OPTICAL SCIENCES THE UNIVERSITY OF ARIZONA TUCSON. ARIZONA

TECHNICAL REPORT NO, 13

OPTICAL TRANSFER OF THE

THREE-DIMENSIONAL OBJECT*
B. ROY FRIEDEN

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\text { January, } 1967
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*This paper has been accepted for publication in the Journal of the Optical Society of America.


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## ABSTRACT

This paper presents a transfer theory for determination of the image space and image spectrum of a three-dimensional object. The theory assumes the existence of volumes of stationarity, called "isotomes," into which the object must be divided. Isotomicity is approximated, over sufficiently small volumes, in the diffraction-limited case.

The main development assumes the object to be radiating incoherently; the results are as follows:

The image (irradiance) distribution $i(x, y, z)$ is the threedimensional convolution of the point spread function $s(\alpha, \beta, \gamma)$ with the object distribution $o\left(x^{\prime}, y^{\prime}, z^{\prime}\right)$. The image spectrum $I\left(\omega_{1}, \omega_{2}, \omega_{3}\right)$ is defined as the three-dimensional Fourier transform of $i(x, y, z)$. It is found that I obeys a transfer theorem, $I=F \cdot 0$, where $F\left(\omega_{1}, \omega_{2}, \omega_{3}\right)$ is the three-dimensional Fourier transform of $s(\alpha, \beta, \gamma)$ and $O\left(\omega_{1}, \omega_{2}, \omega_{3}\right)$ is an integral transform of $o\left(x^{\prime}, y^{\prime}, z^{\prime}\right)$. This transfer theorem establishes the value of using $F$ as a criterion of optical design. In the Fraunhofer approximation, F may be represented as a line integral across the pupil U. This shows that $F$ contains a simple pole at $\omega_{1}=\omega_{2}=0$. Nevertheless, all integrals involving F are convergent. The pupil representation for F also shows that $F$ is zero outside a restricted volume $\Sigma$ of ( $\omega_{1}, \omega_{2}, \omega_{3}$ )-space. Because F is bandwidth-limited, $F, I, s, i$ and $\tau$ (the optical transfer function) individually obey sampling theorems. These theorems imply that if each point in image space is regarded as an independent degree of freedom, there can be no more than $1 / \lambda^{3} f_{\#}^{4}$ degrees of freedom/volume in image space.

For coherent object radiation, analogous theorems of convolution, transfer, and sampling can be constructed. In addition, the "amplitude"
transfer function $W\left(\omega_{1}, \omega_{2}, \omega_{3}\right)$, defined as the Fourier transform of the point amplitude distribution $u(\alpha, \beta, \gamma)$, is proportional to the pupil function and to $\delta\left[\omega_{3}-\left(\omega_{1}^{2}+\omega_{2}^{2}\right) / 2 k\right]$, where $\delta$ is the Dirac delta function and $k=2 \pi / \lambda$. This relation is used to establish sampling theorems for $u$ and for $g$ (the image amplitude) and to express $g(x, y, z)$ as a double integral over $U$ and 0 .

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## I. INTRODUCTION

A real object, such as a ball, has three-dimensional extent in space. At present, the image of a real object cannot be described by a transfer theory unless the object approximates one of two idealized limits. These are the "transverse" object and the "longitudinal" object.

The "transverse" object ${ }^{1}$ is defined as the radiance distribution within a given plane normal to the optical axis. The transfer theory that applies, due to Duffieux, ${ }^{2}$ is familiar to workers in the field of image evaluation. ${ }^{3}$ For purposes of distinction, this theory will be called the "lateral" or "transverse" transfer theory. In an approximate sense, lateral transfer theory also applies to any three-dimensional object that is sufficiently distant from the optics. In this case the opticsdetector arrangement has a depth of focus that is sufficiently large to mask any spatial departures, over the object, from the transverse plane. Hence, the measured image gives no information about depth variation across the object.

The "longitudinal" object ${ }^{1}$ is defined as the radiance distribution that exists along a line which is directed in an arbitrary field direction (from the nodal point for that direction). The irradiance distribution that exists along the Gaussian image of the longitudinal object obeys a "longitudinal" transfer theory. Longitudinal transfer theory also applies if the object has a finite thickness, but then the radiance must be assumed to be uniform within any cross section of the object, even if it is not. Consequently, all information is lost about the radiance variation within any cross section.

The aim of this paper is to describe the image and image spectrum of a three-dimensional object by means of a suitably generalized transfer
theory. A surface object is probably the most common type of object observed by the unaided eye. When the object is itself an aerial image, which has three-dimensional extent, it must be represented as a volume distribution. The generalized transfer theory will apply to objects of both types.

The development is similar to that of lateral transfer theory, ${ }^{3}$ and corresponding results of the two developments are frequently compared. Since the generalized transfer theory does not require the object to approximate an idealized model, such as a plane or a line, there are no "information losses" of the types previously considered.


Fig. 1. Object and image spaces for three-dimensional image formation. The region of isotomicity is exaggerated for emphasis. Image irradiance $i(r)$ is shown to arise from overlap of the point spread functions from differential areas $d \sigma_{i}$ of the object.

Fig. 1 identifies the main parameters of the problem. The optics are of focal length $f$ and radius $r_{0} ; H$ and $H^{\prime}$ are the principal points.

We introduce the vector notation

$$
\begin{align*}
\overrightarrow{\mathrm{r}}^{\prime} & =\left(\mathrm{x}^{\prime}, y^{\prime}, z^{\prime}\right) \\
\overrightarrow{\mathrm{r}}_{\mathrm{G}} & =\left(\mathrm{x}_{\mathrm{G}}, \mathrm{y}_{\mathrm{G}}, \mathrm{z}_{\mathrm{G}}\right) \\
\overrightarrow{\mathrm{r}} & =(\mathrm{x}, \mathrm{y}, \mathrm{z})  \tag{1.1}\\
\overrightarrow{\mathrm{d}}^{\prime} & =\mathrm{d} x^{\prime} \mathrm{dy}
\end{align*} \mathrm{~d}^{\prime}{ }^{\prime} .
$$

In the above, $\vec{r}^{\prime}$ is a general object point, $\vec{r}_{G}$ is the corresponding Gauss point, and $\vec{r}$ is a general point in the image space. The origins for primed and unprimed coordinates are $\mathrm{H}^{\prime}$ and H , respectively.

Any given object may generally be described as either a surface distribution of radiance $a_{\sigma}\left(\vec{r}^{\prime}\right)$ (as illustrated in Fig. 1), or a volume distribution of radiance $a_{V}\left(\vec{r}^{\prime}\right)$. The differential radiance $d N$ from object point $\vec{r}^{\prime}$ is then given as either

$$
\begin{equation*}
d N\left(\vec{r}^{\prime}\right)=a_{\sigma}\left(\vec{r}^{\prime}\right) d \sigma\left(\vec{r}^{\prime}\right) \tag{1.2a}
\end{equation*}
$$

or

$$
\begin{equation*}
\mathrm{dN}\left(\vec{r}^{\prime}\right)=a_{\mathrm{V}}\left(\vec{r}^{\prime}\right) \mathrm{d} \vec{r}^{\prime}, \tag{1.2b}
\end{equation*}
$$

where d $\sigma$ is a differential area on the surface $\sigma$. Comparing Eqs. (1.2a) and (1.2b), it is noted that $a_{\sigma}$ and $a_{V}$ are dimensionally different quantities.

## II. POSTULATES

Lateral transfer theory is based on two assumptions: stationarity and superposition. ${ }^{3}$ As is well known, these conditions are met by an object which radiates incoherently from an "isoplanatic" ${ }^{4}$ area of the object plane. It will be seen that three-dimensional generalizations of these assumptions imply a three-dimensional transfer theory.

## A. Stationarity

Let $s\left(\vec{r}^{\prime} ; \vec{r}\right)$ represent the irradiance at $\vec{r}$ due to a point source at $\vec{r}^{\prime}$. Let $\vec{r}_{G}$ locate the Gauss point corresponding to $\vec{r}^{\prime}$. We shall require, as our condition of stationarity, that

$$
\begin{equation*}
s\left(\vec{r}^{\prime} ; \vec{r}\right)=s\left(\vec{r}-\vec{r}_{G}\right) \tag{2.1}
\end{equation*}
$$

The portion $V$ of the object space that obeys (2.1) will be called an "isotome" (from the Greek "iso" and "tomos," meaning "equal" and "volume section," respectively). A surface $\sigma$ that lies within $V$, as in Fig. 1, will be called "isotomic." If the entire object is not "isotomous," it must be subdivided into volumes $V$ which are individually isotomous. Each volume V can then be separately treated by the generalized transfer theory. This technique is especially useful if each $V$ is a meaningful fraction of the object.

Physically, (2.1) requires the point spread function to remain invariant under changes in position of the Gauss point. This will be true if each point within $V$ has essentially the same wave aberrations. The existence of stationarity in two dimensions, within an "isoplanatic" patch, has been established; ${ }^{5}$ hence, the existence of usefully large (as described above) volumes of stationarity seems likely, in particular cases. Appendix $I$ verifies isotomicity in the diffraction-limited case.

For a material object with longitudinal extent, some radiation from distant object points might be scattered or reflected by nearer object points. Such "radiation blockage" will affect $s(\vec{r}$ '; $\vec{r}$ ) and consequently must be taken into account when determining the isotomous volume. Unless the object is an aerial image, "radiation blockage" will always be present for object types $a_{V}$ and will sometimes be present (depending on the shape of surface $\sigma$ ) when the object is of type $a_{\sigma}$.

## B. Superposition

The condition of superposition requires that the irradiance $\mathbf{i}$ at $\overrightarrow{\mathbf{r}}$ due to point sources at $\vec{r}_{1}^{\prime}$ and $\vec{r}_{2}^{\prime}$ be given as the sum

$$
\begin{equation*}
i(\vec{r})=s\left(\vec{r}_{1}^{\prime} ; \vec{r}\right)+s\left(\vec{r}_{2}^{\prime} ; \vec{r}\right) \tag{2.2}
\end{equation*}
$$

As in the transverse case, this condition is satisfied by point sources that radiate in a mutually incoherent manner. From (2.2), if $\vec{r}_{1}^{\prime}=\vec{r}_{2}^{\prime}$, that is, if a "strength" of two point sources is present at $\vec{r}_{1}^{\prime}$, then

$$
\begin{equation*}
i(\vec{r})=2 s\left(\vec{r}_{1}^{\prime} ; \vec{r}\right) \tag{2.3}
\end{equation*}
$$

By continuing the process we establish that source strengths o( $r_{i}^{\prime}$ ), where $i=1,2, \ldots, N$, cause $a n$

$$
\begin{equation*}
i(\vec{r})=\sum_{i=1}^{N} o\left(\vec{r}_{i}^{\prime}\right) s\left(\vec{r}_{i}^{\prime} ; \vec{r}\right) \tag{2.4}
\end{equation*}
$$

The $o\left(\vec{r}_{i}^{\prime}\right)$ are necessarily unitless, so that $i$ has the (irradiance) unit of s. Eq. (2.4) is sometimes called the principal of linearity, and may be generalized to include a continuum of source, or object, strengths. If the object is characterized by a volume distribution $o_{V}\left(\vec{r}^{\prime}\right)$ over volume $V$,

$$
\begin{equation*}
i(\vec{r})=\int_{V} o_{V}\left(\vec{r}^{\prime}\right) s\left(\vec{r}^{\prime} ; \vec{r}\right) d \vec{r}^{\prime} \tag{2.5a}
\end{equation*}
$$

whereas an object distribution $o_{\sigma}\left(\vec{r}^{\prime}\right)$ on surface $\sigma$ causes an

$$
\begin{equation*}
i(\vec{r})=\int_{\sigma} o_{\sigma}\left(\vec{r}^{\prime}\right) s\left(\vec{r}^{\prime} ; \vec{r}\right) d \sigma\left(\vec{r}^{\prime}\right) . \tag{2.5b}
\end{equation*}
$$

Objects $o_{V}$ and $o_{\sigma}$ must have units of (volume) ${ }^{-1}$ and (area) ${ }^{-1}$, respectively, so that $i$ and $s$ have identical units.

On physical grounds,

$$
\begin{array}{ll} 
& o_{V} \propto a_{V}  \tag{2.6}\\
\text { and } & o_{\sigma} \propto a_{\sigma}
\end{array}
$$

that is, object strength is proportional to object radiance. Hence $o_{V}$ and $o_{\sigma}$ may be experimentally determined by measuring the radiance distribution of the object.

Since $i$ is due to the superposition of real images, any object must obey either

$$
\begin{align*}
o_{\sigma}\left(\vec{r}^{\prime}\right) & =0  \tag{2.7}\\
\text { or } \quad o_{V}\left(\vec{r}^{\prime}\right) & =0
\end{align*}
$$

Constraint (2.7) will be assumed throughout this paper.
An object which is partially type $o_{V}$ and partially type $o_{\sigma}$ produces an $i(\vec{r})$ which is the sum of Eqs. (2.5a) and (2.5b).
III. CONVOLUTION THEOREMS

Assuming an isotomous, incoherently radiating object, we may use identity (2.1) in Eqs. (2.5). Then

$$
i(\vec{r})=\left\{\begin{array}{l}
\int_{V} o_{V}\left(\vec{r}^{\prime}\right) s\left(\vec{r}-\vec{r}_{G}\right) d \vec{r}^{\prime} \quad \text { or }  \tag{3.1a}\\
\int_{\sigma} o_{\sigma}\left(\vec{r}^{\prime}\right) s\left(\vec{r}^{\prime}-\vec{r}_{G}\right) d \sigma\left(\vec{r}^{\prime}\right),
\end{array}\right.
$$

depending on the object type.
In order to integrate over the primed coordinates it is necessary to express $\vec{r}_{G}$ in terms of $\vec{r}^{\prime}$. From first-order optics,

$$
\begin{equation*}
\vec{r}_{G}=\vec{f}^{\prime}\left(z^{\prime}-f\right)^{-1} \tag{3.1c}
\end{equation*}
$$

Eqs. (3.1) are the required convolution integrals. They may be used in the mathematical determination of $i(\vec{r})$ when the object and point spread function are given.

In the case of an object $o_{\sigma}$ located in a $p l a n e z^{\prime}=$ constant, Eq.
(3.1b) becomes

$$
\begin{align*}
& i(\vec{r})=\iint_{-\infty}^{\infty} o_{\sigma}\left(\vec{r} \vec{r}^{\prime}\right) s\left(\vec{r}-m \vec{r}^{\prime}\right) d x^{\prime} d y^{\prime}  \tag{3.2a}\\
& \text { where } m=f\left(z^{\prime}-f\right)^{-1} \tag{3.2b}
\end{align*}
$$

Eqs. (3.2) describe transverse image formation in any plane $z$.
IV. IMAGE SPECTRUM

Continuing a three-dimensional development, we define the image spectrum $I(\vec{\Omega})$ as

$$
\begin{align*}
& \qquad \begin{array}{l}
I(\vec{\Omega})=(2 \pi)^{-3 / 2} \int \underset{\text { Image }}{i(\vec{r})} \exp (-j \vec{\Omega} \cdot r) \mathrm{d} \vec{r} \\
\text { where } \\
\text { and } \quad \vec{\Omega}=\left(\omega_{1}, \omega_{2}, \omega_{3}\right) \\
\\
\text { an }=(-1)^{1 / 2} .
\end{array} \text {. } l \tag{4.1a}
\end{align*}
$$

Parameter $\vec{\Omega}$ represents a triplicate of spatial frequencies. For later use, we define

$$
\begin{equation*}
\vec{\omega}=\left(\omega_{1}, \omega_{2}\right) \tag{4.2}
\end{equation*}
$$

Since $i(\vec{r})$ is generally a function of three spatial coordinates, $I(\vec{\Omega})$ represents a spatially "complete" Fourier spectrum of $i(\vec{r})$. It will be shown in Section $V$ that definition (4.1a) implies a three-dimensional transfer theorem.

In any plane $z$ beyond the exit pupil, the total image power $P_{i}$ must be conserved. Algebraically,

$$
\begin{equation*}
\iint_{-\infty}^{\infty} i(x, y, z) d x d y=P_{i} \tag{4.3}
\end{equation*}
$$

for all $z \geq 0$. Using (4.3) and (4.1a) we find that, in particular,

$$
\begin{equation*}
I\left(0,0, \omega_{3}\right)=(2 \pi)^{-1 / 2} P_{i} \delta\left(\omega_{3}\right) \tag{4.4}
\end{equation*}
$$

where $\delta$ is the Dirac delta function. ${ }^{6}$ Eq. (4.4) shows that $I$ is singular at $\vec{\Omega}=(0,0,0)$, and that $I\left(0,0, \omega_{3}\right)$ is independent of the nature of the object and of the point spread function. This independence will not be true at any other triplicate $\vec{\Omega}$ of frequencies.

By the Fourier inversion theorem, ${ }^{7}$ Eq. (4.1a) yields

$$
\begin{align*}
& \quad \mathrm{i}(\overrightarrow{\mathrm{r}})=(2 \pi)^{-3 / 2} \int_{\Sigma} \mathrm{I}(\vec{\Omega}) \exp (\mathrm{j} \vec{\Omega} \cdot \overrightarrow{\mathrm{r}}) \mathrm{d} \vec{\Omega},  \tag{4.5a}\\
& \text { where } \quad \mathrm{d} \vec{\Omega}=\mathrm{d} \omega_{1} \mathrm{~d} \omega_{2} \mathrm{~d} \omega_{3}, \tag{4.5b}
\end{align*}
$$

and $\Sigma$ represents the three-dimensional "passband" for frequencies $\vec{\Omega}$, which is established in Section VIII. Eq. (4.5a) allows $i(\vec{r})$ to be computed from a known $I(\vec{\Omega})$. It is proven in Appendix $I I$ that integral (4.5a) generally converges, in the Fraunhofer approximation, ${ }^{8}$ even though $I(0,0,0)$ is singular.

## V. TRANSFER THEOREM

We shall consider the three-dimensional frequency spectrum of Eq. (3.1b). Using definition (4.1a) on the left side, and rearranging integration orders on the right side, we obtain

$$
I(\vec{\Omega})=(2 \pi)^{-3 / 2} \int_{\sigma} \operatorname{d\sigma }\left(\vec{r}^{\prime}\right) \circ\left(\vec{r}^{\prime}\right) \int \begin{align*}
& \mathrm{drs}\left(\vec{r}-\vec{r}_{G}\right) \exp (-j \vec{\Omega} \cdot \overrightarrow{\mathbf{r}}) .  \tag{5.1}\\
& \text { All space }
\end{align*}
$$

The change of variable,

$$
\begin{equation*}
\vec{\rho}=\vec{r}-\vec{r}_{G} \equiv(\alpha, \beta, \gamma), \quad \quad \overrightarrow{d \rho} \equiv \operatorname{d} \alpha \mathrm{~d} \beta \mathrm{~d} \gamma, \tag{5.2}
\end{equation*}
$$

is employed in (5.1), along with (3.1c). There results

$$
\begin{equation*}
\mathrm{I}(\vec{\Omega})=O(\vec{\Omega}) \cdot F(\vec{\Omega}) \tag{5.3}
\end{equation*}
$$

which is the required transfer theorem. In (5.3)

$$
\begin{align*}
& O(\vec{\Omega})=\int_{\sigma} \mathrm{d} \sigma\left(\overrightarrow{\mathrm{r}}^{\prime}\right) o_{\sigma}\left(\overrightarrow{\mathrm{r}}^{\prime}\right) \exp \left[-j f\left(z^{\prime}-f\right)^{-1} \vec{\Omega} \cdot \vec{r}^{\prime}\right]  \tag{5.4}\\
& F(\vec{\Omega})=(2 \pi)^{-3 / 2} \int_{\text {All space }}^{\operatorname{dips}(\vec{\rho}) \exp (-j \vec{\Omega} \cdot \vec{\rho})} \tag{5.5}
\end{align*}
$$

Eq. (5.3) therefore exhibits a separation in the effects of the object and the optics upon I. Were I defined as a one- or two-dimensional transform of $i$, this separation would not generally occur.

In the case of a volume distribution $o_{V}\left(\vec{r}^{\prime}\right)$, Eq. (5.4) is replaced by

$$
\begin{equation*}
O(\vec{\Omega})=\int_{V} d^{\vec{r}^{\prime}} o_{V}\left(\vec{r}^{\prime}\right) \exp \left[-j f\left(z^{\prime}-f\right)^{-1} \vec{\Omega}^{\prime} \cdot \vec{r}^{\prime}\right] \tag{5.6}
\end{equation*}
$$

We note from Eqs. (5.4) and (5.6) that $O(\vec{\Omega})$ is not the true Fourier spectrum of $o\left(\vec{r}^{\prime}\right)$ in general. Eqs. (5.4) and (5.6) define a new type of transform. Because of constraint (2.7) the quantity ( $\left.z^{\prime}-f\right)^{-1}$ cannot be singular, so that 0 is well defined.

The transfer theorem in the transverse case is well known:

$$
\begin{gather*}
I_{t}(\vec{\omega} ; z)=o_{t}\left(\vec{\omega} ; z^{\prime}\right) \tau\left(\vec{\omega} ; z-m z^{\prime}\right),  \tag{5.7a}\\
\text { where } I_{t}(\vec{\omega} ; z)=(2 \pi)^{-1} \iint_{-\infty}^{\infty} i(\vec{r}) \exp \left[-j\left(\omega_{1} x+\omega_{2} y\right)\right] d x d y,  \tag{5.7b}\\
o_{t}\left(\vec{\omega} ; z^{\prime}\right)=\iint_{-\infty}^{\infty} o\left(\vec{r}^{\prime}\right) \exp \left[-j m\left(\omega_{1} x^{\prime}+\omega_{2} y^{\prime}\right)\right] d x^{\prime} d y^{\prime},  \tag{5.7c}\\
\tau\left(\vec{\omega} ; z-m z^{\prime}\right)=(2 \pi)^{-1} \iint_{-\infty}^{\infty} s\left(\alpha, \beta, z-m z^{\prime}\right) \exp \left[-j\left(\omega_{1} \alpha+\omega 2^{\beta}\right)\right] d \alpha d \beta, \tag{5.7d}
\end{gather*}
$$

and $z^{\prime}$ locates the object plane. Eq. (5.7a) results from the use of definition (5.7b) in Eq. (3.2a). Eq. (5.7a) cannot be sequentially used in the optical relay of an image through a series of lens systems. This is because all successive images of a plane object have three-dimensional extent, whereas Eq. (5.7a) assumes the object to be confined in one plane z'. Hence, Eq. (5.3) is the only transfer theorem which can be sequentially used in the case of serial imagery as described.
VI. TRANSFER FUNCTION: $F(\vec{\Omega})$

The "complete" optical transfer function $F(\vec{\Omega})$ is observed, in Eq. (5.5), to be the three-dimensional Fourier transform of $s$. The origin of coordinates $(\alpha, \beta, \gamma)$ is the Gauss point.

As in derivation of Eq. (4.4), we have

$$
\begin{equation*}
F\left(0,0, \omega_{3}\right)=(2 \pi)^{-1 / 2}{\mathrm{P} \delta\left(\omega_{3}\right)} \tag{6.1}
\end{equation*}
$$

where $P$ is the total power in any receiving plane $\gamma$ within the spread function. Eq. (6.1) is consistent with the implication of Eqs. (4.4), (5.3), and either (5.4) or (5.6).

By using Eqs. (5.7d) and (5.5), as supplemented by identities (3.1c), (3.2b) and (5.2), we have

$$
\begin{equation*}
F(\vec{\Omega})=(2 \pi)^{-1 / 2} \int_{-\infty}^{\infty} d \gamma \tau(\vec{\omega} ; \gamma) \exp \left(-j \omega_{3} \gamma\right) \tag{6.2a}
\end{equation*}
$$

Then by the Fourier inversion theorem,

$$
\begin{equation*}
\tau(\vec{\omega} ; \gamma)=(2 \pi)^{-1 / 2} \int_{-\infty}^{\infty} \mathrm{d} \omega_{3} F(\vec{\Omega}) \exp \left(j \omega_{3} \gamma\right) \tag{6.2b}
\end{equation*}
$$

which establishes that $F$ and $\tau$ are Fourier transform pairs. Although $F(0,0,0)$ is infinite (as discussed previously), the integral (6.2b) generally converges. This is proved in Appendix II.

Eq. (6.2a), or an equivalent sampling expression in Section IX, may be used to compute $F(\vec{\Omega})$ from knowledge of $\tau(\vec{\omega} ; \gamma)$. Other methods for the determination of $F$ are derived in Sections VII and IX.

From definition (5.5), $F(\vec{\Omega})$ is a generally complex function. Accordingly, $F(\vec{\Omega})$ has a modulus $|F(\vec{\Omega})|$ and a phase $\Lambda(\vec{\Omega})$ defined by

$$
\begin{equation*}
F(\vec{\Omega})=|F(\vec{\Omega})| \exp [-\mathrm{j} \Lambda(\vec{\Omega})], \tag{6.3}
\end{equation*}
$$

where $F$ and $\Lambda$ are real.
By applying the Fourier inversion theorem to Eq. (5.5), we have

$$
\begin{equation*}
s(\vec{\rho})=(2 \pi)^{-3 / 2} \int_{\Sigma} F(\vec{\Omega}) \exp (j \vec{\Omega} \cdot \vec{\rho}) d \vec{\Omega} . \tag{6.4}
\end{equation*}
$$

It is shown in Appendix II that integral (6.4) converges generally.
VII. DEPENDENCE OF $\mathrm{F}(\vec{\Omega})$ ON PUPIL FUNCTION

## A. Derivation



Fig. 2. Parameters of the pupil and of the point spread function space.

Referring to Fig. 2, let axes $\alpha^{\prime} \beta$ ' $\gamma$ ' have their origin at Gauss point $G$. Let axis $\gamma^{\prime}$ lie along $\vec{r}_{G}$, and let axis $\beta^{\prime}$ be in the plane that passes through $\vec{r}_{G}$ and the optical axis. A reference sphere of radius $r_{G}$ intersects the optical axis at $H^{\prime}$. Axes $P Q$ are oriented such that $P$ is perpendicular to the $\alpha^{\prime} \gamma$ ' plane. As is customary, we assume field angle $\left(x_{G}^{2}+y_{G}^{2}\right)^{1 / 2} / z_{G}$ to be so small that axes $\alpha^{\prime} \beta^{\prime} \gamma^{\prime}$ coincide with axes $\alpha \beta \gamma$, respectively. ${ }^{9}$ Then $\alpha^{\prime}=\alpha, \beta^{\prime}=\beta, \gamma^{\prime}=\gamma$, and the image amplitude $u$ is given by the Fraunhofer approximation ${ }^{8}$ as
$u(\alpha, \beta, \gamma)=\left(\lambda r_{G}\right)^{-1} \iint_{-\infty}^{\infty} \operatorname{dpdqU}(p, q) \exp \left[j k\left(2 r_{G}^{2}\right)^{-1}\left(p^{2}+q^{2}\right) \gamma+j k r_{G}^{-1}(p \alpha+q \beta)\right]$,
where $\lambda$ is the wavelength of light, and $k=2 \pi / \lambda$. Using (7.1) and the well-known relation

$$
\begin{equation*}
s=u \cdot u * \tag{7.2}
\end{equation*}
$$

in definition (5.7d), we establish the well-known autocorrelation representation

$$
\begin{align*}
& \tau(\vec{\omega} ; \gamma)=(2 \pi)^{-1} \int_{-\infty}^{\infty} \int_{-\infty} \operatorname{dpdqU}(p, q) U^{*}\left[p-r_{G} \omega_{1} / k, q-r_{G} \omega_{2} / k\right] \\
& X \exp \left[j \gamma\left(p \omega_{1} / r_{G}+q \omega_{2} / r_{G}-\omega^{2} / 2 k\right)\right] . \tag{7.3}
\end{align*}
$$

Eq. (7.3) differs from the usual autocorrelation formula ${ }^{10}$ in its explicit presentation of the $\gamma$ dependence. This dependence is traditionally absorbed into the pupil function dependence of Eq. (7.3) because the $\gamma$ dependence is usually of secondary interest. We now substitute Eq. (7.3) into Eq. (6.1a). After an interchange of the orders of integration,

$$
\begin{gather*}
\mathrm{F}(\vec{\Omega})=(2 \pi)^{-3 / 2} \int^{\infty} \int_{-\infty}^{\infty} \operatorname{dpdqU}(\mathrm{p}, \mathrm{q}) \mathrm{U} *\left(\mathrm{p}-\mathrm{r}_{\mathrm{G}} \omega_{1} / \mathrm{k}, \mathrm{q}-\mathrm{r}_{\mathrm{G}} \omega_{2} / \mathrm{k}\right) \\
\mathrm{X} \int_{-\infty}^{\infty} \operatorname{d\gamma } \exp \left[j \gamma\left(\mathrm{p} \omega_{1} / r_{G}+q \omega_{2} / r_{G}-\omega^{2} / 2 k-\omega_{3}\right)\right] \tag{7.4}
\end{gather*}
$$

By two identities ${ }^{11}$ the $\gamma$ integral in (7.4) is

$$
\begin{equation*}
2 \pi r_{G}\left|\omega_{1}^{-1}\right| \delta\left[p+\omega_{1}^{-1}\left(\omega_{2} q-r_{G} \omega^{2} / 2 k-r_{G} \omega_{3}\right)\right] \tag{7.5}
\end{equation*}
$$

Substitution of (7.5) into (7.4) yields

$$
\begin{gather*}
F(\vec{\Omega})=(2 \pi)^{-1 / 2} r_{G}\left|\omega_{1}\right|^{-1} \int_{-\infty}^{\infty} d q U[p(q), q] U *\left[p(q)-r_{G} \omega_{1} k^{-1}, q-r_{G} \omega_{2} k^{-1}\right]  \tag{7.6a}\\
\text { where } p(q)=\omega_{1}^{-1}\left(r_{G} \omega^{2} / 2 k+r_{G} \omega_{3}-q \omega_{2}\right) . \tag{7.6b}
\end{gather*}
$$

In Eq. (7.6a), $U$ and $U *$ are evaluated along two parallel straight lines in the pupil. This may be further simplified by evaluating $U$ in the $P Q$ coordinate system and $U^{*}$ in a system $P^{\prime} Q^{\prime}$ such that

$$
\begin{align*}
p^{\prime} & =p-r_{G} \omega_{1} k^{-1} \\
\text { and } \quad q^{\prime} & =q-r_{G} \omega_{2} k^{-1} \tag{7.7}
\end{align*}
$$



Fig. 3. Transfer function $F$ represented as a line integral across the pupil. Integration path $\Gamma$ is confined to overlap region $R$.

Referring to Fig. 3, $F(\vec{\Omega})$ is then evaluated as one line integral along the straight path $\Gamma$ between points $A$ and $A^{\prime}$. Path $\Gamma$ extends over overlap region $R$ of the (generally vignetted) pupil $P Q$ and its displaced equivalent $P^{\prime} Q^{\prime}$. From Eq. (7.6b), $\Gamma$ is perpendicular to $00^{\prime}$ and is located distance $b(\vec{\Omega})$ from origin $O$, where

$$
\begin{equation*}
\mathrm{b}(\vec{\Omega})=\mathrm{r}_{\mathrm{G}} \omega(2 \mathrm{k})^{-1}+\mathrm{r}_{\mathrm{G}}\left(\omega_{3} / \omega\right) \tag{7.8}
\end{equation*}
$$

Integration path $\Gamma$ is thereby determined for any $\vec{\Omega}$. It is interesting to note that

$$
\begin{equation*}
\mathrm{b}\left(\omega_{1}, \omega_{2}, 0\right)=\frac{1}{2} 00^{\prime} \tag{7.9}
\end{equation*}
$$

in terms of the relative displacement $00^{\prime}$ of the two pupils.

Result (7.6a) may be further simplified through the replacement of integration variable $q$ by a coordinate $\ell$ along $\Gamma$. From Fig. 3 and Eq. (7.6b),

$$
\begin{equation*}
\mathrm{dq}=\left(\omega_{1} / \omega\right) \mathrm{d} l . \tag{7.10}
\end{equation*}
$$

Hence

$$
\begin{equation*}
F(\vec{\Omega})=(2 \pi)^{-1 / 2} r_{G} \omega^{-1} \int_{A}^{A^{\prime}} \mathrm{d} \ell(U \cdot U *)_{\Gamma} \tag{7.11a}
\end{equation*}
$$

The notation in (7.11a) is meant to imply that $U$ and $U^{*}$ are evaluated in their respective coordinate systems.

In summary, Eqs. (7.8) and (7.11) determine $F(\vec{\Omega})$ for a general pupil function. Examination of these equations shows that $\omega_{3}$ contributes to $F$ only in its capacity to locate integration path $\Gamma$. We see from (7.11a) that $F$ contains a simple pole at $\omega=0$; this is the essential reason for the behavior noted in Eq. (6.1).
B. Experimental Determination of $F(\vec{\Omega})$ by Use of a Shearing Interferometer

Eq. (7.11a) suggests that $F(\vec{\Omega})$ may be experimentally determined by the use of a shearing interferometer. ${ }^{12}$ Pupil function $U$ is physically sheared, or displaced, from itself by an amount $r_{G} \omega / k$, and a slit is placed upon path $\Gamma$, as located by Eq. (7.8). The light flux passing through the slit is, according to Eq. (7.11a), proportional to $F(\vec{\Omega})$. This method is already being used in experimental determination of $\tau(\vec{\omega} ; \gamma),{ }^{12}$ where the light flux passing through overlap region $R$ is instead measured. C. Numerical Calculation of $\mathrm{F}(\vec{\Omega})$

Eq. (7.11a) may be numerically processed by an electronic computer. The speed of computation is facilitated by the one-dimensional nature of
the integral. By contrast, the pupil function representation (7.3) for $\tau(\vec{\omega} ; \gamma)$ involves a double integration and, hence, considerably more computation time.
D. Normalization of $\mathrm{F}(\vec{\Omega})$

The normalized transfer function $\tau(\vec{\omega} ; \gamma)$ has several properties which facilitate its practical use: $\tau$ is (1) solely a function of the optics; (2) unitless, and (3) bounded at all $\vec{\omega}$. We note from definition (5.5) and Eq. (6.1) that $F$ does not obey any of these properties. For example, (1) is violated because $F$ is proportional to the source strength, according to (5.5).

Using Eq. (7.11a), we can define a new function $F(\vec{\Omega})$ that obeys all three properties and is proportional to $\mathrm{F}(\vec{\Omega})$ :

$$
\begin{align*}
F(\vec{\Omega}) & =\omega K(\vec{\omega}) F(\vec{\Omega}),  \tag{7.12a}\\
\text { where } \quad K(\vec{\omega}) & =(2 \pi)^{1 / 2} r_{G}^{-1} \int_{-\infty}^{\infty}|U(\eta, \theta)|^{2} d \eta .
\end{align*}
$$

In the above, $(n, \theta)$ are polar coordinates corresponding to ( $p, q$ ), and

$$
\begin{equation*}
\theta=\tan ^{-1}\left(\omega_{1} / \omega_{2}\right) \tag{7.12c}
\end{equation*}
$$

By substitution of (7.11a) into (7.12a),

$$
\begin{equation*}
F(\vec{\Omega})=\int_{A}^{A^{\prime}} \mathrm{d} \ell\left(U \cdot U^{*}\right)_{\Gamma} / \int_{-\infty}^{\infty}|U(\eta, \theta)|^{2} \mathrm{~d} \eta \tag{7.13}
\end{equation*}
$$

It can be seen from this relation that $F(\vec{\Omega})$ obeys requirements (1)-(3).
In general, the denominator $D(\theta)$ of (7.13) must be evaluated at all azimuths $\theta$. However, because $|\mathrm{U}|^{2}=|T|^{2}$ (the energy transmittance), $D(\theta)$ does not depend on the aberrations and should be simple to compute in general.

In most instances $T=1$; that is, the optics are uncoated. Then $D$ is independent of $\theta$, unless there is vignetting. If the pupil is round and of radius $r_{0}$, then $D(\theta)=2 r_{0}$.

From Eq. (7.13), $F(\vec{\Omega})<1$ for a general function $U$, and $F(0,0,0)=1$. Also, by Schwarz' inequality,

$$
\begin{equation*}
|F(\vec{\Omega})|<\left|F_{0}(\vec{\Omega})\right|, \tag{7.14}
\end{equation*}
$$

where subscript o indicates diffraction-limited optics. The optics are assumed to be uncoated.
E. Diffraction-Limited Case


Fig. 4. Determination of $F$ for a diffraction-limited, circular pupil. $F=A A^{\prime} / B B^{\prime}$.

In the diffraction limit

$$
U(\eta, \theta)=\left\{\quad \begin{array}{l}
1 \text { for }|\eta| \leq r_{0}  \tag{7.15}\\
0 \text { for }|\eta|>r_{0},
\end{array}\right.
$$

independent of $\theta$. The pupil is assumed to be nonvignetted, with radius
$r_{0}$. For a general $\vec{\omega}$ we may shift pupils along the $P$ axis because, by (7.15), U is radially symmetric. Then, by substituting (7.15) into Eq. (7.13),

$$
\begin{equation*}
F_{\mathrm{o}}(\vec{\Omega})=\mathrm{AA} \mathrm{~A}^{\prime} / \mathrm{BB} B^{\prime}, \tag{7.16}
\end{equation*}
$$

as defined in Fig. 4. By sight,

$$
\begin{equation*}
\mathrm{BB}^{\prime}=2 \mathrm{r}_{\mathrm{o}} . \tag{7.17}
\end{equation*}
$$

By elementary geometry

$$
\begin{equation*}
A A^{\prime}=2\left[r_{o}^{2}-\left(r_{G} \omega / 2 k+r_{G}\left|\omega_{3}\right| / \omega\right)^{2}\right]^{1 / 2} \tag{7.18}
\end{equation*}
$$

where $\left|\omega_{3}\right|$ is taken because, by inspection of Fig. 4, AA' is even in $\omega_{3}$. By substitution of (7.17) and (7.18) into Eq. (7.16),

$$
F_{\mathrm{o}}(\vec{\Omega})=\left\{\begin{array}{l}
{\left[1-\left(\mathrm{r}_{\mathrm{G}} / \mathrm{r}_{\mathrm{o}}\right)^{2}\left(\omega / 2 \mathrm{k}+\left|\omega_{3}\right| / \omega\right)^{2}\right]^{1 / 2} \text { for } \vec{\Omega} \text { in } \Sigma}  \tag{7.19}\\
0 \text { for } \vec{\Omega} \text { not in } \Sigma .
\end{array}\right.
$$

Bandpass region $\Sigma$ is derived in Section VIII.
VIII. BANDPASS REGION FOR F AND F

We seek a region $\Sigma$ in $\vec{\Omega}$-space within which $F$ (or $F$ ) is generally zero. From Eqs. (7.8) and (7.11), F is zero whenever (1) region $R$ is zero, or (2) $\Gamma$ falls outside of $R$.

Condition (1) is first considered. From Fig. 3,

$$
\begin{equation*}
F(\vec{\Omega})=0 \text { when } \omega \geq 2 \alpha_{0} \text {, where } \alpha_{0}=k r_{0} / r_{G} . \tag{8.1}
\end{equation*}
$$

In the above, $r_{o}$ is either the pupil radius or a radius that contains a generally vignetted pupil. We note that (8.1) is independent of $\omega_{3}$.

Condition (2) is accomplished, according to Eq. (7.8), when, for a given $\omega \leq 2 \alpha_{0}$,

$$
\begin{equation*}
\left|\omega_{3}\right| \geq(\omega / 2 k)\left(2 \alpha_{o}-\omega\right) ; \tag{8.2a}
\end{equation*}
$$

or, for a given $\omega_{3}$ that obeys $\left|\omega_{3}\right| \leq \alpha_{o}^{2} / 2 k$, when either

$$
\begin{align*}
\omega & \leq \alpha_{0}-\left(\alpha_{o}^{2}-2 k \omega_{3}\right)^{1 / 2} \\
\text { or } \quad & \omega \geq \alpha_{0}+\left(\alpha_{o}^{2}-2 k \omega_{3}\right)^{1 / 2}
\end{align*}
$$

Eqs. (8.2b) indicate that, except at $\omega_{3}=0, F$ is bandwidth-limited at both large and small values of $\omega$. The latter property has no parallel in the dependence of $\tau(\omega ; \gamma)$ upon $\vec{\omega}$.

Conditions (8.2) may be combined to define a volume $\Sigma$ in $\vec{\Omega}$-space within which $F$ is generally nonzero. As shown in Fig. 5, region $\Sigma$ is enclosed within the surfaces

$$
\begin{equation*}
\omega_{3}= \pm(\omega / 2 k)\left(2 \alpha_{0}-\omega\right) \tag{8.3}
\end{equation*}
$$



Fig．5．Bandpass volume $\Sigma$ in $\vec{\Omega}$－space for $F(\vec{\Omega})$ and $I(\vec{\Omega})$ ． Region $\Sigma$ is generated by rotation of the curve $\omega_{3}=\omega(2 k)^{-1}\left(2 \alpha_{o}-\omega\right)$ about the $\omega_{3}$ axis．

From Eqs．（8．1）and（8．2），$F$ is zero when independently

$$
\begin{align*}
&\left|\omega_{1}\right|>2 \pi R \\
& \text { or }\left|\omega_{2}\right|>2 \pi R  \tag{8.4}\\
& \text { or }\left|\omega_{3}\right|>2 \pi R_{3}, \\
& \text { where } R=\alpha_{0} / \pi \quad \text { and } \quad R_{3}=\alpha_{0}^{2} /(4 \pi k) . \tag{8.5}
\end{align*}
$$

Eqs．（8．4）will be used later in the derivation of sampling expressions．
It is interesting to note from（8．5）that ratio

$$
\left.\begin{array}{l}
R_{3} / R=\alpha_{o} / 4 k=(8 f ⿰ ⿰ 三 丨 ⿰ 丨 三 ⿻
\end{array}\right)^{-1} .
$$

Hence in all practical cases

$$
\begin{equation*}
\mathrm{R}_{3} \ll \mathrm{R} \tag{8.8}
\end{equation*}
$$

Consequences of（8．8）are discussed in Section X．

## IX. SAMPLING THEOREMS

## A. Derivations

By use of Eqs. (8.4) and (5.5), we have the following sampling theorems : ${ }^{13}$

$$
F(\vec{\Omega})=\left\{\begin{array}{l}
(2 \pi)^{-3 / 2} \sum_{\ell, m, n} \mathrm{~s}\left(\ell \pi / 2 \alpha_{0}, m \pi / 2 \alpha_{0}, 2 n \pi k / \alpha_{0}^{2}\right) \\
\times \exp \left[-j \pi\left(\ell \omega_{1} / 2 \alpha_{0}+m \omega_{2} / 2 \alpha_{0}+2 n k \omega_{3} / \alpha_{o}^{2}\right)\right]  \tag{9.1}\\
\text { for }\left|\omega_{1}\right| \leq 2 \alpha_{o},\left|\omega_{2}\right| \leq 2 \alpha_{o},\left|\omega_{3}\right| \leq \alpha_{o}^{2} / 2 k ; \\
\text { or } 0, \text { for } \vec{\Omega} \text { not in } \Sigma ;
\end{array}\right.
$$

and

$$
\begin{align*}
s(\alpha, \beta, \gamma)= & \sum_{\ell, m, n} s\left(\ell \pi / 2 \alpha_{o}, m \pi / 2 \alpha_{o}, 2 n \pi k / \alpha_{o}^{2}\right)  \tag{9.2}\\
& x \operatorname{sinc}\left(2 \alpha_{o} \alpha-\ell \pi\right) \operatorname{sinc}\left(2 \alpha_{o} \beta-m \pi\right) \operatorname{sinc}\left(\alpha_{o}^{2} \gamma / 2 k-n \pi\right) .
\end{align*}
$$

All summation indices are integers that range from $-\infty$ to ${ }^{+\infty}$. Spatial points ( $\ell \pi / 2 \alpha_{0}, m \pi / 2 \alpha_{0}, 2 n \pi k / \alpha_{0}^{2}$ ) will be called "sampling points," as is customary.

Eq. (9.2) shows that $s$ is known throughout all space once it is determined at the sampling points. From (9.2),

$$
\begin{equation*}
s(\alpha, \beta, 0)=\sum_{\ell, m} s\left(\ell \pi / 2 \alpha_{0}, m \pi / 2 \alpha_{0}, 0\right) \operatorname{sinc}\left(2 \alpha_{0} \alpha-\ell \pi\right) \operatorname{sinc}\left(2 \alpha_{0} \beta-m \pi\right) \tag{9.3}
\end{equation*}
$$

and

$$
\begin{equation*}
s(0,0, \gamma)=\sum_{n} s\left(0,0,2 n \pi k / \alpha_{o}^{2}\right) \operatorname{sinc}\left(\alpha_{o}^{2} \gamma / 2 k-n \pi\right) . \tag{9.4}
\end{equation*}
$$

Eq. (9.3) is known from transverse theory, and Eq. (9.4), which relates values of the Streh1 definition, has recently been derived. ${ }^{1}$

By use of Eqs. (8.4) and (6.1), we have the sampling expressions: ${ }^{13}$

$$
\begin{equation*}
F(\vec{\Omega})=(2 \pi)^{-1 / 2} \sum_{n} \tau\left(\vec{\omega} ; 2 k \pi n / \alpha_{o}^{2}\right) \exp \left(-2 j k \pi n \omega_{3} / \alpha_{o}^{2}\right) \tag{9.5}
\end{equation*}
$$

and

$$
\begin{equation*}
\tau(\vec{\omega} ; \gamma)=\sum_{n} \tau\left(\vec{\omega} ; 2 k \pi n / \alpha_{o}^{2}\right) \operatorname{sinc}\left(\alpha_{o}^{2} \gamma / 2 k-n \pi\right) . \tag{9.6}
\end{equation*}
$$

Eq. (9.6) indicates that the curves $\tau(\vec{\omega})$, as determined in different receiving planes $\gamma$, are interrelated by a sampling theorem.

Owing to relations (4.1a) and (5.3), I is limited to the same passband as is $F$, and we may make the replacements

$$
\begin{align*}
(\alpha, \beta, \gamma) & \rightarrow(x, y, z) \\
\text { s } & \rightarrow i  \tag{9.7}\\
\text { and } \quad \mathrm{F} & \rightarrow \mathrm{I}
\end{align*}
$$

in Eqs. (9.1) through (9.6). Eq. (9.2) then shows that any image space is degenerate, and is, in fact, known everywhere if it is known at the sampling points. This is an important limitation on, for example, the information content within an emulsion.

## B. Experimental Determination of $F(\vec{\Omega})$ by Use of Sampling Theorems

Eq. (9.1) may be used to determine $F(\vec{\Omega})$ for a fabricated optical system if the sampled values of $s$ are experimentally determined. Because (9.1) is a summation, which may be exactly evaluated on the electronic computer, the error in any resulting value of $F$ is due solely to experimental error in values of $s$; there is no computational error.

Alternatively, Eq. (9.5) shows that $F(\vec{\Omega})$ may be experimentally determined by determining $\tau(\vec{w} ; \gamma)$ in the "sampling planes" $\gamma_{n}=2 k \pi n / \alpha_{o}^{2}$.

## X. SUMMARY AND DISCUSSION

A transfer theory has been established to describe the image of a three-dimensional object. This theory is based on the existence of "isotomicity," as defined in Eq. (2.1). According to Eqs. (3.1) and (5.3), the image-forming ability of an optical system is determined by either the point spread function $s(\rho)$ or the "complete" optical transfer function $F\left(\omega_{1}, \omega_{2}, \omega_{3}\right)$. Function $F$ is shown, in Eq. (6.2a), to be the Fourier transform with respect to the $\gamma$ (longitudinal) direction of the "lateral" transfer function $\tau(\vec{\omega} ; \gamma)$.

The generalized transfer theory becomes identical with the lateral transfer theory in the limit

$$
\begin{equation*}
o_{V}\left(\vec{r}^{\prime}\right) \rightarrow o\left(x^{\prime}, y^{\prime}\right) \delta\left(z^{\prime}-z_{o}^{\prime}\right) \tag{10.1}
\end{equation*}
$$

of a transverse object $o\left(x^{\prime}, y^{\prime}\right)$, where $z^{\prime}=z_{o}^{\prime}$ is the object plane. This can be shown by substituting (10.1) into Eq. (5.6), with subsequent use of Eqs. (5.3), (4.5a), and (6.2b). In this case, $i(\vec{r})$ is determined by the Fourier transform of $F$, as in Eq. (6.2b), and therefore by $\tau(\vec{\omega} ; \gamma)$.

Practical use can be made of the convolution theorem (3.1) and the transfer property (5.3) if the object can be subdivided into isotomes which are not too numerous. (The order of 10 would be desirable.) Appendix I establishes a method for estimating the size of isotomes in the diffraction-1imited case.

Eq. (9.2) and its analogy in image space (based on substitutions [9.7]) establish a limit on the information density of three-dimensional image space. Regarding each independent image value as a degree of freedom for the image, there can be no higher information density than

$$
\begin{equation*}
1 / \lambda^{3} f_{\#}^{4} \text { degrees of freedom/volume } \tag{10.2}
\end{equation*}
$$

in image space. The value (10.2) corresponds to a diffraction-limited system, and is based on cutoff frequencies $R$ and $R_{3}$ of Eq. (8.5). Because $\mathrm{R}_{3} \ll \mathrm{R}$ in general, the z -distance $1 / 2 \mathrm{R}_{3}$ between sampling points is much greater than the corresponding $x$ and $y$ distances of $1 / 2 R$. Hence, a much higher density of image information is contained in a direction normal to the optical axis than parallel to it.

A lens system designed such that $F$ is constant at all $\vec{\Omega}$ will, according to Eqs. (5.3) and (4.5a) (with infinite limits), produce an image that is the Gaussian distortion of the object. Hence, any transverse object is perfectly imaged, and with a magnification determined by position $z$ of the object plane. This property would be most advantageous for a lens that is to operate at a variety of conjugates (e.g., the relay lens). Use of $F(\vec{\Omega})$ as a design criterion seems, therefore, to be indicated. In this regard, it is interesting to note relations between moments of the $F(\vec{\Omega})$ distribution and derivatives of $s(\vec{\rho})$ at $\vec{\rho}=0$, as implied by Eq. (6.4). Also, to compute $F(\bar{\Omega})$ from ray-trace data, the geometrical approximation to $F(\vec{\Omega})$ must be expressed in terms of ray intercepts in the spot diagram. This may be accomplished by taking the limit $\lambda \rightarrow 0$ in Eq. (7.11a) and using 1'Hôpital's rule.

Three-dimensional image processing is suggested by the following. We may invert Eq. (5.6), expressing object $o_{V}$ in terms of its spectrum 0 :

$$
\begin{equation*}
o_{V}\left(\vec{r}^{\prime}\right)=(2 \pi)^{-3} f^{4}\left(z^{\prime}-f\right)^{-4} \int_{-\infty}^{\infty} d \vec{\Omega} 0(\vec{\Omega}) \exp \left[j f\left(z^{\prime}-f\right)^{-1} \vec{\Omega} \cdot \vec{r}{ }^{\prime}\right] \tag{10.3}
\end{equation*}
$$

If $O(\vec{\Omega})$ of Eq. (5.3) is substituted into Eq. (10.3), and the infinite limits in (10.3) are replaced by bandpass region $\Sigma$, a method is indicated for the approximate reconstruction of a three-dimensional object by observation of its image.

All of the foregoing has assumed the object to be radiating incoherently. If the opposite assumption is made, a transfer theory for image amplitude $g(\vec{r})$ can be constructed, where

$$
\begin{equation*}
i(\vec{r})=|g(\vec{r})|^{2} \tag{10.4}
\end{equation*}
$$

A condition of stationarity and convolution, transfer, and sampling theorems can be constructed which are completely analogous to those of the incoherent case. Parameter $F$ of the latter case corresponds to an "amplitude transfer function" $W(\vec{\Omega})$ of the coherent case, ${ }^{14}$ defined as

$$
\begin{equation*}
W(\vec{\Omega})=(2 \pi)^{-3 / 2} \int \mathrm{~d} \vec{\rho} \mathrm{u}(\vec{\rho}) \exp (-j \vec{\Omega} \cdot \vec{\rho}) \tag{10.5}
\end{equation*}
$$

It is interesting to obtain a pupil-representation for $W(\vec{\Omega})$. By substituting Eq. (7.1) into Eq. (10.5), we find that $W(\vec{\Omega})$ is proportional to

$$
\begin{equation*}
U\left(r_{G} \omega_{1} / k, \quad r_{G} \omega_{2} / k\right) \delta\left(\omega^{2} / 2 k-\omega_{3}\right) \tag{10.6}
\end{equation*}
$$

This shows that $W$ is bandwidth-1imited to a region

$$
\begin{equation*}
\omega \leq \alpha_{o}, \omega_{3} \leq \alpha_{o}^{2} / 2 k \tag{10.7}
\end{equation*}
$$

Therefore, both $u(\vec{\rho})$ and $g(\vec{r})$ obey sampling theorems that correspond to (9.2), once the sampling intervals are adjusted.

We may also use relation (10.6) in the coherent analog to Eqs. (4.5a) and (5.3) in combination. Amplitude $g(\vec{r})$ is found to be proportional to

$$
\begin{equation*}
\iint_{-\infty}^{\infty} d \omega_{1} d \omega_{2} U\left(r_{G} \omega_{1} / k, r_{G} \omega_{2} / k\right) 0\left(\omega_{1}, \omega_{2}, \omega^{2} / 2 k\right) \exp \left[j\left(\omega_{1} x+\omega_{2} y\right)+j z \omega^{2} / 2 k\right] \tag{10.8}
\end{equation*}
$$

This equation is especially useful because it involves only a double integration and depends directly upon the pupil function.
XI. ACKNOWLEDGMENTS

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Appendix I. ISOTOMICITY IN THE DIFFRACTION-LIMITED CASE
We are to establish the extent to which condition (2.1) is obeyed as the position $\vec{r}^{\prime}$ of a point source is changed. Eqs. (7.1) and (7.2) will be used in establishing $s\left(\vec{r}^{\prime} ; \vec{r}\right)$.

Assume a point source, of total power output E , to be located at general field position $\vec{r}^{\prime}$. Then the irradiance at the pupil is approximately $\mathrm{E} / 4 \pi \mathrm{r}^{\prime 2}$, so that

$$
\begin{equation*}
U(p, q)=(E / 4 \pi)^{1 / 2} r^{\prime-1} . \tag{11.1}
\end{equation*}
$$

Substituting (11.1) into (7.1) leads to

$$
\begin{equation*}
u=u(\nu, \gamma)=(\pi E)^{1 / 2}\left(\lambda r_{G} r^{\prime}\right)^{-1} \int_{0}^{r_{o}} d \mu \cdot \mu J_{o}\left(k \mu \nu / r_{G}\right) \exp \left(j k \mu^{2} \gamma / 2 r_{G}^{2}\right) \tag{11.2a}
\end{equation*}
$$

$$
\begin{equation*}
\text { where } \nu=\left(\alpha^{2}+\beta^{2}\right)^{1 / 2} \text { and } \mu=\left(p^{2}+q^{2}\right)^{1 / 2} \tag{11.2b}
\end{equation*}
$$

Since (11.2a) is dependent upon $r^{\prime}$ and $r_{G}$ in modulus only, we suspect that any isotome will be in the form of a spherical shell of thickness $\Delta r^{\prime}$. This is shown, below, to be true at small field angle positions.

We now observe the effect upon $|u|^{2}$ of a radial change $\Delta r^{\prime}$ in position of the point source. In particular, we examine resulting changes in the $|u(v, 0)|^{2}$ and $|u(0, \gamma)|^{2}$ distributions, since they are easy to estimate and, at the same time, correlate with changes in the overall $|u(\nu, \gamma)|^{2}$ distribution.

From Eq. (11.2a), we have

$$
\begin{equation*}
|u(\nu, 0)|^{2}=s(\nu, 0)=\left(\pi E / 4 \lambda^{2}\right)\left(r_{o} / r^{\prime}\right)^{4}\left(r^{\prime}-f_{\phi}\right)^{2} f_{\phi}^{-2}\left[2 J_{1}\left(\alpha_{o} \nu\right) / \alpha_{o} \nu\right]_{(11}^{2}, \tag{11.3a}
\end{equation*}
$$

where $\quad f_{\phi}=f \sec (\phi)$,
$\phi$ is the field angle between $r^{\prime}$ and the optical axis, and $J_{1}$ is the firstorder Bessel function. Relation (3.1c) was used. Also, from Eq. (11.2a), $\left.|u(0, \gamma)|^{2}=s(0, \gamma)=\left(\pi E / 4 \lambda^{2}\right)\left(r_{o} / r^{\prime}\right)^{4}\left(r^{\prime}-f_{\phi}\right)^{2} f_{\phi}{ }^{-2}\left[\sin \left(\alpha_{o}^{2} \gamma / 4 k\right) /\left(\alpha_{o}^{2} \gamma / 4 k\right)\right]^{2} . \dot{11.4}\right)$

The invariance of $s(\nu, 0)$ and $s(0, \gamma)$ under changes $\Delta r^{\prime}$ can be observed through behavior of the central maximum $s(0,0)$, the radial distance $R_{\nu}$ to the first zero of $s(\nu, 0)$, and the axial distance $Z_{\gamma}$ to the first zero of $s(0, \gamma) . \quad$ From (11.3) and (11.4),

$$
\begin{align*}
s(0,0) & =\left(\pi E / 4 \lambda^{2}\right)\left(r_{o} / r^{\prime}\right)^{4}\left(r^{\prime}-f_{\phi}\right)^{2} f_{\phi}^{-2}  \tag{11.5a}\\
R_{V} & =\left(1.64 \lambda / r_{o}\right) f_{\phi} r^{\prime}\left(r^{\prime}-f_{\phi}\right)^{-1}  \tag{11.5b}\\
\text { and } Z_{\gamma} & =\left(2 \lambda / r_{o}^{2}\right)\left(f_{\phi} r^{\prime}\right)^{2}\left(r^{\prime}-f_{\phi}\right)^{-2} \tag{11.5c}
\end{align*}
$$

Using these equations we can compute the relative changes in $s(0,0), R_{V}$, and $Z_{\gamma}$ due to changes $\Delta r^{\prime}$. We note that these relative changes will depend upon field angle $\phi$ to a certain extent, because of relation (11.3b). However, in many cases $\phi$ is sufficiently small (less than about $20^{\circ}$ ) that $f_{\phi} \simeq f$ over the object. In these cases the isotomes are then spherical shells. When $\phi$ is appreciable, relative changes in parameters (11.5) must be evaluated due to changes $\Delta \phi$ as well as changes $\Delta r^{\prime}$.

As an example, we use relations (11.5) for the case $r^{\prime}=2 f$, $\Delta r^{\prime} / r^{\prime}=5 \%$, and $\phi$ small. Using differentials we find that $\Delta s(0,0) / s(0,0)$ $=0$ identically; $\Delta R_{V} / R_{V}=-5 \%$, and $\Delta Z_{\gamma} / Z_{\gamma}=-10 \%$. This indicates that isotomicity holds fairly well within the shell $r^{\prime} \pm 0.05 r^{\prime}, \phi$ small, in this case.

Appendix II. CONVERGENCE OF THE INVERSE FOURIER TRANSFORMS
Functions I and F are "generalized" 15 in the limit of unit "modifying factors" ${ }^{15}$; this is indicated by Eqs. (4.4) and (6.1). For a generalized function $f(x)$ the repeated Fourier integral

$$
(2 \pi)^{-1} \int_{-\infty}^{\infty} d \omega \exp \left(j \omega x^{\prime}\right) \int_{-\infty}^{\infty} d x^{\prime} f\left(x^{\prime}\right) \exp \left(-j \omega x^{\prime}\right)
$$

equals $f(x)$ at each value of $x$ for which it converges. ${ }^{15}$ Hence, in order to prove that inverse transforms ( 6.4 ) , ( 6.2 b ) and (4.5a) converge to $s(\vec{\rho})$, $\tau(\vec{\omega} ; \gamma)$, and $i(\vec{r})$, respectively, we need only establish that the inverse transforms converge. For example, in the diffraction-limited case it can be shown that the known functions $s(\vec{\rho})$ and $\tau(\vec{\omega} ; 0)$ are converged upon by integrals (6.4) and (6.2b), respectively. The general convergences of transforms (6.4), (6.2b), and (4.5a) are shown below.

## A. Transform (6.4)

By defining

$$
\begin{equation*}
\varepsilon(\theta)=\int_{-\infty}^{\infty}|U(\eta, \theta)|^{2} d \eta \tag{11.6}
\end{equation*}
$$

and substituting Eqs. (7.12) into Eq. (6.4), we have

$$
\begin{equation*}
s(\vec{\rho})=(2 \pi)^{-2} r_{G} \int_{\Sigma} F(\vec{\Omega})_{\omega}^{-1} \varepsilon(\theta) \exp (\vec{j} \vec{\Omega} \cdot \vec{\rho}) \mathrm{d} \vec{\Omega} . \tag{11.7}
\end{equation*}
$$

Owing to factor $\omega^{-1}$, the integrand is singular at $\omega=0$. We show below that integral (11.7) converges at all $\vec{\rho}$, despite this singularity. Replace Cartesian coordinates $\vec{\rho}$ and $\vec{\Omega}$ in (11.7) by cy1indrical coordinates ( $\nu, \Phi, \gamma$ ) and ( $\omega, \theta, \omega_{3}$ ), respectively, where

$$
\left.\begin{array}{l}
\alpha=v \sin \Phi  \tag{11.8a}\\
\beta=v \cos \Phi
\end{array}\right\}
$$

and

$$
\left.\begin{array}{l}
\omega_{1}=\omega \sin \theta  \tag{11.8b}\\
\omega_{2}=\omega \cos \theta
\end{array}\right\}
$$

Bandpass $\Sigma$ is determined by Eqs. (8.1) and (8.3). Eq. (11.7) then becomes

$$
\begin{gather*}
s(\nu, \Phi, \gamma)=(2 \pi)^{-2} r_{G} \int_{0}^{2 \pi} \int_{0}^{2 \alpha_{0}} \int_{-g(\omega)}^{g(\omega)} d \omega_{3} d \omega d \theta F\left(\omega, \theta, \omega_{3}\right) \varepsilon(\theta) \\
x \exp \left[j \omega \nu \cos (\Phi-\theta)+j \omega_{3} \gamma\right]  \tag{11.9a}\\
\text { where } g(\omega)=(\omega / 2 k)\left(2 \alpha_{0}-\omega\right) \text { and } \omega \leq 2 \alpha_{0} . \tag{11.9b}
\end{gather*}
$$

By definitions (7.13) and (11.6), F and $\varepsilon$ are bounded at all $\vec{\Omega}$. In addition, the integration limits in (11.9a) are finite. Hence, integral (11.9a) must be finite at all $\vec{p}$.
B. Transform (6.2b)

By the use of Eq. (8.3), integral transform (6.2b) becomes

$$
\begin{equation*}
\tau(\vec{\omega} ; \gamma)=(2 \pi)^{-1 / 2} \int_{-g(\omega)}^{g(\omega)} d \omega_{3} F(\vec{\Omega}) \exp \left(j \omega_{3} \gamma\right) \tag{11.10}
\end{equation*}
$$

By Eq. (7.11a), $F$ is nonsingular, except at $\omega=0$. Now

$$
\begin{equation*}
g(\omega) \leq \alpha_{o}^{2} / 2 k \tag{11.11}
\end{equation*}
$$

because $\tau(\vec{\omega} ; \gamma)$ is nonzero only when $\omega \leq 2 \alpha_{0}$. Since integrand and limits in (11.10) are then finite, except at $\omega=0$, (11.10) must be finite for $\omega \neq 0$.

The case $\omega=0$ is now considered. Substitution of Eq. (6.1) into (11.10) yields

$$
\begin{equation*}
\tau(0,0 ; \gamma)=(2 \pi)^{-1} P \tag{11.12}
\end{equation*}
$$

a finite number.
C. Transform (4.5a)

By using polar coordinates ( $r^{\prime} ; \psi$ ), defined by

$$
\left.\begin{array}{l}
x=r^{\prime} \sin \psi  \tag{11.13}\\
y=r^{\prime} \cos \psi
\end{array}\right\}
$$

and Eqs. (5.3), (7.12), (11.6), and (11.8a), Eq. (4.5a) becomes

$$
\begin{equation*}
i(\vec{r})=(2 \pi)^{-2} r_{G} \int_{0}^{2 \pi} \int_{0}^{2 \alpha_{0}} \int_{-g(\omega)}^{g(\omega)} d \omega_{3} d \omega d \theta \varepsilon(\theta) a\left(\omega, \theta, \omega_{3}\right) F\left(\omega, \theta, \omega_{3}\right) \tag{11.14}
\end{equation*}
$$

$$
x \exp \left[j \omega r^{\prime} \cos (\theta-\psi)+j \omega_{3} z\right] .
$$

By either definition (5.4) or (5.6), any 0 due to a real object must be nonsingular at all $\vec{\Omega}$. Also, Eq. (7.13) shows that $F$ is nonsingular at all $\vec{\Omega}$. The integral (11.14) must then be convergent, because its integrand and limits are finite.


