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# Optical operations on wave functions as the Abelian subgroups of the special affine Fourier transformation

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The special affine Fourier transformation (SAFT) is a generalization of the fractional Fourier transformation (FRT) and represents the most general lossless inhomogeneous linear mapping, in phase space, as the integral transformation of a wave function. Here we first summarize the most well-known optical operations on light-wave functions (i.e., the FRT, lens transformation, free-space propagation, and magnification), in a unified way, from the viewpoint of the one-parameter Abelian subgroups of the SAFT. Then we present a new operation, which is the Lorentz-type hyperbolic transformation in phase space and exhibits squeezing. We also show that the SAFT including these five operations can be generated from any two independent operations.

In recent years fractional Fourier transformation<sup>1,2</sup> (FRT) has attracted much attention in the field of optical signal processing.<sup>3-5</sup> Geometrically the FRT of a wave function is the  $\theta$  rotation of the corresponding Wigner distribution function in phase space<sup>4</sup> and reduces to the classical Fourier transformation when  $\theta = \pi/2$ .

In a recent paper<sup>6</sup> we generalized the FRT to the integral transformation that is associated with the following most general inhomogeneous lossless linear mapping in phase space:

$$\begin{bmatrix} x \\ k \end{bmatrix} \rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ k \end{bmatrix} + \begin{bmatrix} m \\ n \end{bmatrix}, \quad (1)$$

$$ad - bc = 1. \quad (2)$$

Here  $x$  and  $k$  are the position and wave-number variables, respectively. The parameters  $\{a, b, c, d, m, n\}$  are constants independent of the phase-space point  $(x, k)$ . This transformation is referred to as the special affine transformation. It transforms any convex body into another convex body (e.g., any triangular area into another triangular area) under any set of linear deformations, rotations, and translations in phase space. Equation (2) guarantees that the area of the body is preserved by the transformation. Such a set of transformations forms the inhomogeneous special linear group  $ISL(2, \mathbf{R})$  [the homogeneous  $SL(2, \mathbf{R})$  of Ref. 7], which is, in general, non-Abelian (i.e., the product of two transformations are noncommutative). In Ref. 6 we derived the integral representation of the wave-function transformation associated with the mapping [Eq. (1)] under condition (2). We refer to it as the special affine Fourier transformation (SAFT). In terms of the parameters in Eq. (1), the SAFT of a function  $u(x)$  is given as follows:

$$u^{\text{SAFT}}(x) = \frac{1}{\sqrt{2\pi|b|}} \exp\left\{\frac{i}{2b}[dx^2 + 2(bn - dm)x]\right\} \times \int dx' \exp\left\{\frac{i}{2b}[ax'^2 - 2(x - m)x']\right\} u(x'). \quad (3)$$

In Eq. (3) an overall constant phase factor is omitted for simplicity.

In this Letter, first we show how the SAFT offers a novel unified viewpoint of the known optical operations on light waves. In particular, we discuss the operations of the FRT, lens transformation, free-space propagation, and magnification as the one-parameter Abelian subgroups (i.e., the product of two transformations are commutative) of the SAFT. Then we introduce a new operation, which is analogous to the Lorentz transformation and exhibits squeezing. We also show that a SAFT including these five operations can be obtained by the combination of any two, in particular, lens transformation and free-space propagation.

Let us consider the following matrices in phase space<sup>8,9</sup>:

$$g_1(\theta) = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \quad (\text{rotation}), \quad (4)$$

$$g_2(\xi) = \begin{bmatrix} 1 & 0 \\ \xi & 1 \end{bmatrix} \quad (\text{lens transformation}), \quad (5)$$

$$g_3(\eta) = \begin{bmatrix} 1 & \eta \\ 0 & 1 \end{bmatrix} \quad (\text{free-space propagation}), \quad (6)$$

$$g_4(\alpha) = \begin{bmatrix} e^\alpha & 0 \\ 0 & e^{-\alpha} \end{bmatrix} \quad (\text{magnification}). \quad (7)$$

All these matrices satisfy condition (2) and therefore are the elements of the special linear group  $SL(2, \mathbf{R})$ .

What is special about them is that each of them forms a one-parameter Abelian subgroup of  $SL(2, \mathbf{R})$ :

$$g_A(t_1)g_A(t_2) = g_A(t_2)g_A(t_1) = g_A(t_1 + t_2),$$

$$g_A(-t) = g_A^{-1}(t), \quad g_A(0) = I \quad (A = 1, 2, 3, 4), \quad (8)$$

where  $I$  is the  $2 \times 2$  identity matrix. Applying the SAFT to Eqs. (4)–(7), we obtain the following transformation formulas including translation ( $m, n$ ):

$$u^{\text{FRT}}(x) = \frac{1}{\sqrt{2\pi|\sin\theta|}} \exp[(i/2)x^2 \cot\theta + ixm \cos e\theta] \\ \times \int dx' \exp[(i/2)x'^2 \cot\theta - i(x \cos e\theta \\ + m \cot\theta + n)x']u(x')$$

(the FRT, i.e., rotation), (9)

$$u^{\text{LENS}}(x) = \exp\left[\frac{1}{2}\xi x^2 - i(m\xi - n)x\right]u(x - m)$$

(lens transformation), (10)

$$u^{\text{FSP}}(x) = \frac{1}{\sqrt{2\pi|\eta|}} \exp(inx) \int dx' \\ \times \exp\left[\frac{i}{2\eta}(x' - x + m)^2\right]u(x')$$

(free-space propagation), (11)

$$u^{\text{MGN}}(x) = \exp\left(-\frac{\alpha}{2} + inx\right)u\left[e^{-\alpha}(x - m)\right]$$

(magnification), (12)

provided that the overall constant phase factors have been omitted. To obtain Eq. (12) we used the following formula:

$$\lim_{b \rightarrow 0} \frac{1}{\sqrt{2\pi ib}} \exp\left[i\frac{(x_1 - x_2)^2}{2b}\right] = \delta(x_1 - x_2), \quad (13)$$

whereas to derive Eq. (10) requires treatment by the degenerate SAFT.<sup>6</sup>

Two obvious questions that now arise are the following: (a) Are the above operations independent of one another, that is, is it possible to express any of them in terms of some combinations of the others? (b) Are there any other optically interesting simple operations expressible as one-parameter Abelian subgroups of the SAFT? We will answer question (a) below in a more general context.

To answer question (b) we introduce the matrices

$$L_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

(14)

which satisfy the following commutation relations ( $[A, B] \equiv AB - BA$ ):

$$[L_1, L_2] = 2L_3, \quad [L_2, L_3] = 2L_1, \quad [L_3, L_1] = -2L_2$$

(15)

and the anticommutation relations ( $\{A, B\} \equiv AB + BA$ )

$$\{L_1, L_1\} = -\{L_2, L_2\} = \{L_3, L_3\} = 2I, \\ \{L_i, L_j\} = 0 \quad (i \neq j; \quad i, j = 1, 2, 3). \quad (16)$$

With  $I$ , these matrices form a basis of the space of all  $2 \times 2$  real matrices. It is also useful to introduce the ladder matrix operators

$$L_{\pm} = \frac{1}{2}(L_1 \mp L_2), \quad (17)$$

which are nilpotent, that is,  $L_{\pm}^2 = 0$ . From Eqs. (15) and (16) it follows that

$$[L_+, L_-] = L_3, \quad [L_3, L_{\pm}] = \pm 2L_{\pm}, \quad (18)$$

$$\{L_{\pm}, L_{\mp}\} = I, \quad \{L_3, L_{\pm}\} = 0. \quad (19)$$

In the closed algebra [Eqs. (15) or (18)], only two of  $\{L_1, L_2, L_3\}$  (or, equivalently, of  $\{L_+, L_-, L_3\}$ ) are independent. In terms of these matrices,  $g_1, \dots, g_4$  are expressed as follows:

$$g_1(\theta) = \exp(-\theta L_2), \quad (20)$$

$$g_2(\xi) = I + \xi L_- = \exp(\xi L_-), \quad (21)$$

$$g_3 = I + \eta L_+ = \exp(\eta L_+), \quad (22)$$

$$g_4(\alpha) = \exp(\alpha L_3). \quad (23)$$

Examining these forms, we find another one-parameter Abelian operation:

$$g_5(\phi) = \exp(\phi L_1) = \begin{bmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{bmatrix}. \quad (24)$$

This is a hyperbolic transformation, which is analogous to the Lorentz transformation in special relativity<sup>10</sup> and exhibits squeezing different from that of magnification. To our knowledge, it was not previously discussed in the literature of geometrical optics. The integral transformation associated with  $g_5$  is found to be

$$u^{\text{HYP}}(x) = \frac{1}{\sqrt{2\pi|\sinh\phi|}} \exp\left[\frac{i}{2}x^2 \coth\phi \\ + i(n - m \coth\phi)x\right] \int dx' \exp\left[\frac{i}{2}x'^2 \coth\phi \\ - i(x - m)x' \cos e\phi\right] u(x'). \quad (25)$$

Now we come to the question (a) of the (in)dependence of the above operations. It can be shown that only two of them are independent, or, more generically, it is possible to express any  $2 \times 2$  matrix that satisfies condition (2) by a combination of any other two. Let us illustrate this explicitly, using the optical operations. For this purpose, we

consider the decomposition of an arbitrary element  $g$  of  $SL(2, \mathbf{R})$  as follows<sup>7</sup>:

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & s^{-1} \end{bmatrix} \begin{bmatrix} 1 & \eta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \\ = \text{sgn}(s)g_4(\ln|s|)g_3(\eta)g_1(\theta), \quad (26)$$

with the parameterization

$$a = s(\cos \theta - \eta \sin \theta), \\ b = s(\sin \theta + \eta \cos \theta), \\ c = -\sin \theta/s, \quad d = \cos \theta/s. \quad (27)$$

$s$ ,  $\eta$ , and  $\theta$  play the roles of the three independent parameters of  $g$  under condition (2). When  $\text{sgn}(s) = -1$ ,  $\text{sgn}(s)g_4(\ln|s|) = -g_4(\ln|s|) = g_1(\pi)g_4(\ln|s|) = g_1^2(\pi/2)g_4(\ln|s|)$ . Next we consider the combination

$$g_3(\eta)g_2(\xi)g_3(\eta) = \begin{bmatrix} 1 + \xi\eta & \eta(2 + \xi\eta) \\ \xi & 1 + \xi\eta \end{bmatrix}. \quad (28)$$

If  $\xi$  and  $\eta$  satisfy the condition  $|1 + \xi\eta| < 1$ , we have

$$g_3(\eta)g_2(\xi)g_3(\eta) = g_1(\theta) \\ [\xi = -\sin \theta, \quad \eta = \sin \theta/(1 + \cos \theta)], \quad (29)$$

where  $\theta \neq (2n + 1)\pi$  ( $n = 0, \pm 1, \pm 2, \dots$ ). {In the case when  $\theta = (2n + 1)\pi$ ,  $g_1[(2n + 1)\pi] = -I = g_1^2(\pi/2)$ .} On the other hand, if  $\xi$  and  $\eta$  satisfy  $\xi\eta > 0$ , then we obtain

$$g_3(\eta)g_2(\xi)g_3(\eta) = g_5(\phi) \\ [\xi = \sinh \phi, \quad \eta = \sinh \phi/(1 + \cosh \phi)]. \quad (30)$$

In addition,

$$g_4(\alpha) = g_1(\pi/4)g_5(\phi)g_1^{-1}(\pi/4) \quad (\alpha = \phi). \quad (31)$$

Thus  $g_1$  and  $g_4$  can be expressed in terms of  $g_2$  and  $g_3$ , and therefore the combinations of the lens transformation ( $g_2$ ) and free-space propagation ( $g_3$ ) can describe all transformations of  $SL(2, \mathbf{R})$ . This generalizes the result recently obtained by Lohmann,<sup>11</sup> who has shown that the FRT can be realized by use of combinations of the lens transformation and free-space propagation.

Combining  $g_1, \dots, g_5$ , we can derive several other interesting operations. An example presented in Ref. 9 is

$$g_4(\alpha/2)g_1(\theta)g_4^{-1}(\alpha/2) = \begin{bmatrix} \cos \theta & e^\alpha \sin \theta \\ -e^{-\alpha} \sin \theta & \cos \theta \end{bmatrix}, \quad (32)$$

which gives rise to the squeezed FRT. Another example is

$$g_4(\alpha/2)g_5(\phi)g_4^{-1}(\alpha/2) = \begin{bmatrix} \cosh \phi & e^\alpha \sinh \phi \\ e^{-\alpha} \sinh \phi & \cosh \phi \end{bmatrix}, \quad (33)$$

which is a combination of two different kinds of squeezing. This type of similarity transformation,  $g_B g_A(t)g_B^{-1}$ , preserves the one-parameter Abelian-subgroup nature of  $g_A(t)$  with fixed  $g_B$ .

In conclusion, we have discussed some optical operations and their corresponding wave-function transformations based on the special affine Fourier transformation. In these analyses we have used the one-parameter Abelian-subgroup nature of the operations. We have proposed a new operation of the hyperbolic type. We have also shown that the SAFT can be realized by the combination of only two independent transformations, in particular, those of lens transformation and free-space propagation. Previously<sup>5,12</sup> the relationships among the FRT, Fresnel diffraction, and wavelet transformation were indicated. Clearly such a discussion can be carried out more generally in the context of the SAFT.

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