes a system of pseudodifferential operators. This case is treated in Eškin [6].

Example 3. Let P be as in Theorem 1. If we choose R^+ and R^{-*} with $\mathrm{WF}(R^+)$ and $\mathrm{WF}(R^{-*})$ close to Σ_0^+ and Σ_0^- respectively, it follows from (iii) in Theorem 1 that $\mathrm{WF}'(E)$ is close to

$$\Delta T^*(\Omega)\setminus 0 = \{(\varrho, \varrho) \in (T^*(\Omega)\setminus 0) \times (T^*(\Omega)\setminus 0)\}.$$

We shall now construct operators A^+ and A^- such that $\tilde{\mathcal{P}}$ has a parametrix

$$\mathcal{F} = \begin{pmatrix} F & F^+ \\ F^- & 0 \end{pmatrix}$$

where

$$\mathrm{WF}'(F) \subset \Delta T^*(\Omega)\setminus 0$$
$$\mathrm{WF}'(F^+) \subset \{(\varrho, \mathcal{G}_+\varrho);\ \varrho \in \Sigma^+\}$$
$$\mathrm{WF}'(F^-) \subset \{(\mathcal{G}_-\varrho, \varrho);\ \varrho \in \Sigma^-\}.$$

LEMMA 5.2. *If $1/2 < \varrho < 1$, there exists a properly supported $P' \in L_\varrho^{-m+1-\varrho}(\Omega)$ such that* $(\mathrm{WF}(P'P - I) \cup \mathrm{WF}(PP' - I)) \subset \Sigma$.

Proof. Since P is elliptic outside Σ it suffices to find $P' \in L_\varrho^{-m+1-\varrho}(\Omega)$ such that $\mathrm{WF}(PP' - I) \subset \Sigma$. Clearly it suffices to construct P' locally. We can therefore assume that $\Omega = \mathbf{R}^n$ and

$$\Sigma = \{(x, \xi) \in T^*(\mathbf{R}^n)\setminus 0; \quad x_n = \xi_n - \tau(x', \xi') = 0\}.$$

Let $\psi \in C_0^\infty(\mathbf{R})$ be 1 near the origin and let $\chi \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ be equal to

$$\psi((\xi_n - \tau(x', \xi'))|\xi'|^{-\varrho})\,\psi(x_n|\xi'|^{1-\varrho}) \quad \text{for} \quad |\xi| > 2$$

and equal to 1 near Σ. Then $\chi \in S_\varrho^0(\mathbf{R}^n \times \mathbf{R}^n)$ and the restriction to $(T^*(\mathbf{R}^n)\setminus 0)\setminus\Sigma$ belongs to $S^{-\infty}((T^*(\mathbf{R}^n)\setminus 0)\setminus\Sigma)$. If $p(x, \xi)$ is the homogeneous principal symbol of P we have

$$(1-\chi(x, \xi))p(x, \xi)^{-1} \in S_\varrho^{-m+1-\varrho}(\mathbf{R}^n \times (\mathbf{R}^n\setminus\{0\})).$$

In fact, we have

$$|p(x, \xi)| > C_K(|\xi|^m|x_n| + |\xi|^{m-1}|\xi_n - \tau(x', \xi')|), \quad x \in K \subset\subset \mathbf{R}^n,$$

where $C_K > 0$ and thus

$$|p(x, \xi)| > C'_K|\xi|^{m+\varrho-1} \text{ in supp } (1-\chi) \text{ when } |\xi| > 2.$$

The derivatives of $(1-\chi)p^{-1}$ can now be estimated inductively if we take the derivatives of the identity

$$p((1-\chi)p^{-1}) = 1-\chi$$

and use Leibniz' formula (cf. [8]).

Let $P'_0 \in L_\varrho^{-m+1-\varrho}(\mathbf{R}^n)$ be properly supported with symbol $(1-\chi)p^{-1}$ mod $(S^{-\infty})$. Then by the formula for composition of two pseudodifferential operators, we get

$$PP'_0 = I - \chi(x, D) + A,$$

where $A \in L_\varrho^{\max(-\varrho, 1-2\varrho)}$ and $\mathrm{WF}(\chi(x, D)) \subset \Sigma$. In fact, if $p + p_{m-1}$ is the symbol of P, then A has the symbol

$$\sim p_{m-1}(1-\chi)\,p^{-1} + \sum_{|\alpha|>0} (p+p_{m-1})^{(\alpha)} D_x^\alpha (1-\chi)\,p^{-1}/\alpha!$$

Since $\frac{1}{2} < \varrho < 1$, we see that $I + A$ has a properly supported parametrix; $(I+A)^{-1} \sim I - A + A^2 - A^3 + \ldots \in L_\varrho^0$ and the lemma follows if we put $P' = P'_0(I+A)^{-1}$.

Now take R^+ and R^- as in Theorem 1 and let E, E^+, E^- be the corresponding parametrix operators. Put

$$A^+ = R^+(I - P'P), \quad A^- = (I - PP')R^-.$$

Then

$$\mathrm{WF}'(A^+) \subset \{(\mathcal{G}_+\varrho, \varrho); \quad \varrho \in \Sigma^+\} \tag{5.4}$$

$$\mathrm{WF}'(A^-) \subset \{(\varrho, \mathcal{G}_-\varrho); \quad \varrho \in \Sigma^-\} \tag{5.5}$$

Moreover, since $PE^+ \equiv 0$;

$$A^+E^+ = R^+E^+ - R^+P'PE^+ \equiv I$$

and similarly $E^-A^- \equiv I$. Thus we can apply Proposition 5.1 and find that

$$\tilde{\mathcal{P}} = \begin{pmatrix} P & A^- \\ A^+ & 0 \end{pmatrix}$$

has a parametrix

$$\mathcal{F} = \begin{pmatrix} F & F^+ \\ F^- & 0 \end{pmatrix},$$

where $F^+ = E^+$, $F^- = E^-$ and

$$F = E - EA^-E^- - E^+A^+E. \tag{5.6}$$

If $B \in L^0(\Omega)$ is properly supported and $\mathrm{WF}(I-B) \cap \Sigma = \emptyset$, we have

$$A^+ \equiv R_B^+(I - P'P), \quad A^- \equiv (I - PP')R_B^-,$$

where $R_B^+ = R^+B$ and $R_B^- = BR^-$ satisfy the conditions of Theorem 1. Let E_B, E_B^+, E_B^- be the corresponding solution operators. Since the parametrix of $\tilde{\mathcal{P}}$ is unique mod. $(L^{-\infty})$, we have

$$F \equiv E_B - E_BA^-E_B^- - E_B^+A^+E_B. \tag{5.7}$$

By choosing B with $\mathrm{WF}(B)$ arbitrarily close to Σ, it then follows from (iii) of Theorem 1 and (5.4), (5.5) that $\mathrm{WF}'(F) \subset \Delta T^*(\Omega) \setminus 0$.

Since Theorem 2.1 is a local version of Theorem 1, one can modify the operator Q there in such a way that for the corresponding solution operator we have $\mathrm{WF}'(E) \subset \Delta T^*(\mathbf{R}^n) \setminus 0$. Using this modified version of Theorem 2.1 in the proof of Theorem 2, we find that it is possible to choose the operators R_ϱ, E_ϱ, E_ϱ^+ in Theorem 2 such that $\mathrm{WF}'(E_\varrho) \subset \Delta T^*(\Omega) \setminus 0$.

Finally we claim that the inclusions (iii) in Theorem 1 are actually equalities. To prove this one has to prove the opposite inclusions. To illustrate the ideas we shall only prove that $A \subset \mathrm{WF}'(E)$, where we have put

$$A = \{(f_\Omega \varrho, f_\Omega \mu) \in (T^*(\Omega) \setminus 0) \times (T^*(\Omega) \setminus 0);\ \varrho \in \Sigma_0^+,\ \mu \in \mathrm{WF}(R^+),\ f_{\Gamma^+} \varrho = f_{\Gamma^+} \mu\}.$$

We have

$$\mathrm{WF}'(R^+) = \{(f_{\Gamma^+} \mu, f_\Omega \mu);\ \mu \in \mathrm{WF}(R^+)\} = \mathcal{G}_+ \circ A, \tag{5.8}$$

where $\circ$ means composition of relations. Since $R^+E^+ \equiv I$ we have $\mathrm{WF}'(R^+) = \mathrm{WF}'(R^+E^+R^+) \subset \mathrm{WF}'(R^+) \circ \mathrm{WF}'(E^+R^+) = \mathcal{G}_+ \circ \mathrm{WF}'(E^+R^+)$, where the last equality follows from the fact that $\varrho \in \Sigma^+$ if $(\varrho, \mu) \in \mathrm{WF}'(E^+R^+)$. Thus (5.8) gives $\mathcal{G}_+ \circ A \subset \mathcal{G}_+ \circ \mathrm{WF}'(E^+R^+)$ and since $\mathcal{G}_+$ is bijective we get $A \subset \mathrm{WF}'(E^+R^+)$.

Since $\mathrm{WF}'(I) = \Delta T^*(\Omega) \setminus 0$ if I is the identity in $\mathcal{D}'(\Omega)$, we see that $\mathrm{WF}'(I - E^+R^+) \supset \mathrm{WF}'(E^+R^+) \setminus \{(\varrho, \varrho) \in \mathrm{WF}'(E^+R^+)\} \supset A \setminus \{(\varrho, \varrho) \in A\}$. Since $\mathrm{WF}'(I - E^+R^+)$ is closed and the closure of $A \setminus \{(\varrho, \varrho) \in A\}$ is A, it follows that $A \subset \mathrm{WF}'(I - E^+R^+)$. Now $EP \equiv I - E^+R^+$ by Theorem I and since $\mathrm{WF}'(P) \subset \Delta T^*(\Omega) \setminus 0$, it follows that $A \subset \mathrm{WF}'(E)$ as asserted.

Appendix

Here we shall define and investigate a certain type of pseudodifferential operators. We let $T^m(\mathbf{R}^n)$ be the set of operators $T: C_0^\infty(\mathbf{R}^n) \to C^\infty(\mathbf{R}^n)$ which can be written in the form

$$Tu(x) = \iint s(x, y', \xi')\, e^{i\langle x'-y', \xi'\rangle} u(y', x_n)\, dy'\, d\xi'/(2\pi)^{n-1}, \quad u \in C_0^\infty(\mathbf{R}^n), x \in \mathbf{R}^n, y', \xi' \in \mathbf{R}^{n-1},$$

for some $s \in S^m_{10}((\mathbf{R}^n \times \mathbf{R}^{n-1}) \times \mathbf{R}^{n-1})$.

If $T \in T^m(\mathbf{R}^n)$, we can regard T as a family of pseudodifferential operators in $\mathbf{R}^{n-1}$ depending on the parameter x_n. Using this observation it is easy to show that T is continuous $H^{\text{comp}}_{(r,s)} \to H^{\text{loc}}_{(r,s-m)}$ for all $x \in R$ and integers $r > 0$. Using the same observation one shows, exactly as for pseudodifferential operators, that every properly supported $T \in T^m(\mathbf{R}^n)$ can be given by the formula

$$Tu(x) = \int t(x, \xi')\, e^{i\langle x', \xi'\rangle} \tilde{u}(\xi', x_n)\, d\xi'/(2\pi)^{n-1}, \quad u \in C_0^\infty(\mathbf{R}^n),$$

where $\tilde{u}$ denotes the partial Fourier transform of u with respect to x' and $t(x, \xi') \in S^m_{10}(\mathbf{R}^n \times \mathbf{R}^{n-1})$. t is uniquely determined by T and will be called the symbol of T. If $\hat{u}$ is the Fourier transform of u, we have

$$\tilde{u}(\xi', x_n) = \int \exp(ix_n \xi_n)\, \hat{u}(\xi)\; d\xi_n/(2\pi),$$

thus we have

$$Tu(x) = \int t(x, \xi')\, e^{i\langle x, \xi\rangle} \hat{u}(\xi)\; d\xi/(2\pi)^n, \quad u \in C_0^\infty(\mathbf{R}^n). \tag{A.1}$$

PROPOSITION A.1. *Let $T \in T^{m'}(\mathbf{R}^n)$ and $Q \in L^{m''}(\mathbf{R}^n)$ be properly supported with symbols t and q respectively. Suppose* $\mathrm{WF}(Q) \cap \{(x, \xi) \in T^*(\mathbf{R}^n) \setminus 0;\ \xi' = 0\} = \emptyset$. *Then QT and TQ belong to $L^{m'+m''}(\mathbf{R}^n)$. Their symbols are asymtotically $\sum q^{(\alpha)}(x, \xi)\, D_x^\alpha t(x, \xi')/\alpha'!$ and $\sum t^{(\alpha')}(x, \xi')\, D_{x'}^{\alpha'} q(x, \xi)/\alpha'!$ respectively. Here $q^{(\alpha)} = (\partial/\partial\xi)^\alpha q$ and $t^{(\alpha')} = (\partial/\partial\xi')^{\alpha'} t$.*

Proof. By the same argument as used in [8] to prove the composition formula for pseudodifferential operators, we are allowed to apply Q under the sign of integration in (A.1) and get:

$$QTu(x) = \int s(x, \xi)\, e^{i\langle x, \xi\rangle} \hat{u}(\xi)\, d\xi/(2\pi)^n, \quad u \in C_0^\infty(\mathbf{R}^n),$$

where $s(x, \xi) = e^{-i\langle x, \xi\rangle} Q(x, D_x)(t(x, \xi')\, e^{i\langle x, \xi\rangle})$. Wellknown estimates (see for instance [8] th. 2.6) for the expression

$$e^{-i\langle x,\xi\rangle}Q(x,D_x)\,(v(x)\,e^{i\langle x,\xi\rangle})-\sum_{|\alpha|<N}q^{(\alpha)}(x,\xi)\,D_x^\alpha v(x)/\alpha!$$

show that $$s(x,\xi)-\sum_{|\alpha|<N}q^{(\alpha)}(x,\xi)\,D_x^\alpha t(x,\xi')/\alpha! = O(|\xi|^{-N+|m'|+m''}),\quad \xi\to\infty,$$

uniformly when $x\in K\subset\subset\mathbf{R}^n$.

Since Q is continuous $C^\infty(\mathbf{R}^n)\to C^\infty(\mathbf{R}^n)$, we see that for all $K\subset\subset\mathbf{R}^n$ and multiindices α and β, there exists $M=M_{\alpha\beta K}\in\mathbf{R}$, such that $D_x^\alpha D_\xi^\beta s(x,\xi)=O(|\xi|^M)$, $\xi\to\infty$, uniformly when $x\in K$.

Since by assumption $q(x,\xi)$ is rapidly decreasing as a function of ξ in a conic neighbourhood of $\{(x,\xi)\in T^*(\mathbf{R}^n)\setminus 0;\ \xi'=0\}$, we have $q^{(\alpha)}D_x^\alpha t\in S_{10}^{m'+m''-|\alpha|}(\mathbf{R}^n\times\mathbf{R}^n)$.

Combination of these three observations with Theorem 2.9 in [8] gives that $s\in S_{10}^{m'+m''}(\mathbf{R}^n\times\mathbf{R}^n)$ and $s\sim\Sigma q^{(\alpha)}D_x^\alpha t/\alpha!$. This proves all the statements about QT.The statements about TQ can be proved similarly and we omit the details.

PROPOSITION A.2. *Let $Q\in L^m(\mathbf{R}^n)$ be such that $\mathrm{WF}(Q)\cap\{(x,\xi)\in T^*(\mathbf{R}^n)\setminus 0;\xi'=0\}=\emptyset$. Then Q is continuous $H_{(r,s)}^{\mathrm{comp}}\to H_{(r-m+N,s-N)}^{\mathrm{loc}}$ for all r, s, $N\in\mathbf{R}$.*

Proof. For all $r\in\mathbf{R}$ let $\Lambda_{(r,0)}\in L^r(\mathbf{R}^n)$ and $\Lambda_{(0,r)}\in T^r(\mathbf{R}^n)$ be properly supported convolution operators with symbols asymtotically equal to $(1+|\xi|)^r$ and $(1+|\xi'|)^r$ respectively. Put $\Lambda_{(r,s)}=\Lambda_{(r,0)}\Lambda_{(0,s)}$. Then

$$\Lambda_{(r,s)}\text{ is continuous }H_{(\mu,\nu)}^{\mathrm{loc(comp)}}\to H_{(\mu-r,\nu-s)}^{\mathrm{loc(comp)}}\text{ for all }\mu,\nu\in\mathbf{R}. \tag{A.2}$$

Proposition A.1 shows that $Q'=\Lambda_{(r-m+N,s-N)}Q\Lambda_{(-r,-s)}$ belongs to $L^0(\mathbf{R}^n)$ and that

$$Q-\Lambda_{(-(r-m+N),-(s-N))}Q'\Lambda_{(r,s)}\text{ has }C^\infty\text{ kernel.} \tag{A.3}$$

Since $Q'\in L^0(\mathbf{R}^n)$ it is continuous $H_{(0,0)}^{\mathrm{comp}}\to H_{(0,0)}^{\mathrm{loc}}$. Now the proposition follows if we combine this with (A.3) and (A.2).

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