OPERATORS WHICH ARE ANNIHILATED BY ANALYTIC FUNCTIONS AND INVARIANT SUBSPACES

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1. Introduction and main results

In what follows the term operator will mean a bounded linear operator T which acts on some infinite dimensional complex Banach space **B**. The spectrum of T will be denoted by $\sigma(T)$. An invariant subspace for T will always mean here a closed subspace $M\subset \mathbf{B}$ such that $TM\subset M$. We say that M is hyperinvariant for T if it is invariant for every operator that commutes with T. A subspace M is called nontrivial if $M\neq\{0\}$ and $M\neq\mathbf{B}$.

J. Wermer [27] proved that if T is an invertible operator which satisfies

$$\sum_{n=-\infty}^{\infty}\frac{\log\|T^n\|}{1+n^2}<\infty \tag{1}$$

and $\sigma(T)$ contains more than one point then T has a nontrivial hyperinvariant subspace. In the case where $\sigma(T)$ consists of a single point λ_0, the existence of nontrivial invariant subspaces was proved in [27] only for invertible operators T which satisfy $\|T^n\|=O(|n|^k)$, $n\to\pm\infty$, for some integer $k\geqslant 0$, by noticing that in this case, a theorem of Hille [14] (see also [15, p. 60]) implies that $(T-\lambda_0 I)^{k+1}=0$ where I is the identity operator.

We prove the following result which is also valid in the case where $\sigma(T)$ consists of a single point:

THEOREM 1. *Let T be an invertible operator which satisfies the following conditions:*

$$\|T^n\|=O(n^k),\ n\to\infty \quad \textit{for some integer} \quad k\geqslant 0 \tag{2}$$

$$\|T^n\|=O(\exp c|n|^{\frac{1}{2}}),\ n\to-\infty \quad \textit{for some constant} \quad c>0. \tag{3}$$

Then T has a nontrivial invariant subspace. Moreover, if T is not a scalar multiple of the identity operator, then T admits a nontrivial hyperinvariant subspace. If $\sigma(T)$ consists of a *single point λ_0 then either $(T-\lambda_0 I)^{k+1}=0$ or T admits an uncountable chain of hyperinvariant subspaces.*

If $\sigma(T)$ contains more than one point, then the conclusion of Theorem 1 follows from the above mentioned theorem of Wermer. Actually Wermer does not state in [27] the existence of hyperinvariant subspaces but this follows easily from his proof. This fact is also proved in [6, p. 154, Corollary 3.3]. We also give a short proof of Wermer's theorem in section 6 which establishes this fact. In the case that $\sigma(T)$ consits of a single point, we prove the existence of invariant subspaces by showing first that there exists a function $f \not\equiv 0$ bounded and analytic in the open unit disc, such that $f(T)=0$. This fact is also true in the more general case when $\sigma(T)$ consists of finitely many points. More precisely we have:

THEOREM 2. *Let T be an operator whose spectrum $\sigma(T)$ is a finite set and which satisfies conditions* (2) *and* (3). *Then there exists an analytic function $f(z)=\sum_{n=0}^{\infty} a_n z^n$, $f\not\equiv 0$ such that $\sum_{n=0}^{\infty}|a_n|n^k<\infty$ and $f(T)=\sum_{n=0}^{\infty} a_n T^n=0$. More concretely if $\sigma(T)=\{z_1, \ldots, z_\nu\}$, one can choose f to be the function*

$$f(z)=\prod_{j=1}^{\nu}(z-z_j)^m \exp\left(a\sum_{q=1}^{\nu}\frac{z+z_q}{z-z_q}\right)$$

with $m=4k+5$ and $a=2c^2$ where c is the constant in (3).

We note here, that using the spectral radius formula, one can easily see that if T is an invertible operator which satisfies (1), then $\sigma(T)$ is contained in the unit circle. Thus in Theorem 2, we have $|z_j|=1$, $j=1, \ldots, \nu$.

If $\sigma(T)$ consists of a single point λ_0, and T satisfies (2) and (3), it is not true in general that $T-\lambda_0 I$ is nilpotent. However if we replace (3) by the stronger condition:

$$\|T^n\| = O(\exp \varepsilon |n|^{\frac{1}{2}}), \quad n\to -\infty \quad \text{for every } \varepsilon>0, \tag{4}$$

this is the case.

More generally we can deduce from Theorem 2 the following:

COROLLARY 1. *Let T be an invertible operator such that $\sigma(T)$ is a finite set $\{z_1, \ldots, z_\nu\}$, and assume that T satisfies condition* (2) *and condition* (4) (*for every $\varepsilon>0$*). *Then if p denotes the polynomial $p(z)=\prod_{j=1}^{\nu}(z-z_j)^{k+1}$, we have that $p(T)=0$; that is, T is an algebraic operator*. (For such operators, see [24, p. 63].)

This extends the above mentioned theorem of Hille [14] and is used in section 6 to obtain and extension of a theorem of Nagy, Foiaş, and Colojoară [22, p. 54] and [6, p. 134].

In trying to extend Theorem 1 to operators T which satisfy less restrictive conditions than (3) by producing a nontrivial analytic functions f such $f(T)=0$, we found, somewhat surprisingly, that condition (3) is necessary for the existence of such a function. Moreover, it turns out that condition (3) is a necessary condition for the existence of such an annihilating functions for a large class of operators, including all C_0 operators in Hilbert space [22, p. 114] whose spectrum is contained in the unit circle.

Before stating our results in this direction we introduce some notations and definitions.

If f is an analytic function in the open unit disc $D=\{z: |z|<1\}$ and $w\in D$, we shall denote by $L_w f$ the analytic function in D defined by

$$L_w f(z)=\frac{f(z)-f(w)}{z-w}, \quad z\in D\setminus\{w\}.$$

Definition 1. A Banach space $\mathbf{A}$ which consists of analytic functions in D will be called admissible if:

(i) For every $w\in D$ the evaluation map $f\to f(w)$, $f\in\mathbf{A}$, is a bounded functional on $\mathbf{A}$.

(ii) For every $f\in\mathbf{A}$ and $w\in D$, $L_w f\in\mathbf{A}$.

If $\mathbf{A}$ is an admissible Banach space, we shall denote, for every $w\in D$, by L_w the operator on $\mathbf{A}$ defined by $f\to L_w f$, $f\in\mathbf{A}$.

We give now some examples of admissible Banach spaces, some of which, will be used in the sequel.

1. Let H^∞ denote as usual the space of bounded analytic functions on D, and for a positive integer k, let H_k^∞ denote the space of functions f in H^∞ such that $f^{(j)}\in H^\infty$, $j=1, 2, \dots, k$. We also adopt the notation $H_0^\infty=H^\infty$. With norm

$$\|f\|=\sum_{j=0}^{k}\frac{\|f^{(j)}\|_\infty}{j!}$$

(where $\| \|_\infty$ denotes the sup norm on D) the space H_k^∞ forms an admissible Banach space, and it is easy to verify, that in this case, $\|L_w\|=O((1-|w|)^{-k-1})$, $|w|\to 1-$.

2. The usual Hardy spaces on D, H^p, $1\leqslant p\leqslant\infty$, are admissible Banach spaces and for all of them,

$$\|L_w\|\leqslant\frac{3}{(1-|w|)^2}, \quad w\in D.$$

3. If $(p_n)_{n=0}^\infty$ is an increasing sequence of positive numbers, then the space of all functions $f(z)=\sum_{n=0}^\infty a_n z^n$, $z\in\bar{D}$ such that $\|f\|=\sum_{n=0}^\infty |a_n| p_n<\infty$, is an admissible Banach space and a simple computation shows that in this case $\|L_w\|\leqslant(1-|w|)^{-1}$, $\forall w\in D$.

If $f(z)=\sum_{n=0}^{\infty} a_n z^n$, $z\in D$, is an analytic function in D, and T is an operator with $\sigma(T)\subseteq \bar{D}$, then the spectral radius formula implies that for every $0<\varrho<1$, the series $\sum_{n=0}^{\infty} a_n \varrho^n T^n$ is convergent in the operator norm, to an operator which we denote by $f(\varrho T)$. If $\lim_{\varrho\to 1-} f(\varrho T)$ exists in the strong operator topology, we shall denote this operator by $f(T)$.

Definition 2. If $\mathbf{A}$ is an admissible Banach space and T is an operator with $\sigma(T)\subseteq \bar{D}$, acting on a Banach space $\mathbf{B}$, we shall say that $\mathbf{A}$ operates on T if $\lim_{\varrho\to 1-} f(\varrho T)$ exists for every f in $\mathbf{A}$, and the mapping $f\to f(T)$ from $\mathbf{A}$ into $L(\mathbf{B})$ (the Banach space of all bounded linear operators on $\mathbf{B}$) is bounded.

Definition 3. If $\mathbf{A}$ is an admissible Banach space which operates on an operator T, we shall say that T is in $C_0(\mathbf{A})$, if there exists a function $f\not\equiv 0$, $f\in\mathbf{A}$, such that $f(T)=0$.

Remark. According to Definition 3, every C_0 contraction in Hilbert space [22, p. 114] is in $C_0(H^\infty)$.

We recall that a function f which is analytic in D is said to be of bounded characteristic, if

$$\sup_{0<r<1}\int_{-\pi}^{\pi}\log^+|f(re^{i\theta})|\,d\theta<\infty.$$

THEOREM 3. *Let* $\mathbf{A}$ *be an admissible Banach space, and* T *an operator in* $C_0(\mathbf{A})$. *Then:*

(a) *If the set* $W=\sigma(T)\cap D$ *is not empty, it is at most countable and consists of eigenvalues of* T.

(b) $$\text{If}\quad \|L_w\|=O\left(\exp\frac{c}{(1-|w|)^k}\right),\quad |w|\to 1-$$

for some constants $c>0$, $k>0$, *and*

$$\sum_{n=0}^{\infty}\frac{\log\|T^n\|}{1+n^2}<\infty,$$

then:

(b_1) *If* $\sigma(T)$ *is contained in the unit circle* ∂D, *then* T *is a* $\mathfrak{U}$ *unitary operator in the sense of* [6, *p.* 127], *thus in particular* T *is decomposable* [6, *p.* 30].

(b_2) *If* $\sigma(T)$ *contains more than one point, then* T *admits a non-trivial hyperinvariant subspace.*

(c) *If* $\mathbf{A}$ *consists of functions of bounded characteristic and*

$$\|L_w\|=O\left(\exp\frac{c}{1-|w|}\right),\quad |w|\to 1-$$

for some constant $c>0$, *then:*

(c_1) *If* $\sigma(T)$ *is contained in* ∂D, *then* T *satisfies condition* (3).

(c_2) *If the set* $W=\sigma(T)\cap D$ *is not empty, it is at most countable, consists of eigenvalues of* T *and* $\sum_{\lambda\in W}(1-|\lambda|)<\infty$.

(d) *If* $\mathbf{A}$ *satisfies the hypotheses of* (c) *and* $\|T^n\|=O(n^k)$, $n\to\infty$ *for some integer* $k\geqslant 0$, *then* T *admits a non-trivial invariant subspace.*

COROLLARY 2. *If* T *is an operator in* $C_0(H_k^\infty)$ (*for some* $k\geqslant 0$) *then:*

(a) T *admits a non-trivial invariant subspace.*

(b) *If the set* $W=\sigma(T)\cap D$ *is not empty, it is at most countable, consists of eigenvalues of* T *and* $\sum_{\lambda\in W}(1-|\lambda|)<\infty$.

(c) *If* $\sigma(T)\subseteq\partial D$, *then* T *satisfies condition* (3), *and is* $\mathfrak{U}$ *unitary* (*thus in particular decomposable*).

Remarks. 1. For C_0 contractions in Hilbert space, conclusions (a) and (b) of Corollary 2 are proved in [18, Cor. 5.2 and Th. 6.3] by methods which are different from ours. These methods are not applicable to the classes $C_0(H_k^\infty)$ for $k>0$.

2. By conclusion (c) of Corollary 2, every C_0 operator with spectrum in ∂D is $\mathfrak{U}$ unitary. In this connection we mention that C. Foiaş [10] proved that all C_0 operators are decomposable.

The following is an immediate consequence of Theorem 3 and Theorem 1:

COROLLARY 3. *Let* T *be an operator with* $\sigma(T)\subseteq\bar{D}$ *and let* $(p_n)_{n=0}^\infty$ *be an increasing sequence of positive numbers such that* $\|T^n\|=O(p_n)$, $n\to\infty$. *Suppose that there exists a function* $f(z)=\sum_{n=0}^\infty a_n z^n$, $z\in\bar{D}$, $f\not\equiv 0$ *such that* $\sum_{n=0}^\infty |a_n|p_n<\infty$ *and* $f(T)=0$. *Then:*

1) T *satisfies the conclusions of parts* (a) *and* (c) *in Theorem* 3.

2) *If* $p_n=O(n^k)$, $n\to\infty$ *for some integer* $k\geqslant 0$, *then* T *admits a non-trivial invariant subspace.*

3) *If* $\sum_{n=1}^\infty \log\|T^n\|/n^2<\infty$ *and* $\sigma(T)$ *contains more than one point, then* T *has a non-trivial hyperinvariant subspace.*

Under some assumptions on an admissible Banach space $\mathbf{A}$, one can deduce that if T is a $C_0(\mathbf{A})$ operator, then $\sigma(T)\cap\partial D$ must be in a certain sense a thin set. This is the content of the next result.

THEOREM 4. *Let* $\mathbf{A}$ *be an admissible Banach space, such that every function* f *in* $\mathbf{A}$ *has a continuous extension to* $\bar{D}$. (*We shall denote this extension also by* f.) *Assume also that the polynomials* (*in* z) *are dense in* $\mathbf{A}$ *and that for every* $w\in\bar{D}$ *the evaluation map* $f\to f(w)$, $f\in A$, *is a bounded functional on* $\mathbf{A}$. *If* T *is in* $C_0(\mathbf{A})$, *then the set* $E=\sigma(T)\cap\partial D$ *is of measure* *zero (with respect to Lebesque measure on ∂D). If in addition the functions in* **A** *satisfy a Lipshitz condition on ∂D, then E is a Carleson set (that is, $\int_{-\pi}^{\pi} |\log \varrho(e^{i\theta}, E)|\, d\theta < \infty$ where $\varrho(e^{i\theta}, E)$ denotes the distance of $e^{i\theta}$ from E).*

COROLLARY 4. *If T satisfies the hypotheses of Corollary 3, then $\sigma(T) \cap \partial D$ is of measure zero.*

The organization of the paper is as follows:

In section 2, we prove the results stated in this section.

In section 3, we prove some lemmas which are used to prove the theorems in section 2.

In section 4, we consider invertible operators T such that $\sigma(T) = \{1\}$ and $\|T^n\| = O(\exp c|n|^{\alpha})$, $n \to \pm\infty$ for some $c > 0$ and $0 < \alpha < 1$. We show that the problem of existence of nontrivial hyperinvariant subspaces for these classes is equivalent to their existence for a single class determined by some $\alpha_0 \in (0, 1)$. We also show that if T satisfies the above conditions for some $0 < \alpha < \frac{1}{2}$, then T admits a nontrivial analytic annihilating function. However, we are not able to deduce from this fact the existence of nontrivial invariant subspaces for these operators.

In section 5, we prove the existence of nontrivial invariant subspaces for some classes of quasinilpotent operators, by using the results of Section 1 and Section 4. We obtain, in particular, an extension of a result of Isaev [17].

In section 6, we extend a result of Nagy, Foiaş, and Colojoară mentioned before, and give a short proof of Wermer's Theorem [27].

In section 7, we use the results of sections 1 and 4 to prove some results on closed primary ideals and restriction algebras of some Banach algebras of continuous functions on the unit circle. We thereby obtain in particular an extension of the results of [18] and [1].

In Section 8, we mention some extensions of the results of section 1 and pose some problems.

For various results on invariant subspaces for operators on Hilbert space, including extensions of the above mentioned result of Wermer, we refer to [24]. The basic reference for results on contractions on Hilbert space is [22]. For results on invariant subspaces for operators on Banach spaces, which are related to the results of this paper, we refer to [6].

2. Proofs of main results

We shall now state several lemmas and use them to prove the theorems stated in section 1. In section 3 we shall prove the lemmas.

If T is an operator we shall denote its resolvent $(T - zI)^{-1}$, $z \notin \sigma(T)$, as usual by $R(T, z)$.

LEMMA 1. *Let T be an operator with $\sigma(T)\subseteq \bar{D}$ and assume that $\mathbf{A}$ is an admissible Banach space which operates on T. Let $f\in\mathbf{A}$ and set $G(T,z)=(L_z f)(T)$, $z\in D$. Then the identity*

$$f(T)\,R(T,z)-f(z)\,R(T,z)=G(T,z) \tag{5}$$

holds for every $z\in D\setminus\sigma(T)$.

LEMMA 2. *Let T be an operator with $\sigma(T)\subseteq\partial D$. Then*

(a) *If there exist constants $c>0$ and $0<\alpha<1$ such that*

$$\|T^{-n}\| = O\,(\exp\,(cn^{\alpha})),\quad n\to\infty. \tag{6}$$

Then

$$\|R(T,z)\| = O\left(\exp\frac{d}{(1-|z|)^{\beta}}\right),\quad |z|\to 1- \tag{7}$$

where $\beta=\alpha/(1-\alpha)$ and $d=4^{\beta}c^{\alpha/\beta}$.

(b) *If (7) holds with some constants $d>0$ and $\beta>0$, then (6) holds with $\alpha=\beta/(1+\beta)$ and $c=3d^{\alpha/\beta}$.*

Remark. Part (b) of Lemma 2 is proved in [6, p. 155] with different relations between the constants c and d. Since we will need in the sequel the more precise relation between these constants, we shall include the proof.

LEMMA 3. *Let E be a finite set contained in ∂D, and let φ be an analytic function on $\mathbf{C}\setminus E$ with values in some Banach space. Assume that there exist constants $M>0$, $N>0$, $K>0$ and $d>0$ such that*

$$\|\varphi(z)\|\leqslant M\,\exp\frac{d}{1-|z|},\quad z\in D\quad \textit{and} \tag{8}$$

$$\|\varphi(z)\|\leqslant K(|z|-1)^{-N},\ \textit{for}\ |z|>1. \tag{9}$$

Then there exist constants $M_1>0$, $K_1>0$ and $b>0$ such that

$$\|\varphi(z)\|\leqslant M_1\exp\,\{b\varrho(z,E)^{-1}\},\ z\in D\quad \textit{and} \tag{10}$$

$$\|\varphi(z)\|\leqslant K_1\varrho(z,E)^{-2N},\ |z|>1. \tag{11}$$

LEMMA 4. *Let E and φ be as in the statement of Lemma 3. Assume that φ satisfies condition (10) and also:*

$$\sup\,\{\|\varphi(e^{i\theta})\|,\,e^{i\theta}\notin E\}<\infty\quad \textit{and} \tag{12}$$

$$\sup_{0<r<1}\|\varphi(rw)\|<\infty\quad \textit{for all}\ w\in E. \tag{13}$$

Then $\sup_{|z|<1}\|\varphi(z)\|<\infty$.

LEMMA 5. *Let Φ be an analytic function in D with values in some Banach space* **B** *and let $f \not\equiv 0$ be a (complex) analytic function in D.*

Denote:

$$K(r) = \sup_{0 \leqslant \theta \leqslant 2\pi} \|f(re^{i\theta})\Phi(re^{i\theta})\|, \quad 0<r<1$$

and

$$m(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta, \quad 0<r<1.$$

Then:

There exists a constant $c>0$ such that

$$\text{(a)} \qquad \|\Phi(z)\| = O\left(K\left(\frac{1+|z|}{2}\right)\exp\left\{\frac{6m\left(\frac{1+|z|}{2}\right)+c}{1-|z|}\right\}\right), \quad |z|\to 1-.$$

Suppose in addition that $f \in H^\infty$, and let μ be the singular measure on $[-\pi, \pi)$ which defines the singular inner part of f [16, p. 67]. *Let ν denote the discrete part of μ and set*

$$V(z) = \exp\left\{\int \frac{z+e^{it}}{z-e^{it}}\, d\nu(t)\right\}, \quad z\in D.$$

Then for every $\varepsilon>0$,

$$\text{(b)} \qquad \|\Phi(z)\| = O\left(K\left(\frac{1+|z|}{2}\right)|V(z)|^{-1}\exp\frac{\varepsilon}{1-|z|}\right), \quad |z|\to 1-$$

and

$$\text{(c)} \qquad \|\Phi(z)\| = O\left(K\left(\frac{1+|z|}{2}\right)\exp\left\{\frac{2\|\nu\|+\varepsilon}{1-|z|}\right\}\right), \quad |z|\to 1-$$

where $\|\nu\|$ denotes the total variation of ν.

We shall prove first Theorem 2 and then use it to prove Theorem 1. First, we introduce some notations and make some preliminary observations.

For an integer $k \geqslant 0$ we shall denote by B_k the Banach space of analytic functions $f(z) = \sum_{n=0}^\infty a_n z^n$, $z \in D$, such that $\sum_{n=0}^\infty |a_n|(n+1)^k < \infty$, the latter being the norm of f in B_k. It is known and easy to verify that B_k are Banach algebras (with respect to pointwise multiplication) and that the continuous imbeding $H_{k+1}^\infty \subset B_k$ holds for every integer $k \geqslant 0$.

If T is an operator acting on a Banach space **B** and $\|T^n\| = O(n^k)$, $n\to\infty$, for some integer $k\geqslant 0$, it is clear that B_k operates on T, and that the mapping $f \to f(T)$, $f\in B_k$, establishes a continuous homomorphism of the Banach algebra B_k into the Banach algebra of bounded linear operators on **B**.

Let k be an integer, and for every $a \geqslant 0$, let ψ_a denote the function on D defined by

$$\psi_a(z) = (z-1)^{2k+3}\exp\left\{a\frac{z+1}{z-1}\right\}, \quad z\in D.$$

It is easy to verify that $\psi_a \in H^\infty_{k+1}$ and that for every $b \geq 0$, $\lim_{a\to b} \psi_a = \psi_b$ in the norm of H^∞_{k+1} and therefore also in the norm of B_k. If $w \in \partial D$, then the same remarks apply to the functions $\psi_{a,w}$ defined by: $\psi_{a,w}(z) = \psi_a(wz)$, $z \in D$. We shall use these facts in the following proof.

Proof of Theorem 2. Let T be an operator which satisfies the hypotheses of Theorem 2. Assume that $\sigma(T) = \{z_1, ..., z_\nu\} = E$. Set $m = 4k+5$ (where k is the integer that appears in (2)) and for every $a \geq 0$ denote by f_a the function defined by

$$f_a(z) = \prod_{j=1}^{\nu} (z - z_j)^m \exp\left\{a \sum_{q=1}^{\nu} \frac{z + z_q}{z - z_q}\right\}.$$

It follows from the preceding remarks that f_a is in B_{2k} and that $\lim_{a\to b} f_a(T) = f_b(T)$ in the operator norm. Let c be the constant that appears in (3). We claim that if $a \geq 2c^2$, then $f_a(T) = 0$. It is easy to see that condition (2) implies (see [6, p. 132]) that there exists a constant $C > 0$ such that for $|z| > 1$,

$$\|R(T, z)\| \leq C(|z| - 1)^{-k-1}. \tag{14}$$

Condition (3) and Lemma 2 imply that there exists a constant $K > 0$ such that

$$\|R(T, z)\| \leq K \exp\{4c^2/(1 - |z|)\}, \quad z \in D, \tag{15}$$

and therefore using Lemma 3, we deduce that there exist constants $M_1 > 0$, $K_1 > 0$ and $b > 0$ such that

$$\|R(T, z)\| \leq M_1 \exp(b\varrho(z, E)^{-1}) \quad \text{and} \tag{16}$$

$$\|R(T, z)\| \leq K_1 \varrho(z, E)^{-2k-2}. \tag{17}$$

Noticing that $|f_a(rw)| \leq 2^m \exp(2a(r-1)^{-1})$, $0 < r < 1$, for all $w \in E$ and

$$|f_a(e^{i\theta})| \leq \textstyle\prod_{j=1}^{\nu} |e^{i\theta} - z_j|^m, \quad 0 < \theta \leq 2\pi$$

and remembering that $a \geq 2c^2$, we deduce from (17) and (15) that

$$\sup\{\|f_a(e^{i\theta}) R(T, e^{i\theta})\| : e^{i\theta} \notin E\} < \infty \text{ and}$$
$$\sup\{\|f_a(rw) R(T, rw)\| : 0 < r < 1, w \in E\} < \infty,$$

and therefore, taking into account (16) and using Lemma 4 we obtain that

$$\sup_{|z|<1} \|f_a(z) R(T, z)\| < \infty. \tag{18}$$

Let $f(z)=\sum_{n=0}^{\infty}a_n z^n$, $z\in D$, be a function in B_{k+1}. A simple computation shows that for every $w\in D$ and $z\in D$, $L_w f(z)=\sum_{n=0}^{\infty} g_n(z)w^n$ where $g_n(z)=\sum_{j=0}^{\infty} a_{n+j+1}z^j$, $n=0, 1, \dots$

Therefore

$$\|L_w f\|_{B_k}\leqslant \sum_{n=0}^{\infty}\|g_n\|_{B_k}\leqslant \sum_{j=0}^{\infty}|a_j|(j+1)^{k+1}=\|f\|_{B_{k+1}}$$

and conseqeuently $\sup_{w\in D}\|(L_w f)(T)\|<\infty$.

Thus using Lemma 1 we obtain that

$$\sup_{|z|<1}\|f(T)R(T,z)-f(z)R(T,z)\|<\infty$$

and applying this to $f=f_a$ we deduce from (18) that

$$\sup_{|z|<1}\|f_a(T)R(T,z)\|<\infty. \tag{19}$$

To obtain a similar estimate for $1<|z|<2$, we define $u(z)=\prod_{j=1}^{\nu}(z-z_j)^{2k+2}$ and

$$h(z)=\prod_{j=0}^{\nu}(z-z_j)^{2k+3}\exp\left\{a\sum_{q=1}^{\nu}\frac{z+z_q}{z-z_q}\right\}.$$

Then $h\in B_k$, $f_a(z)=u(z)h(z)$, and $f_a(T)=u(T)h(T)$. Using identity (5) with u we obtain:

$$u(T)R(T,z)=u(z)R(T,z)+Q(T,z)$$

where $Q(T,z)$ is a polynomial in z (with operator coefficients). Multiplying both sides by $h(T)$ we obtain that

$$f_a(T)R(T,z)=h(T)u(z)R(T,z)+h(T)Q(T,z).$$

Thus taking into consideration (17) and the definition of u, we obtain that

$$\sup\{\|f_a(T)R(T,z)\|:\ 1<|z|<2\}<\infty.$$

Combining this with (19), we see that

$$\sup\{\|f_a(T)R(T,z)\|:\ |z|<2,\ z\notin E\}<\infty.$$

Therefore the operator valued analytic function $z\to f_a(T)R(T,z)$, $z\notin\sigma(T)$, has only removable singularities at the points of $\sigma(T)$, and remembering that $\lim_{|z|\to\infty}\|R(T,z)\|=0$, we deduce from Liouville's Theorem for vector valued functions [15, p. 100] that $f_a(T)R(T,z)\equiv 0$, and consequently $f_a(T)=0$. This completes the proof of Theorem 2.

Proof of Corollary 1. If T satisfies condition (4), then by Lemma 2, we obtain that for every $\varepsilon>0$, there exists a constant $M_\varepsilon>0$ such that $\|R(T,z)\|\leqslant M_\varepsilon\exp(\varepsilon/(1-|z|))$, $z\in D$, and therefore by Theorem 2, $f_a(T)=0$ for every $a>0$. Thus, by the remarks in the beginning of the proof of Theorem 2, we see that $\lim_{a\to 0+} f_a(T)=f_0(T)=0$ where $f_0(z)=\prod_{j=0}^{\nu}(z-z_j)^m$. Using identity (5) with f_0, we obtain that $f_0(z)R(T,z)$ is a polynomial in z (with operator coefficients) and therefore $R(T,z)$ is a rational function with poles at the points $z_1, \ldots, z_\nu$. Remembering that by (2), $\|R(T,z)\|=0((|z|-1)^{-k-1})$, $|z|\to 1+$, we obtain that $R(T,z)$ has poles of order not exceeding $k+1$, at $z_1, \ldots, z_\nu$, and therefore if $p(z)=\prod_{j=1}^{\nu}(z-z_j)^{k+1}$, then $p(z)R(T,z)$ is a polynomial in z, and using identity (5) with p we obtain that $p(T)R(T,z)$ is a polynomial, and as in the proof of Theorem 2, we conclude that $p(T)=0$.

Proof of Theorem 1. As already remarked after the statement of the theorem, it suffices to consider the case where $\sigma(T)$ consists of a single point λ_0. Since $|\lambda_0|=1$, conditions (2) and (3) are not changed if T is replaced by $\lambda_0^{-1}T$, and we may assume, as we shall, that $\sigma(T)=\{1\}$. For $a\geqslant 0$, let f_a be the function defined in the proof of Theorem 2, with $\nu=1$, and $z_1=1$, that is,

$$f_a(z)=(z-1)^m \exp\left\{a\frac{z+1}{z-1}\right\}$$

where $m=4k+5$. By Theorem 2 we know that $f_a(T)=0$, if $a\geqslant 2c^2$. Let $\alpha=\inf\{a>0: (T-I)^m f_a(T)=0\}$. Since $\lim_{a\to\alpha} f_a(T)=f_\alpha(T)$, we obtain that $(T-I)^m f_\alpha(T)=0$. Therefore if $\alpha=0$, $(T-I)^{2m}=0$, (and by Corollary 1, also $(T-I)^{k+1}=0$). Thus in this case, either $T=I$, or $T-I$ is a nontrivial nilpotent operator and its kernel is a nontrivial hyperinvariant subspace for T. Assume now that $\alpha>0$. Since $f_a(z)f_b(z)=(z-1)^m f_{a+b}(z)$, for all $a\geqslant 0$, $b\geqslant 0$, we obtain from the definition of α that

$$f_a(T)f_b(T)\neq 0 \quad \text{if} \quad a+b<\alpha \tag{20}$$

and

$$f_a(T)f_b(T)=0 \quad \text{if} \quad a+b\geqslant\alpha. \tag{21}$$

For every $0<s<\alpha$, let M_s denote the kernel of the operator $f_s(T)$. It is clear that M_s is a hyperinvariant subspace for T. From (20) we see that $M_s\neq B$ and from (21) we see that M_s contains the range of $f_{\alpha-s}(T)$ which is not $\{0\}$, since $\alpha-s<\alpha$. Thus M_s, $0<s<\alpha$, are nontrivial hyperinvariant subspaces for T. Also if $0<a<b<\alpha$, if follows from (20) and (21) that the range of $f_{\alpha-b}(T)$ is contained in M_b but not in M_a, thus $M_a\neq M_b$. To show that the subspaces M_s, $0<s<\alpha$, form an uncountable chain, it remains to show that if $0<a<b<\alpha$ then $M_a\subset M_b$. Assume that $0<a<b<\alpha$ and that $v\in M_a$, that is $f_a(T)v=0$. The argument in the proof of Theorem 2 shows that if $G_a(T,z)$ is the function which is associated with f_a by Lemma 1, then $\sup_{|z|<1}\|G_a(T,z)\|<\infty$, and therefore using (5) with

f_a, we see that the hypothesis $f_a(T)v=0$, implies that $\sup_{|z|<1}\|f_a(z)R(T,z)v\|<\infty$. Since $a<b$, we have $|f_b(z)|\leqslant|f_a(z)|$, $z\in D$, and therefore $\sup_{|z|<1}\|f_b(z)R(T,z)v\|<\infty$. Using (5) with f_b, and the fact that $\sup_{|z|<1}\|G_b(T,z)\|<\infty$ (where $G_b(T,z)$ is the function associated with f_b by Lemma 1), we obtain that $\sup_{|z|<1}\|f_b(T)R(T,z)v\|<\infty$. Repeating the argument (after (19)) in the proof of Theorem 2, (with $f_a(T)$ replaced by $f_b(T)v$) we conclude that $f_b(T)v=0$, thus $v\in M_b$. This shows that $M_a\subset M_b$, and completes the proof of Theorem 1.

Proof of Theorem 3.

Proof of (a). Let $f\not\equiv 0$ be a function in $\mathbf{A}$ such that $f(T)=0$. Since $\mathbf{A}$ is an admissible Banach space which operates on T, the identity

$$(\lambda-z)L_\lambda f(z)=f(\lambda)-f(z),\quad \lambda, z\in D$$

implies that for every $\lambda\in D$,

$$(\lambda I-T)(L_\lambda f)(T)=f(\lambda)I;$$

therefore, if $f(\lambda)\neq 0$, then $\lambda\notin\sigma(T)$. Thus $f=0$ on W, and since f is analytic in D and $f\not\equiv 0$, W is at most countable. To show that W consists of eigenvalues of T, let f be as above and $\lambda\in W$. Then λ is a zero of f, and denoting its order by s, we have that

$$f(z)=(z-\lambda)^s g(z),\quad z\in D$$

where g is an analytic function on D such that $g(\lambda)\neq 0$. Since $\mathbf{A}$ is admissible, $g\in\mathbf{A}$, and the identity above implies that

$$(T-\lambda I)^s g(T)=f(T)=0. \tag{22}$$

Since $\lambda\in\sigma(T)$ and $g(\lambda)\neq 0$, it follows from the first part of the proof that $g(T)\neq 0$, and therefore by (22), λ is an eigenvalue of T.

Proof of (b_1). Since the functional $f\to f(0)$, $f\in\mathbf{A}$ is bounded, there exists a constant $M>0$ such that for every $f\in\mathbf{A}$ and $w\in D$

$$|f(w)|=|w|\,|L_w f(0)|\leqslant M\|f\|\,\|L_w\|.$$

Therefore, by the hypothesis on $\|L_w\|$

$$|f(w)|=O\left(\exp\frac{c}{(1-|w|)^k}\right),\quad |w|\to 1- \tag{23}$$

for every $f \in \mathbf{A}$. The hypothesis on $\|L_w\|$ and Lemma 1, imply that, if $\sigma(T) \subseteq \partial D$, $f \in \mathbf{A}$ and $f(T)=0$, then

$$\|f(z)\,R(T, z)\| = O\left(\exp \frac{c}{(1-|z|)^k}\right), \quad |z| \to 1-$$

and therefore if $f \not\equiv 0$, we obtain from Lemma 5 and (23) that

$$\|R(T, z)\| = O\left(\exp \frac{b}{(1-|z|)^{k+1}}\right), \quad |z| \to 1-$$

for some constant $b>0$. Thus by Lemma 2 there exists a constant $d>0$ such that

$$\|T^{-n}\| = O\,(\exp\,(dn^{\alpha})), \quad n \to \infty \tag{24}$$

with $\alpha = (k+1)/(k+2)$. Consequently the assumption $\sum_{n=0}^{\infty} \log \|T^n\|/(1+n^2) < \infty$ and (24) imply that $\sum_{n=-\infty}^{\infty} \log \|T^n\|/(1+n^2) < \infty$, and therefore by [6, p. 154] T is a $\mathfrak{U}$ unitary operator.

Proof of (b_2). If $\sigma(T)$ contains more than one point and λ is an eigenvalue of T, then Ker $(T-\lambda I)$ is a nontrivial hyperinvariant for T. If T has no eigenvalues, then by (a), $\sigma(T) \subseteq \partial D$ and therefore by (b_1), T satisfies (1), and since $\sigma(T)$ contains more than one point, Wermer's theorem [27] implies that T has a non-trivial hyperinvariant subspace.

Proof of (c_1). If $\mathbf{A}$ consists of functions of bounded characteristic, then the hypothesis on $\|L_w\|$ implies by Lemma 1 and Lemma 5 (as in the proof of (b_1)) that if $\sigma(T) \subseteq \partial D$, then

$$\|R(T, z)\| = O\left(\exp \frac{d}{1-|z|}\right), \quad |z| \to 1-$$

for some constant $d>0$, and therefore by Lemma 2, T satisfies condition (3).

Proof of (c_2). The proof of (a) shows that $W = \sigma(T) \cap D$ consists of eigenvalues of T and is contained in $f^{-1}(0)$ for some non-identically zero function $f \in \mathbf{A}$. Therefore by the well-known theorem on the zeros of functions of bounded characteristic [20, p. 85], $\sum_{\lambda \in W} (1-|\lambda|) < \infty$.

Proof of (d). (d) is an immediate consequence of (c) and Theorem 1.

Proof of Corollary 2. Let $\|\ \|_k$ denote the H_k^∞ norm. Since $\|z^n\|_k = O(n^k)$, $n \to \infty$, it follows from condition 2 of Definition 1, that if T is an operator in $C_0(H_k^\infty)$, then $\|T^n\| = O(n^k)$, $n \to \infty$. Since $H_k^\infty \subset H^\infty$ and every H^∞ function is of bounded characteristic [20, p. 90] the corollary follows from parts (c) and (d) of Theorem 3.

Proof of Corollary 3. If (p_n) is an increasing sequence of positive numbers, then the space of analytic functions $f(z)=\sum_{n=0}^{\infty} a_n z^n$, $z\in D$ such that $\|f\|=\sum_{n=0}^{\infty}|a_n|p_n<\infty$ forms an admissible Banach space $\mathbf{A}$ for which $\|L_w\|\leqslant 1/(1-|w|)$, $\forall w\in D$. It is clear that $\mathbf{A}\subseteq H^\infty$ and if T is an operator such that $\|T^n\|=O(p_n)$, $n\to\infty$, then $\mathbf{A}$ operates on T. Thus the corollary follows from Theorem 3: (1) from parts (a) and (c), (2) from part (d) and (3) from part (b).

Proof of Theorem 4. Let $\mathbf{A}$ be an admissible Banach space which satisfies the hypotheses of the theorem and assume that $\mathbf{A}$ operates on some operator T acting on a Banach space $\mathbf{B}$. We claim that for every $f\in\mathbf{A}$, $f(\sigma(T))\subseteq\sigma(f(T))$. To prove this, let $\lambda\in\sigma(T)$ and let $(p_n)_{n=1}^{\infty}$ be a sequence of polynomials such that $\lim_{n\to\infty}p_n=f$ in the norm of $\mathbf{A}$. Let $(q_n)_{n=1}^{\infty}$ be the polynomials such that $p_n(z)-p_n(\lambda)=(z-\lambda)q_n(z)$, $z\in D$, $n=1, 2, \ldots$. The hypotheses on $\mathbf{A}$ imply that $\lim_{n\to\infty}(T-\lambda I)q_n(T)=f(T)$ in the operator norm. Therefore since $\lambda\in\sigma(T)$, and the set of non-invertible operators is closed (in the Banach space of bounded linear operators on $\mathbf{B}$) we obtain that $f(\lambda)\in\sigma(f(T))$ and the claim is proved. Consequently, if $f\in\mathbf{A}$ and $f(T)=0$, then $\sigma(T)\subseteq f^{-1}(0)$. Thus if T is in $C_0(\mathbf{A})$, there exists a function $f\in\mathbf{A}$, $f\not\equiv 0$ such that $f=0$ on $E=\sigma(T)\cap\partial D$. Therefore since f is continuous in $\bar{D}$ and analytic in D, E is of measure zero [20, p. 90]. The last assertion of the theorem follows in the same way from [5].

Proof of Corollary 4. Let $\mathbf{A}$ be the Banach space introduced in the proof of Corollary 3. The conclusion of Corollary 4 follows from Theorem 4 by observing that $\mathbf{A}$ satisfies the hypotheses of the theorem.

3. Proofs of the lemmas

Proof of Lemma 1. Since $\mathbf{A}$ is an admissible Banach space which operates on T, we have for every $z\in D$ the identity

$$f(T)-f(z)I=G(T,z)(T-zI)$$

and therefore for every $z\in D\setminus\sigma(T)$

$$f(T)R(T,z)-f(z)R(T,z)=G(T,z)$$

and the Lemma is proved.

Proof of Lemma 2. Assume that T satisfies condition (6). Let $z\in D$ be fixed and set $|z|=r$. Let N be the smallest integer such that

$$N>\left(\frac{c}{\log(2-r)}\right)^{1/(1-\alpha)}$$

(where c is the constant appearing in (6)). Consider the two sums:

$$\Sigma_1 = \sum_{n=0}^{N} r^n \exp(cn^\alpha), \quad \Sigma_2 = \sum_{n=N+1}^{\infty} r^n \exp(cn^\alpha).$$

For Σ_1 we have the obvious estimate

$$\Sigma_1 < (1-r)^{-1} \exp\{cN^\alpha\}.$$

Using the fact that $cN^{\alpha-1} < \log(2-r) < \log(1/r)$, we obtain that

$$\Sigma_2 < \sum_{n=N+1}^{\infty} \exp\{-n(\log(1/r) - c/N^{1-\alpha})\} < \frac{\exp(cN^\alpha)}{1 - r\exp\dfrac{c}{N^{1-\alpha}}} \leqslant \frac{\exp(cN^\alpha)}{(1-r)^2}.$$

Combining all these estimates, noticing that $R(T,z) = \sum_{n=0}^{\infty} T^{-n-1}z^n$, $z \in D$, and that the choice of N implies that $N < [2c/(1-r)]^{1/(1-\alpha)} + 1$, we obtain (7) with $d = 4^\beta c^{\beta/\alpha}$. This completes the proof of (a).

Proof of (b). Since $\sigma(T) \subseteq \partial D$ the identity $R(T,z) = \sum_{n=0}^{\infty} T^{-n-1}z^n$, $z \in D$ holds, and therefore for every $0 < r < 1$,

$$T^{-n} = \frac{1}{2\pi i} \int_{|z|=r} R(T,z)\, z^{-n} dz.$$

Consequently, if (7) holds, there exists a constant $M > 0$ such that

$$\|T^{-n}\| \leqslant Mr^{-n} \exp\frac{d}{(1-r)^\beta}, \quad 0 < r < 1$$

and setting $r = 1 - (d/n)^{1/(1+\beta)}$ we obtain (6) with $\alpha = \beta/(\beta+1)$ and $c = 3d^{\alpha/\beta}$.

Proof of Lemma 3. For complex valued functions φ, the conclusion of the lemma follows from [26, Lemma 5.8 and Lemma 5.9]. The vector valued version can be deduced from the scalar case as follows: The proof in [26] shows that in the scalar case, the constants M_1, K_1 and b depend only on M, N, K and d, and not on the function φ. Therefore if $\mathbf{B}$ is the Banach space which contains the range of φ (in the vector valued case), and v is any element in $\mathbf{B}^*$ (the dual of $\mathbf{B}$), such that $\|v\|_{\mathbf{B}^*} = 1$, we obtain from the scalar case, applied to the function $(\varphi(z), v)$ that

$$|(\varphi(z), v)| \leqslant M_1 \exp\{b\varrho(z,E)^{-1}\}, \quad z \in D$$

and

$$|\varphi(z), v)| \leqslant K_1 \varrho(z,E)^{-2N}, \quad |z| > 1,$$

where M_1, K_1 and b do not depend on v, and the conclusion of the lemma follows from the fact that

$$\|\varphi(z)\| = \sup \{|(\varphi(z), v)|: v \in \mathbf{B}^*, \|v\|_{\mathbf{B}^*} = 1\}.$$

Proof of Lemma 4. Let $E = \{e^{i\theta_1}, ..., e^{i\theta_k}\}$ be the set in the hypotheses of the lemma, and choose $0 < \delta < \pi/2$ such that $\delta < \min \{|\theta_i - \theta_j|: i \neq j, i, j = 1, 2, ..., k\}$. For every $j = 1, ..., k$ consider the two sectors

$$S_j^+ = \{z: |z| < 1, \theta_j \leqslant \arg z \leqslant \theta_j + \delta\}$$

and

$$S_j^- = \{z: |z| < 1, \theta_j - \delta \leqslant \arg z \leqslant \theta_j\}.$$

It suffices to show that φ is bounded (in norm) in each of these $2k$ sectors. For convenience of notation we assume that $\theta_1 = 0$ and show that φ is bounded in S_1^+. The proof for the other sectors is the same. Let U be the image of S_1^+ by the conformal map

$$z \to w = i\frac{1-z}{1+z},$$

and consider the function ψ defined on U by

$$\psi(w) = \varphi\left(\frac{i-w}{i+w}\right), \quad w \in U.$$

By the hypotheses φ is bounded on (boundary $S_1^+) \setminus \{1\}$ and therefore ψ is bounded on (boundary $U) \setminus \{0\}$. By Lemma 3, there exist constants M_1 and b such that $|\varphi(z)| \leqslant M_1 \exp(b/|1-z|)$, for $z \in S_1^+ \setminus \{1\}$, and this implies that $\|\psi(w)\| \leqslant M_1 \exp(b/|w|)$, for $w \in U \setminus \{0\}$. Noticing that the part of the boundary of U which contains the point $w = 0$ consists of two perpendicular line segments which intersect at $w = 0$, we deduce from (the vector valued analog of) the Phragmen-Lindelöf principle that ψ is bounded in $U \setminus \{0\}$ and therefore φ is bounded in $S_1^+ \setminus \{1\}$. This completes the proof of the lemma.

Proof of Lemma 5. Let U^* denote the unit ball of $\mathbf{B}^*$ (the dual of $\mathbf{B}$) and consider $L \in U^*$. Since the function $z \to (\Phi(z), L)$ $z \in D$ is analytic, it follows from the Poisson-Jensen formula [23, p. 22] that for every $z \in D$ and $|z| < \varrho < 1$,

$$\log |(\Phi(z), L)| \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\varrho^2 - |z|^2}{|\varrho e^{it} - z|^2} \log |(\Phi(\varrho e^{it}), L)| \, dt$$

and therefore using the identity

$$\int_{-\pi}^{\pi} \frac{\varrho^2 - |z|^2}{|\varrho e^{it} - z|^2} dt = 1,$$

$0 \leqslant |z| < \varrho < 1$ and the fact that $\|\Phi(z)\| = \sup \{|(\Phi(z), L)|: L \in U^*\}$ we deduce that

$$\log \|\Phi(z)\| \leqslant \log K(\varrho) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\varrho^2 - |z|^2}{|\varrho e^{it} - z|^2} \log |f(\varrho e^{it})| \, dt \tag{25}$$

for every $z \in D$ and $|z| < \varrho < 1$. Thus using the estimate

$$\frac{\varrho^2 - |z|^2}{|\varrho e^{it} - z|^2} \leqslant \frac{\varrho + |z|}{\varrho - |z|}, \quad 0 \leqslant |z| < \varrho < 1,$$

the identity

$$\int_{-\pi}^{\pi} |\log |f(\varrho e^{it})|| \, dt = 2 \int_{-\pi}^{\pi} \log^+ |f(\varrho e^{it})| \, dt - \int_{-\pi}^{\pi} \log |f(\varrho e^{it})| \, dt,$$

Jensen's inequality [20, p. 82]

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(\varrho e^{it})| \, dt \geqslant \log |a_s| + s \log \varrho$$

(where a_s is the first non-zero Taylor coefficient of f) and setting $\varrho = (1 + |z|)/2$ we obtain that for some constant $c > 0$.

$$\|\Phi(z)\| = O\left(K\left(\frac{1+|z|}{2}\right) \exp\left\{\frac{6m\left(\frac{1+|z|}{2}\right) + c}{1 - |z|}\right\}\right), \quad |z| \to 1 -$$

and part (a) of the lemma is proved. To prove part (b), assume that $f \in H^\infty$, $f \not\equiv 0$, and let ν denote the discrete part of the measure which defines the singular inner part of f and let

$$V(z) = \exp\left\{\int \frac{z + e^{it}}{z - e^{it}} \, d\nu(t))\right\}, \quad z \in D.$$

Using [16, pp. 67–68] we see that there exists a continuous measure τ on $[-\pi, \pi)$ such that f admits a factorization $f(z) = B(z) G(z) V(z)$, $z \in D$, where B is a Blaschke product and

$$G(z) = \exp\left\{\int \frac{z + e^{it}}{z - e^{it}} \, d\tau(t)\right\}, \quad z \in D.$$

Noticing that $|B(z)| < 1$, $z \in D$ and using once again the estimate

$$\frac{\varrho^2 - |z|^2}{|\varrho e^{it} - z|^2} \leqslant \frac{\varrho + |z|}{\varrho - |z|} \quad z \in D, |z| < \varrho < 1,$$

we obtain from (25) that for every $z \in D$ and $0 < |z| < 1$,

$$\begin{aligned} \log \|\Phi(z)\| \leqslant \log K(\varrho) &- \frac{1}{2\pi} \frac{\varrho + |z|}{\varrho - |z|} \int_{-\pi}^{\pi} \log |B(\varrho e^{it})| \, dt \\ &- \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\varrho^2 - |z|^2}{|\varrho e^{it} - z|^2} \log |G(\varrho e^{it}) \, V(\varrho e^{it})| \, dt. \end{aligned} \tag{26}$$

Since $G(z)V(z) \neq 0$, for all $z \in D$, the function $\log |G(z)V(z)|$ is harmonic in D, and therefore the Poisson formula implies that the last term in the right hand side of (26) is equal to $-\log |G(z)V(z)|$. Thus remembering that $\lim_{\varrho\to 1-} \int_{-\pi}^{\pi} \log |B(\varrho e^{it})| dt = 0$ [23, Chapter I] and setting $\varrho = (1+|z|)/2$, we obtain from (26) that

$$\log \|\Phi(z)\| \leqslant \log K\left(\frac{1+|z|}{2}\right) + \frac{v(|z|)}{1-|z|} + |\log|G(z)|| + |\log|V(z)|| \tag{27}$$

where v is a function on $(0, 1)$ which satisfies $\lim_{r\to 1-} v(r) = 0$. It is clear that (b) will follow from (27) if we show that for every $\varepsilon > 0$ there exists a constant M_ε such that

$$|\log|G(z)|| \leqslant \frac{\varepsilon}{1-|z|} + M_\varepsilon, \quad z \in D.$$

Let $\varepsilon > 0$. Since the measure τ is continuous, there exists a number $\delta > 0$ such that for every interval $I \subset [-\pi, \pi)$ with length less than δ we have that $|\tau|(I) < \varepsilon/2$, and therefore for every $z = re^{i\theta} \in D$,

$$|\log|G(z)|| \leqslant \int \frac{1-|z|^2}{|e^{it}-z|^2} d|\tau|(t) = \int \frac{1-r^2}{(1-r)^2 + 4r\sin^2\left(\frac{\theta-t}{2}\right)} d|\tau|(t)$$

$$\leqslant \frac{2}{1-r}\int_{|t-\theta|<\delta} d|\tau|(t) + \frac{1}{\sin^2\frac{\delta}{2}}\int_{|t-\theta|\geqslant\delta} d|\tau|(t) < \frac{\varepsilon}{1-r} + M_\varepsilon$$

where $M_\varepsilon = \|\tau\|/\sin^2(\delta/2)$. This completes the proof of (b). (c) follows from (b) by observing that for all $z \in D$,

$$|V(z)|^{-1} = \left|\exp \int \frac{e^{it}+z}{e^{it}-z} d\nu(t)\right| \leqslant \exp \frac{2\|\nu\|}{1-|z|}.$$

4. Operators which satisfy $\|T^n\| = O(\exp(c|n|^\alpha))$ $n \to \pm\infty$ for some $0 < \alpha < 1$ and $c > 0$

In this section we consider invertible operators T which for some $0 < \alpha < 1$ and $c > 0$ satisfy the condition

$$\|T^n\| = 0\,(\exp c|n|^\alpha), \quad n \to \pm\infty. \tag{28}$$

Since $\alpha < 1$, it is clear that if T satisfies (28), then T also satisfies condition (1) and therefore by Wermer's Theorem, if $\sigma(T)$ contains more than one point, T has a nontrivial hyperinvariant subspace. The existence of nontrivial invariant subspace in the case where $\sigma(T)$ consists of a single point remains open. We present here some partial results for this case.

It will be convenient to introduce the following notation: If $\mathbf{B}$ is a Banach space and $0<\alpha<1$, we denote by $L_\alpha(\mathbf{B})$ the set of all bounded invertible linear operators on $\mathbf{B}$ such that $\sigma(T)=\{1\}$ which satisfy (28). In what follows $\mathbf{B}$ will be fixed and we set: $L_\alpha=L_\alpha(\mathbf{B})$. We have the following results:

PROPOSITION 1. *If for some $0<\gamma<1$, every operator in L_γ has a nontrivial hyperinvariant subspace, then the same is true for all the classes L_α, $0<\alpha<1$.*

PROPOSITION 2. *If $0<\alpha<\frac{1}{2}$, and T is in L_α, then there exists an analytic function $f(z)=\sum_{n=0}^\infty a_n z^n$, $f(z)\not\equiv 0$, such that $\sum_{n=0}^\infty |a_n| \exp(cn^\alpha)<\infty$ and $f(T)=0$ (c is the constant associated with T by (28)).*

As already noticed in section 1, we are not able to deduce from Proposition 2 the existence of nontrivial invariant subspaces for operator in L_α, for $0<\alpha<\frac{1}{2}$.

For the proof of Propositions 1 and 2, we shall need some equivalent characterizations of the classes L_α. This is given in:

PROPOSITION 3. *Let $0<\alpha<1$ and set $\beta=\alpha/(1-\alpha)$. The following are equivalent:*

(I) *T is in L_α*

(II) *There exists a constant $R>0$ such that*

$$\|(T-I)^n\|\leqslant R^n n^{-n/\beta},\quad n=0,1,\ldots.$$

(III) *There exist constants $K>0$ and $b>0$ such that*

$$\|R(T,z)\|\leqslant K\exp\frac{b}{|1-z|^\beta},\quad z\in\mathbf{C}\setminus\{1\}.$$

Proof of (I) ⇒ (II). Since $\sigma(T)=\{1\}$, there exists a bounded linear operator A such that $T=\exp A$ and $\sigma(A)=\{0\}$ (we may take $A=\log(I-(I-T))=-\sum_{j=1}^\infty (I-T)^j/j$. Consider the entire operator valued function $\Phi(z)=\exp(Az)$, $z\in\mathbf{C}$. We claim that Φ is of order α and finite type (see definition in [15, p. 104]). First Φ is of order 1 and minimal type: Let $\varepsilon>0$; since A is quasi-nilpotent, $\lim_{n\to\infty}\|A^n\|^{1/n}=0$, and therefore there exists a constant C_ε such that $\|A^n\|\leqslant C_\varepsilon\varepsilon^n$, $n=0,1,\ldots$, and

$$\|\varphi(z)\|\leqslant\sum_{n=0}^\infty\frac{\|A^n\|\,|z|^n}{n!}\leqslant C_\varepsilon\exp(\varepsilon|z|).$$

Next, (I) implies that $\|\varphi(x)\|\leqslant M_1\exp(c|x|^\alpha)$, $-\infty<x<\infty$, where c is the constant given by (28) and $M_1=M\sup_{|t|\leqslant 1}\|\exp(tA)\|$. Thus, using the analog for vector valued functions of [3, p. 97, Theorem 6.69] (which is obtained from the scalar case by an argument which is similar to the one in the proof of Lemma 3), we deduce that there exist contants $K_1>0$

and $b>0$ such that $\|\varphi(z)\| \leqslant K_1 \exp(b|z|^\alpha)$, $z \in \mathbf{C}$, and by using the vector valued analog (which is proved in the same way as the scalar case) of the known relation between the order of an entire function and the magnitude of its Taylor coefficients [3, p. 11, Theorem 22.10] we obtain that there exists a constant $C>0$ such that

$$\left\|\frac{A^n}{n!}\right\| \leqslant C^n n^{-n/\alpha}, \quad n=0, 1, \ldots,$$

and therefore by Stirling's formula there exists a constant $C_1>0$ such that

$$\|A^n\| \leqslant C_1^n n^{-n/\beta}, \quad n=0, 1, \ldots .$$

Noticing that $T-I=A\cdot B$ where $B=\sum_{j=1}^{\infty} A^{j-1}/j!$, we obtain (II) with $R=C_1\|B\|$.

Proof of (II)⇒(III). Observing that

$$R(T, z)=\sum_{n=0}^{\infty} \frac{(I-T)^n}{(1-z)^{n+1}}, \quad z \neq 1,$$

is an entire vector function of $1/(1-z)$, we obtain (III) from (II) by using once again the relation between the magnitude of Taylor coefficients and the order of entire functions (this time in the other direction).

Proof of (III)⇒(I). This is an immediate consequence of Lemma 2.

COROLLARY 5. *Let T be an operator such that $\sigma(T)=\{1\}$, let $0<\alpha<1$ and $\beta=\alpha/(1-\alpha)$. The following conditions* (a), (b) *and* (c) *are equivalent:*

(a) $\|T^n\|=O(n^k)$, $n\to\infty$, *for some integer $k \geqslant 0$ and* $\|T^n\|=O(\exp(c|n|^\alpha))$, $n\to-\infty$ *for some constant $c>0$.*

(b) $\|T^n\|=O(n^k)$, $n\to\infty$ *for some integer $k\geqslant 0$ and* $\|(T-I)^n\|=O(R^n n^{-n/\beta})$, $n\to\infty$ *for some constant $R>0$.*

(c) $\|R(T, z)\|=O(\exp(d/|1-z|^\beta))$, $|z|\to 1-$ *for some constant $d>0$, and* $\|R(T,z)\|=O(|1-|z||^{-N})$, $|z|\to 1+$, *for some integer $N>0$.*

Proof. It follows from Proposition 3 that (a) and (b) are equivalent; and from Proposition 3 and [6, p. 131, Proposition 1.6] that (b) and (c) are equivalent. More precisely (b)⇒(c) with $N=k+1$ and (c)⇒(b) with $k=N$.

COROLLARY 6. *Let T be an operator such that $\sigma(T)=\{1\}$ and $\|T^n\|=O(n^k)$, $n\to\infty$, for some integer $k\geqslant 0$. A necessary and sufficient condition for the existence of an analytic function $f(z)=\sum_{n=0}^{\infty} a_n z^n$, $f \not\equiv 0$, such that $\sum_{n=0}^{\infty}|a_n| n^k<\infty$ and $f(T)=0$ is that*

$$\|(T-I)^n\|^{1/n}=O(1/n), \quad n\to\infty.$$

Proof. This is an immediate consequence of Theorem 2, Corollary 3 and Corollary 5.

COROLLARY 7. *If T is an operator such that $\|T^n\| = O(n^k)$, $n \to \infty$, for some integer $k \geq 0$, and $\|(T-I)^n\|^{1/n} = o(1/n)$, $n \to \infty$, then $(T-I)^{k+1} = 0$.*

Proof. The condition $\|(T-I)^n\|^{1/n} = o(1/n)$ implies that $\sigma(T) = \{1\}$, and that if A is a quasinilpotent operator such that $T = e^A$, then the function $\Phi(z) = e^{Az}$, $z \in \mathbf{C}$, is an entire function of order $\frac{1}{2}$ and minimal type. (This follows by the arguments of Proposition 3.) Therefore the function $\Psi(z) = \Phi(z^2)$, $z \in \mathbf{C}$, is of order 1, and minimal type, and the hypothesis $\|T^n\| = O(n^k)$, $n \to \infty$ implies that $\|\Psi(x)\| = O(|x|^{2k})$, $x \to \pm\infty$ and therefore by [15, p. 104, Th. 3.13.8] Ψ is a polynomial of degree $\leq 2k$, hence Φ is a polynomial of degree $\leq k$, thus $A^{k+1} = 0$. Since

$$T - I = A \sum_{j=1}^{\infty} \frac{A^{j-1}}{j!},$$

we obtain that $(T-I)^{k+1} = 0$.

Proof of Proposition 1. Let $0 < \gamma < 1$, and let n be a positive integer. It follows from Proposition 3(II) that if T is in L_α, for some $0 < \alpha < 1$, then $T_1 = I + (T-I)^n$ is in L_β where $\beta = \alpha/(n(1-\alpha)+\alpha)$, and therefore for n large enough $\beta < \gamma$ and T_1 will be in L_γ, and since any hyperinvariant subspace for T_1 is also a hyperinvariant subspace for T the conclusion of Proposition 1 follows.

Proof of Proposition 2. Assume that T is in L_α for some $0 < \alpha < \frac{1}{2}$. Let $2\alpha < \varrho < 1$. It is known ([20, p. 118, Ex. 7], or [7]) that there exists a function $h \not\equiv 0$ of the form $h(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$, $-\infty < t < \infty$, such that $\sum_{n=-\infty}^{\infty} |c_n| \exp(n^\varrho) < \infty$ and $h^{(j)}(0) = 0$, $j = 0, 1, \ldots$. It is easy to see that we also may assume, (as we shall), that h is even, or equivalently, that $c_{-n} = c_n$, $n = 1, 2, \ldots$. Let f be the function defined by $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $|z| \leq 1$, where $a_n = c_k$ if $n = k^2$ for some integer k, and $a_n = 0$, otherwise. Noticing that $f(z) = \sum_{n=0}^{\infty} c_n z^{n^2}$, and $c_{-n} = c_n$, $n = 1, 2, \ldots$, it is clear that since $h \not\equiv 0$ also $f \not\equiv 0$. Let $\gamma = \varrho/2$; then the hypothesis $\sum_{n=-\infty}^{\infty} |c_n| \exp(n^\varrho) < \infty$ implies that:

$$\sum_{n=0}^{\infty} |a_n| \exp(n^\gamma) < \infty \tag{29}$$

and therefore in particular that $\sum_{n=0}^{\infty} |a_n| \exp(cn^\alpha) < \infty$ for every $c > 0$. Let $F(t) = f(e^{it}) = \sum_{n=0}^{\infty} a_n e^{int}$, $-\infty < t < \infty$. The hypothesis $h^{(j)}(0) = 0$, $j = 0, 1, \ldots$, and the fact that $c_{-n} = c_n$, $n = 1, 2, \ldots$, imply that:

$$F^{(k)}(0) = \frac{i^k}{2} h^{(2k)}(0) = 0, \quad k = 0, 1, \ldots. \tag{30}$$

It follows from (29) (see [20, p. 26, Ex. 7.]) that there exists a constant $R_1>0$ such that $|f^{(n)}(e^{it})|\leqslant R_1^n n^{n/\gamma}$, $n=0, 1, \ldots$; $|t|\leqslant\pi$. And therefore using (30) together with Taylor's and Stirling's formulas we obtain that there exists a constant $R_2>0$, such that for all $|t|\leqslant\pi$, $|f(e^{it})|\leqslant\inf_{n\geqslant 0} R_2^n n^{n/\delta}|e^{it}-1|^n$, $n=0, 1, \ldots$, where $\delta=(1-\gamma)/\gamma$, and this implies (see [12, p. 170]) that for some constants $K>0$ and $d>0$ we have:

$$(31)\qquad |f(e^{it})|\leqslant K\exp\left\{-\frac{d}{|e^{it}-1|^{\delta}}\right\},\quad 0<|t|\leqslant\pi.$$

Using the fact that $\alpha<\frac{1}{2}$ and $\varrho<1$ we see that $\delta>\alpha/(1-\alpha)$ and therefore from (31) and Proposition 3 (III) we obtain that

$$(32)\qquad \sup\{\|f(e^{it})R(T, e^{it})\|: 0<|t|\leqslant\pi\}<\infty.$$

Let $G(T, z)$ be the function associated with f by Lemma 1. Since $\gamma>\alpha$ we obtain from (29) that $\sum_{n=0}^{\infty}|a_n|n\exp(cn^{\alpha})<\infty$, and therefore by the argument used in the proof of the second part of Theorem 2 we deduce that $\sup_{|z|<1}\|G(T, z)\|<\infty$, and this together with (32) and identity (5) imply that

$$(33)\qquad \sup\{\|f(T)R(T, e^{it})\|: 0<|t|\leqslant\pi\}<\infty.$$

Let Φ be the operator function $\Phi(z)=f(T)R(T, z)$, $z\in\mathbf{C}\setminus\{1\}$, and let

$$\Psi(w)=\Phi\left(\frac{w+i}{w-i}\right),\quad w\in\mathbf{C}.$$

Ψ is an entire operator function, and from Proposition 3(III) we deduce that there exist constants $M>0$, $b>0$ such that $\|\Psi(w)\|\leqslant M\exp(b|w|^{\beta})$ where $\beta=\alpha/(1-\alpha)$, and from (33) we obtain that $\sup_{-\infty<x<\infty}\|\Psi(x)\|<\infty$. Since $\alpha<\frac{1}{2}$ we see that $\beta<1$, thus Ψ is an entire (vector) function of order less than 1 which is bounded on the real axis and therefore by a well known theorem [15, p. 104, Th. 3.13.8] we obtain that Ψ is a constant operator, and therefore the same holds for Φ. Using the same argument as in the proof of Theorem 2, we conclude that $f(T)=0$.

5. Quasinilpotent operators

In this section we apply the results of section 2 and section 4 to some classes of quasinilpotent operators. Our first result is:

THEOREM 5. *Let A be a quasinilpotent operator on a Banach space. Assume that there exist constants $M>0$, $N>0$, $K>0$ and $c>0$ such that*

$$(34)\qquad \|R(A, z)\|\leqslant M\exp\frac{c}{|\operatorname{Im} z|},\quad \textit{for } \operatorname{Im} z>0$$

and

$$\|R(A, z)\| \leqslant K|\operatorname{Im} z|^{-N}, \quad \textit{for } \operatorname{Im} z<0. \tag{35}$$

If $A \neq 0$, then A has a non trivial hyperinvariant subspace, and if A is not nilpotent then A has an uncountable chain of hyperinvariant subspaces.

Proof. Let T be the Cayley transform of A, that is $T=(I+iA)(I-iA)^{-1}$. Then $\sigma(T)=\{1\}$ and a direct computation gives that

$$R(T, z)=(i+z)^{-1}(A-iI)\, R\left(A, \frac{i(1-z)}{1+z}\right),$$

and from this we see that if A satisfies (34) and (35) then T satisfies:

$$\begin{aligned} \|R(T, z)\| &= O\left(\exp \frac{a}{1-|z|}\right), \quad |z| \rightarrow 1- \quad \text{and} \\ \|R(T, z)\| &= O((|z|-1)^{-N}), \quad |z| \rightarrow 1+ \end{aligned}$$

and therefore by [6, p. 131, Prop. 16 and p. 155, Prop. 3.5] we obtain that T satisfies the hypothesis of Theorem 1, and from this the conclusion of the theorem follows by observing that $A=i(I-T)(I+T)^{-1}$ and therefore A and T have the same invariant and hyperinvariant subspaces.

COROLLARY 8. *Let A be a quasinilpotent operator acting on a Hilbert space H. If A is dissipative* (*that is* $\operatorname{Im}(Ax, x) \geqslant 0$, $\forall x \in H$) *and satisfies* (34), *then the conclusions of Theorem 5 hold for A. In particular, A has a non trivial invariant subspace.*

Proof. It is well known and easy to verify that if A is dissipative then $\|R(A, z)\| \leqslant |\operatorname{Im} z|^{-1}$, for $\operatorname{Im} z<0$, and therefore the corollary is an immediate consequence of Theorem 5.

Remarks. 1. Corollary 8 is proved in [17] by using the theory of characteristic operator functions in Hilbert space under the seemingly stronger hypothesis

$$\|R(A, z)\|=O\left(\exp \frac{c}{|z|}\right), \quad |z| \rightarrow 0.$$

However by Proposition 3 one can show that for these operators (34) implies this hypothesis. (For a study of this class of operators see also [4, p. 52].)

2. Corollary 8 can also be proved by the same argument for quasinilpotent dissipative operators in Banach spaces in the sense of [21], which satisfy condition (34). (Notice

that the definition of dissipative in [21] is that Re $(Ax, x) \leqslant 0$, hence in this case we have to consider the half planes Re $z<0$ and Re $z>0$ in the hypotheses of Theorem 5, or to replace A by iA).

3. If condition (34) holds for every $c>0$, it follows from the proof of Corollary 1 that A is nilpotent. This was proved under stronger hypotheses in [21] and [25].

4. If A is an operator such that $\sigma(A)$ is contained in the real line, contains more than one point, and satisfies for some constants $M>0$, $K>0$ and $\beta>0$, the condition

$$\|R(A, z)\| \leqslant M \exp \frac{K}{|\operatorname{Im} z|^{\beta}}, \quad \operatorname{Im} z \neq 0,$$

the existence of non trivial hyperinvariant subspaces is proved in [6, p. 159, Th. 4.3]. Thus the first conclusion of Theorem 5 remains true if the hypothesis that A is quasinilpotent is replaced by the hypothesis that $\sigma(A)$ is contained in the real line. (Notice that conditions (34) and (35) remain unchanged if A is replaced by $A-\lambda I$ for some real number λ.)

Our next observation is that the existence of non trivial invariant subspaces for quasinilpotent operators whose resolvent is of finite exponential type is equivalent to the existence of non trivial invariant subspaces for operators in the classes L_α considered in section 4. To state this more precisely we introduce the following:

Notation. If $0<\beta<\infty$, we denote by Q_β the class of all quasinilpotent operators A which satisfy for some constants $M>0$ and $c>0$ the condition:

$$\|R(A, z)\| \leqslant M \exp \frac{c}{|z|^{\beta}}, \quad z \neq 0.$$

PROPOSITION 4. *Let $0<\alpha<1$, and set $\beta=\alpha/(1-\alpha)$. The existence of non trivial invariant (hyperinvariant) subspaces for all operators in the class L_α is equivalent to the existence of non trivial invariant (hyperinvariant) subspaces for all operators in the class Q_β.*

Proof. It follows from Proposition 3 (III) that T is in L_α if and only if $A=T-I$ is in Q_β, and this proves Proposition 4.

Remark. It follows from Proposition 1 and Proposition 4 that the existence of non trivial hyperinvariant subspaces for all operators in one of the classes Q_β is equivalent to the existence of non trivial hyperinvariant subspaces for all operators in all these classes. This can also be seen directly by observing that if A is in Q_β then A^2 is in $Q_{\beta/2}$.

6. An extension of a theorem of Nagy, Foiaş and Colojoară and a short proof of Wermer's theorem

By a theorem of Nagy and Foiaş [22, p. 74, Th. 5.4], if T is a power bounded operator on a Hilbert space and neither $\{T^n\}_{n=0}^{\infty}$ nor $\{T^{*n}\}_{n=0}^{\infty}$ converges strongly to 0, then either $T=cI$ where $|c|=1$ or T has a hyperinvariant subspace. This theorem was extended by Colojoară and Foiaş [6, p. 134, Th. 1.9] as follows: If T is an operator on a reflexive Banach space and $\|T^n\|=O(\varrho_n)$, $n\to\infty$, where $\{\varrho_n\}_{n=0}^{\infty}$ is a sequence which satisfies $\overline{\lim}_{n\to\infty}\varrho_{n+m}/\varrho_n\leqslant Km^{\alpha}$, $m=0, 1, \ldots$, for some $K>0$ and $\alpha>0$ and neither $\varrho_n^{-1}T^n$ nor $\varrho_n^{-1}T^{*n}$ converges strongly to zero, the same conclusion holds.

The next result shows that the same conclusion holds for much faster growing sequences $\{\varrho_n\}_{n=0}^{\infty}$.

PROPOSITION 5. *Let T be a bounded linear operator on a reflexive Banach space and let $\{\varrho_n\}_{n=0}^{\infty}$ be a sequence such that for some constants $0<\alpha<\frac{1}{2}$, $K>0$ and $a>0$,*

$$\overline{\lim_{n\to\infty}}\frac{\varrho_{n+m}}{\varrho_n}\leqslant K\exp(am^{\alpha}),\quad m=0, 1, \ldots.$$

Then if $\|T^n\|=O(\varrho_n)$, $n\to\infty$, and neither $\varrho_n^{-1}T^n$ nor $\varrho_n^{-1}T^{*n}$ converges strongly to 0, the same conclusion as in the above mentioned theorems holds for T.

Proof. The proof follows the same lines as the proof of [6, p. 134, Th. 1.9] except for the following changes: In the first part of the argument, replace the fact that every invertible operator T_1 such that $\|T_1^n\|=O(|n|^{\alpha})$, $n\to\pm\infty$ for some $\alpha>0$, is decomposable by the fact that the same is true if $\|T_1^n\|=O(\exp c|n|^{\beta})$, $n\to\pm\infty$, for some $c>0$, $0<\beta<1$, (Wermer's Theorem or Theorem 3.2, p. 154 in [6]). In the last step of the argument, where (an equivalent form of) the theorem of Hille [14] is used, use instead, the extension of Hille's Theorem given in Corollary 1 of the present paper.

We give now a short proof of

THEOREM (Wermer [27]). *Let T be an invertible operator on a Banach space which satisfies $\sum_{n=-\infty}^{\infty}\log\|T^n\|/(1+n^2)<\infty$. If $\sigma(T)$ contains more than one point then T has a non trivial hyperinvariant subspace.*

Proof. Let $\omega_n=\|T^n\|$, $n=0, \pm1, \pm2, \ldots$, and consider the Banach algebra $\mathbf{A}$ which consists of all functions $f(e^{it})=\sum_{n=-\infty}^{\infty}a_ne^{int}$, $0\leqslant t\leqslant 2\pi$, such that $\|f\|_{\mathbf{A}}=\sum_{n=-\infty}^{\infty}|a_n|\omega_n<\infty$. (This is a Banach algebra since $\omega_{n+m}\leqslant\omega_n\omega_m$ for all integers n, m). It is clear that the mapping $f\to f(T)=\sum_{n=-\infty}^{\infty}a_nT^n$, $f\in\mathbf{A}$, establishes a continuous homomorphism of $\mathbf{A}$ into the Banach algebra of bounded linear operators on the space on which T acts. We claim that for every f in $\mathbf{A}$, $f(\sigma(T))\subset\sigma(f(T))$. (We recall here again, that (1) implies that $\sigma(T)$ is contained in the unit circle.) Let $\lambda\in\sigma(T)$, $f(e^{it})=\sum_{n=-\infty}^{\infty}a_n e^{int}\in\mathbf{A}$ and $P_N(e^{it})=\sum_{-N}^{N}a_n e^{int}$, $N=1,2,\ldots$. Let $q_N(e^{it})$ be the trigonometric polynomials such that $P_N(e^{it})-P_N(\lambda)=(e^{it}-\lambda)q_N(e^{it})$, $0\leqslant t\leqslant 2\pi$. Then $\lim_{N\to\infty}(e^{it}-\lambda)q_N(e^{it})=f(e^{it})-f(\lambda)$ in the norm of $\mathbf{A}$, and therefore $f(T)-f(\lambda)I=\lim_{N\to\infty}(T-\lambda)q_N(T)$ in the operator norm. Since $\lambda\in\sigma(T)$, and the set of non invertible elements in a Banach algebra is closed, it follows that $f(\lambda)\in\sigma(f(T))$. Let now $\lambda_1\neq\lambda_2$ be two points in $\sigma(T)$. Since $\sum_{n=-\infty}^{\infty}\log\omega_n/(1+n^2)<\infty$, the algebra $\mathbf{A}$ satisfies the Beurling condition for regularity ([20, p. 118, Ex. 7] or [7]), and there exists functions f_1, f_2 in $\mathbf{A}$, with disjoint supports such that $f_j(\lambda_j)=1$, $j=1,2$. Thus, $f_1(e^{it})\cdot f_2(e^{it})\equiv 0$, and therefore $f_1(T)\cdot f_2(T)=0$, but $f_j(T)\neq 0$, $j=1,2$, since $f_j(\lambda_j)=1\in\sigma(f_j(T))$, $j=1,2$. Consequently Kernel $(f_1(T))$ is a non trivial hyperinvariant subspace for T.

7. Applications to Banach algebras

Most of our previous results could be stated and proved in the context of Banach algebras. For instance, it is clear that Corollaries 1, 6, and 7 are statements about elements in Banach algebras. In this section we apply the results of the previous sections to prove results on closed primary ideals and restriction algebras of certain Banach algebras of continuous functions on the unit circle. We obtain in particular some of the results of [18] and [2] and an extension of the results of [1].

We recall first some definitions and known facts from [11, pp. 214–215] and introduce some notations.

Definition. Let $\mathbf{R}$ be a Banach algebra of continuous functions on the unit circle ∂D. $\mathbf{R}$ is called a homogeneous Banach algebra (in the sense of Shilov) if:

(H_1) $\mathbf{R}$ is generated by the functions 1, e^{it}. e^{-it} and its maximal ideal space is ∂D.

(H_2) For every $f\in\mathbf{R}$ and $e^{i\tau}\in\partial D$, $\|f_\tau\|_{\mathbf{R}}=\|f\|_{\mathbf{R}}$ where $f_\tau(e^{it})=f(e^{i(t-\tau)})$, $e^{it}\in\partial D$. (It follows from H_1 that $f_\tau\in\mathbf{R}$ for every $f\in\mathbf{R}$ and $e^{i\tau}\in\partial D$.)

If $\mathbf{R}$ is a homogeneous Banach algebra on ∂D, we shall denote by $\mathbf{R}^+$ the closed subalgebra of $\mathbf{R}$ which is generated by the functions 1 and e^{it}. $\mathbf{R}^+$ is a homogeneous Banach space on ∂D in the more general sense of [20, p. 14] and it follows from [20, Th. 2.12] that $\mathbf{R}^+$ consists of all functions f in $\mathbf{R}$ such that $\hat{f}(n)=0$, $n=-1,-2,\ldots$. (Where for every integer j, $\hat{f}(j)$ denotes the jth Fourier coefficient of f, that is, $\hat{f}(j)=(1/2\pi)\int_{-\pi}^{\pi}f(e^{i\theta})e^{-ij\theta}d\theta$.)

Every function f in $\mathbf{R}^+$ admits an extension to a continuous function $\tilde{f}$ on $\bar{D}$ which is analytic in D. This extension is given by $\tilde{f}(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n$, $z\in D$ or equivalently by $\tilde{f}(\mathrm{r}e^{i\theta})=(P_r*f)(e^{i\theta}\in\partial D$, $0\leqslant r<1$, where P_r denotes the Poisson kernel and $*$ denotes convolution on ∂D. It follows from the properties of $\mathbf{R}$ that the Gelfand representation of the Banach algebra $\mathbf{R}^+$ is given by the mapping $f \to \tilde{f}$ and that the maximal ideal space of $\mathbf{R}^+$ is $\bar{D}$. We shall denote by $\tilde{\mathbf{R}}^+$ the algebra of functions on $\bar{D}$ which are Gelfand transforms of elements of $\mathbf{R}^+$. With norm $\|\tilde{f}\|_{\tilde{\mathbf{R}}^+} = \|f\|_{\mathbf{R}}$, $\tilde{\mathbf{R}}^+$ becomes a Banach algebra which is isometrically isomorphic to $\mathbf{R}^+$. We shall need in the sequel the following two facts:

(a) $\tilde{\mathbf{R}}^+$ is an admissible Banach space and $\|L_w\| \leqslant 2\|(e^{i\theta}-w)^{-1}\|_{\mathbf{R}}$, $\forall w \in D$.

(b) $\tilde{\mathbf{R}}^+$ operates on the multiplier operator T defined on $\mathbf{R}^+$ by: $Tf(e^{i\theta}) = e^{i\theta}f(e^{i\theta})$, $e^{i\theta} \in \partial D$, $f \in \mathbf{R}^+$.

We show first that (a) holds:

1) Since $\mathbf{R}^+$ is a Banach algebra and for every $w \in D$, the mapping $f \to \tilde{f}(w)$, $f \in \mathbf{R}^+$ is a complex homomorphism of $\mathbf{R}^+$, it follows [20, p. 201] that $|\tilde{f}(w)| \leqslant \|f\|_{\mathbf{R}^+} = \|\tilde{f}\|_{\tilde{\mathbf{R}}^+}$, for every $\tilde{f} \in \tilde{\mathbf{R}}^+$.

2) Assume that $f \in \mathbf{R}^+$ and $w \in D$. To show that $L_w\tilde{f} \in \tilde{\mathbf{R}}^+$, consider the function

$$g(e^{it}) = (f(e^{it}) - \tilde{f}(w))(e^{it}-w)^{-1}, \; e^{it} \in \partial D.$$

Since the maximal ideal space of $\mathbf{R}$ is ∂D and $w \in D$, it follows that $g \in \mathbf{R}$, and since $f \in \mathbf{R}^+$, the function $L_w\tilde{f}$ is a continuous extension of g to $\bar{D}$, which is analytic in D. Thus $\hat{g}(n) = 0$ for $n < 0$, and therefore, $g \in \mathbf{R}^+$. Consequently $L_w\tilde{f} = \tilde{g} \in \tilde{\mathbf{R}}^+$ and

$$\|L_w\tilde{f}\|_{\tilde{\mathbf{R}}^+} = \|g\|_{\mathbf{R}} \leqslant 2\|\tilde{f}\|_{\tilde{\mathbf{R}}^+}\|(e^{i\theta}-w)^{-1}\|_{\mathbf{R}}$$

and (a) is proved.

(b) follows from the fact that for very f in $\mathbf{R}^+$, $\lim_{\varrho\to 1} \tilde{f}(\varrho e^{i\theta}) = f(e^{i\theta})$ in the norm of $\mathbf{R}$ [20, p. 16].

We recall that a primary ideal in a commutative Banach algebra with unit, is an ideal that is contained in a single maximal ideal. A primary ideal is called trivial if it is the zero ideal or a maximal ideal.

If $\mathbf{R}$ is a homogeneous Banach algebra on ∂D, we shall denote by $\mathbf{R}_0$ the maximal ideal which consists of all functions f in $\mathbf{R}$ such that $f(1) = 0$. It follows from the homogeneity of $\mathbf{R}$, that it suffices to consider primary ideals contained in $\mathbf{R}_0$. The next result gives some information on these ideals, for certain homogeneous Banach algebras.

Proposition 6. *Let* $\mathbf{R}$ *be a homogeneous Banach algebra on* ∂D, *which for some integer* $k \geqslant 0$ *and constant* $c > 0$, *satisfies the conditions*

$$\|e^{int}\|_{\mathbf{R}} = O(n^k), \quad n \to \infty$$

and

$$\|e^{-int}\|_{\mathbf{R}} = O(\exp(cn^{\frac{1}{2}})), \quad n \to \infty.$$

Let I be a closed primary ideal in $\mathbf{R}$ *which is contained in* $\mathbf{R}_0$. *Then either I is of finite codimension not exceeding $k+1$ and is the closure of the ideal generated in* $\mathbf{R}$ *by the function* $(e^{i\theta}-1)^j$ *for some* $1 \leqslant j \leqslant k$, *or there exists an uncountable chain of closed ideals between I and* $\mathbf{R}_0$. *If in addition*

$$\|e^{-int}\|_{\mathbf{R}} = O(\exp(\varepsilon n^{\frac{1}{2}})), \quad n \to \infty$$

for every $\varepsilon > 0$, *then every closed primary ideal contained in* $\mathbf{R}_0$ *is of the first kind, and if also* $k=0$, *there are no non-trivial closed primary ideals in* $\mathbf{R}$.

Proof. Consider the quotient algebra $\mathbf{B} = \mathbf{R}/I$ endowed with the canonical quotient norm. Let u be the image of e^{it} under the canonical map of $\mathbf{R}$ onto $\mathbf{B}$. Since $e^{it}-1 \in \mathbf{R}_0$, $u-1$ is contained in the single maximal ideal of B, and therefore $\sigma(u) = \{1\}$, [11, p. 34]. Consequently, $u^{-1} = \sum_{j=0}^{\infty}(1-u)^j$, the series being convergent in the norm of $\mathbf{B}$. Therefore, since 1, u and u^{-1} generate $\mathbf{B}$, 1 and u also generate $\mathbf{B}$. Let T be the operator defined on $\mathbf{B}$ by $Tx = ux$, $x \in \mathbf{B}$. Since $\|T^n\| = \|u^n\|_{\mathbf{B}} \leqslant \|e^{int}\|_{\mathbf{R}}$, $n = 0, \pm 1, \pm 2, \ldots$, the assumptions on $\mathbf{R}$ imply that $\|T^n\| = O(n^k)$, $n \to \infty$ and $\|T^n\| = O(\exp(c|n|^{1/2}))$, $n \to -\infty$, and since $\sigma(T) = \sigma(u) = \{1\}$, it follows from Theorem 1 that if $(T-I)^{k+1} \neq 0$, then T has an uncountable chain of invariant subspaces. Since 1 and u generate $\mathbf{B}$, the invariant subspaces of T coincide with the closed ideals in $\mathbf{B}$, and since the pre-images by the canonical map of the closed ideals in $\mathbf{B}$ are the closed ideals between I and $\mathbf{R}_0$, we obtain that, if $(u-1)^{k+1} \neq 0$, then there exists an uncountable chain of closed ideals between I and $\mathbf{R}_0$. If $(u-1)^{k+1} = 0$, then the fact that $\mathbf{B}$ is generated by 1 and u implies that $\mathbf{B}$ is of finite dimension not exceeding $k+1$, and it is easy to see that if j is the smallest integer such that $(u-1)^j = 0$, then I is the closure of the ideal generated in $\mathbf{R}$ by the function $(e^{i\theta}-1)^j$. The last assertion of the Proposition is proved in the same way, by using Corollary 1.

The following examples illustrate the two different possibilities described by Proposition 6

1) Let $\mathbf{R}$ be the Banach algebra of all continuous funtions f on ∂D such that

$$\|f\|_{\mathbf{R}} = \sum_{n=-\infty}^{-1} |\hat{f}(n)| \exp |n|^{1/2} + \sum_{n=0}^{\infty} |\hat{f}(n)| < \infty.$$

Then $\mathbf{R}$ is a homogeneous Banach algebra on ∂D [10, p. 120] which clearly satisfies the hypotheses of Proposition 6. One can show that if I is a non-trivial closed primary ideal contained in $\mathbf{R}_0$, then there exists an uncountable chain of closed ideals between I and $\mathbf{R}_0$. Moreover, one can prove that there exists a positive number a such that I is the closure of the ideal generated in $\mathbf{R}$ by the function

$$\psi(e^{i\theta}) = (1-e^{i\theta})^2 \exp\left\{a\,\frac{e^{i\theta}+1}{e^{i\theta}-1}\right\}$$

for $e^{i\theta}\neq 1$ and $\psi(1)=0$. (It is easy to verify that $\psi\in\mathbf{R}^+\subset\mathbf{R}$.) A similar result for the analogous weighted Fourier algebra on the real line is announced in [13, Th. 8.1.].

2) If $0\leqslant\alpha<\frac{1}{2}$ and $\mathbf{R}$ is the Banach algebra of all continuous functions f on ∂D such that

$$\|f\|_{\mathbf{R}}=\sum_{n=-\infty}^{-1}|\hat{f}(n)|\exp|n|^{\alpha}+\sum_{n=0}^{\infty}|\hat{f}(n)|<\infty,$$

then the last assertion of Proposition 6 implies that $\mathbf{R}$ has no non-trivial closed primary ideals.

Other results on the structure of closed primary ideas in Beurling algebras are given in [8] and [9].

Next we consider closed primary ideals in the Banach algebras $\mathbf{R}^+$ where $\mathbf{R}$ is a homogeneous Banach algebra on ∂D which, for some positive integer $k>0$, contains the Banach algebra $C^k(\partial D)$ of k-times continuously differentiable functions on ∂D. The structure of closed primary ideals in some of these algebras was determined in [18] and also follows from the more general result proved in [2]. The proofs in [18] and [2] are based on the Carleman transform of elements in the dual of $\mathbf{R}^+$. We shall obtain these results on closed primary ideals, from Theorem 1 and Lemma 5, without using the Carleman transform or duality.

We recall [11, p. 215] that the assumptions on $\mathbf{R}$ imply that there exists a constant $c>0$ such that $\|g\|_{\mathbf{R}}\leqslant c\|g\|_{C^k(\partial D)}$ for every $g\in C^k(\partial D)$ and therefore $\|e^{int}\|_{\mathbf{R}}=O(n^k)$, $n\to\infty$. Using the fact that

$$\|(e^{i\theta}-w)^{-1}\|_{C^k(\partial D)}=O((1-|w|)^{-k-1}),\ |w|\to 1+$$

and property (a) of $\tilde{\mathbf{R}}^+$, we obtain also that

$$\|L_w\| = O((1-|w|)^{-k-1}),\ |w|\to 1-$$

(L_w being regarded here as an operator on $\tilde{\mathbf{R}}$).

In what follows we shall denote for every real number α by v_α the function defined on $\bar{D}\setminus\{1\}$ by

$$v_\alpha(e^{i\theta})=\exp\left\{\alpha\frac{e^{i\theta}+1}{e^{i\theta}-1}\right\},$$

and by g_α the function defined on ∂D by $g_\alpha(e^{i\theta})=(e^{i\theta}-1)^{2k+1}v_\alpha(e^{i\theta})$, $e^{i\theta}\neq 1$ and $g_\alpha(1)=0$. It is easy to see that g_α belongs to $C^k(\partial D)$ and therefore also to $\mathbf{R}$. For $\alpha\geqslant 0$, g_α admits a continuous extension to $\bar{D}$ which is analytic in D given by

$$g_\alpha(z)=(z-1)^{2k+1}\exp\left\{\alpha\frac{z+1}{z-1}\right\},\quad z\in\bar{D}$$

and therefore $g_\alpha \in \mathbf{R}^+$. For every $\alpha \geqslant 0$, we shall denote by I_α the closure of the ideal generated in $\mathbf{R}^+$ by g_α. We shall denote by $\mathbf{R}_0^+$ the maximal ideal in $\mathbf{R}^+$, which consists of all functions f in $\mathbf{R}^+$ such that $f(1)=0$.

The following definition is introduced in [2, part II]: One says that $\mathbf{R}$ satisfies the analytic Ditkin condition if for every $f \in \mathbf{R}_0^+$ there exists a sequence $(g_n) \subset \mathbf{R}^+$ such that $g_n(0)=1$, $n=1, 2, \ldots$, and $\lim_{n\to\infty} g_n f = 0$ in the norm of $\mathbf{R}$.

Finally we adopt the following notation: For every $f \in \mathbf{R}^+$ we shall denote by S_f the singular inner part of $\tilde{f}$ and by μ_f the positive singular measure on $[-\pi, \pi)$ that defines S_f [16, p. 66]. For a closed ideal $I \subset \mathbf{R}^+$ we denote by

$$\alpha(I) = \inf \{\mu_f(\{0\}): f \in I\}.$$

The following contains the results on closed primary ideals in the algebras $\mathbf{R}^+$ that are proved in [18] and [2]:

Proposition 7. *Let* $\mathbf{R}$ *be a homogeneous Banach algebra on* ∂D *which contains* $C^k(\partial D)$ *for some positive integer* k. *Let* I *be a closed primary ideal in* $\mathbf{R}^+$ *which is contained in* $\mathbf{R}_0^+$ *and let* $\alpha = \alpha(I)$. *Then* $\overline{(e^{i\theta}-1)^{4k+1} I} \subseteq I_\alpha$ *and* $\overline{(e^{i\theta}-1)^{2k+4} I_\alpha} \subseteq I$. *If in addition* $\mathbf{R}$ *satisfies the analytic Ditkin condition, then* $I = I_\alpha$.

Proof. To prove the inclusion $\overline{(e^{i\theta}-1)^{4k+1} I} \subseteq I_\alpha$ consider $f \in I$. Since $f \in \mathbf{R}_0^+$, it follows from the definition of α that $f = v_\alpha F$ where $F \in C^+(\partial D)$. ($C(\partial D)$ is the algebra of continuous functions on ∂D.) Therefore $(e^{i\theta}-1)^{2k+1} F = g_{-\alpha} f \in R \cap C^+(\partial D) = \mathbf{R}^+$ and consequently $(e^{i\theta}-1)^{4k+1} f = g_\alpha (e^{i\theta}-1)^{2k+1} F \in I_\alpha$. Since this is true for every $f \in I$ and I_α is closed, the required inclusion is proved. To prove the inclusion $\overline{(e^{i\theta}-1)^{2k+4} I_\alpha} \subset I$, consider the quotient algebra $\mathbf{B} = \mathbf{R}^+/I$, and let u be the image of e^{it} under the canonical map of $\mathbf{R}^+$ onto $\mathbf{B}$. Since $e^{it}-1 \in \mathbf{R}_0^+$, $u-1$ is contained in the single maximal ideal of $\mathbf{B}$ and therefore $\sigma(u) = \{1\}$, [11, p. 34]. Thus the vector valued function $R(u, z) = (u-z)^{-1}$ is defined and analytic in $\mathbf{C} \setminus \{1\}$. We claim that for every $\beta > \alpha$, $\|R(u, z)\|_{\mathbf{B}} = O\ (\exp 2\beta/(1-|z|))$, $|z| \to 1-$. Let $\beta > \alpha$. By the definition of α there exists a function $f \in I$ such that $\mu_f(\{0\}) < \beta$. Therefore by the regularity of μ_f there exists $\delta \in (0, \pi)$ such that $\mu_f(-\delta, \delta) < \beta$. Thus remembering that

$$S_f^{-1}(z) = \exp \left\{ \int \frac{e^{it}+z}{e^{it}+z} \, d\mu_f(t) \right\}$$

[16, p. 66] we obtain that there exists a constant $K > 0$ such that for $z \in D$ with $|\arg z| < \delta$

$$|S_f(z)|^{-1} \leqslant K \exp \frac{2\beta}{1-|z|}.$$

Property (b) of $\tilde{\mathbf{R}}^+$ implies that $\tilde{\mathbf{R}}^+$ operates on u (regarded as a multiplier operator on $\mathbf{B}$) and since $f \in I$ we obtain that $\tilde{f}(u)=0$. Thus remembering that $\|L_w\| = O((1-|w|)^{-k-1})$, $|w| \to 1-$, we obtain from Lemma 5 part (b) that

$$\|R(u, z)\|_{\mathbf{B}} = O((1-|z|)^{-k} |S_f(z)|^{-1}), \quad |z| \to 1-$$

and since $R(u, z)$ is analytic for $z \neq 1$ he above estimate on $|S_f(z)|^{-1}$ implies that

$$\|R(u, z)\| = O\left(\exp \frac{2\beta}{1-|z|}\right), \quad |z| \to 1-.$$

Therefore, by the proof of Theorem 2, $(u-1)^{2k+4} g_\alpha(u) = 0$, which means that $(e^{i\theta}-1)^{2k+4} \tilde{g}_\alpha \in I$ and since I is closed the required inclusion is proved.

Assume now that $\mathbf{R}$ also satisfies the analytic Ditkin condition. It is clear that the second conclusion of the proposition will follow from the first conclusion if we show that every function in $\mathbf{R}_0^+$ belongs to the closure of the ideal generated in $\mathbf{R}^+$ by $(e^{i\theta}-1)f$. Let $f \in \mathbf{R}_0^+$; the analytic Ditkin condition implies that there exists a sequence $(f_n)_{n=1}^{\infty} \subset \mathbf{R}_0^+$ such that $\lim_{n\to\infty} f_n f = f$ in the norm of $\mathbf{R}$. Since $\mathbf{R}^+$ is generated by 1 and e^{it}, we may also assume that f_n are trigonometric polynomials which belong to $\mathbf{R}_0^+$, and therefore there exist trigonometric polynomials $q_n \in \mathbf{R}^+$, $n=1, 2, \ldots$, such that $f_n = (e^{i\theta}-1) q_n$, $n=1, 2, \ldots$. This shows that f belongs to the closure of the ideal generated in $\mathbf{R}^+$ by $(e^{i\theta}-1)f$ and the proposition is proved.

Remark. If $\mathbf{R}$ is a homogeneous Banach algebra on ∂D such that $\|e^{int}\|_{\mathbf{R}} = O(1)$, $n \to \infty$ and $v_n(e^{i\theta}) = (\frac{1}{2}(1+e^{i\theta}))^n$, $e^{i\theta} \in \partial D$, $n=1, 2, \ldots$, then one can show that $\lim_{n\to\infty} \|v_n f\|_{\mathbf{R}} = 0$ for every $f \in \mathbf{R}_0^+$. Thus all these algebras satisfy the analytic Ditkin condition; they include in particular the algebra $C(\partial D)$, the algebra of absolutely convergent Fourier series, and more generally the algebras considered in [18]. For many other examples of homogeneous Banach algebras which satisfy the analytic Ditkin condition we refer to [2, part II].

Our next application is to restriction algebras of the algebras $\mathbf{R}^+$. We extend and improve the results of [1], thereby answering (in the negative) a question raised in that paper. Before stating our result, we recall some definitions and known facts and introduce some notations.

Definition. If $\mathbf{R}$ is a homogeneous Banach algebra on ∂D, we say that E is a $Z\mathbf{R}^+$ set if there exists a non-identically zero function in $\mathbf{R}^+$ which vanishes on E.

If E is a closed subset of ∂D, we shall denote by $\mathbf{R}^+(E)$ the restriction algebra of $\mathbf{R}^+$ to E, which can be identified with the quotient algebra $\mathbf{R}^+/I(E)$, where $I(E)$ denotes the ideal of functions in $\mathbf{R}^+$ which vanish on E. If E is not a $Z\mathbf{R}^+$ set, then $I(E)=\{0\}$ and $\mathbf{R}^+(E)$ is isometrically isomorphic to $\mathbf{R}^+$. On the other hand, if E is a $Z\mathbf{R}^+$ set, then the maximal ideal space of $\mathbf{R}^+(E)$ is E. This is proved in [19] in the case where $\mathbf{R}$ is the algebra of absolutely convergent Fourier series and the same proof carries over to the general case. We shall use this fact in the proof of the next result.

PROPOSITION 8. *Let* $\mathbf{R}$ *be a homogeneous Banach algebra on* ∂D *and let* E *be a* $Z\mathbf{R}^+$ *set. Then*

(a) *If* $\|e^{-int}\|_{\mathbf{R}}=O(n^k)$, $n\to\infty$, *for some integer* $k\geqslant 0$, *then there exists a constant* $c>0$ *such that if* ψ *is a function in* $C(\partial D)$ *which satisfies*

$$\sum_{n=-\infty}^{-1}|\hat{\psi}(n)|\exp(c|n|^{1/2})+\sum_{n=0}^{\infty}|\hat{\psi}(n)|\,\|e^{int}\|_{\mathbf{R}}<\infty,$$

then $\psi|_E$ *(the restriction of* ψ *to* E*) belongs to* $\mathbf{R}^+(E)$.

(b) *If for some positive integer* k, $C^k(\partial D)\subset\mathbf{R}$ *and* ψ *is a function in* $C(\partial D)$ *such that for some constant* $c>0$, $\sum_{n=-\infty}^{\infty}|\hat{\psi}(n)|\exp(c|n|^{1/2})<\infty$, *then* $\psi|_E\in\mathbf{R}^+(E)$.

Proof of (a). First we note that property (a) of $\tilde{\mathbf{R}}^+$, the identity

$$(e^{i\theta}-w)^{-1}=\sum_{n=0}^{\infty}e^{-i(n-1)\theta}w^n,\quad w\in D,\ e^{i\theta}\in\partial D,$$

and the assumption $\|e^{-in\theta}\|_{\mathbf{R}}=O(n^k)$, $n\to\infty$ imply that $\|L_w\|=O((1-|w|)^{-k-1})$, $|w|\to 1-$. Let $u=e^{i\theta}|_E$. Since the maximal ideal space of $\mathbf{R}^+(E)$ is E, it follows that $\sigma(u)\subset E\subset\partial D$. Property (b) of $\tilde{\mathbf{R}}^+$ implies that $\tilde{\mathbf{R}}^+$ operates on u (regarded as a multiplier operator on $\mathbf{R}^+(E)$) and therefore $\tilde{f}(u)=0$ for every $f\in I(E)$. Since E is a $Z\mathbf{R}^+$ set, there exists a function $f\in I(E)$ such that $f\not\equiv 0$, that is, there exists a function $\tilde{f}\not\equiv 0$ in $\tilde{\mathbf{R}}^+$ such that $\tilde{f}(u)=0$. Therefore by Theorem 3 part (c_1), there exists a constant $c>0$ such that

$$\|u^{-n}\|_{\mathbf{R}^+(E)}=O(\exp(cn^{\frac{1}{2}})),\quad n\to\infty.$$

Consequently, if ψ is a continuous function on ∂D which satisfies

$$\sum_{n=-\infty}^{-1}|\hat{\psi}(n)|\exp(c|n|^{1/2})+\sum_{n=0}^{\infty}|\hat{\psi}(n)|\,\|e^{in\theta}\|_{\mathbf{R}}<\infty,$$

then also $\sum_{n=-\infty}^{\infty}|\hat{\psi}(n)|\,\|u^n\|_{\mathbf{R}^+(E)}<\infty$ and therefore $\psi|_E=\sum_{n=-\infty}^{\infty}\hat{\psi}(n)u^n\in\mathbf{R}^+(E)$. Thus part (a) of the proposition is proved.

Proof of (b). Let u be as in part (a). The assumption that $C^k(\partial D)\subset \mathbf{R}$ implies that if ψ is a function in $C(\partial D)$ such that for some constant $c>0$, $\sum_{n=-\infty}^{\infty}|\hat\psi(n)|\exp c|n|^{\frac{1}{2}}<\infty$, then $\sum_{n=0}^{\infty}|\hat\psi(n)|\,\|u^n\|_{\mathbf{R}^+(E)}<\infty$. Therefore (b) will be proved if we show that

$$\|u^{-n}\|_{\mathbf{R}^+(E)}=O\,(\exp\,(\varepsilon n^{\frac{1}{2}})),\quad n\to\infty$$

for every $\varepsilon>0$. Setting $R(u,z)=(u-z)^{-1}$, $z\notin E$ it follows from Lemma 2 part (b) that it suffices to show that

$$\|R(u,z)\|_{\mathbf{R}^+(E)}=O\left(\exp\frac{\varepsilon}{1-|z|}\right),\quad |z|\to 1-$$

for every $\varepsilon>0$. Since $\tilde f(u)=0$ for every $f\in I(E)$, Lemma 1 and the fact that $\|L_w\|=O((1-|w|)^{-k-1})$, $|w|\to 1-$ imply that

$$\|\tilde f(z)\,R(u,z)\|_{\mathbf{R}^+(E)}=O((1-|z|)^{-k-1}),\quad |z|\to 1-$$

for every $f\in I(E)$. Thus by virtue of Lemma 5 part (c). The required estimate of $\|R(u,z)\|_{\mathbf{R}^+(E)}$ will follow, if we show that for every $\varepsilon>0$ there exists a function $f\in I(E)$, $f\not\equiv 0$, such that $\|\mu_d\|<\varepsilon$, where μ_d deotes the discrete part of the measure which defines the singular inner part of $\tilde f$. To show this, consider a function $g\in I(E)$, $g\not\equiv 0$, (such a function exists since E is a $\mathbf{ZR}^+$ set). Let ν be the discrete part of the measure which defines the singular inner part of $\tilde g$. Since ν is a positive discrete measure on $[-\pi,\pi)$, there exists a positive measure ν_1 supported on a finite set $\{\tau_1,\dots,\tau_q\}\subset[-\pi,\pi)$, and a positive discrete measure on $[-\pi,\pi)$, such that $\|\nu_2\|<\varepsilon$ and $\nu=\nu_1+\nu_2$. Consider the function v defined on ∂D by

$$v(e^{i\theta})=\prod_{j=1}^{q}(e^{i\theta}-e^{i\tau_j})^{2k+3}\exp\left\{\int\frac{e^{it}+e^{i\theta}}{e^{it}-e^{i\theta}}\,d\nu_1(t)\right\}$$

for $e^{i\theta}\neq e^{i\tau_j}$, $j=1,2,\dots,q$ and $v(e^{i\tau_j})=0$, $j=1,\dots,q$. It is easily seen that the assumption on ν_1 implies that $v\in C^k(\partial D)$, and therefore $v\in\mathbf{R}$. Let $f=v\cdot g$. Then $f\in\mathbf{R}$, and since the singular inner function

$$\exp\left\{\int\frac{z+e^{i\theta}}{z-e^{i\theta}}\,d\nu_1(t)\right\},\quad z\in D,$$

divides the singular inner part of $\tilde g$, f has a continuous extension to $\bar D$ which is analytic in D, and therefore $f\in\mathbf{R}^+$. Since $g\in I(E)$, $g\not\equiv 0$, also $f\in I(E)$, $f\not\equiv 0$. Noticing that the discrete part of the measure which defines the singular inner part of $\tilde f$ is ν_2, we see that f has the required property, and the proof of the proposition is complete.

The following result is proved in [1, Th. 3]: Let $p=(p_n)_{n=0}^{\infty}$ be an increasing sequence of positive numbers such that $p_0=1$, $p_{n+m}\leqslant p_n p_m$, $m,n=0,1,\dots$, and

$\lim_{n\to\infty} (\log p_n)/n = 0$, and let $\mathbf{B}_p$ be the Banach algebra of functions f in $C^+(\partial D)$ such that $\|f\|_{\mathbf{B}_p} = \sum_{n=0}^{\infty} |\hat{f}(n)| p_n < \infty$. Then if E is a $Z\mathbf{B}_p$ set (the definition is the same as for $Z\mathbf{R}^+$ sets) there exists a constant $c > 0$, such that if ψ is a function in $C(\partial D)$ which satisfies $\sum_{n=-\infty}^{-1} |\hat{\psi}(n)| \exp(c|n|^{\frac{1}{2}}) + \sum_{n=0}^{\infty} |\hat{\psi}(n)| p_n < \infty$, then $\psi|_E \in \mathbf{B}_p(E)$ (the restriction algebra of $\mathbf{B}_p$ to E).

This result follows from part (a) of Proposition 8 by the following simple observations: If $p = (p_n)_{n=0}^{\infty}$ is a sequence with the above properties and $\mathbf{R}$ is the Banach space of all functions $g \in \partial D$ such that $\|g\|_{\mathbf{R}} = \sum_{n=-\infty}^{-1} |\hat{g}(n)| + \sum_{n=0}^{\infty} |\hat{g}(n)| p_n < \infty$, then $\mathbf{R}$ is a homogeneous Banach algebra on ∂D [11, p. 120–121], $\|e^{-int}\|_{\mathbf{R}} = 1$, $n = 0, 1, \ldots$, and $\mathbf{R}^+ = \mathbf{B}_p$. If in addition we assume that $p_n = O(n^k)$, $n \to \infty$, for some integer $k \geq 0$, it is easily seen that $\mathbf{R}$ contains the algebra $C^{k+2}(\partial D)$ (cf. [11, p. 215]) and therefore part (b) of Proposition 8 applies to $\mathbf{B}_p$. In particular we obtain that if $\mathbf{A}$ is the algebra of absolutely convergent Fourier series and ψ is a function in $C(\partial D)$ such that for some constant $c > 0$, $\sum_{n=-\infty}^{\infty} |\hat{\psi}(n)| \exp(c|n|^{\frac{1}{2}}) < \infty$, then $\psi|_E \in \mathbf{A}^+(E)$ for every $Z\mathbf{A}^+$ set E. This improves the result of [1, Th. 1] where a weaker assertion is proved, namely, that for every $Z\mathbf{A}^+$ set E there exists a constant $c > 0$ such that if ψ is in $C(\partial D)$ and $\sum_{n=-\infty}^{\infty} |\hat{\psi}(n)| \exp(c|n|^{\frac{1}{2}}) < \infty$, then $\psi|_E \in \mathbf{A}^+(E)$. In the same way, an analogous improvement of [1, Th. 2] is obtained.

If $\mathbf{R}$ is a homogeneous Banach algebra on ∂D which contains $C^k(\partial D)$ for some integer $k > 0$, then the proof of part (b) of Proposition 8 shows that for every $Z\mathbf{R}^+$ set E, $\lim_{n\to\infty} n^{-\frac{1}{2}} \log \|e^{-in\theta}\|_{\mathbf{R}^+(E)} = 0$. Since this is in particular true for the algebra $\mathbf{A}$ of absolutely convergent Fourier series, we obtain a negative answer to the question raised in [1]. In this connection we mention that a $Z\mathbf{A}^+$ set E for which

$$\lim_{n\to\infty} n^{-\frac{1}{2}} (\log n)^{\beta} \log \|e^{-in\theta}\|_{\mathbf{A}^+(E)} = \infty, \quad \forall \beta > 1,$$

is constructed in [19]. By virtue of these results one can ask whether for every sequence of positive numbers $(c_n)_{n=1}^{\infty}$ such that $\lim c_n = \infty$, there exists a $Z\mathbf{A}^+$ set E such that $\lim_{n\to\infty} n^{-\frac{1}{2}} c_n \log \|e^{-in\theta}\|_{\mathbf{A}^+(E)} = \infty$. We do not know the answer.

8. Extensions of results and problems

If T is a contraction in Hilbert space such that $\sigma(T) = \{1\}$ and $\lim_{n\to-\infty} \log \|T^n\| / |n|^{\frac{1}{2}} = \infty$, then by virtue of Theorems 3 and 4, it is no more possible to prove the existence of non trivial invariant subspaces for T by constructing a bounded analytic function $f \not\equiv 0$, such that $f(T) = 0$. In this connection the following extension of Theorem 1 might be useful:

THEOREM 1*. *Let T be an operator on a reflexive Banach space* $\mathbf{B}$ *with dual* $\mathbf{B}^*$. *Assume that* $\sigma(T)=\{1\}$ *and that* $\|T^n\|=0(n^k)$, $n\to\infty$ *for some integer* $k\geqslant 0$. *If there exist non zero vectors* $x_0\in\mathbf{B}$ *and* $y_0\in\mathbf{B}^*$ *such that*

$$|(T^n x_0, y_0)| = O(\exp(c|n|^{1/2})), \quad n\to-\infty \tag{36}$$

for some constant $c>0$, then T has a non trivial invariant subspace.

Condition (36) is much weaker than (3) and can be established in some concrete cases where (3) does not hold. It is also clear that condition (36) is necessary for the existence of a non trivial invariant subspace for T, since if T has a non trivial invariant subspace so does T^{-1}, and if $x_0\in B$ is any non cyclic vector for T^{-1}, then there exists $y_0\neq 0$ in $\mathbf{B}^*$ such that $(T^n x_0, y_0)=0$, $n=0, -1, -2, \ldots$.

The proof of Theorem 1* depends on methods which are different from those used in this paper, and it will be given elsewhere. We only mention here that in the proof of Theorem 1* we use also Theorem 1.

Another result which we mention here without proof is the following:

THEOREM. *Let T be an operator on a Banach space such that* $\sigma(T)=\{1\}$ *and assume that*

$$0<\overline{\lim_{n\to-\infty}}\frac{\log\|T^n\|}{|n|^{1/2}}<\infty. \tag{37}$$

If $\|T^n\|=O(p_n)$, $n\to\infty$, *where* $(p_n)_{n=0}^{\infty}$ *is a sequence of positive numbers such that* $\sum_{n=1}^{\infty}\log p_n/n^{3/2}<\infty$, *then T has an uncountable chain of hyperinvariant subspaces.*

Thus in Theorem 1, condition (2) can be replaced for example by the condition: $\|T^n\|=O(\exp(n^\alpha))$, $n\to\infty$ for some $0<\alpha<\frac{1}{2}$, provided that condition (37) holds. We do not know whether operators T which satisfy the above conditions admit non trivial analytic annihilating functions. The proof of the Theorem uses a different method, and will also be given elsewhere.

We conclude with two problems. It is natural to ask, to what extent the converses of Theorem 3(c_1), Corollary 2(c) and Corollary 3(1) are true. More precisely we ask:

Problem 1. Suppose that T is a c.n.u. contraction acting on a Hilbert space, which satisfies condition (3) and assume that $\sigma(T)$ is of measure zero (with respect to Lebesque measure on ∂D). Is T a C_0 operator?

Problem 2. Let T be an operator acting on a Banach space and assume that T satisfies conditions (2) and (3) and that $\sigma(T)$ is a Carleson set. Is T in $C_0^1(H_m^\infty)$ for some positive integer m?

Remark. In an earlier version of this paper we also asked whether every c.n.u. contraction acting on a Hilbert space, which satisfies condition (3) is in C_0. To this Professor Cirpian Foiaş gave the following counter example:

Let $\mathbf{H}$ be the Hilbert space of all functions f in $L^2(\partial D)$ such that

$$\|f\|_{\mathbf{H}} = \left(\sum_{n=-\infty}^{-1} |\hat{f}(n)|^2 n^2 + \sum_{n=0}^{\infty} |\hat{f}(n)|^2\right)^{1/2} < \infty,$$

and let T be the operator defined on $\mathbf{H}$ by $Tf(e^{i\theta}) = e^{i\theta} f(e^{i\theta})$, $e^{i\theta} \in \partial D$, $f \in \mathbf{H}$. Then T is a c.n.u. contraction (cf. [22, Theorem 3.2.]) and $\|T^{-n}\| = O(n)$, $n \to \infty$. However, for every $\varphi \in H^\infty$, $\varphi(T)f = \varphi f$, $f \in \mathbf{H}$ and therefore if $\varphi(T) = 0$, then $\varphi = 0$ a.e, that is T is not in C_0. It is easy to see that in this case $\sigma(T) = \partial D$.

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