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J. F. Gómez-Aguilar,[1] M. S. Osman,[2,3] Nauman Raza,[4] Asad Zubair,[4] Saima Arshed,[4] Mohamed E. Ghoneim,[3] Emad E. Mahmoud,[5,6] and Abdel-Haleem Abdel-Aty[7,8,a)]

AFFILIATIONS

[1] Tecnológico Nacional de México/CENIDET, Interior Internado Palmira S/N, Col. Palmira, Mexico Universidad Virtual CNCI, C.P. 62490 Cuernavaca, Morelos, Mexico
[2] Department of Mathematics, Faculty of Science, Cairo University, Giza 12613, Egypt
[3] Department of Mathematics, Faculty of Applied Science, Umm Alqura University, Makkah 21955, Saudi Arabia
[4] Department of Mathematics, Namal Institute, Talagang Road, Mianwali 42250, Pakistan
[5] Department of Mathematics and Statistics, College of Science, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia
[6] Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt
[7] Department of Physics, College of Sciences, University of Bisha, P.O. Box 344, Bisha 61922, Saudi Arabia
[8] Physics Department, Faculty of Science, Al-Azhar University, Assiut 71524, Egypt

[a)] Author to whom correspondence should be addressed: amabdelaty@ub.edu.sa

ABSTRACT

In this work, the nonlinear Schrödinger's equation is studied for birefringent fibers incorporating four-wave mixing. The improved $\tan\left(\frac{\phi(\xi)}{2}\right)$-expansion, first integral, and $\frac{G'}{G^2}$-expansion methods are used to extract a novel class of optical solitons in the quadratic-cubic nonlinear medium. The extracted solutions are dark, periodic, singular, and dark-singular, along with other soliton solutions. These solutions are listed with their respective existence criteria. The recommended computational methods here are uncomplicated, outspoken, and consistent and minimize the computational work size, which give it a wide range of applicability. A detailed comparison with the results that already exist is also presented.

I. INTRODUCTION

Optical solitons are a valuable accretion in the field of fiber optic communications.[1–3] The nonlinear Schrödinger's equation (NLSE) is the governing model that describes the propagation of optical solitons with different forms of nonlinear media.[4–14] The nonlinear media based on the Kerr law have been extensively studied through various research papers.[12–15] Nowadays, a growing interest to study optical solitons in the non-Kerr law medium can be observed. There are various forms of nonlinearities that are studied in the context of the non-Kerr law, i.e., polynomial law, parabolic law, power law, dual-power law, saturable law, triple power law, among others.[16–20] For more than a couple of decades, the study of optical solitons has been carried out with quadratic-cubic (QC) nonlinearity. This form of nonlinearity first appeared during 1994.[21] Later, interest was rekindled with this model during 2011.[22] There are several results with a variety of mathematical methods that are reported.[23–33] These include the traveling wave hypothesis, semi-inverse variational principle, method of undetermined coefficients, conservation laws, the unified method and its generalized technique, and various other aspects.[34–48]

In this study, in the presence of birefringent fibers and the four-wave mixing effect, we consider the NLSE with quadratic-cubic nonlinearity (4WM). It is likely that birefringence exists in an optical waveguide. By splitting pulses into two, this phenomenon steers toward differential party latency. Birefringence implies the mutual reaction to this pause.

Refractive indices caused by orthogonally polarized materials vary significantly in the case of fiber birefringence in optical modes. Nonlinear structures are significantly affected by this refractive index, which plays a significant role in the field of fiber optics. If Bragg gratings are rendered within the center of polarization-maintaining fibers, their effects should be included. It is worthwhile extending the coupled-mode principle to account for fiber birefringence.[49–51] In this case, a series of four coupled equations comprising forward and backward propagating waves explain the evolution of two orthogonally polarized elements, which makes the topic very complicated. A major class of nonlinear phenomena of functional implementation, such as optical logic gates, is given by this ramification. To extract soliton solutions to the models, multiple integration schemes are available.[17,18,52–54] The first integral approach is an exceptionally desirable and stable method for scrutinizing reliable evolution equation solutions since it copes fairly well with both integrable and non-integrable equations. Feng[55] introduced this method by solving the Burgers KdV equation. This approach is fundamental to the ring theory of commutative algebra and is used all over the world.[56,57]

The $\frac{G'}{G^2}$-expansion method[58] is a novel approach for calculating the soliton solutions of single and combined nonlinear equations that exist in different fields of physics, fluid mechanics, problems of wave propagation, population dynamics, etc. Owing to its easy thinking and suitability, it has become very popular. In the form of hyperbolic, trigonometric, and rational functions, the solutions obtained using the approach proposed can be expressed.

The present model, however, has been already studied in the past with the extended trial function method,[59] extended Jacobi's elliptic function expansion method,[60] (G'/G)-expansion method,[61] and F-expansion method.[62] The current paper will apply the improved $\tan\left(\frac{\phi(\xi)}{2}\right)$-expansion method,[63] first integral method,[55] and $\frac{G'}{G^2}$-expansion method[58] to extract a novel class of soliton solutions to the model. Dark, singular, periodic, dark-singular, and some other solitons are the part of this novel class. After a quick intro to the model, the details are enumerated in the rest of the paper.

A. Governing model

The governing equation for the propagation of solitons through optical fibers with QC nonlinearity is given by the following dimensionless form of the NLSE:[38]

$$i\psi_t + \alpha\psi_{xx} + (\beta_1|\psi| + \beta_2|\psi|^2)\psi = 0. \tag{1}$$

In Eq. (1), the independent variables are t and x that represent temporal and spatial variables, respectively. The dependent variable $\psi(x,t)$ gives the complex valued wave profile. The coefficient of the real-valued constant α is group velocity dispersion (GVD). The two nonlinear terms are with coefficients β_1 and β_2, which are both real-valued constants. This leads to the formulation of the NLSE with QC nonlinearity. With birefringence, Eq. (1) splits into two components as follows:[59]

$$ir_t + \alpha_1 r_{xx} + \beta_1 r\sqrt{|r|^2 + |s|^2 + rs^* + r^*s} + \left(\mu_1|r|^2 + \gamma_1|s|^2\right)r + \tau_1 s^2 r^* = 0, \tag{2}$$

$$is_t + \alpha_2 s_{xx} + \beta_2 s\sqrt{|s|^2 + |r|^2 + sr^* + s^*r} + \left(\mu_2|s|^2 + \gamma_2|r|^2\right)s + \tau_2 r^2 s^* = 0. \tag{3}$$

In Eqs. (2) and (3), α_j, for $j = 1$ or 2, gives GVD for the two components. The coefficients of β_j, the first two terms in the radical, account for self-phase modulation (SPM) and cross-phase modulation (XPM) terms, while the third and fourth terms stem from 4WM due to quadratic nonlinearity. Next, for the cubic nonlinear term, μ_j and ν_j are from SPM and XPM, respectively, while τ_j gives the effect of 4WM. With this complete picture to the model, the rest of the paper explores the mathematical scheme to enlist the solution spectrum.

II. MATHEMATICAL PRELIMINARIES

In order to solve Eqs. (2) and (3), we first adopt the following transformations:

$$r(x,t) = U_1(\xi)e^{i\phi}, \tag{4}$$

$$s(x,t) = U_2(\xi)e^{i\phi}, \tag{5}$$

where

$$\xi = x - vt, \tag{6}$$

while the phase component ϕ is given as

$$\phi = -\kappa x + \omega t + \theta. \tag{7}$$

Here, v is the velocity of the soliton, κ refers to the frequency of the solitons in each of the two components while θ is the phase constant and ω is the soliton wave number. Substituting Eqs. (4) and (5) into Eqs. (2) and (3) and splitting into real and imaginary parts, respectively, yield

$$\alpha_l U_l'' - (\alpha_l\kappa^2 + \omega)U_l + \beta_l U_l^2 + \beta_l U_l U_{\bar{l}} + \mu_l U_l^3 + (\gamma_l + \tau_l)U_l U_{\bar{l}}^2 = 0, \tag{8}$$

$$(2\alpha_l\kappa + v)U_l' = 0, \tag{9}$$

for $l = 1, 2$ and $\bar{l} = 3 - l$. From Eq. (9), the soliton speed is

$$v = -2\alpha_l\kappa. \tag{10}$$

On comparing both soliton speed expressions, we get

$$\alpha_l = \alpha_2 = \alpha. \tag{11}$$

Thus, the speed of the solitons becomes

$$v = -2\alpha\kappa. \tag{12}$$

Therefore, Eqs. (2) and (3) can be rewritten as

$$\begin{aligned} &ir_t + \alpha r_{xx} + \beta_1 r\sqrt{|r|^2 + |s|^2 + rs^* + r^* s} \\ &\quad +\left(\mu_1 |r|^2 + \gamma_1 |s|^2\right) r + \tau_1 s^2 r^* = 0, \\ &is_t + \alpha s_{xx} + \beta_1 s\sqrt{|s|^2 + |r|^2 + sr^* + s^* r} \\ &\quad +\left(\mu_1 |s|^2 + \gamma_1 |r|^2\right) s + \tau_1 r^2 s^* = 0. \end{aligned} \tag{13}$$

In this case, the real part (8) changes to

$$\alpha U_l'' - (\alpha\kappa^2 + \omega)U_l + \beta_l U_l^2 + \beta_l U_l U_{\bar{l}} + \mu_l U_l^3 + (\gamma_l + \tau_l)U_l U_{\bar{l}}^2 = 0. \tag{14}$$

Next, using the balancing principle, a relationship is acquired, which is given by

$$U_l = U_{\bar{l}}. \tag{15}$$

As a consequence, Eq. (16) turns into

$$\alpha U_l'' - (\alpha\kappa^2 + \omega)U_l + 2\beta_l U_l^2 + (\mu_l + \gamma_l + \tau_l)U_l^3 = 0. \tag{16}$$

III. EXTRACTION OF SOLITONS BY THE IMPROVED $\tan\left(\frac{\phi(\xi)}{2}\right)$-EXPANSION METHOD

This section deals with the integration of Eqs. (2) and (3) by the aid of a mathematical tool called the improved $\tan\left(\frac{\phi(\xi)}{2}\right)$-expansion method.[63] Suppose that Eq. (16) has a solution in terms of the $\tan\left(\frac{\phi(\xi)}{2}\right)$ function,

$$U(\xi) = \sum_{k=0}^{m} A_k\left[p + \tan\left(\frac{\phi(\xi)}{2}\right)\right]^k + \sum_{k=1}^{m} B_k\left[p + \tan\left(\frac{\phi(\xi)}{2}\right)\right]^{-k}, \tag{17}$$

where A_k and B_k are constants to be determined, such that $A_m \neq 0$, $B_m \neq 0$, and $\phi(\xi)$ satisfies the following differential equation:

$$\phi'(\xi) = E\sin(\phi(\xi)) + F\cos(\phi(\xi)) + G. \tag{18}$$

Balancing U'' and U^3 in Eq. (16), yields $m = 1$. Hence, Eq. (17) along with $p = 0$ takes the following form:

$$U(\xi) = A_0 + A_1\left[\tan\left(\frac{\phi(\xi)}{2}\right)\right] + B_1\left[\tan\left(\frac{\phi(\xi)}{2}\right)\right]^{-1}. \tag{19}$$

Here, the objective is to find the values of A_0, A_1, and B_1. In order to find these values, Eq. (19) is substituted into Eq. (16) and all the coefficients of $\left(\tan\left(\frac{\phi}{2}\right)\right)^n$ are compared, where $n = -3, -2, -1, 0, 1, 2,$ and3 with zero providing the following set of algebraic equations:

$$\mu_l {A_1}^3 + \tau_l {A_1}^3 - \alpha F A_1 G + \gamma_l {A_1}^3 + \frac{1}{2}\alpha A_1 G^2 + \frac{1}{2}F^2\alpha A_1 = 0, \tag{20}$$

$$\begin{aligned} &3\gamma_l A_0 {A_1}^2 + \frac{3}{2}\alpha G A_1 E + 3\mu_l A_0 {A_1}^2 + 3\tau_l A_0 {A_1}^2 \\ &\quad + 2\beta_l {A_1}^2 - \frac{3}{2}\alpha F A_1 E = 0, \end{aligned} \tag{21}$$

$$\begin{aligned} &4\beta_l A_0 A_1 - \frac{1}{2}F^2\alpha A_1 + 3\gamma_l {A_0}^2 A_1 + 3\tau_l {A_0}^2 A_1 + 3\mu_l {A_0}^2 A_1 \\ &\quad + \frac{1}{2}\alpha A_1 G^2 + 3\tau_l {A_1}^2 B_1 + \alpha E^2 A_1 + 3\mu_l {A_1}^2 B_1 \\ &\quad - \omega A_1 - \alpha\kappa^2 A_1 + 3\gamma_l {A_1}^2 B_1 = 0, \end{aligned} \tag{22}$$

$$\begin{aligned} &6\gamma_l A_0 A_1 B_1 - \frac{1}{2}\alpha E B_1 F + \frac{1}{2}\alpha E B_1 G + \frac{1}{2}\alpha F A_1 E + \frac{1}{2}\alpha G A_1 E \\ &\quad + \mu_l {A_0}^3 + 4\beta_l A_1 B_1 + \gamma_l {A_0}^3 + 2\beta_l {A_0}^2 + \tau_l {A_0}^3 - \alpha\kappa^2 A_0 \\ &\quad + 6\mu_l A_0 A_1 B_1 + 6\tau_l A_0 A_1 B_1 - \omega A_0 = 0, \end{aligned} \tag{23}$$

$$\begin{aligned} &4\beta_l A_0 B_1 - \omega B_1 + \alpha E^2 B_1 + \frac{1}{2}\alpha B_1 G^2 - \frac{1}{2}\alpha B_1 F^2 + 3\gamma_l A_1 {B_1}^2 \\ &\quad + 3\gamma_l {A_0}^2 B_1 + 3\tau_l {A_0}^2 B_1 + 3\mu_l {A_0}^2 B_1 + 3\mu_l A_1 {B_1}^2 \\ &\quad - \alpha\kappa^2 B_1 + 3\tau_l A_1 {B_1}^2 = 0, \end{aligned} \tag{24}$$

$$\begin{aligned} &\frac{3}{2}\alpha E B_1 G + 3\tau_l A_0 {B_1}^2 + 2\beta_l {B_1}^2 + 3\gamma_l A_0 {B_1}^2 \\ &\quad + \frac{3}{2}\alpha E B_1 F + 3\mu_l A_0 {B_1}^2 = 0, \end{aligned} \tag{25}$$

$$\frac{1}{2}\alpha B_1 F^2 + \frac{1}{2}\alpha B_1 G^2 + \gamma_l {B_1}^3 + \tau_l {B_1}^3 + \mu_l {B_1}^3 + \alpha F B_1 G = 0. \tag{26}$$

On solving the above-mentioned system of algebraic equations with the help of Maple, we get the subsequent cases for values of A_0, A_1, B_1, α, and ω.

Case 1:

$$\begin{aligned} &A_0 = -\frac{4}{3}\frac{\beta_l}{\mu_l + \gamma_l + \tau_l}, \quad A_1 = \frac{2}{3}\frac{(-G + F)\beta_l}{E(\mu_l + \gamma_l + \tau_l)}, \\ &B_1 = -\frac{2}{3}\frac{(G + F)\beta_l}{E(\mu_l + \gamma_l + \tau_l)}, \quad \alpha = -\frac{8}{9}\frac{{\beta_l}^2}{(\mu_l + \gamma_l + \tau_l)E^2}, \\ &\omega = -\frac{8}{9}\frac{\left(E^2 - G^2 + F^2 - \kappa^2\right){\beta_l}^2}{(\mu_l + \gamma_l + \tau_l)E^2}. \end{aligned} \tag{27}$$

Substituting the above-mentioned values in Eq. (19) and using the relation in Eq. (18) yield the following soliton solutions for systems (2) and (3).

When $E^2 + F^2 - G^2 = -M^2 < 0$ and $F - G \neq 0$, then the following periodic soliton solutions are obtained:

$$\begin{aligned} r(x,t) = &-\frac{2\beta_1}{3(\mu_1 + \gamma_1 + \tau_1)}\left[1 + \frac{M}{E}\tan\left(\frac{M}{2}(x + 2\alpha\kappa t + C)\right)\right. \\ &\left. + \frac{(F + G)(F - G)}{E^2 - ME\tan\left(\frac{M}{2}(x + 2\alpha\kappa t + C)\right)}\right] \\ &\times e^{i\left(-\kappa x + \frac{8}{9}\frac{(M^2 + \kappa^2){\beta_l}^2}{(\mu_l + \gamma_l + \tau_l)E^2}t + \theta\right)}, \end{aligned} \tag{28}$$

$$s(x,t) = -\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)}\Bigg[1+\frac{M}{E}\tan\Big(\frac{M}{2}(x+2\alpha\kappa t+C)\Big) + \frac{(F+G)(F-G)}{E^2 - ME\tan(\frac{M}{2}(x+2\alpha\kappa t+C))}\Bigg] \times e^{i\left(-\kappa x+\frac{8}{9}\frac{(M^2+\kappa^2)\beta_l{}^2}{(\mu_l+\gamma_l+\tau_l)E^2}t+\theta\right)}. \tag{29}$$

When $E^2+F^2-G^2 = R^2 > 0$ and $F-G \neq 0$, then the subsequent dark-singular soliton solutions are obtained,

$$r(x,t) = -\frac{2\beta_1}{3(\mu_1-\gamma_1+\tau_1)}\Bigg[1+\frac{R}{E}\tanh\Big(\frac{R}{2}(x+2\alpha\kappa t+C)\Big) + \frac{(F+G)(F-G)}{E^2 + RE\tanh(\frac{R}{2}(x+2\alpha\kappa t+C))}\Bigg] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(R^2-\kappa^2)\beta_l{}^2}{(\mu_l+\gamma_l+\tau_l)E^2}t+\theta\right)}, \tag{30}$$

$$s(x,t) = -\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)}\Bigg[1-\frac{R}{E}\tanh\Big(\frac{R}{2}(x+2\alpha\kappa t+C)\Big) + \frac{(F+G)(F-G)}{E^2 + ER\tanh\frac{R}{2}(x+2\alpha\kappa t+C))}\Bigg] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(R^2-\kappa^2)\beta_l{}^2}{(\mu_l+\gamma_l+\tau_l)E^2}t+\theta\right)}. \tag{31}$$

When $E^2+F^2-G^2 > 0$, $F \neq 0$, and $G = 0$, then the subsequent dark-singular soliton solutions are obtained,

$$r(x,t) = -\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)}\Bigg[1-\frac{\sqrt{F^2+E^2}}{E} \times \tanh\left(\frac{\sqrt{F^2+E^2}}{2}(x+2\alpha\kappa t+C)\right) + \frac{F^2}{E}\left(\frac{1}{E+\sqrt{F^2+E^2}\tanh\Big(\frac{\sqrt{F^2+E^2}}{2}(x+2\alpha\kappa t+C)\Big)}\right)\Bigg] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(E^2+F^2-\kappa^2)\beta_l{}^2}{(\mu_l+\gamma_l+\tau_l)E^2}t+\theta\right)}, \tag{32}$$

$$s(x,t) = -\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)}\Bigg[1-\frac{\sqrt{F^2+E^2}}{E} \times \tanh\left(\frac{\sqrt{F^2+E^2}}{2}(x+2\alpha\kappa t+C)\right) + \frac{F^2}{E}\left(\frac{1}{E+\sqrt{F^2+E^2}\tanh\Big(\frac{\sqrt{F^2+E^2}}{2}(x+2\alpha\kappa t+C)\Big)}\right)\Bigg] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(E^2+F^2-\kappa^2)\beta_l{}^2}{(\mu_l+\gamma_l+\tau_l)E^2}t+\theta\right)}. \tag{33}$$

When $E^2+F^2-G^2 < 0$, $G \neq 0$, and $F = 0$, then the following periodic soliton solutions are obtained:

$$r(x,t) = -\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)}\Bigg[1+\frac{\sqrt{G^2-E^2}}{E} \times \tan\left(\frac{\sqrt{G^2-E^2}}{2}(x+2\alpha\kappa t+C)\right) + \frac{G^2}{E} \times \left(\frac{1}{E-\sqrt{G^2-E^2}\tan\Big(\frac{\sqrt{G^2-E^2}}{2}(x+2\alpha\kappa t+C)\Big)}\right)\Bigg] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(E^2-G^2-\kappa^2)\beta_l{}^2}{(\mu_l+\gamma_l+\tau_l)E^2}t+\theta\right)}, \tag{34}$$

$$s(x,t) = -\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)}\Bigg[1+\frac{\sqrt{G^2-E^2}}{E} \times \tan\left(\frac{\sqrt{G^2-E^2}}{2}(x+2\alpha\kappa t+C)\right) + \times\frac{G^2}{E}\left(\frac{1}{E-\sqrt{G^2-E^2}\tan\Big(\frac{\sqrt{G^2-E^2}}{2}(x+2\alpha\kappa t+C)\Big)}\right)\Bigg] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(E^2-G^2-\kappa^2)\beta_l{}^2}{(\mu_l+\gamma_l+\tau_l)E^2}t+\theta\right)}. \tag{35}$$

When $E^2+F^2 = G^2$, then the following rational soliton solutions are obtained:

$$r(x,t) = -\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)}\left[2-\frac{E(x+2\alpha\kappa t+C)+2}{E(x+2\alpha\kappa t+C)} - \frac{E(x+2\alpha\kappa t+C)}{E(x+2\alpha\kappa t+C)+2}\right]e^{i\left(-\kappa x+\frac{8}{9}\frac{\kappa^2\beta_l{}^2}{(\mu_l+\gamma_l+\tau_l)E^2}t+\theta\right)}, \tag{36}$$

$$s(x,t) = -\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)}\left[2-\frac{E(x+2\alpha\kappa t+C)+2}{E(x+2\alpha\kappa t+C)} - \frac{E(x+2\alpha\kappa t+C)}{E(x+2\alpha\kappa t+C)+2}\right]e^{i\left(-\kappa x+\frac{8}{9}\frac{\kappa^2\beta_l{}^2}{(\mu_l+\gamma_l+\tau_l)E^2}t+\theta\right)}. \tag{37}$$

When $E = F = G = kE$, then the subsequent periodic soliton solutions are obtained,

$$r(x,t) = -\frac{4\beta_1}{3(\mu_1+\gamma_1+\tau_1)}\left[1+\frac{1}{e^{kE(x+2\alpha\kappa t+C)}-1}\right] \times e^{i\left(-\kappa x+\frac{8}{9}\frac{\kappa^2\beta_l{}^2}{(\mu_l+\gamma_l+\tau_l)k^2E^2}t+\theta\right)}, \tag{38}$$

$$s(x,t) = -\frac{4\beta_2}{3(\mu_2+\gamma_2+\tau_2)}\left[1+\frac{1}{e^{kE(x+2\alpha\kappa t+C)}-1}\right] \times e^{i\left(-\kappa x+\frac{8}{9}\frac{\kappa^2\beta_l{}^2}{(\mu_l+\gamma_l+\tau_l)k^2E^2}t+\theta\right)}. \tag{39}$$

When $E = G = kE$ and $F = -kE$, then the subsequent periodic soliton solutions are obtained,

$$r(x,t) = -\frac{4\beta_1}{3(\mu_1+\gamma_1+\tau_1)}\left[1+\frac{e^{kE(x+2\alpha\kappa t+C)}}{1-e^{kE(x+2\alpha\kappa t+C)}}\right] \times e^{i\left(-\kappa x+\frac{8}{9}\frac{\kappa^2\beta_l^2}{(\mu_l+\gamma_l+\tau_l)k^2E^2}t+\theta\right)}, \tag{40}$$

$$s(x,t) = -\frac{4\beta_2}{3(\mu_2+\gamma_2+\tau_2)}\left[1+\frac{e^{kE(x+2\alpha\kappa t+C)}}{1-e^{kE(x+2\alpha\kappa t+C)}}\right] \times e^{i\left(-\kappa x+\frac{8}{9}\frac{\kappa^2\beta_l^2}{(\mu_l+\gamma_l+\tau_l)k^2E^2}t+\theta\right)}. \tag{41}$$

When $G = E$, then the following periodic soliton solutions are obtained:

$$r(x,t) = -\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)} \times\left[2+\frac{(F-E)}{E}\left(\frac{(E+F)e^{F(x+2\alpha\kappa t+C)}-1}{(E-F)e^{F(x+2\alpha\kappa t+C)}-1}\right) - \frac{(F+E)}{E}\left(\frac{(E-F)e^{F(x+2\alpha\kappa t+C)}-1}{(E+F)e^{F(x+2\alpha\kappa t+C)}-1}\right)\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(F^2-\kappa^2)\beta_l^2}{(\mu_l+\gamma_l+\tau_l)E^2}t+\theta\right)}, \tag{42}$$

$$s(x,t) = -\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)} \times\left[2+\frac{(F-E)}{E}\left(\frac{(E+F)e^{F(x+2\alpha\kappa t+C)}-1}{(E-F)e^{F(x+2\alpha\kappa t+C)}-1}\right) - \frac{(F+E)}{E}\left(\frac{(E-F)e^{F(x+2\alpha\kappa t+C)}-1}{(E+F)e^{F(x+2\alpha\kappa t+C)}-1}\right)\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(F^2-\kappa^2)\beta_l^2}{(\mu_l+\gamma_l+\tau_l)E^2}t+\theta\right)}. \tag{43}$$

When $G = -E$, then the following periodic soliton solutions are obtained:

$$r(x,t) = -\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)} \times\left[2-\frac{(F-E)}{E}\left(\frac{e^{F(x+2\alpha\kappa t+C)}+F-E}{e^{F(x+2\alpha\kappa t+C)}+F-E}\right) + \frac{(F+E)}{E}\left(\frac{e^{F(x+2\alpha\kappa t+C)}+F-E}{e^{F(x+2\alpha\kappa t+C)}+F-E}\right)\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(F^2-\kappa^2)\beta_l^2}{(\mu_l+\gamma_l+\tau_l)E^2}t+\theta\right)}, \tag{44}$$

$$s(x,t) = -\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)} \times\left[2-\frac{(F-E)}{E}\left(\frac{e^{F(x+2\alpha\kappa t+C)}+F-E}{e^{F(x+2\alpha\kappa t+C)}+F-E}\right) + \frac{(F+E)}{E}\left(\frac{e^{F(x+2\alpha\kappa t+C)}+F-E}{e^{F(x+2\alpha\kappa t+C)}+F-E}\right)\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(F^2-\kappa^2)\beta_l^2}{(\mu_l+\gamma_l+\tau_l)E^2}t+\theta\right)}. \tag{45}$$

When $F = -G$, then the subsequent periodic soliton solutions are obtained,

$$r(x,t) = -\frac{4\beta_1}{3(\mu_1+\gamma_1+\tau_1)}\left[1-\frac{Ge^{E(x+2\alpha\kappa t+C)}}{e^{kE(x+2\alpha\kappa t+C)}-1}\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(E^2-\kappa^2)\beta_l^2}{(\mu_l+\gamma_l+\tau_l)E^2}t+\theta\right)}, \tag{46}$$

$$s(x,t) = -\frac{4\beta_2}{3(\mu_2+\gamma_2+\tau_2)}\left[1-\frac{Ge^{E(x+2\alpha\kappa t+C)}}{e^{kE(x+2\alpha\kappa t+C)}-1}\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(E^2-\kappa^2)\beta_l^2}{(\mu_l+\gamma_l+\tau_l)E^2}t+\theta\right)}. \tag{47}$$

When $F = 0$ and $E = G$, then the subsequent rational soliton solutions are obtained,

$$r(x,t) = -\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)}\left[2-\frac{G(x+2\alpha\kappa t+C)+2}{G(x+2\alpha\kappa t+C)} - \frac{G(x+2\alpha\kappa t+C)}{G(x+2\alpha\kappa t+C)+2}\right] e^{i\left(-\kappa x+\frac{8}{9}\frac{\kappa^2\beta_l^2}{(\mu_l+\gamma_l+\tau_l)G^2}t+\theta\right)}, \tag{48}$$

$$s(x,t) = -\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)}\left[2-\frac{G(x+2\alpha\kappa t+C)+2}{G(x+2\alpha\kappa t+C)} - \frac{G(x+2\alpha\kappa t+C)}{G(x+2\alpha\kappa t+C)+2}\right] e^{i\left(-\kappa x+\frac{8}{9}\frac{\kappa^2\beta_l^2}{(\mu_l+\gamma_l+\tau_l)G^2}t+\theta\right)}. \tag{49}$$

When $F = G$, then the subsequent periodic soliton solutions are obtained,

$$r(x,t) = -\frac{4\beta_1}{3(\mu_1+\gamma_1+\tau_1)}\left[1+\frac{G}{e^{E(x+2\alpha\kappa t+C)}-F}\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(E^2-\kappa^2)\beta_l^2}{(\mu_l+\gamma_l+\tau_l)E^2}t+\theta\right)}, \tag{50}$$

$$s(x,t) = -\frac{4\beta_2}{3(\mu_2+\gamma_2+\tau_2)}\left[1+\frac{G}{e^{E(x+2\alpha\kappa t+C)}-F}\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(E^2-\kappa^2)\beta_l^2}{(\mu_l+\gamma_l+\tau_l)E^2}t+\theta\right)}. \tag{51}$$

Case 2:

$$\begin{aligned} A_0 &= -\frac{2}{3}\frac{\left(F^2+E^2-G^2-E\sqrt{F^2+E^2-G^2}\right)\beta_l}{(F^2+E^2-G^2)(\mu_l+\gamma_l+\tau_l)}, \\ B_1 &= \frac{2}{3}\frac{(G+F)\beta_l}{\sqrt{F^2+E^2-G^2}(\mu_l+\gamma_l+\tau_l)}, \\ A_1 &= 0,\ \alpha = -\frac{8}{9}\frac{\beta_l^2}{(F^2+E^2-G^2)(\mu_l+\gamma_l+\tau_l)}, \\ \omega &= -\frac{8}{9}\frac{\left(E^2-G^2+F^2-\kappa^2\right)\beta_l^2}{(F^2+E^2-G^2)(\mu_l+\gamma_l+\tau_l)}. \end{aligned} \tag{52}$$

Substituting the above-mentioned values in Eq. (19) and using the relation in Eq. (18) yield the following soliton solutions for systems (2) and (3).

When $E^2 + F^2 - G^2 < 0$ and $F - G \neq 0$, then the following singular periodic soliton solutions are obtained:

$$r(x,t) = -\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)}\left[\frac{\left(F^2+E^2-G^2-E\sqrt{F^2+E^2-G^2}\right)}{(F^2+E^2-G^2)} - \frac{(F^2-G^2)}{\sqrt{F^2+E^2-G^2}} \times \left(\frac{1}{E-\sqrt{G^2-F^2-E^2}\tan\left(\frac{\sqrt{G^2-F^2-E^2}}{2}(x+2\alpha\kappa t+C)\right)}\right)\right] e^{i\left(-\kappa x-\frac{8}{9}\frac{(E^2-G^2+F^2-\kappa^2)\beta_l{}^2}{(F^2+E^2-G^2)(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}, \tag{53}$$

$$s(x,t) = -\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)}\left[\frac{\left(F^2+E^2-G^2-E\sqrt{F^2+E^2-G^2}\right)}{(F^2+E^2-G^2)} - \frac{(F^2-G^2)}{\sqrt{F^2+E^2-G^2}} \times \left(\frac{1}{E-\sqrt{G^2-F^2-E^2}\tan\left(\frac{\sqrt{G^2-F^2-E^2}}{2}(x+2\alpha\kappa t+C)\right)}\right)\right] e^{i\left(-\kappa x-\frac{8}{9}\frac{(E^2-G^2+F^2-\kappa^2)\beta_l{}^2}{(F^2+E^2-G^2)(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}. \tag{54}$$

When $E^2+F^2-G^2>0$ and $F-G\neq 0$, then the subsequent singular soliton solutions are obtained,

$$r(x,t) = -\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)}\left[\frac{\left(F^2+E^2-G^2-E\sqrt{F^2+E^2-G^2}\right)}{(F^2+E^2-G^2)} - \frac{(F^2-G^2)}{\sqrt{F^2+E^2-G^2}} \times \left(\frac{1}{E+\sqrt{F^2+E^2-G^2}\tanh\left(\frac{\sqrt{F^2+E^2-G^2}}{2}(x+2\alpha\kappa t+C)\right)}\right)\right] e^{i\left(-\kappa x-\frac{8}{9}\frac{(E^2-G^2+F^2-\kappa^2)\beta_l{}^2}{(F^2+E^2-G^2)(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}, \tag{55}$$

$$s(x,t) = -\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)}\left[\frac{\left(F^2+E^2-G^2-E\sqrt{F^2+E^2-G^2}\right)}{(F^2+E^2-G^2)} - \frac{(F^2-G^2)}{\sqrt{F^2+E^2-G^2}} \times \left(\frac{1}{E+\sqrt{F^2+E^2-G^2}\tanh\left(\frac{\sqrt{F^2+E^2-G^2}}{2}(x+2\alpha\kappa t+C)\right)}\right)\right] e^{i\left(-\kappa x-\frac{8}{9}\frac{(E^2-G^2+F^2-\kappa^2)\beta_l{}^2}{(F^2+E^2-G^2)(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}. \tag{56}$$

When $E^2+F^2-G^2>0$, $F\neq 0$, and $G=0$, then the subsequent singular soliton solutions are obtained,

$$r(x,t) = -\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)}\left[\frac{\left(F^2+E^2-E\sqrt{F^2+E^2}\right)}{(F^2+E^2)} - \frac{(F^2)}{\sqrt{F^2+E^2}}\left(\frac{1}{E+\sqrt{F^2+E^2}\tanh\left(\frac{\sqrt{F^2+E^2}}{2}(x+2\alpha\kappa t+C)\right)}\right)\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(E^2+F^2-\kappa^2)\beta_l{}^2}{(F^2+E^2)(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}, \tag{57}$$

$$s(x,t) = -\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)}\left[\frac{\left(F^2+E^2-E\sqrt{F^2+E^2}\right)}{(F^2+E^2)} - \frac{(F^2)}{\sqrt{F^2+E^2}}\left(\frac{1}{E+\sqrt{F^2+E^2}\tanh\left(\frac{\sqrt{F^2+E^2}}{2}(x+2\alpha\kappa t+C)\right)}\right)\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(E^2+F^2-\kappa^2)\beta_l{}^2}{(F^2+E^2)(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}. \tag{58}$$

When $E^2+F^2-G^2<0$, $G\neq 0$, and $F=0$, then the following singular periodic soliton solutions are obtained:

$$r(x,t) = -\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)}\left[\frac{\left(E^2-G^2-E\sqrt{E^2-G^2}\right)}{(E^2-G^2)} + \frac{(G^2)}{\sqrt{E^2-G^2}}\left(\frac{1}{E-\sqrt{G^2-E^2}\tan\left(\frac{\sqrt{G^2-E^2}}{2}(x+2\alpha\kappa t+C)\right)}\right)\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(E^2-G^2-\kappa^2)\beta_l{}^2}{(E^2-G^2)(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}, \tag{59}$$

$$s(x,t) = -\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)}\left[\frac{\left(E^2-G^2-E\sqrt{E^2-G^2}\right)}{(E^2-G^2)} + \frac{(G^2)}{\sqrt{E^2-G^2}}\left(\frac{1}{E-\sqrt{G^2-E^2}\tan\left(\frac{\sqrt{G^2-E^2}}{2}(x+2\alpha\kappa t+C)\right)}\right)\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{\left(E^2-G^2-\kappa^2\right)\beta_l^2}{(E^2-G^2)(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}. \quad (60)$$

When $E^2 + F^2 - G^2 > 0$, $F - G \neq 0$, and $E = 0$, then the subsequent singular soliton solutions are obtained,

$$r(x,t) = -\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)} \times \left[1-\left(\frac{1}{\tanh\left(\frac{\sqrt{F^2-G^2}}{2}(x+2\alpha\kappa t+C)\right)}\right)\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{\left(F^2-G^2-\kappa^2\right)\beta_l^2}{(F^2-G^2)(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}, \quad (61)$$

$$s(x,t) = -\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)} \times \left[1-\left(\frac{1}{\tanh\left(\frac{\sqrt{F^2-G^2}}{2}(x+2\alpha\kappa t+C)\right)}\right)\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{\left(F^2-G^2-\kappa^2\right)\beta_l^2}{(F^2-G^2)(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}. \quad (62)$$

When $E = F = G = kE$, then the subsequent periodic soliton solutions are obtained,

$$r(x,t) = \frac{4\beta_1}{3(\mu_1+\gamma_1+\tau_1)}\left[\frac{1}{e^{kE(x+2\alpha\kappa t+C)}}\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{\left(k^2E^2-\kappa^2\right)\beta_l^2}{(K^2E^2)(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}, \quad (63)$$

$$s(x,t) = \frac{4\beta_2}{3(\mu_2+\gamma_2+\tau_2)}\left[\frac{1}{e^{kE(x+2\alpha\kappa t+C)}}\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{\left(k^2E^2-\kappa^2\right)\beta_l^2}{(K^2E^2)(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}. \quad (64)$$

When $G = E$, then the following periodic soliton solutions are obtained:

$$r(x,t) = -\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)}\left[\frac{F-E}{F} + \left(\frac{F+E}{F}\right)\left(\frac{(E-F)e^{F(x+2\alpha\kappa t+C)}-1}{(E+F)e^{F(x+2\alpha\kappa t+C)}-1}\right)\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{\left(F^2-\kappa^2\right)\beta_l^2}{F^2(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}, \quad (65)$$

$$s(x,t) = -\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)}\left[\frac{F-E}{F} + \left(\frac{F+E}{F}\right)\left(\frac{(E-F)e^{F(x+2\alpha\kappa t+C)}-1}{(E+F)e^{F(x+2\alpha\kappa t+C)}-1}\right)\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{\left(F^2-\kappa^2\right)\beta_l^2}{F^2(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}. \quad (66)$$

When $E = G$, then the subsequent periodic soliton solutions are obtained,

$$r(x,t) = -\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)}\left[\frac{F-G}{F} - \left(\frac{F+G}{F}\right)\left(\frac{(F-G)e^{F(x+2\alpha\kappa t+C)}-1}{(F+G)e^{F(x+2\alpha\kappa t+C)}+1}\right)\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{\left(F^2-\kappa^2\right)\beta_l^2}{F^2(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}, \quad (67)$$

$$s(x,t) = -\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)}\left[\frac{F-G}{F} - \left(\frac{F+G}{F}\right)\left(\frac{(F-G)e^{F(x+2\alpha\kappa t+C)}-1}{(F+G)e^{F(x+2\alpha\kappa t+C)}+1}\right)\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{\left(F^2-\kappa^2\right)\beta_l^2}{F^2(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}. \quad (68)$$

When $G = -E$, then the following periodic soliton solutions are obtained:

$$r(x,t) = -\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)}\left[\frac{F-E}{F} - \left(\frac{F-E}{F}\right)\left(\frac{e^{F(x+2\alpha\kappa t+C)}-F-E}{e^{F(x+2\alpha\kappa t+C)}+F-E}\right)\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{\left(F^2-\kappa^2\right)\beta_l^2}{F^2(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}, \quad (69)$$

$$s(x,t) = -\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)}\left[\frac{F-E}{F} - \left(\frac{F-E}{F}\right) \times \left(\frac{e^{F(x+2\alpha\kappa t+C)}-F-E}{e^{F(x+2\alpha\kappa t+C)}+F-E}\right)\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{\left(F^2-\kappa^2\right)\beta_l^2}{F^2(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}. \quad (70)$$

When $F = G$, then the subsequent periodic soliton solutions are obtained,

$$r(x,t) = \frac{4\beta_1}{3(\mu_1+\gamma_1+\tau_1)}\left[\frac{G}{e^{E(x+2\alpha\kappa t+C)}-F}\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(E^2-\kappa^2)\beta_l^2}{E^2(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}, \tag{71}$$

$$s(x,t) = \frac{4\beta_2}{3(\mu_2+\gamma_2+\tau_2)}\left[\frac{G}{e^{E(x+2\alpha\kappa t+C)}-F}\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(E^2-\kappa^2)\beta_l^2}{E^2(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}. \tag{72}$$

Case 3:

$$\begin{aligned} A_0 &= -\frac{2}{3}\frac{\left(F^2+E^2-G^2-E\sqrt{F^2+E^2-G^2}\right)\beta_l}{(F^2+E^2-G^2)(\mu_l+\gamma_l+\tau_l)},\\ A_1 &= -\frac{2}{3}\frac{(F-G)\beta_l}{\sqrt{F^2+E^2-G^2}(\mu_l+\gamma_l+\tau_l)},\\ B_1 &= 0,\ \alpha = -\frac{8}{9}\frac{\beta_l^2}{(F^2+E^2-G^2)(\mu_l+\gamma_l+\tau_l)},\\ \omega &= -\frac{8}{9}\frac{(E^2-G^2+F^2-\kappa^2)\beta_l^2}{(F^2+E^2-G^2)(\mu_l+\gamma_l+\tau_l)}. \end{aligned} \tag{73}$$

Substituting the above-mentioned values in Eq. (19) and using the relation in Eq. (18) yield the following soliton solutions for systems (2) and (3).

When $E^2 + F^2 - G^2 < 0$ and $F - G \neq 0$, then the following periodic soliton solutions are obtained:

$$r(x,t) = -\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)}\left[\frac{\left(F^2+E^2-G^2-E\sqrt{F^2+E^2-G^2}\right)}{(F^2+E^2-G^2)} + \frac{1}{\sqrt{F^2+E^2-G^2}}\left(E-\sqrt{G^2-F^2-E^2} \times \tan\left(\frac{\sqrt{G^2-F^2-E^2}}{2}(x+2\alpha\kappa t+C)\right)\right)\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(E^2-G^2+F^2-\kappa^2)\beta_l^2}{(F^2+E^2-G^2)(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}, \tag{74}$$

$$s(x,t) = -\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)}\left[\frac{\left(F^2+E^2-G^2-E\sqrt{F^2+E^2-G^2}\right)}{(F^2+E^2-G^2)} + \frac{1}{\sqrt{F^2+E^2-G^2}}\left(E-\sqrt{G^2-F^2-E^2} \times \tan\left(\frac{\sqrt{G^2-F^2-E^2}}{2}(x+2\alpha\kappa t+C)\right)\right)\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(E^2-G^2+F^2-\kappa^2)\beta_l^2}{(F^2+E^2-G^2)(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}. \tag{75}$$

When $E^2 + F^2 - G^2 > 0$ and $F - G \neq 0$, then the subsequent dark soliton solutions are obtained,

$$r(x,t) = -\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)}\left[\frac{\left(F^2+E^2-G^2-E\sqrt{F^2+E^2-G^2}\right)}{(F^2+E^2-G^2)} + \frac{1}{\sqrt{F^2+E^2-G^2}}\left(E+\sqrt{F^2+E^2-G^2} \times \tanh\left(\frac{\sqrt{F^2+E^2-G^2}}{2}(x+2\alpha\kappa t+C)\right)\right)\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(E^2-G^2+F^2-\kappa^2)\beta_l^2}{(F^2+E^2-G^2)(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}, \tag{76}$$

$$s(x,t) = -\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)}\left[\frac{\left(F^2+E^2-G^2-E\sqrt{F^2+E^2-G^2}\right)}{(F^2+E^2-G^2)} + \frac{1}{\sqrt{F^2+E^2-G^2}}\left(E+\sqrt{F^2+E^2-G^2} \times \tanh\left(\frac{\sqrt{F^2+E^2-G^2}}{2}(x+2\alpha\kappa t+C)\right)\right)\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(E^2-G^2+F^2-\kappa^2)\beta_l^2}{(F^2+E^2-G^2)(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}. \tag{77}$$

When $E^2 + F^2 - G^2 > 0$, $F \neq 0$, and $G = 0$, then the subsequent dark soliton solutions are obtained,

$$r(x,t) = -\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)}\left[\frac{\left(F^2+E^2-E\sqrt{F^2+E^2}\right)}{(F^2+E^2)} + \frac{1}{\sqrt{F^2+E^2}}\left(E+\sqrt{F^2+E^2} \times \tanh\left(\frac{\sqrt{F^2+E^2}}{2}(x+2\alpha\kappa t+C)\right)\right)\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(E^2+F^2-\kappa^2)\beta_l^2}{(F^2+E^2)(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}, \tag{78}$$

$$s(x,t) = -\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)}\left[\frac{\left(F^2+E^2-E\sqrt{F^2+E^2}\right)}{(F^2+E^2)} + \frac{1}{\sqrt{F^2+E^2}}\left(E+\sqrt{F^2+E^2} \times \tanh\left(\frac{\sqrt{F^2+E^2}}{2}(x+2\alpha\kappa t+C)\right)\right)\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(E^2+F^2-\kappa^2)\beta_l^2}{(F^2+E^2)(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}. \tag{79}$$

When $E^2 + F^2 - G^2 < 0$, $G \neq 0$, and $F = 0$, then the following periodic soliton solutions are obtained:

$$r(x,t) = -\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)}\left[\frac{\left(E^2-G^2-E\sqrt{E^2-G^2}\right)}{(E^2-G^2)} + \frac{1}{\sqrt{E^2-G^2}}\left(E-\sqrt{G^2-E^2} \times \tan\left(\frac{\sqrt{G^2-E^2}}{2}(x+2\alpha\kappa t+C)\right)\right)\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(E^2-G^2-\kappa^2)\beta_l^2}{(E^2-G^2)(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}, \quad (80)$$

$$s(x,t) = -\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)}\left[\frac{\left(E^2-G^2-E\sqrt{E^2-G^2}\right)}{(E^2-G^2)} + \frac{1}{\sqrt{E^2-G^2}}\left(E-\sqrt{G^2-E^2} \times \tan\left(\frac{\sqrt{G^2-E^2}}{2}(x+2\alpha\kappa t+C)\right)\right)\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(E^2-G^2-\kappa^2)\beta_l^2}{(E^2-G^2)(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}. \quad (81)$$

When $E^2 + F^2 - G^2 > 0$, $F - G \neq 0$, and $E = 0$, then the subsequent dark soliton solutions are obtained,

$$r(x,t) = -\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)} \times \left[1+\left(\tanh\left(\frac{\sqrt{F^2-G^2}}{2}(x+2\alpha\kappa t+C)\right)\right)\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(F^2-G^2-\kappa^2)\beta_l^2}{(F^2-G^2)(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}, \quad (82)$$

$$s(x,t) = -\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)} \times \left[1+\left(\tanh\left(\frac{\sqrt{F^2-G^2}}{2}(x+2\alpha\kappa t+C)\right)\right)\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(F^2-G^2-\kappa^2)\beta_l^2}{(F^2-G^2)(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}. \quad (83)$$

When $E = G = kE$ and $F = -kE$, then the subsequent periodic soliton solutions are obtained,

$$r(x,t) = -\frac{4\beta_1}{3(\mu_1+\gamma_1+\tau_1)}\left[\frac{e^{kE(x+2\alpha\kappa t+C)}}{1-e^{kE(x+2\alpha\kappa t+C)}}\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(k^2E^2-\kappa^2)\beta_l^2}{K^2E^2(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}, \quad (84)$$

$$s(x,t) = -\frac{4\beta_2}{3(\mu_2+\gamma_2+\tau_2)}\left[\frac{e^{kE(x+2\alpha\kappa t+C)}}{1-e^{kE(x+2\alpha\kappa t+C)}}\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(k^2E^2-\kappa^2)\beta_l^2}{K^2E^2(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}. \quad (85)$$

When $G = E$, then the following periodic soliton solutions are obtained:

$$r(x,t) = -\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)}\left[\frac{F-E}{F}-\left(\frac{F-E}{F}\right) \times \left(\frac{(E+F)e^{F(x+2\alpha\kappa t+C)}-1}{(E-F)e^{F(x+2\alpha\kappa t+C)}-1}\right)\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(F^2-\kappa^2)\beta_l^2}{F^2(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}, \quad (86)$$

$$s(x,t) = -\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)}\left[\frac{F-E}{F}-\left(\frac{F-E}{F}\right) \times \left(\frac{(E+F)e^{F(x+2\alpha\kappa t+C)}-1}{(E-F)e^{F(x+2\alpha\kappa t+C)}-1}\right)\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(F^2-\kappa^2)\beta_l^2}{F^2(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}. \quad (87)$$

When $E = G$, then the subsequent periodic soliton solutions are obtained,

$$r(x,t) = -\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)}\left[\frac{F-G}{F}+\left(\frac{F-G}{F}\right) \times \left(\frac{(F+G)e^{F(x+2\alpha\kappa t+C)}+1}{(F-G)e^{F(x+2\alpha\kappa t+C)}-1}\right)\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(F^2-\kappa^2)\beta_l^2}{F^2(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}, \quad (88)$$

$$s(x,t) = -\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)}\left[\frac{F-G}{F}+\left(\frac{F-G}{F}\right) \times \left(\frac{(F+G)e^{F(x+2\alpha\kappa t+C)}+1}{(F-G)e^{F(x+2\alpha\kappa t+C)}-1}\right)\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(F^2-\kappa^2)\beta_l^2}{F^2(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}. \quad (89)$$

When $G = -E$, then the following periodic soliton solutions are obtained:

$$r(x,t) = -\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)}\left[\frac{F-E}{F}+\left(\frac{E+F}{F}\right) \times \left(\frac{e^{F(x+2\alpha\kappa t+C)}+F-E}{e^{F(x+2\alpha\kappa t+C)}-F-E}\right)\right] \times e^{i\left(-\kappa x-\frac{8}{9}\frac{(F^2-\kappa^2)\beta_l^2}{F^2(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}, \quad (90)$$

$$s(x,t) = -\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)}\left[\frac{F-E}{F}+\left(\frac{E+F}{F}\right)\times\left(\frac{e^{F(x+2\alpha\kappa t+C)}+F-E}{e^{F(x+2\alpha\kappa t+C)}-F-E}\right)\right]\times e^{i\left(-\kappa x-\frac{8}{9}\frac{(F^2-\kappa^2)\beta_l^2}{F^2(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}. \tag{91}$$

When $F = -G$, then the subsequent periodic soliton solutions are obtained,

$$r(x,t) = -\frac{4\beta_1}{3(\mu_1+\gamma_1+\tau_1)}\left[\frac{Ge^{E(x+2\alpha\kappa t+C)}}{Ge^{E(x+2\alpha\kappa t+C)}-1}\right]\times e^{i\left(-\kappa x-\frac{8}{9}\frac{(E^2-\kappa^2)\beta_l^2}{E^2(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}, \tag{92}$$

$$s(x,t) = -\frac{4\beta_2}{3(\mu_2+\gamma_2+\tau_2)}\left[\frac{Ge^{E(x+2\alpha\kappa t+C)}}{Ge^{E(x+2\alpha\kappa t+C)}-1}\right]\times e^{i\left(-\kappa x-\frac{8}{9}\frac{(E^2-\kappa^2)\beta_l^2}{E^2(\mu_l+\gamma_l+\tau_l)}t+\theta\right)}. \tag{93}$$

It is important to mention that all the above-mentioned solutions are valid for

$$\mu_1+\gamma_1+\tau_1 > 0, \tag{94}$$

$$\mu_2+\gamma_2+\tau_2 > 0. \tag{95}$$

A. The first integral method

In order to solve Eq. (16) by using the first integral method we first introduce the following transformations:[55–57]

$$U(\xi) = X(\xi), \qquad Y(\xi) = X'(\xi). \tag{96}$$

Equation (16) takes the following form:

$$\alpha Y_l' - \left(\alpha\kappa^2+\omega\right)X_l + 2\beta_l X_l^2 + (\mu_l+\gamma_l+\tau_l)X_l^3 = 0. \tag{97}$$

Equation (97) can be rewritten as

$$Y_l'(\xi) = \frac{\alpha\kappa^2+\omega}{\alpha}X_l - \frac{2\beta_l}{\alpha}X_l^2 - \frac{\mu_l+\gamma_l+\tau_l}{\alpha}X_l^3. \tag{98}$$

According to the first integral method, it is assumed that $X_l(\xi)$ and $Y_l(\xi)$ are non-trivial solutions of Eq. (98) and the polynomial $Q(X_l, Y_l) = \sum_{j=0}^{m} a_j(X_l)Y_l^j(\xi)$ is an irreducible polynomial in the complex domain $C[X_l, Y_l]$ such that

$$Q(X_l(\xi), Y_l(\xi)) = \sum_{j=0}^{m} a_j(X_l)Y_l^j = 0, \tag{99}$$

where $a_j(X_l)(j = 0,1,2,3,\ldots,m)$ are polynomials in X_l and $a_m(X_l) \neq 0$. By the division theorem, there exists a polynomial $[g(X_l) + h(X_l)Y_l]$ in the complex domain $C[X_l, Y_l]$ such that

$$\frac{dQ}{d\xi} = \frac{\partial Q}{\partial X_l}\frac{dX_l}{d\xi} + \frac{\partial Q}{\partial Y_l}\frac{dY_l}{d\xi} = [g(X_l) + h(X_l)Y_l(\xi)]\sum_{j=0}^{m} a_j(X_l)Y_l^j. \tag{100}$$

For $m = 1$, Eq. (99) becomes

$$Q(X_l, Y_l) = a_0(X_l) + a_1(X_l)Y_l = 0. \tag{101}$$

Equation (100) becomes

$$\frac{da_0(X_l)}{dX_l}Y_l + \frac{da_1(X_l)}{dX_l}Y_l^2 + a_1(X_l)\times\left[\frac{\omega+a\kappa^2}{\alpha}X_l - \frac{2\beta_l}{\alpha}X_l^2 - \frac{\mu_l+\gamma_l+\tau_l}{\alpha}X_l^3\right] = a_0(X_l)g(X_l) + [a_1(X_l)g(X_l) + a_0(X_l)h(X_l)]Y_l + a_1(X_l)h(X_l)Y_l^2. \tag{102}$$

Equating the coefficients of $Y_l^j(j = 0,1,2)$ on both sides of Eq. (102) gives

$$Y_l^0: \quad a_0(X_l)g(X_l) = a_1(X_l)\left[\frac{\omega+a\kappa^2}{\alpha}X_l - \frac{2\beta_l}{\alpha}X_l^2 - \frac{\mu_l+\gamma_l+\tau_l}{\alpha}X_l^3\right], \tag{103}$$

$$Y_l^1: \quad \frac{da_0(X_l)}{dX_l} = a_1(X_l)g(X_l) + a_0(X_l)h(X_l), \tag{104}$$

$$Y_l^2: \quad \frac{da_1(X_l)}{dX_l} = a_1(X_l)h(X_l). \tag{105}$$

Since $a_j(X_l)(j = 0,1)$ are assumed to be polynomials, if we choose $h(X_l) = 0$, Eq. (105) yields $a_1(X_l) =$ constant. For simplicity, take $a_1(X_l) = 1$. Balancing the degrees of $a_0(X_l)$ and $g(X_l)$ gives a degree of $g(X_l) = 1$. Suppose that

$$g(X_l) = AX_l + B, \tag{106}$$

where $A \neq 0$. From Eq. (104), we obtain

$$a_0(X_l) = \frac{AX_l^2}{2} + BX_l + C, \tag{107}$$

where C is a constant of integration. When substituting $a_0(X_l), a_1(X_l)$ and $g(X_l)$ in Eq. (103) and equating the coefficients of $X_l^j(j = 0,1,2,3)$ on both sides, the system of nonlinear algebraic equations is obtained. After solving the system, the following values of constants are obtained:

$$A = \pm\frac{\sqrt{2}\sqrt{\mu_l+\gamma_l+\tau_l}}{\sqrt{-\alpha}}, \qquad B = \frac{2\sqrt{2}\beta_l}{3\sqrt{-\alpha}\sqrt{\mu_l+\gamma_l+\tau_l}}, \qquad C = 0. \tag{108}$$

Using the above-mentioned values in Eq. (101), we obtain

$$Y(\xi) = \mp\frac{2\sqrt{2}\beta_l}{3\sqrt{-\alpha}\sqrt{\mu_l+\gamma_l+\tau_l}}X_l \mp \frac{\sqrt{\mu_l+\gamma_l+\tau_l}}{\sqrt{-2\alpha}}X_l^2, \tag{109}$$

In order to solve Eq. (109), we refer to the Bernoulli equation,

$$v'(\xi) = l_1 v(\xi) + l_2 v^\beta(\xi), \tag{110}$$

where $l_1, l_2,$ andβ are real integers and $l_1 l_2 \neq 0$and$\beta \neq 1$. Its general solution is of the form

$$v(\xi) = \left[\frac{-l_1 l_2}{\epsilon_0 \exp(l_1(1-\beta)\xi) + 1}\right]^{\frac{1}{\beta-1}}$$

or

$$v(\xi) = \left[\frac{-l_1}{2l_2}\left[1 + \tanh\left(\frac{l_1(1-\beta)}{2}\xi - \frac{\ln\epsilon_0}{2}\right)\right]\right]^{\frac{1}{\beta-1}}, \text{ if } \epsilon_0 > 0$$
$$= \left[\frac{-l_1}{2l_2}\left[1 + \coth\left(\frac{l_1(1-\beta)}{2}\xi - \frac{\ln(-\epsilon_0)}{2}\right)\right]\right]^{\frac{1}{\beta-1}}, \text{ if } \epsilon_0 < 0$$
$$= (-l_1 l_2)^{\frac{1}{\beta-1}}, \text{ if } \epsilon_0 = 0. \tag{111}$$

If we choose $l_1 = \mp\frac{2\sqrt{2}\beta_l}{3\sqrt{-\alpha}\sqrt{\mu_l+\gamma_l+\tau_l}}$, $l_2 = \mp\frac{\sqrt{\mu_l+\gamma_l+\tau_l}}{\sqrt{-2\alpha}}$, and $\beta = 2$ in Eq. (111), then we get our required solutions.

If $\epsilon_0 > 0$, the dark soliton solution is obtained as

$$r(x,t) = \left[\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)} \times \left[1 \pm \tanh\left(\frac{\sqrt{2}\beta_1}{3\sqrt{-\alpha}\sqrt{\mu_1+\gamma_1+\tau_1}}\xi - \frac{\ln\epsilon_0}{2}\right)\right]\right] \times e^{i(-\kappa x+\omega t+\theta)}, \tag{112}$$

$$s(x,t) = \left[\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)} \times \left[1 \pm \tanh\left(\frac{\sqrt{2}\beta_2}{3\sqrt{-\alpha}\sqrt{\mu_2+\gamma_2+\tau_2}}\xi - \frac{\ln\epsilon_0}{2}\right)\right]\right] \times e^{i(-\kappa x+\omega t+\theta)}. \tag{113}$$

If $\epsilon_0 < 0$, we obtain the singular soliton solution,

$$r(x,t) = \left[\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)} \times \left[1 \pm \coth\left(\frac{\sqrt{2}\beta_1}{3\sqrt{-\alpha}\sqrt{\mu_1+\gamma_1+\tau_1}}\xi - \frac{\ln(-\epsilon_0)}{2}\right)\right]\right] \times e^{i(-\kappa x+\omega t+\theta)}, \tag{114}$$

$$s(x,t) = \left[\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)} \times \left[1 \pm \coth\left(\frac{\sqrt{2}\beta_2}{3\sqrt{-\alpha}\sqrt{\mu_2+\gamma_2+\tau_2}}\xi - \frac{\ln(-\epsilon_0)}{2}\right)\right]\right] \times e^{i(-\kappa x+\omega t+\theta)}. \tag{115}$$

If $\epsilon_0 = 0$, we obtain the plane wave solution,

$$r(x,t) = \left[\frac{4\beta_1}{3(\mu_1+\gamma_1+\tau_1)}\right] e^{i(-\kappa x+\omega t+\theta)}, \tag{116}$$

$$s(x,t) = \left[\frac{4\beta_2}{3(\mu_2+\gamma_l+\tau_2)}\right] e^{i(-\kappa x+\omega t+\theta)}. \tag{117}$$

The constraint conditions for the existence of soliton solutions obtained by the first integral method are $\alpha < 0$, $\mu_1 + \gamma_1 + \tau_1 > 0$, and $\mu_1 + \gamma_1 + \tau_1 > 0$.

B. $\frac{G'}{G^2}$-Expansion method

In this section, the $\frac{G'}{G^2}$-expansion method[58] is employed to solve Eq. (16). According to this method, the traveling wave solution can be expressed as

$$U(\xi) = a_0 + \sum_{n=1}^{N}\left(a_n\left(\frac{G'}{G^2}\right)^n + b_n\left(\frac{G'}{G^2}\right)^{-n}\right), \tag{118}$$

where $G = G(\xi)$ satisfies

$$\left(\frac{G'}{G^2}\right)' = \epsilon + \delta\left(\frac{G'}{G^2}\right)^2, \tag{119}$$

in which $\delta \neq 0$ and $\epsilon \neq 1$ are integers. The unknown constants a_0, a_n, and $b_n (n = 1, 2, 3, \ldots, N)$ are to be determined. Balancing the terms U_l'' and U_l^3 in Eq. (16) by using the homogeneous principle yields $N = 1$. As a result, Eq. (118) takes the form

$$U(\xi) = a_0 + a_1\left(\frac{G'}{G^2}\right) + b_1\left(\frac{G'}{G^2}\right)^{-1}. \tag{120}$$

Now putting Eq. (120) into Eq. (16) and then comparing the coefficients of same powers of $\left(\frac{G'}{G^2}\right)^j$, $(j = 0, \pm1, \pm2, \pm3, \pm4)$, to 0 provide a system of algebraic equations. The following set of solutions are retrieved on solving the system of algebraic equations.

Set 1:

$$a_0 = -\frac{2\beta_l}{3(\mu_l+\gamma_l+\tau_l)}, \quad a_1 = 0, \quad b_1 = \pm\frac{\sqrt{2}\sqrt{-\alpha\epsilon}}{\sqrt{\mu_l+\gamma_l+\tau_l}},$$

$$\delta = \frac{2\beta_l^2}{9\alpha(\mu_l+\gamma_l+\tau_l)\epsilon}, \quad \omega = -\alpha\kappa^2 - \frac{8\beta_l^2}{9(\mu_l+\gamma_l+\tau_l)}.$$

Set 2:

$$a_0 = -\frac{2\beta_l}{3(\mu_l+\gamma_l+\tau_l)}, \quad a_1 = \pm\frac{\sqrt{2}\sqrt{-\alpha\delta}}{\sqrt{\mu_l+\gamma_l+\tau_l}}, \quad b_1 = 0,$$

$$\epsilon = \frac{2\beta_l^2}{9\alpha(\mu_l+\gamma_l+\tau_l)\delta}, \quad \omega = -\alpha\kappa^2 - \frac{8\beta_l^2}{9(\mu_l+\gamma_l+\tau_l)}.$$

Set 3:

$$a_0 = -\frac{2\beta_l}{3(\mu_l+\gamma_l+\tau_l)}, \quad a_1 = \pm\frac{\sqrt{2}\sqrt{-\alpha\delta}}{\sqrt{\mu_l+\gamma_l+\tau_l}},$$
$$b_1 = \pm\frac{\beta_l^2}{9\sqrt{2}\sqrt{-\alpha}\sqrt{(\mu_l+\gamma_l+\tau_l)^3}\delta},$$

$$\epsilon = \frac{\beta_l^2}{18\alpha(\mu_l + \gamma_l + \tau_l)\delta}, \quad \omega = -\alpha\kappa^2 - \frac{8\beta_l^2}{9(\mu_l + \gamma_l + \tau_l)}.$$

According to set 1, the subsequent solutions are retrieved.

If $\epsilon\delta > 0$,

$$r(x,t) = e^{i(-\kappa x+\omega t+\theta)}\left[-\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)} \pm \frac{\sqrt{2\delta\epsilon}\sqrt{-\alpha}\left(M\cos[\sqrt{\delta\epsilon}\xi] - P\sin[\sqrt{\delta\epsilon}\xi]\right)}{\sqrt{\mu_1+\gamma_1+\tau_1}\left(P\cos[\sqrt{\delta\epsilon}\xi] + M\sin[\sqrt{\delta\epsilon}\xi]\right)}\right], \tag{121}$$

$$s(x,t) = e^{i(-\kappa x+\omega t+\theta)}\left[-\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)} \pm \frac{\sqrt{2\delta\epsilon}\sqrt{-\alpha}\left(M\cos[\sqrt{\delta\epsilon}\xi] - P\sin[\sqrt{\delta\epsilon}\xi]\right)}{\sqrt{\mu_2+\gamma_2+\tau_2}\left(P\cos[\sqrt{\delta\epsilon}\xi] + M\sin[\sqrt{\delta\epsilon}\xi]\right)}\right]. \tag{122}$$

If $\epsilon\delta < 0$,

$$r(x,t) = e^{i(-\kappa x+\omega t+\theta)}\left[-\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)} \pm \frac{\sqrt{2}\sqrt{-\alpha\epsilon}}{\sqrt{\mu_1+\gamma_1+\tau_1}} \times \left(-\frac{\sqrt{|\epsilon\delta|}}{\delta}\left(\frac{P\sinh(2\sqrt{|\epsilon\delta|}\xi) + P\cosh(2\sqrt{|\epsilon\delta|}\xi) + M}{P\sinh(2\sqrt{|\epsilon\delta|}\xi) + P\cosh(2\sqrt{|\epsilon\delta|}\xi) - M}\right)\right)^{-1}\right], \tag{123}$$

$$s(x,t) = e^{i(-\kappa x+\omega t+\theta)}\left[-\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)} \pm \frac{\sqrt{2}\sqrt{-\alpha\epsilon}}{\sqrt{\mu_2+\gamma_2+\tau_2}} \times \left(-\frac{\sqrt{|\epsilon\delta|}}{\delta}\left(\frac{P\sinh(2\sqrt{|\epsilon\delta|}\xi) + P\cosh(2\sqrt{|\epsilon\delta|}\xi) + M}{P\sinh(2\sqrt{|\epsilon\delta|}\xi) + P\cosh(2\sqrt{|\epsilon\delta|}\xi) - M}\right)\right)^{-1}\right]. \tag{124}$$

Upon choosing $P = M$, the dark soliton solution is obtained as

$$r(x,t) = -e^{i(-\kappa x+\omega t+\theta)}\left[-\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)} \mp \frac{\sqrt{2|\delta\epsilon|}\sqrt{-\alpha}\tanh\left[\sqrt{|\delta\epsilon|}\xi\right]}{\sqrt{\mu_1+\gamma_1+\tau_1}}\right], \tag{125}$$

$$s(x,t) = -e^{i(-\kappa x+\omega t+\theta)}\left[-\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)} \mp \frac{\sqrt{2|\delta\epsilon|}\sqrt{-\alpha}\tanh\left[\sqrt{|\delta\epsilon|}\xi\right]}{\sqrt{\mu_2+\gamma_2+\tau_2}}\right]. \tag{126}$$

If $\epsilon = 0, \delta \neq 0$, the plane wave solution is obtained as

$$r(x,t) = -e^{i(-\kappa x+\omega t+\theta)}\left[-\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)}\right], \tag{127}$$

$$s(x,t) = -e^{i(-\kappa x+\omega t+\theta)}\left[-\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)}\right]. \tag{128}$$

According to set 2, the following solutions are retrieved.

If $\epsilon\delta > 0$,

$$r(x,t) = e^{i(-\kappa x+\omega t+\theta)}\left[-\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)} \pm \frac{\sqrt{2\delta\epsilon}\sqrt{-\alpha}\left(P\cos\left[\sqrt{\delta\epsilon}\xi\right] + M\sin\left[\sqrt{\delta\epsilon}\xi\right]\right)}{\sqrt{\mu_1+\gamma_1+\tau_1}\left(M\cos\left[\sqrt{\delta\epsilon}\xi\right] - P\sin\left[\sqrt{\delta\epsilon}\xi\right]\right)}\right], \tag{129}$$

$$s(x,t) = e^{i(-\kappa x+\omega t+\theta)}\left[-\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)} \pm \frac{\sqrt{2\delta\epsilon}\sqrt{-\alpha}\left(P\cos\left[\sqrt{\delta\epsilon}\xi\right] + M\sin\left[\sqrt{\delta\epsilon}\xi\right]\right)}{\sqrt{\mu_2+\gamma_2+\tau_2}\left(M\cos\left[\sqrt{\delta\epsilon}\xi\right] - P\sin\left[\sqrt{\delta\epsilon}\xi\right]\right)}\right]. \tag{130}$$

If $\epsilon\delta < 0$,

$$r(x,t) = e^{i(-\kappa x+\omega t+\theta)}\left[-\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)} \pm \frac{\sqrt{2}\sqrt{-\alpha\delta}}{\sqrt{\mu_1+\gamma_1+\tau_1}} \times \left(-\frac{\sqrt{|\epsilon\delta|}}{\delta}\left(\frac{P\sinh(2\sqrt{|\epsilon\delta|}\xi) + P\cosh(2\sqrt{|\epsilon\delta|}\xi) + M}{P\sinh(2\sqrt{|\epsilon\delta|}\xi) + P\cosh(2\sqrt{|\epsilon\delta|}\xi) - M}\right)\right)\right], \tag{131}$$

$$s(x,t) = e^{i(-\kappa x+\omega t+\theta)}\left[-\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)} \pm \frac{\sqrt{2}\sqrt{-\alpha\delta}}{\sqrt{\mu_2+\gamma_2+\tau_2}} \times \left(-\frac{\sqrt{|\epsilon\delta|}}{\delta}\left(\frac{P\sinh(2\sqrt{|\epsilon\delta|}\xi) + P\cosh(2\sqrt{|\epsilon\delta|}\xi) + M}{P\sinh(2\sqrt{|\epsilon\delta|}\xi) + P\cosh(2\sqrt{|\epsilon\delta|}\xi) - M}\right)\right)\right]. \tag{132}$$

To obtain the soliton solution, $P = M$ is chosen; we get the singular soliton as

$$r(x,t) = -e^{i(-\kappa x+\omega t+\theta)}\left[-\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)} \mp \frac{\sqrt{2|\delta\epsilon|}\sqrt{-\alpha}\coth\left[\sqrt{|\delta\epsilon|}\xi\right]}{\sqrt{\mu_1+\gamma_1+\tau_1}}\right], \tag{133}$$

$$s(x,t) = -e^{i(-\kappa x+\omega t+\theta)}\left[-\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)} \mp \frac{\sqrt{2|\delta\epsilon|}\sqrt{-\alpha}\coth\left[\sqrt{|\delta\epsilon|}\xi\right]}{\sqrt{\mu_2+\gamma_2+\tau_2}}\right]. \tag{134}$$

If $\epsilon = 0, \delta \neq 0$, the plane wave solution is obtained as

$$r(x,t) = -e^{i(-\kappa x+\omega t+\theta)}\left[-\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)}\right], \tag{135}$$

$$s(x,t) = -e^{i(-\kappa x+\omega t+\theta)}\left[-\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)}\right]. \tag{136}$$

According to set 3, the subsequent solutions are retrieved.

If $\epsilon\delta > 0$,

$$r(x,t) = e^{i(-\kappa x+\omega t+\theta)}\Bigg[-\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)} \pm \frac{\sqrt{2\delta\epsilon}\sqrt{-\alpha}\left(P\cos\left[\sqrt{\delta\epsilon}\xi\right]+M\sin\left[\sqrt{\delta\epsilon}\xi\right]\right)}{\sqrt{\mu_1+\gamma_1+\tau_1}\left(M\cos\left[\sqrt{\delta\epsilon}\xi\right]-P\sin\left[\sqrt{\delta\epsilon}\xi\right]\right)} \pm \frac{\beta_1^2}{9\sqrt{2\delta\epsilon}\sqrt{-\alpha}\sqrt{(\mu_1+\gamma_1+\tau_1)^3}}\frac{\left(M\cos[\sqrt{\delta\epsilon}\xi]-P\sin[\sqrt{\delta\epsilon}\xi]\right)}{\left(P\cos[\sqrt{\delta\epsilon}\xi]+M\sin[\sqrt{\delta\epsilon}\xi]\right)}\Bigg], \tag{137}$$

$$s(x,t) = e^{i(-\kappa x+\omega t+\theta)}\Bigg[-\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)} \pm \frac{\sqrt{2\delta\epsilon}\sqrt{-\alpha}\left(P\cos\left[\sqrt{\delta\epsilon}\xi\right]+M\sin\left[\sqrt{\delta\epsilon}\xi\right]\right)}{\sqrt{\mu_2+\gamma_2+\tau_2}\left(M\cos\left[\sqrt{\delta\epsilon}\xi\right]-P\sin\left[\sqrt{\delta\epsilon}\xi\right]\right)} \pm \frac{\beta_2^2}{9\sqrt{2\delta\epsilon}\sqrt{-\alpha}\sqrt{(\mu_2+\gamma_2+\tau_2)^3}}\frac{\left(M\cos[\sqrt{\delta\epsilon}\xi]-P\sin[\sqrt{\delta\epsilon}\xi]\right)}{\left(P\cos[\sqrt{\delta\epsilon}\xi]+M\sin[\sqrt{\delta\epsilon}\xi]\right)}\Bigg]. \tag{138}$$

If $\epsilon\delta < 0$,

$$r(x,t) = e^{i(-\kappa x+\omega t+\theta)}\Bigg[-\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)} \pm \frac{\sqrt{2}\sqrt{-\alpha}\delta}{\sqrt{\mu_1+\gamma_1+\tau_1}}\left(-\frac{\sqrt{|\epsilon\delta|}}{\delta}\left(\frac{P\sinh(2\sqrt{|\epsilon\delta|}\xi)+P\cosh(2\sqrt{|\epsilon\delta|}\xi)+M}{P\sinh(2\sqrt{|\epsilon\delta|}\xi)+P\cosh(2\sqrt{|\epsilon\delta|}\xi)-M}\right)\right) \pm \frac{\beta_1^2}{9\sqrt{2}\sqrt{-\alpha}\sqrt{(\mu_1+\gamma_1+\tau_1)^3}\delta}\left(-\frac{\sqrt{|\epsilon\delta|}}{\delta}\left(\frac{P\sinh(2\sqrt{|\epsilon\delta|}\xi)+P\cosh(2\sqrt{|\epsilon\delta|}\xi)+M}{P\sinh(2\sqrt{|\epsilon\delta|}\xi)+P\cosh(2\sqrt{|\epsilon\delta|}\xi)-M}\right)\right)^{-1}\Bigg], \tag{139}$$

$$s(x,t) = e^{i(-\kappa x+\omega t+\theta)}\Bigg[-\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)} \pm \frac{\sqrt{2}\sqrt{-\alpha}\delta}{\sqrt{\mu_2+\gamma_2+\tau_2}}\left(-\frac{\sqrt{|\epsilon\delta|}}{\delta}\left(\frac{P\sinh(2\sqrt{|\epsilon\delta|}\xi)+P\cosh(2\sqrt{|\epsilon\delta|}\xi)+M}{P\sinh(2\sqrt{|\epsilon\delta|}\xi)+P\cosh(2\sqrt{|\epsilon\delta|}\xi)-M}\right)\right) \pm \frac{\beta_l^2}{9\sqrt{2}\sqrt{-\alpha}\sqrt{(\mu_2+\gamma_2+\tau_2)^3}\delta}\left(-\frac{\sqrt{|\epsilon\delta|}}{\delta}\left(\frac{P\sinh(2\sqrt{|\epsilon\delta|}\xi)+P\cosh(2\sqrt{|\epsilon\delta|}\xi)+M}{P\sinh(2\sqrt{|\epsilon\delta|}\xi)+P\cosh(2\sqrt{|\epsilon\delta|}\xi)-M}\right)\right)^{-1}\Bigg]. \tag{140}$$

Upon choosing $P = M$, the dark singular combo soliton solution is obtained as

$$r(x,t) = e^{i(-\kappa x+\omega t+\theta)}\Bigg[-\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)} \mp \frac{\sqrt{2|\delta\epsilon|}\sqrt{-\alpha}}{\sqrt{\mu_1+\gamma_1+\tau_1}}\coth\left[\sqrt{|\delta\epsilon|}\xi\right] \mp \frac{\beta_1^2}{9\sqrt{2|\delta\epsilon|}\sqrt{-\alpha}\sqrt{(\mu_1+\gamma_1+\tau_1)^3}}\tanh\left[\sqrt{|\delta\epsilon|}\xi\right]\Bigg], \tag{141}$$

$$s(x,t) = e^{i(-\kappa x+\omega t+\theta)}\Bigg[-\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)} \mp \frac{\sqrt{2|\delta\epsilon|}\sqrt{-\alpha}}{\sqrt{\mu_l+\gamma_l+\tau_l}}\coth\left[\sqrt{|\delta\epsilon|}\xi\right] \mp \frac{\beta_2^2}{9\sqrt{2|\delta\epsilon|}\sqrt{-\alpha}\sqrt{(\mu_2+\gamma_2+\tau_2)^3}}\tanh\left[\sqrt{|\delta\epsilon|}\xi\right]\Bigg]. \tag{142}$$

If $\epsilon = 0$, $\delta \neq 0$, the plane wave solution is obtained as

TABLE I. Comparison of current paper solutions with already existing solutions.

Parametric values	Solutions of current paper	Solutions by the JEF-expansion method[60]	Solutions by the $\left(\frac{G'}{G}\right)$ method[61]	Solutions by the F-expansion method[62]
F = 2, E, G, C = 0	(76), (77), (78), (79), (82), (83)	(29), (30)	(48), (49)	(84), (85)
F = 2, E, G, C = 0	(55), (56), (57), (58), (61), (62)	(31), (32)	(52), (53)	(88), (89)
F = 4, E, G, C = 0	(55), (56), (57), (58), (61), (62)	(33), (34)	...	...
G = 2, E, F, C = 0	(53), (54), (59), (60)	...	(62), (63)	(120), (121)
G = 2, E, F, C = 0	(74), (75), (80), (81)	...	(58), (59)	(118), (119)
F = 1, E, G, C = 0	(55), (56), (57), (58), (61), (62)	...	...	(108), (109)
F = 1, E, G, C = 0	(76), (77), (78), (79), (82), (83)	...	...	(110), (111)

$$r(x,t) = -e^{i(-\kappa x+\omega t+\theta)}\left[-\frac{2\beta_1}{3(\mu_1+\gamma_1+\tau_1)} \mp \frac{\sqrt{2}\sqrt{-\alpha}P}{\sqrt{\mu_1+\gamma_1+\tau_1}(M+P\xi)} \mp \frac{\beta_1^2(M+P\xi)}{9\sqrt{2}\sqrt{-\alpha}P\sqrt{(\mu_1+\gamma_1+\tau_1)}^3}\right], \tag{143}$$

$$s(x,t) = -e^{i(-\kappa x+\omega t+\theta)}\left[-\frac{2\beta_2}{3(\mu_2+\gamma_2+\tau_2)} \mp \frac{\sqrt{2}\sqrt{-\alpha}P}{\sqrt{\mu_2+\gamma_2+\tau_2}(M+P\xi)} \mp \frac{\beta_2^2(M+P\xi)}{9\sqrt{2}\sqrt{-\alpha}P\sqrt{(\mu_2+\gamma_2+\tau_2)}^3}\right]. \tag{144}$$

The constraint conditions for the existence of soliton solutions given above using the $\frac{G'}{G^2}$–expansion method are $\alpha < 0$, $\mu_1 + \gamma_1 + \tau_1 > 0$, and $\mu_2 + \gamma_2 + \tau_2 > 0$.

IV. COMPARISON

This section deals with the comparison of obtained results with some previously known ones. A detailed analysis is made on the basis of different values of parameters E, F, and G, which is represented through equation numbers of the present article and already exists in the literature, as shown in Table I.

In Table I, the first column shows different values of E, F, and G, and the second column represents the equation numbers of the concerned article; third, fourth, and fifth columns denote the equation numbers of the articles.[60–62] It is observed that for different values of E, F, and G, most of our results match exactly the solutions[60–62] mentioned in Table I, which shows the effectiveness of our results as a generalized case of the mentioned published results. The values of $E, F,$ and$G$, which are taken from Table I, satisfy their corresponding constraints.

From Table I, we find that the techniques implemented in this study, over all the other methods, provide further new computational solutions, including additional free parameters. Most of the obtained solutions in the literature are taken via these applied approaches as a particular case, and we receive some new solutions as well.

Remarks:

(1) To the best of our knowledge, the obtained results in this article are new, except those mentioned in Table I.
(2) The CPU time of the computation is ~0.2189 s.

V. CONCLUSION

This paper reveals a plethora of solutions to the coupled NLSE in birefringent fibers with 4WM incorporating QC nonlinearity. The improved $\tan\left(\frac{\phi(\xi)}{2}\right)$-expansion method, first integral method, and $\frac{G'}{G^2}$-expansion method are used to procure these soliton solutions. Dark, periodic, singular, dark-singular, and a plenty of other soliton solutions have been successfully yielded through this process. A detailed comparison of solutions with the already published results reveals that our techniques are not only reliable but also fruitful as they provide us with a bunch of new solutions. The results being reported in this work are an excellent addition to the existing literature. In the near future, we will modify the algorithms presented here to deal with different NLEEs when their coefficients are variables, for exhaling nonautonomous wave solutions.

AUTHORS' CONTRIBUTIONS

All authors contributed equally to this work. All authors read and approved the final manuscript.

All persons who meet authorship criteria are listed as authors, and all authors certify that they have participated sufficiently in the work to take public responsibility for the content, including participation in the concept, design, analysis, writing, or revision of the manuscript. Furthermore, each author certifies that this material or similar material has not been and will not be submitted to or published in any other publication before.

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DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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