

OPERATORS WHICH HAVE A CLOSED QUASI-NILPOTENT PART

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(Communicated by Joseph A. Ball)

ABSTRACT. We find several conditions for the quasi-nilpotent part of a bounded operator acting on a Banach space to be closed. Most of these conditions are established for semi-Fredholm operators or, more generally, for operators which admit a generalized Kato decomposition. For these operators the property of having a closed quasi-nilpotent part is related to the so-called single valued extension property.

1. THE QUASI-NILPOTENT PART OF AN OPERATOR AND THE SVEP

The single valued extension property was first introduced by Dunford [5], [6] and has, successively, received a more systematic treatment in Dunford-Schwartz [7]. It also plays an important role in local spectral theory; see the monograph of Laursen and Neumann [14]. The following local version of this property has been studied in recent papers by Aiena and Monsalve [1], [2] and previously by Finch [8].

Definition 1.1. Let X be a complex Banach space and $T \in L(X)$. The operator T is said to have the *single valued extension property* at $\lambda_o \in \mathbb{C}$ (abbreviated SVEP at λ_o), if for every open disc \mathbf{D}_{λ_o} centered at λ_o the only analytic function $f : \mathbf{D}_{\lambda_o} \rightarrow X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in \mathbf{D}_{\lambda_o}$ is the function $f \equiv 0$.

An operator $T \in L(X)$ is said to have the SVEP if T has the SVEP at every point $\lambda \in \mathbb{C}$.

Let us consider the so-called *local resolvent set* $\rho_T(x)$ of T at the point $x \in X$, defined as the union of all open subsets \mathcal{U} of \mathbb{C} such that there exists an analytic function $f : \mathcal{U} \rightarrow X$ which satisfies $(\lambda I - T)f(\lambda) = x$ for all $\lambda \in \mathcal{U}$. The *local spectrum* $\sigma_T(x)$ of T at x is the set defined by $\sigma_T(x) := \mathbb{C} \setminus \rho_T(x)$. Obviously, $\sigma_T(x) \subseteq \sigma(T)$, where $\sigma(T)$ denotes the spectrum of T .

For every subset Ω of \mathbb{C} , the *analytic spectral subspace* of T associated with Ω is the set

$$X_T(\Omega) := \{x \in X : \sigma_T(x) \subseteq \Omega\}.$$

Received by the editors December 8, 2000 and, in revised form, April 20, 2001.

2000 *Mathematics Subject Classification.* Primary 47A10, 47A11; Secondary 47A53, 47A55.

Key words and phrases. Quasi-nilpotent part, single valued extension property, operators with a generalized Kato decomposition.

The research of the first two authors was supported by the International Cooperation Project between the University of Palermo (Italy) and Conicit-Venezuela.

The research of the third author was supported by DGICYT, Spain.

It is easy to see from the definition that $X_T(\Omega)$ is a T -hyperinvariant linear subspace of X [14].

The SVEP, as well as the SVEP at a point $\lambda_o \in \mathbf{C}$, may be characterized in a very simple way.

Theorem 1.2. *Let $T \in L(X)$, X a Banach space. Then:*

- (i) *T has the SVEP at λ_o if and only if $\ker(\lambda_o I - T) \cap X_T(\emptyset) = \{0\}$; see [1, Theorem 1.9].*
- (ii) *T has the SVEP if and only if $X_T(\emptyset) = \{0\}$, and this is the case if and only if $X_T(\emptyset)$ is closed; see [14, Proposition 1.2.16].*

Definition 1.3. Let X be a Banach space and $T \in L(X)$. The *analytical core* of T is the set $K(T)$ of all $x \in X$ such that there exists a sequence $(u_n)_{n \in \mathbb{N}} \subset X$ and $\delta > 0$ for which $x = u_o$, $Tu_{n+1} = u_n$ and $\|u_n\| \leq \delta^n \|x\|$, for every $n \in \mathbb{N} = \{0, 1, \dots\}$.

It is easy to check that $K(T)$ is a linear, generally not closed, subspace of X . Furthermore, $T(K(T)) = K(T)$ and if F is a closed subspace of X for which $T(F) = F$, then $F \subseteq K(T)$; see [19, Proposition 2]. Note that if T is quasi-nilpotent, then $K(T) = \{0\}$; see [16, Remarque 1.1].

Definition 1.4. Let $T \in L(X)$, X a Banach space. The *quasi-nilpotent part* of T is the set

$$H_o(T) := \{x \in X : \lim_{n \rightarrow \infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

Also $H_o(T)$ is a linear subspace of X , generally not closed. Furthermore, $\ker(T^m) \subseteq H_o(T)$ for every $m \in \mathbb{N}$, and T is quasi-nilpotent if and only if $H_o(T) = X$; see [20, Theorem 1.5].

The next result may be found in Vrbová [20], or Mbekhta [16]; see also [14, Propositions 3.3.7 and 3.3.13].

Theorem 1.5. *For a bounded operator $T \in L(X)$, X a Banach space, we have:*

- (i) $K(\lambda_o I - T) = X_T(\mathbf{C} \setminus \{\lambda_o\})$.
- (ii) $H_o(\lambda_o I - T) \subseteq X_T(\{\lambda_o\})$ and the equality holds whenever T has the SVEP.

In the sequel by M^\perp we shall denote the *annihilator* of the subset $M \subseteq X$, and by ${}^\perp N$ the *pre-annihilator* of the subset $N \subseteq X^*$.

Theorem 1.6. *For a bounded operator $T \in L(X)$, X a Banach space, the following implications hold:*

- (i) $H_o(\lambda_o I - T)$ closed $\Rightarrow H_o(\lambda_o I - T) \cap K(\lambda_o I - T) = \{0\} \Rightarrow T$ has the SVEP at λ_o .
- (ii) $X = H(\lambda_o I - T) + K(\lambda_o I - T) \Rightarrow T^*$ has the SVEP at λ_o .

Proof. Without loss of generality, we may consider $\lambda_o = 0$.

(i) Assume that $H_o(T)$ is closed. Let \tilde{T} denote the restriction of T to the Banach space $H_o(T)$. Clearly, $H_o(T) = H_o(\tilde{T})$, thus \tilde{T} is quasi-nilpotent. Therefore $K(\tilde{T}) = \{0\}$. It is easy to see that $H_o(T) \cap K(T) = K(\tilde{T})$. This shows the first implication.

The second implication of (i) is an immediate consequence of Theorem 1.5. Indeed, we have

$$\ker(\lambda_o I - T) \cap X_T(\emptyset) \subseteq H_o(\lambda_o I - T) \cap K(\lambda_o I - T),$$

so, if the last intersection is $\{0\}$, then T has the SVEP at λ_o , by Theorem 1.2.

(ii) From [15, Proposition 1.8] we know that $H_o(T) \subseteq^\perp K(T^*)$ and therefore $K(T^*) \subseteq H_o(T)^\perp$. We also have $H_o(T^*) \subseteq K(T)^\perp$. Indeed, let $\varphi \in H_o(T^*)$ and consider an arbitrary element $x \in K(T)$. According to the definition of $K(T)$, there is a sequence $(u_n)_{n \in \mathbf{N}} \subset X$, and a $\delta > 0$, such that $u_o = x$, $Tu_{n+1} = u_n$ and $\|u_n\| \leq \delta^n \|x\|$ for every $n \in \mathbf{N}$. Clearly, $T^n u_n = x$ for every $n \in \mathbf{N}$. Consequently,

$$|\varphi(x)| = |\varphi(T^n u_n)| = |(T^{*n} \varphi)(u_n)| \leq \|u_n\| \|T^{*n} \varphi\| \leq \delta^n \|T^{*n} \varphi\|,$$

and hence $|\varphi(x)|^{\frac{1}{n}} \leq \delta \|T^{*n} \varphi\|^{\frac{1}{n}}$ for every $n \in \mathbf{N}$. The last term converges to 0 as $n \rightarrow \infty$, since $\varphi \in H_o(T^*)$, and from this it follows that $\varphi(x) = 0$, i.e. $\varphi \in K(T)^\perp$. Finally, if $X = H_o(T) + K(T)$, then $\{0\} = H_o(T)^\perp \cap K(T)^\perp \supseteq H_o(T^*) \cap K(T^*)$. Thus, by part (i), T^* has the SVEP at 0. □

The two implications of part (i) of Theorem 1.6 were observed in [16, Proposition 1.10]. Later we shall prove that in the case of semi-Fredholm operators, or more generally in the case that $\lambda_o I - T$ admits a generalized Kato decomposition, the implications of Theorem 1.6 are actually equivalences.

Theorem 1.6 suggests in a very natural way the following definition:

Definition 1.7. A bounded operator $T \in L(X)$, X a Banach space, is said to have *property (Q)* if $H_o(\lambda I - T)$ is closed for every $\lambda \in \mathbf{C}$.

Recall that a bounded operator $T \in L(X)$, X a Banach space, is said to have *Dunford's property (C)*, shortly *property (C)*, if the analytic subspace $X_T(\Omega)$ is closed for every closed subset $\Omega \subseteq \mathbf{C}$. From part (ii) of Theorem 1.2 it follows that if $T \in L(X)$ has *property (C)*, then T has the SVEP.

An obvious consequence of part (ii) of Theorem 1.5 is that if T has *property (C)*, then $H_o(\lambda I - T) = X_T(\{\lambda\})$ is closed for every $\lambda \in \mathbf{C}$, so that the following implications hold:

$$(1) \quad T \text{ has property (C)} \Rightarrow T \text{ has property (Q)} \Rightarrow T \text{ has the SVEP.}$$

Note that neither of the implications in (1) may be reversed in general. A first counterexample, which shows that the first implication is not reversed in general, may be found among the convolution operators of group algebras.

Recall that a Banach algebra A is said to be *semi-simple* if the radical $\text{rad } A = \{0\}$ [3]; A is said to be *semi-prime* if there is no non-zero two-sided ideal J for which $J^2 = \{0\}$. Note that A is semi-prime if and only if, for every $x \in A$, the identity $xAx = \{0\}$ implies that $x = 0$, and that a commutative algebra is semi-prime if and only if it contains no non-zero nilpotent elements. Clearly, any semi-simple Banach algebra is semi-prime. A map $T : A \rightarrow A$, A a Banach algebra, is said to be a *multiplier* if $(Tx)y = x(Ty)$ holds for all $x, y \in A$. Note that if T is a multiplier of a semi-prime commutative Banach algebra A , then $(Tx)y = x(Ty) = T(xy)$ for every $x, y \in A$; see the proof of [11, Theorem 1.1.1]. A very important example of a multiplier is given in the case that A is the semi-simple Banach algebra $L_1(G)$, the group algebra of a locally compact abelian group G with convolution as multiplication. Indeed, in this case to any complex Borel measure μ on G there corresponds a multiplier T_μ defined by

$$T_\mu(f) := \mu \star f \quad \text{for all } f \in L_1(G),$$

where

$$(\mu \star f)(t) := \int_G f(t-s)d\mu(s).$$

The classical Helson-Wendel Theorem shows that each multiplier is a convolution operator and the multiplier algebra of $A := L_1(G)$ may be identified with the measure algebra $M(G)$; see [11, Chapter 0].

Theorem 1.8. *Let T be a multiplier of a semi-simple Banach algebra A . Then*

$$H_o(T) = \ker T.$$

In particular, T has property (Q).

Proof. We know that $\ker T \subseteq H_o(T)$, so it remains to prove the inverse inclusion.

Suppose that $x \in H_o(T)$. By an easy inductive argument we have

$$(Ty)^n = (T^n y)y^{n-1} \text{ for every } y \in A \text{ and } n \in \mathbb{N}.$$

From this it follows that

$$\begin{aligned} \|(aTx)^n\| &= \|(T^na x)^n\| = \|T^n(ax)(ax)^{n-1}\| \\ &\leq \|a\| \|T^n x\| \|(ax)^{n-1}\| \end{aligned}$$

for every $a \in A$, so the spectral radius of the element aTx satisfies

$$r(aTx) = \lim_{n \rightarrow \infty} \|(aTx)^n\|^{\frac{1}{n}} = 0$$

for every $a \in A$. This implies that $Tx \in \text{rad } A$; see [3, §25, Proposition 1]. Since A is semi-simple then $Tx = 0$ and therefore $H_o(T) \subseteq \ker T$.

The last assertion is clear, because $\lambda I - T$ is a multiplier of A for every $\lambda \in \mathbf{C}$. \square

Clearly, if T is a quasi-nilpotent multiplier, then $\ker T = H_o(T) = A$, so $T = 0$ ([13]).

The following example shows that the assumption that A is semi-simple in Theorem 1.8 cannot be replaced by the weaker assumption that A is semi-prime.

Example 1.9. Let $\omega := (\omega_n)_{n \in \mathbb{N}}$ be a sequence with the property that $0 < \omega_{m+n} \leq \omega_m \omega_n$ for all $m, n \in \mathbb{N}$. Let $\ell^1(\omega)$ denote the space of all complex sequences $x := (x_n)_{n \in \mathbb{N}}$ for which $\|x\|_\omega := \sum_{n=0}^\infty \omega_n |x_n| < \infty$. The space $\ell^1(\omega)$ equipped with convolution

$$(x \star y)_n := \sum_{j=0}^n x_{n-j} y_j \text{ for all } n \in \mathbb{N}$$

is a commutative unital Banach algebra. Let A_ω denote the maximal ideal of $\ell^1(\omega)$ given by

$$A_\omega := \{(x_n)_{n \in \mathbb{N}} \in \ell^1(\omega) : x_0 = 0\}.$$

The Banach algebra A_ω is an integral domain and hence semi-prime. Moreover, if the weight sequence ω satisfies the condition $\rho_\omega := \lim_{n \rightarrow \infty} \omega_n^{\frac{1}{n}} = 0$, then A_ω is a radical algebra ([14, Example 4.1.9]), i.e. coincides with its radical. Now, fix $0 \neq a \in A_\omega$ and let $T_a(x) := a \star x$, $x \in A_\omega$, denote the multiplication operator by the element a . From the estimate

$$\|T^n x\|_\omega^{\frac{1}{n}} = \|a^n \star x\|_\omega^{\frac{1}{n}} \leq \|a^n\|_\omega^{\frac{1}{n}} \|x\|_\omega^{\frac{1}{n}}$$

we see that T_a is quasi-nilpotent, thus $H_o(T_a) = A_\omega$. On the other hand, A_ω is an integral domain so that $\ker T_a = \{0\}$.

Theorem 1.8 suggests the way of obtaining examples of operators which have property (Q) but not property (C). Indeed, there are convolution operators T_μ , $\mu \in M(G)$, on the group algebra $L_1(G)$ which do not enjoy property (C); see [14, Chapter 4].

The next example, which is obtained by a slight modification of Example 3.9 of [4], shows that also the second implication of (1) may be not reversed in general.

Example 1.10. Let $X := \ell_2 \oplus \ell_2 \cdots$ and define

$$T_n e_i := \begin{cases} e_{i+1} & \text{if } i = 1, \dots, n, \\ \frac{e_{i+1}}{i-n} & \text{if } i > n. \end{cases}$$

Clearly,

$$\|T_n^{n+k}\| = \frac{1}{k!} \text{ and } \left(\frac{1}{k!}\right)^{\frac{1}{n+k}} \rightarrow 0 \text{ as } k \rightarrow \infty.$$

From this it follows that $\sigma(T_n) = \{0\}$. Moreover, T_n is injective and the point spectrum $\sigma_p(T_n)$ is empty. Now, let $T := T_1 \oplus \cdots \oplus T_n \oplus \cdots$. From $\|T_n\| = 1$, for every $n \in \mathbf{N}$, we obtain $\|T\| = 1$. From $\sigma_p(T_n) = \emptyset$ it also follows that $\sigma_p(T) = \emptyset$. Take $x = (x_n) \in X$ with $x_n := \frac{e_n}{n}$. We have

$$\|x\| = \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{\frac{1}{2}} < \infty,$$

thus $x \in X$. Moreover,

$$\|T^n x\|^{\frac{1}{n}} \geq \|T_n^n \frac{e_1}{n}\|^{\frac{1}{n}} = \left(\frac{1}{n}\right)^{\frac{1}{n}}$$

and the last term does not converge to 0. Clearly, $x \notin H_o(T)$. Finally, $\ell_2 \oplus \ell_2 \cdots \oplus \ell_2 \oplus \{0\} \cdots \subset H_o(T)$, where the non-zero terms are n . Since $H_o(T)$ contains all sequences with only finitely many non-zero terms, it follows that $H_o(T)$ is dense in X . Since $H_o(T) \neq X$ then $H_o(T)$ is not closed, thus T has not property (Q). Note that the operator T has the SVEP, since $\sigma_p(T)$ is empty.

2. THE CASE OF SEMI-FREDHOLM OPERATORS

For every linear operator T on a vector space X , let us consider the increasing sequence of kernels $\ker T^n$ and the decreasing sequence of ranges $T^n(X)$.

Definition 2.1. Let T be a linear operator on a vector space X . The *generalized kernel* of T is defined by

$$\mathcal{N}(T) := \bigcup_{n \in \mathbf{N}} \ker T^n.$$

The *hypperrange* of T is defined by

$$T^\infty(X) := \bigcap_{n \in \mathbf{N}} T^n(X).$$

Clearly, $T^\infty(X)$ is a T -invariant subspace and it is easily seen that, if X is a Banach space, $K(T) \subseteq T^\infty(X)$. Moreover, for every $n \in \mathbf{N}$ we have

$$\ker (\lambda_o I - T)^n \subseteq \mathcal{N}(\lambda_o I - T) \subseteq H_o(\lambda_o I - T).$$

Recall that T is said to have *finite ascent* if $\mathcal{N}(T) = \ker T^k$ for some positive integer k . Clearly, in such a case there is a smallest positive integer $p = p(T)$ such that $\ker T^p = \ker T^{p+1}$. The positive integer p is called the *ascent* of T . Analogously, T is said to have *finite descent* if $T^\infty(X) = T^k(X)$ for some k . The smallest integer $q = q(T)$ such that $T^{q+1}(X) = T^q(X)$ is called the *descent* of T . It is possible to prove that if $p(T)$ and $q(T)$ are both finite, then $p(T) = q(T)$; see [9, Proposition 38.3].

Theorem 2.2 ([1]). *For a bounded operator T on a Banach space X the following implications hold:*

- i) $p(\lambda_o I - T) < \infty \Rightarrow T$ has SVEP at λ_o .
- ii) $q(\lambda_o I - T) < \infty \Rightarrow T^*$ has SVEP at λ_o .

Hence each one of the two conditions $p(\lambda_o I - T) < \infty$ or $H_o(\lambda_o I - T)$ closed implies the SVEP at λ_o . In general these two conditions are not related. The operator T of Example 1.10 has ascent $p(T) = 0$ and quasi-nilpotent part $H_o(T)$ not closed. In the following example we find an operator T which has a closed quasi-nilpotent part but ascent $p(T) = \infty$.

Example 2.3. Let $T : \ell_2 \rightarrow \ell_2$ be defined by

$$Tx := \left(\frac{x_2}{2}, \dots, \frac{x_n}{n}, \dots \right), \text{ where } x = (x_1, \dots, x_n, \dots).$$

It is easily seen that $\|T^k\| = \frac{1}{(k+1)!}$ and from this it follows that T is quasi-nilpotent and therefore $H_o(T) = \ell_2$. Obviously, $p(T) = \infty$.

Definition 2.4. An operator $T \in L(X)$, X a Banach space, is said to be *semi-regular* if $T(X)$ is closed and $\ker T \subseteq T^\infty(X)$.

An operator $T \in L(X)$ is said to admit a *generalized Kato decomposition*, abbreviated GKD, if there exists a pair of T -invariant closed subspaces (M, N) such that $X = M \oplus N$, the restriction $T|_M$ is semi-regular and $T|_N$ is quasi-nilpotent.

Remark 2.5. In the sequel we list some examples of operators which admit a GKD.

(i) Every semi-regular operator has the GKD $M = X$ and $N = \{0\}$. Note that if T is semi-regular, then $\overline{H_o(T)} = \overline{\mathcal{N}(T)}$; see [15, Proposition 2.10].

(ii) Every quasi-nilpotent operator has the GKD $M = \{0\}$ and $N = X$.

(iii) An important case is obtained if we assume in the definition above that $T|_N$ is nilpotent. In this case T is said to be of *Kato type*; see [15]. Obviously, any semi-regular operator is of Kato type. Note that if T is of Kato type, then $T^\infty(X) = K(T)$ and $K(T)$ is closed; see [1, Lemma 2.4] or [2, Theorem 2.3 and Theorem 2.4].

(iv) Let

$$\Phi_+(X) := \{T \in L(X) : \dim \ker T < \infty, T(X) \text{ closed}\}$$

denote the class of all *upper semi-Fredholm* operators, and let

$$\Phi_-(X) := \{T \in L(X) : \text{codim } T(X) < \infty\}$$

denote the class of all *lower semi-Fredholm* operators. The class of all *semi-Fredholm operators* is defined as $\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)$ and the class of all *Fredholm operators* is defined as $\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$. A well-known result of Kato [10, Theorem 4] establishes that every $T \in \Phi_\pm(X)$ is of Kato type. More precisely, T admits a GKD (M, N) with $T|_N$ nilpotent and $\dim N < \infty$.

Recall that for $T \in \Phi_{\pm}(X)$ the *index* of T is defined by $\text{ind } T := \dim \ker T - \text{codim } T(X)$. The index is an integer or $\pm\infty$.

Theorem 2.6. *Let $T \in L(X)$, X a Banach space, and assume that $\lambda_o I - T$ has a GKD (M, N) . Then the following properties are equivalent:*

- (i) T has the SVEP at λ_o .
- (ii) $H_o(\lambda_o I - T) \cap K(\lambda_o I - T) = \{0\}$.
- (iii) $H_o(\lambda_o I - T)$ is closed.
- (iv) $H_o(\lambda_o I - T) = N$.

In particular, if $\lambda_o I - T$ is semi-regular the conditions (i)-(iv) are equivalent to the following one.

- (v) $H_o(\lambda_o I - T) = \{0\}$.

Proof. Also here we only consider the case $\lambda_o = 0$. Clearly, (iv) \Rightarrow (iii) and from Theorem 1.6 we know that (iii) \Rightarrow (ii) \Rightarrow (i).

(i) \Rightarrow (iv). First note that if T admits a GKD (M, N) , then $H_o(T) = H_o(T | M) + H_o(T | N)$. The inclusion $H_o(T) \supseteq H_o(T | M) + H_o(T | N)$ is obvious. In order to show the opposite inclusion, consider an arbitrary element $x \in H_o(T)$ and let $x = u + v$, with $u \in M$ and $v \in N$. Since $T | N$ is quasi-nilpotent then $N = H_o(T | N) \subseteq H_o(T)$. Consequently, $u = x - v \in H_o(T) \cap M = H_o(T | M)$ and therefore $H_o(T) \subseteq H_o(T | M) + H_o(T | N)$. Hence $H_o(T) = H_o(T | M) + H_o(T | N) = H_o(T | M) + N$. Now, suppose that T has the SVEP 0. Clearly, the SVEP at a point is inherited by the restrictions to closed invariant subspaces, so $T | M$ has the SVEP at 0 and from the semi-regularity of $T | M$ it follows that $T | M$ is injective; see [1, Theorem 2.14]. From this we obtain

$$H_o(T | M) = \overline{\bigcup_{n=1}^{\infty} \ker (T | M)^n} = \{0\}$$

and therefore $H_o(T) = N$.

The final assertion is clear. □

Corollary 2.7. *Let $T \in L(X)$, X a Banach space, and assume that $\lambda_o I - T$ is of Kato type. Then the conditions (i)-(iv) of Theorem 2.6 are equivalent to the following one:*

- (v') $p(\lambda_o I - T) < \infty$.

In this case, if $p := p(\lambda_o I - T)$, then

$$H_o(\lambda_o I - T) = \mathcal{N}(\lambda_o I - T) = \ker (\lambda_o I - T)^p.$$

Proof. Assume $\lambda_o = 0$. We know that the inclusions $H_o(T) \supseteq \mathcal{N}(T) \supseteq \ker T^n$ hold for every $T \in L(X)$ and for every $n \in \mathbf{N}$. Let (M, N) be a GKD for T such that $(T|N)^k = 0$ for some $k \in \mathbf{N}$. Then $H_o(T) = N \subseteq \ker T^k$ and hence $H_o(T) = \mathcal{N}(T) = \ker T^k$. From this it follows that $p := p(T) \leq k$ and therefore $\ker T^k = \ker T^p$. □

The next result shows that, under the assumption that $\lambda_o I - T$ is semi-Fredholm, another equivalent condition can be added to those given in Corollary 2.7.

Theorem 2.8. *Suppose that $\lambda_o I - T \in L(X)$ is a semi-Fredholm operator. Then the following statements are equivalent:*

- (i) $H_o(\lambda_o I - T)$ is closed.
- (ii) $H_o(\lambda_o I - T)$ is finite dimensional.

Proof. We have only to prove the implication (i) \Rightarrow (ii). This immediately follows from the mentioned Kato decomposition of a semi-Fredholm operator: if (M, N) is a GKD for $\lambda_o I - T$ such that $\lambda_o I - T|_N$ is nilpotent and $\dim N < \infty$, then $H_o(\lambda_o I - T) = N$, by Theorem 2.6. \square

The preceding result was obtained in [12, Theorem 2] under the assumption that $\lambda_o I - T$ is a Fredholm operator.

Theorem 2.9. *Suppose that $\lambda_o I - T \in L(X)$ is of Kato type. Then the following statements are equivalent:*

- (i) T^* has the SVEP at λ_o .
- (ii) $q := q(\lambda_o I - T) < \infty$.
- (iii) $X = H_o(\lambda_o I - T) + K(\lambda_o I - T)$.

Moreover, if any of the equivalent conditions (i)-(iii) holds, then

$$(\lambda_o I - T)^\infty(X) = K(\lambda_o I - T) = (\lambda_o I - T)^q(X).$$

Proof. Assume that $\lambda_o = 0$. The equivalence (i) \Leftrightarrow (ii) has been proved in [2, Theorem 2.6]. The implication (iii) \Rightarrow (i) has been proved in Theorem 1.6.

(ii) \Rightarrow (iii). Assume that $q := q(T) < \infty$. Since T is of Kato type then $K(T) = T^\infty(X) = T^q(X)$. Moreover, $X = \ker T^q + T^n(X)$ for every $n \in \mathbf{N}$ (see [9, Proposition 38.2]), and therefore $X = H_o(T) + T^\infty(X)$. \square

Theorem 2.10. *Suppose that $\lambda_o I - T \in L(X)$ is a semi-Fredholm operator. Then the following statements are equivalent:*

- (i) T^* has the SVEP at λ_o .
- (ii) $K(\lambda_o I - T)$ is finite codimensional.

Proof. Also here we assume that $\lambda_o = 0$.

(i) \Rightarrow (ii). From Fredholm theory we know that T^* is also a semi-Fredholm operator and $\text{ind } T^* = -\text{ind } T$. Now, if T^* has the SVEP at 0, then $\text{ind } T^* \leq 0$ (see [1, Corollary 2.7]) and therefore $\text{ind } T \geq 0$. From this it follows that T is a lower semi-Fredholm and consequently also T^q is lower semi-Fredholm, i.e. $T^q(X) = T^\infty(X) = K(T)$ is finite codimensional.

(ii) \Rightarrow (i). Since $K(T) = T^\infty(X)$, condition (ii) means that $T^\infty(X)$ is of finite codimension. But from this it is immediate that $q(T) < \infty$, so that T^* has the SVEP at 0, by Theorem 2.9. \square

It should be noted that if T is semi-Fredholm, then $T^\infty(X)$ coincides with the so-called algebraic core of T , i.e. the greatest subspace M of X for which $T(M) = M$; see for instance [1, Theorem 2.3].

Corollary 2.11. *Assume that $\lambda_o I - T \in L(X)$ is a semi-Fredholm operator. Then the following statements are equivalent:*

- (i) T and T^* have the SVEP at λ_o .
- (ii) $X = H_o(\lambda_o I - T) \oplus K(\lambda_o I - T)$.
- (iii) $H_o(\lambda_o I - T)$ is closed and $K(\lambda_o I - T)$ is finite-codimensional.
- (iv) λ_o is a pole of $(\lambda I - T)^{-1}$, or equivalently $p(\lambda_o I - T) = q(\lambda_o I - T) < \infty$.
- (v) The spectrum does not cluster at λ_o .

In particular, if any of the equivalent conditions (i)-(v) holds and $p := p(\lambda_o I - T) = q(\lambda_o I - T)$, then

$$H_o(\lambda_o I - T) = \mathcal{N}(\lambda_o I - T) = \ker(\lambda_o I - T)^p$$

and

$$K(\lambda_o I - T) = (\lambda_o I - T)^\infty(X) = (\lambda_o I - T)^p(X).$$

Proof. The equivalences of (i), (ii), (iii), and (iv) are obtained by combining all the results established in this section. The implication (iv) \Rightarrow (v) is obvious. The implication (v) \Rightarrow (i) is an immediate consequence of the fact that both T and T^* have the SVEP at every point of the resolvent, as well as at every isolated point of the spectrum. \square

Note that rather similar results to those of Corollary 2.11 have been established by Mbekhta [15, Théorème 1.6] and Schmoegeer [19], in the case that λ_o is an isolated point of the spectrum.

Remark 2.12. Recall that for every $T \in L(X)$ the *semi-Fredholm region* is defined to be

$$\Sigma(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is semi-Fredholm}\}.$$

It is well-known that $\Sigma(T)$ is an open set and hence it may be decomposed in connected disjoint open nonempty components. Suppose that T has SVEP at some $\lambda_o \in \Omega$, Ω a component of $\Sigma(T)$. Then, by Theorem 2.6 and Corollary 2.7

$$\begin{aligned} \{0\} &= H_o(\lambda_o I - T) \cap K(\lambda_o I - T) = \mathcal{N}(\lambda_o I - T) \cap (\lambda_o I - T)^\infty(X) \\ &= \overline{\mathcal{N}(\lambda_o I - T)} \cap (\lambda_o I - T)^\infty(X), \end{aligned}$$

and from the constancy of the map $\lambda \in \Omega \rightarrow \overline{\mathcal{N}(\lambda I - T)} \cap (\lambda I - T)^\infty(X)$ (see [18, Theorem 4.2]) we conclude that $\mathcal{N}(\lambda I - T) \cap (\lambda I - T)^\infty(X) = \{0\}$ for every point $\lambda \in \Omega$. From [2, Theorem 1.10] it follows that T has the SVEP at every point $\lambda \in \Omega$. Moreover, the set

$$\Gamma := \{\lambda \in \Omega : \text{jump } (\lambda I - T) \neq 0\}$$

is countable ([18]) and is equal to the set of all $\lambda \in \Omega$ such that $\lambda I - T$ is not semi-regular; see [21, Proposition 2.2]. From Theorem 2.6 then $H_o(\lambda I - T) = \{0\}$ for every $\lambda \in \Omega \setminus \Gamma$, while the remaining points $\lambda \in \Gamma$ are eigenvalues with ascent $p := p(\lambda I - T) < \infty$, $H_o(\lambda I - T) = \ker (\lambda I - T)^p$ and $0 < \dim H_o(\lambda I - T) < \infty$, by Corollary 2.7 and Theorem 2.8. In particular this situation occurs for every component of the semi-Fredholm region of an operator which has the SVEP.

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