

OPTIMAL A PRIORI ERROR ESTIMATES
FOR THE hp -VERSION
OF THE LOCAL DISCONTINUOUS GALERKIN METHOD
FOR CONVECTION–DIFFUSION PROBLEMS

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ABSTRACT. We study the convergence properties of the hp -version of the local discontinuous Galerkin finite element method for convection-diffusion problems; we consider a model problem in a one-dimensional space domain. We allow arbitrary meshes and polynomial degree distributions and obtain upper bounds for the energy norm of the error which are explicit in the mesh-width h , in the polynomial degree p , and in the regularity of the exact solution. We identify a special numerical flux for which the estimates are optimal in both h and p . The theoretical results are confirmed in a series of numerical examples.

1. INTRODUCTION

This paper contains the first a priori error estimate of the hp -version of the so-called local discontinuous Galerkin (LDG) finite element method for convection-diffusion problems. Such an error analysis, which takes into account both the mesh-size of the element, h , and the degree of the approximating polynomial in it, p , is quite relevant for the LDG method since, being a *locally conservative* method that does not require any inter-element continuity, it is ideally suited for hp -adaptivity. In this paper, we consider a model convection-diffusion equation in one space dimension with Dirichlet boundary conditions and obtain, for a special choice of the numerical fluxes defining the LDG method, a priori error estimates that are optimal both in h and p , even for $p = 0$; all other error estimates available in the current literature are suboptimal in both h and p and do not give a rate of convergence for $p = 0$.

The LDG method was introduced by Cockburn and Shu in [11] as an extension to general convection-diffusion problems of the numerical scheme for the compressible Navier-Stokes proposed by Bassi and Rebay in [1]. This scheme was in turn an extension of the Runge-Kutta discontinuous Galerkin (RKDG) method developed by Cockburn and Shu [10, 9, 8, 6, 12] for nonlinear hyperbolic systems. For a fairly complete set of references on RKDG and LDG methods, see the short monograph

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by Cockburn [4]; see also the review of the development of discontinuous Galerkin methods by Cockburn, Karniadakis, and Shu [7].

To put our result in proper perspective, let us briefly describe the relevant results available in the current literature. There are only a few a priori error estimates for the LDG method and they are all for the h -version of the method. The first a priori error estimate for the LDG method was obtained in 1998 by Cockburn and Shu [11] who proved that, when polynomials of degree p are used, the LDG method converges in the energy norm at a rate of order h^p . This rate of convergence was obtained for the general form of the so-called *numerical fluxes* that appear in the definition of the LDG method and is sharp since for the numerical flux proposed by Bassi and Rebay [1] this rate is actually achieved. Later, this analysis was extended by Cockburn and Dawson [5] to the case in which the convective velocity and the diffusion tensor depend on x and the domain is bounded; the rate of convergence of order h^p was once again obtained.

Although the rate of convergence of order h^p is sharp for general fluxes, Cockburn and Shu [11] reported numerical experiments in the one-dimensional case indicating that, for a *special* numerical flux, a rate of convergence of order h^{p+1} is achieved for very smooth solutions. This indication was later put on firm mathematical grounds by Castillo [3] who showed, for the model problem of constant-coefficient, linear convection-diffusion in one space dimension, that the LDG method with a particular numerical flux converges with the optimal rate of convergence of order h^{p+1} . Castillo's result can be viewed as an extension to the convection-diffusion setting of the a priori error estimate for the discontinuous Galerkin (DG) method for the purely convective case obtained in 1974 by LeSaint and Raviart [15] who prove that the rate of convergence is of order h^{p+1} .

In this paper, we obtain an a priori error estimate for the hp -version of the LDG method for general numerical fluxes which is explicit in the mesh-width h and the polynomial degree p . Assuming that the $(s + 1)$ -th derivative of the exact solution in the energy norm plus the $L^1(0, T; L^2)$ -norm of the time derivative is finite, we show that, for general numerical fluxes, the energy norm of the error has a rate of convergence of order $h^{p+1/2}/p^{s+1/2}$ in the purely convective case and of $h^p/p^{s-1/2}$ in the convection-diffusion case. Moreover, by using the special numerical flux studied by Castillo [3], we obtain the optimal rate of convergence of order h^{p+1}/p^{s+1} for totally arbitrary meshes and polynomials of degree p in all elements. This result holds in the purely convective case as well as in the purely parabolic case.

Let us give an idea of how the error estimate is obtained. First, using the technique employed by Cockburn and Shu [11], we find an upper bound for the energy norm of a *projection* into the finite element space of the error. Then, following Castillo [3], we eliminate as many as possible terms in the upper bound of the error by carefully defining the numerical flux of the LDG method and by suitably choosing such a projection. Indeed, instead of using the L^2 -projection operator used by Cockburn and Shu [11], the projections used by Houston, Schwab and Süli [14, 30, 29], or the Lagrange interpolation of Gauss-Radau points used by Castillo [3], we pick the more advantageous projection used in 1985 by Thomeé [31] in his study of discontinuous Galerkin time-discretizations for parabolic problems and recently by Schötzau and Schwab [23, 22] in their study of the hp -version of this method. Indeed, with this projection, many terms in the upper bound of the error become identically zero which allows us to obtain an optimal rate of convergence after a simple application of the sharp hp -approximation results for this projection.

Recent work on other discontinuous finite element methods for convection-diffusion (and for pure diffusion) problems has been reviewed by Cockburn, Karniadakis and Shu [7]. See, in particular, the numerical method of Baumann and Oden [2], the optimal error estimates for the method as applied to nonlinear convection-diffusion equations by Rivière and Wheeler [20], the analysis of several versions of the Baumann and Oden method for elliptic problems by Rivière, Wheeler and Girault [21], and the hp -version analyses by Houston, Süli and Schwab [14, 30, 29] of discontinuous Galerkin methods for second-order problems with nonnegative characteristic form. We also mention the recent work of Wihler and Schwab [32] in which robust exponential rates of convergence of DG methods for (stationary) convection-diffusion problems in one space dimension are proven.

The organization of the paper is as follows. In section 2, we describe the LDG method. In section 3, we state and prove the a priori error estimate for the constant-coefficient convection-diffusion problem and the special numerical flux for which the estimates are optimal in both h and p . In section 4, we discuss several extensions and, in section 5, we perform numerical experiments that verify the theoretical results. We end our presentation with some concluding remarks in section 6.

2. THE LDG METHOD

In this section, we introduce and briefly discuss the various key elements of the LDG method for a simple model problem.

2.1. The model problem and its weak formulation. In this paper we consider the following model convection-diffusion equation in one space dimension:

$$(2.1) \quad u_t + (cu - du_x)_x = f \quad \text{in } Q_T = (a, b) \times (0, T),$$

with the initial condition

$$(2.2) \quad u|_{t=0} = u_0 \quad \text{on } \Omega = (a, b),$$

and the Dirichlet boundary conditions

$$(2.3) \quad u(a) = u_D(a), \quad u(b) = u_D(b) \quad \text{on } J = (0, T).$$

The unknown function u is a scalar, and we assume the velocity $c > 0$ to be a positive and the diffusion coefficient $d \geq 0$ to be a nonnegative number; we choose to work with a positive velocity c simply to fix the location of the possible boundary layer at $x = b$. Note that in the purely convective case ($d = 0$), only the Dirichlet boundary condition at $x = a$ is taken into account.

The weak formulation we are going to use is obtained as follows. First, we introduce the new variable $q := \sqrt{d}u_x$ and the “flux” function

$$\mathbf{h} = (h_u, h_q)^\top := (cu - \sqrt{d}q, -\sqrt{d}u)^\top,$$

and rewrite (2.1)–(2.3) in the form

$$\begin{aligned} u_t + (h_u)_x &= f && \text{in } Q_T, \\ q + (h_q)_x &= 0 && \text{in } Q_T, \\ u|_{t=0} &= u_0 && \text{on } \Omega, \\ u(a) = u_D(a), \quad u(b) &= u_D(b) && \text{on } (0, T). \end{aligned}$$

Next, given the nodes $a = x_0 < x_1 < \dots < x_{M-1} < x_M = b$, we define the mesh $\mathcal{T} = \{I_j = (x_{j-1}, x_j), j = 1, \dots, M\}$ and set $h_j := |I_j| = x_j - x_{j-1}$; furthermore, we

define $h := \max_{j=1}^M h_j$. To the mesh \mathcal{T} , we associate the so-called *broken* Sobolev space

$$H^1(\Omega, \mathcal{T}) := \left\{ v : \Omega \rightarrow \mathbb{R} \mid v|_{I_j} \in H^1(I_j), j = 1, \dots, M \right\}.$$

For a function $u \in H^1(\Omega, \mathcal{T})$ the one-sided limits at the nodes $\{x_j\}$ are denoted as follows:

$$(2.4) \quad u^\pm = u(x_j^\pm) := \lim_{x \rightarrow x_j^\pm} u(x).$$

Throughout, we assume the exact solution $\mathbf{w} = (u, q)$ of (2.1)–(2.3) belongs to $H^1(0, T; H^1(\Omega, \mathcal{T})) \times L^2(0, T; H^1(\Omega, \mathcal{T}))$. Then, it satisfies the following equations:

$$\begin{aligned} (u_t, v)_{I_j} - (h_u, v_x)_{I_j} + h_u v \Big|_{x_{j-1}^+}^{x_j^-} &= (f, v)_{I_j}, \\ (q, r)_{I_j} - (h_q, r_x)_{I_j} + h_q r \Big|_{x_{j-1}^+}^{x_j^-} &= 0, \\ (u(\cdot, 0), v)_{I_j} &= (u_0, v)_{I_j}, \end{aligned}$$

for all test functions $v, r \in H^1(\Omega, \mathcal{T})$ and for $j = 1, \dots, M$. Here, the time derivative is to be understood in the weak sense and $(u, v)_I = \int_I u(x) v(x) dx$.

2.2. The method. A discrete version of the above mixed formulation is obtained by restricting the trial and test functions to finite-dimensional subspaces $V_N \subset H^1(\Omega, \mathcal{T})$ and by replacing the flux function \mathbf{h} at the nodes by a *numerical flux* $\hat{\mathbf{h}} = (\hat{h}_u, \hat{h}_q)^\top$: Find $\mathbf{w}_N = (u_N, q_N) \in H^1(0, T; V_N) \times L^2(0, T; V_N)$ such that for all $v, r \in V_N$ and for $j = 1, \dots, M$ the following equations hold:

$$(2.5) \quad \begin{aligned} ((u_N)_t, v)_{I_j} - (h_u, v_x)_{I_j} + \hat{h}_u v \Big|_{x_{j-1}^+}^{x_j^-} &= (f, v)_{I_j}, \\ (q_N, r)_{I_j} - (h_q, r_x)_{I_j} + \hat{h}_q r \Big|_{x_{j-1}^+}^{x_j^-} &= 0, \\ (u_N(\cdot, 0), v)_{I_j} &= (u_0, v)_{I_j}. \end{aligned}$$

Upon a choice of basis for the subspaces V_N , and, more importantly, of the numerical fluxes, the semidiscrete problem (2.5) becomes an ODE system of dimension $2N$ on $J = (0, T)$, where $N = \dim(V_N)$. In what follows we do not consider the impact of the time discretization and refer to Shu [28], Shu and Osher [26, 27] and Gottlieb and Shu [13] for the analysis of certain TVD Runge-Kutta methods for the solution of ODE systems.

Our choice of the space V_N is the space of *discontinuous*, piecewise polynomial functions

$$\left\{ u : \Omega \rightarrow \mathbb{R} \mid u|_{I_j} \in \mathcal{P}^{p_j}(I_j), j = 1, \dots, M \right\},$$

where $\mathcal{P}^{p_j}(I_j)$ denotes the set of all polynomials of degree less or equal than p_j on I_j . Notice that the polynomial degrees can *vary from element to element*.

To complete the definition of the LDG method, it remains to define the numerical flux $\hat{\mathbf{h}}$.

2.3. The numerical flux $\hat{\mathbf{h}}$. Crucial for the stability as well as for the accuracy of the LDG method is the choice of the numerical flux $\hat{\mathbf{h}}$. To define it, we introduce with the notation in (2.4) the following quantities:

$$[u] = u^+ - u^-, \quad \bar{u} = (u^+ + u^-)/2.$$

The numerical flux $\hat{\mathbf{h}}$ has the following general form:

$$\hat{\mathbf{h}}(\mathbf{w}^+, \mathbf{w}^-) = (c\bar{u}, 0)^\top - \sqrt{d}(\bar{q}, \bar{u})^\top - \begin{pmatrix} c_{11} & c_{12} \\ -c_{12} & 0 \end{pmatrix} [\mathbf{w}],$$

which also holds at the boundary if we define

$$(u, q)(a^-) = (u_D(a), q(a^+)), \quad (u, q)(b^+) = (u_D(b), q(b^-)).$$

Let us stress several important points concerning this numerical flux:

- In the purely hyperbolic case, i.e., in the case $d = 0$, if we take $c_{11} = c/2$, we obtain the well-known “upwinding” flux of the original DG method; see, e.g., [4].
- Note that $c_{22} = 0$. This is so because we want to be able to solve for q_N in terms of u_N element by element. This local solvability, which gives the name to the LDG method, is not shared by most mixed methods and allows us to easily eliminate the unknown q_N from the equations.
- The main purpose of the coefficient c_{11} is to *enhance the stability of the method*; that is why it must be a positive number. This results in an improvement of the accuracy of the method too.
- The choice $c_{21} = -c_{12}$ ensures the stability of the LDG method.
- The main purpose of the coefficients c_{12} is to *enhance the accuracy of the method*. Thus, if we take, following Bassi and Rebay [1], $c_{12} = 0$, the rate of convergence of the energy norm is of order h^p for smooth functions. If instead, following Castillo [3], we take $c_{12} = \sqrt{d}/2$, we obtain the optimal rate of order h^{p+1} .

Let us point out that, if we consider the general form of the numerical fluxes, it is possible to obtain exponential convergence for piecewise analytic exact solutions, but not optimality in both h and p . As we shall see, this optimality is *guaranteed for completely arbitrary meshes* if we take (an extension of) Castillo’s [3] choice of the numerical flux $\hat{\mathbf{h}}$, namely,

$$(2.6) \quad \hat{\mathbf{h}}(x_j) = \begin{cases} (cu_D(a) - \sqrt{d}q(a^+), -\sqrt{d}u_D(a))^\top & \text{for } j = 0, \\ (cu(x_j^-) - \sqrt{d}q(x_j^+), -\sqrt{d}u(x_j^-))^\top & \text{for } j = 1, \dots, M-1, \\ (c\hat{u}(b) - \sqrt{d}q(b^-), -\sqrt{d}u_D(b))^\top & \text{for } j = M, \end{cases}$$

where $\hat{u}(b) = cu(b^-) - \max\{c/2, \max\{1, p_M\}d/h_M\}(u_D(b) - u(b^-))$.

This flux is obtained by setting $c_{12} \equiv \sqrt{d}/2$ and

$$c_{11}(x_j) = \begin{cases} c/2 & \text{for } j = 0, \dots, M-1, \\ \max\{c/2, \max\{1, p_M\}d/h_M\} & \text{for } j = M. \end{cases}$$

Note again that in the purely convective case, $d = 0$, this numerical flux is nothing but the standard upwinding flux used by the original DG method. Note also that the fact that the coefficient $c_{11}(b)$ has a special form is a reflection of the fact that at $x = b$ there might be a boundary layer which requires special treatment. The

factor $\max\{1, p_M\}/h_M$ ensures the optimality in both h and p of the energy norm of the error.

The definition of the LDG method is now complete.

3. ERROR ANALYSIS

This section is devoted to our main a priori error estimates. First, we state and briefly discuss the results; the remainder of the section is devoted to their proof.

3.1. A priori estimates. Our main result follows naturally from an estimate of a suitably defined projection π of the error $\mathbf{e} = \mathbf{w} - \mathbf{w}_N$ and from the hp -approximation properties of the projection π . Each of these results is contained in several lemmas that we state next. To do that, we need to introduce the projection π and the norm $\|\cdot\|_T$ in which we measure $\pi\mathbf{e}$.

The projection π is the operator from $H^1(\Omega, \mathcal{T})^2$ to V_N^2 that associates (u, q) to (π^-u, π^+q) where, for each interval $I_j = (x_{j-1}, x_j), j = 1, \dots, M, \pi^\pm$ is defined by the following $p_j + 1$ conditions:

$$(3.1) \quad (\pi^\pm w - w, v)_{I_j} = 0 \quad \forall v \in \mathcal{P}^{p_j-1}(I_j), \quad \text{if } p_j > 0,$$

$$(3.2) \quad \pi^-w(x_j) = w(x_j^-), \quad \pi^+w(x_{j-1}) = w(x_{j-1}^+).$$

Let us now introduce a norm that appears naturally in the analysis of the LDG method. In what follows, we denote by $\|\cdot\|_D$ the L^2 -norm in the subdomain D and omit D when $D = \Omega$. For $\mathbf{v} = (v, r)$ the norm $\|\cdot\|_T$ is defined as follows:

$$(3.3) \quad \|\mathbf{v}\|_T^2 = \|\mathbf{v}\|_{E,T}^2 + \Theta_{T,\mathcal{T}}(v),$$

where the *energy* norm $\|\mathbf{v}\|_{E,T}$ is given by

$$\|\mathbf{v}\|_{E,T}^2 = \|v(T)\|^2 + \|r\|_{Q_T}^2,$$

and

$$\Theta_{T,\mathcal{T}}(v) = \int_0^T \left\{ c_{11}(a) v^2(a^+, t) + \sum_{j=1}^{M-1} c_{11}(x_j) [v]^2(x_j, t) + c_{11}(b) v^2(b^-, t) \right\} dt.$$

Note that information about the numerical flux is contained in the norm $\|\cdot\|_T$ only through $\Theta_{T,\mathcal{T}}(\cdot)$. We are now ready to state our results.

Lemma 3.1 (The basic estimate). *The error \mathbf{e} between the exact solution and the approximation given by the LDG method with numerical flux (2.6) satisfies the inequality*

$$\|\pi\mathbf{e}\|_T \leq A^{1/2}(T) + \int_0^T B(t) dt,$$

where

$$A(T) = \|\pi^-u_0 - u_0\|^2 + \|\pi^+q - q\|_{Q_T}^2 + \frac{d}{c_{11}(b)} \|(\pi^+q - q)(b^-, \cdot)\|_{(0,T)}^2,$$

and

$$B(t) = \|(\pi^-(u_t) - u_t)(\cdot, t)\|.$$

If we combine the above result with the estimates of $w - \pi^\pm w$, for $w = u$ and $w = q$, we immediately obtain our desired error bound. To state the corresponding approximation results, we introduce on the reference interval $I = (-1, 1)$ and for $s \in \mathbb{N}_0$ the following weighted seminorm:

$$|u|_{V^s(I)}^2 := \int_I |u^{(s)}(x)|^2 (1+x)^s (1-x)^s ds.$$

We can now state our approximation result.

Lemma 3.2 (The p -approximation estimates for fixed s). *Let $I = (-1, 1)$ be the reference interval and p a generic polynomial degree on I . Assume $w' \in V^s(I)$ for $s \in \mathbb{N}_0$ and $\pi^\pm w \in \mathcal{P}^p(I)$. Then we have the following estimates:*

$$\begin{aligned} \|\pi^\pm w - w\|_I &\leq \Phi(s) \max\{1, p\}^{-(s+1)} |w'|_{V^s(I)}, \\ |(\pi^\pm w - w)(\pm 1)| &\leq \Phi(s) \max\{1, p\}^{-(s+1/2)} |w'|_{V^s(I)}, \end{aligned}$$

where $\Phi(s)$ depends on s but is independent of p and w .

For standard finite element methods, the introduction of the weighted norms $|\cdot|_{V^s(I)}$ enables one to show that for singular solutions the p -version of the method, i.e., when the mesh \mathcal{T} is fixed and p_j increases unboundedly, yields twice the convergence rate than the h -version provided that the singularity lies at a mesh point x_j ; see, e.g., Schwab [24]. The results in Lemma 3.2 are slightly suboptimal with regard to these aspects as will be shown in the numerical experiments in section 5 below. However, for smooth solutions we obtain optimal approximation properties for π^\pm in h and p . This can be inferred immediately from Lemma 3.2 and standard scaling and interpolation arguments.

Lemma 3.3 (The hp -approximation estimates for fixed s). *Let $w \in H^{s+1}(I_j)$ for $j = 1, \dots, M$ and $s \geq 0$. Then we have the following estimates:*

$$\begin{aligned} \|\pi^\pm w - w\|_{I_j} &\leq \Phi(s) \left(\frac{h_j^{\min(s, p_j)+1}}{\max\{1, p_j\}^{s+1}} \right) \|w\|_{H^{s+1}(I_j)}, \\ |(\pi^+ w - w)(x_j)| &\leq \Phi(s) \left(\frac{h_j^{\min(s, p_j)+1/2}}{\max\{1, p_j\}^{s+1/2}} \right) \|w\|_{H^{s+1}(I_j)}, \\ |(\pi^- w - w)(x_{j-1})| &\leq \Phi(s) \left(\frac{h_j^{\min(s, p_j)+1/2}}{\max\{1, p_j\}^{s+1/2}} \right) \|w\|_{H^{s+1}(I_j)}, \end{aligned}$$

where $\Phi(s)$ depends on s but is independent of p_j , I_j and w .

Now, assume that $\|u^{(s+1)}\|_{\mathcal{E}, T} < \infty$, where

$$\|w\|_{\mathcal{E}, T} = 2 \sup_{0 \leq t \leq T} \|w(t)\| + \int_0^T \|w_t(\cdot, t)\| dt + 3\sqrt{d} \|w_x\|_{Q_T}.$$

Our main result is a simple consequence of the above lemmas. Indeed, since

$$\begin{aligned} \|\mathbf{e}\|_{E, T} &\leq \|\pi \mathbf{e}\|_T + \|\pi \mathbf{w} - \mathbf{w}\|_{E, T} \\ &\leq \|\pi \mathbf{e}\|_T + \sup_{0 \leq t \leq T} \|(\pi^- u - u)(\cdot, t)\| + \|\pi^+ q - q\|_{Q_T}, \end{aligned}$$

from Lemmas 3.1 and 3.3, we obtain the following result.

Theorem 3.4 (The estimate of the energy norm). *Let \mathbf{e} be the error between the exact solution and the approximation given by the LDG method with numerical flux (2.6) and polynomial degree p on each interval. Then, for totally arbitrary meshes, the energy norm of the error satisfies the inequality*

$$\|\mathbf{e}\|_{E,T} \leq \Phi(s) \frac{h^{\min\{s,p\}+1}}{\max\{1,p\}^{s+1}} \|u^{(s+1)}\|_{\mathcal{E},T}.$$

Remark 3.5. The error estimates in Theorem 3.4 are *optimal* in h and p for smooth solutions, even for the case in which piecewise constant approximations ($p = 0$) are used. Note also that in the purely convective case, $d = 0$, the above error estimate is nothing but the extension of the super-convergence error estimate of LeSaint and Raviart [15] for the h -version of the DG method for purely convective problems.

Remark 3.6. The proof of Lemma 3.3 actually gives us estimates which are completely explicit in the mesh-width h , in the polynomial degree p , and in the regularity of the exact solution (see Proposition 3.12 below). Hence, completely explicit error estimates in the energy norm can be obtained. In conjunction with geometric meshes and linearly increasing polynomial degrees, such estimates can be used in the hp -version to prove exponential rates of convergence in the presence of solution singularities; see, e.g., the recent monograph by Schwab [24] and the references therein. However, since the corresponding analytic regularity in space-time still remains to be found, we do not further pursue these issues here.

Remark 3.7. From Theorem 3.4, we conclude that in the p -version of the LDG method, where the mesh is kept fixed and the polynomial degree p is increased, we have $\|\mathbf{e}\|_{E,T} \leq \Psi(s)p^{-(s+1)} \|u^{(s+1)}\|_{\mathcal{E},T}$. Hence, for smooth solutions convergence rates of arbitrarily high algebraic order in p are possible. This is sometimes referred to as spectral convergence. Furthermore, for solutions which are analytic in \overline{Q}_T , even exponential rates of convergence are obtained in the p -version, i.e.,

$$(3.4) \quad \|\mathbf{e}\|_{E,T} \leq C \exp(-bp),$$

with constants $C, b > 0$ independent of p . This result can immediately be derived from Lemma 3.1, properties of π^\pm (see, e.g., (3.15) and (3.16) below) and standard approximation theory for analytic functions. We note that (3.4) holds true for general fluxes as well.

Remark 3.8. For small diffusivities d , i.e., for $d \rightarrow 0$ in (2.1), the solutions typically exhibit viscous boundary layers (or shock profiles) of length scale $O(d)$ or $O(\sqrt{d})$. In principle, layer components in the solutions are still analytic (see the work of Melenk [16] and Melenk and Schwab [19, 18] for a complete characterization of boundary layers in stationary problems with analytic input data) and can thus be approximated at exponential rates of convergence, in agreement with (3.4). However, the estimate (3.4) is not robust with respect to the diffusivity d and deteriorates as $d \rightarrow 0$. A remedy is to employ a needle-element of the appropriate width or geometric mesh refinement near the boundary. It has been shown recently by Melenk [16], Melenk and Schwab [17, 19], Schwab and Suri [25] and Wihler and Schwab [32] that the use of these mesh-design principles yields exponential rates of convergence that are robust with respect to the diffusivity parameter d . We demonstrate this robustness in our numerical examples in section 5.5 below.

3.2. Proof of the basic estimate. This section is devoted to the proof of Lemma 3.1. To do so, we follow the technique used by Cockburn and Shu [11]; see also Cockburn [4], Castillo [3] and Cockburn and Dawson [5].

We start by rewriting the definition of the LDG method in compact form for general numerical fluxes. Integrating (2.5) with respect to t from 0 to T and summing over all elements, it turns out that the LDG solution is defined as follows:

Find $\mathbf{w}_N = (u_N, q_N) \in H^1(0, T; V_N) \times L^2(0, T; V_N)$ such that

$$(3.5) \quad B_N(\mathbf{w}_N, \mathbf{v}) = \mathcal{L}(\mathbf{v}) \quad \forall \mathbf{v} = (v, r) \in H^1(0, T; V_N) \times L^2(0, T; V_N),$$

where the discrete bilinear form $B_N(\cdot, \cdot)$ is given by

$$(3.6) \quad \begin{aligned} B_N(\mathbf{w}_N, \mathbf{v}) &:= (u(0), v(0)) + \int_0^T ((u_N)_t(\cdot, t), v(\cdot, t)) dt \\ &+ \int_0^T (q_N(\cdot, t), r(\cdot, t)) dt \\ &- \int_0^T \sum_{j=1}^M (\mathbf{h}(\mathbf{w}_N(x, t)), \mathbf{v}_x(x, t))_{I_j} dt \\ &- \int_0^T \sum_{j=1}^{M-1} \hat{\mathbf{h}}(\mathbf{w}_N)(x_j, t)^\top [\mathbf{v}](x_j, t) dt \\ &+ \int_0^T \{ (-c/2 + c_{11}(a)) u_N(a^+, t) + \sqrt{d} q_N(a^+, t) \} v(a^+, t) dt \\ &+ \int_0^T (\sqrt{d}/2 - c_{12}(a)) u_N(a^+, t) r(a^+, t) dt \\ &+ \int_0^T \{ (c/2 + c_{11}(b)) u_N(b^-, t) - \sqrt{d} q_N(b^-, t) \} v(b^-, t) dt \\ &+ \int_0^T (-\sqrt{d}/2 - c_{12}(b)) u_N(b^-, t) r(b^-, t) dt, \end{aligned}$$

and the discrete linear form $\mathcal{L}(\cdot, \cdot)$ is given by

$$(3.7) \quad \begin{aligned} \mathcal{L}_N(\mathbf{v}) &:= (u_0, v(0)) + (f, v) \\ &+ \int_0^T (c/2 + c_{11}(a)) u_D(a, t) v(a^+, t) dt \\ &+ \int_0^T (-\sqrt{d}/2 - c_{12}(a)) u_D(a, t) r(a^+, t) dt \\ &+ \int_0^T (-c/2 + c_{11}(b)) u_D(b, t) v(b^-, t) dt \\ &+ \int_0^T (\sqrt{d}/2 - c_{12}(b)) u_D(b, t) r(b^-, t) dt. \end{aligned}$$

The basic error estimate now follows by standard manipulations. Indeed, since we have that

$$B_N(\mathbf{w}, \mathbf{v}) = \mathcal{L}(\mathbf{v}) \quad \forall \mathbf{v} \in H^1(0, T; V_N) \times L^2(0, T; V_N),$$

we obtain that

$$B_N(\mathbf{e}, \mathbf{v}) = 0 \quad \forall \mathbf{v} \in H^1(0, T; V_N) \times L^2(0, T; V_N),$$

where $\mathbf{e} := \mathbf{w} - \mathbf{w}_N$, and this implies that

$$(3.8) \quad B_N(\pi_N \mathbf{e}, \pi_N \mathbf{e}) = B_N(\pi_N \mathbf{e} - \mathbf{e}, \pi_N \mathbf{e}) = B_N(\pi_N \mathbf{w} - \mathbf{w}, \pi_N \mathbf{e}).$$

It only remains to obtain a suitable expression for $B_N(\pi_N \mathbf{e}, \pi_N \mathbf{e})$ and an upper bound for $B_N(\pi_N \mathbf{w} - \mathbf{w}, \pi_N \mathbf{e})$.

Lemma 3.9. *For any $\mathbf{v} = (v, r) \in H^1(0, T; V_N) \times L^2(0, T; V_N)$, it holds that*

$$B_N(\mathbf{v}, \mathbf{v}) = \frac{1}{2} \|\mathbf{v}\|_T^2 + \frac{1}{2} |\mathbf{v}|_T^2,$$

where

$$|\mathbf{v}|_T^2 = \|v(0)\|^2 + \int_0^T \|r(\cdot, t)\|^2 dt + \Theta_{T, \mathcal{T}}(v)$$

and $\|\cdot\|_T$ is defined by (3.3).

Proof. This is a direct consequence of the definition of the form B_N . \square

Lemma 3.10. *For any $\mathbf{v} \in H^1(0, T; V_N) \times L^2(0, T; V_N)$ and the numerical flux (2.6), we have*

$$\begin{aligned} B_N(\pi \mathbf{w} - \mathbf{w}, \mathbf{v}) &\leq \frac{1}{2} \|\pi^- u_0 - u_0\|^2 + \int_0^T \|\pi^-(u_t) - u_t(\cdot, t)\| \|v(\cdot, t)\| dt \\ &\quad + \frac{1}{2} \int_0^T \|(\pi^+ q - q)(\cdot, t)\|^2 dt \\ &\quad + \int_0^T \frac{d}{2c_{11}(b)} |(\pi^+ q - q)(b^-, t)|^2 dt + \frac{1}{2} |\mathbf{v}|_T^2. \end{aligned}$$

Proof. Taking into account the definition of the form B_N , (3.6), and the definition of the projection π , (3.1) and (3.2), we easily get that

$$\begin{aligned} B_N(\pi \mathbf{w} - \mathbf{w}, \mathbf{v}) &= (\pi^- u(0) - u(0), v(0)) + \int_0^T ((\pi^-(u_t) - u_t)(\cdot, t), v(\cdot, t)) dt \\ &\quad + \int_0^T ((\pi^+ q - q)(\cdot, t), r(\cdot, t)) dt \\ &\quad - \int_0^T \sqrt{d} (\pi^+ q - q)(b^-, t) v(b^-, t) dt. \end{aligned}$$

The result follows from simple applications of Cauchy-Schwarz's and Young's inequalities and from the definition of the functional $|\cdot|_T$ defined in Lemma 3.9. \square

Now, inserting the results of Lemmas 3.9 and 3.10 into (3.8), we get the inequality

$$\begin{aligned} \|\pi \mathbf{e}\|^2 &\leq \|\pi^- u_0 - u_0\|^2 + \|\pi^+ q - q\|_{Q_T}^2 + \frac{d}{c_{11}(b)} \|(\pi^+ q - q)(b^-, \cdot)\|_{(0, T)}^2 \\ &\quad + 2 \int_0^T \|(\pi^-(u_t) - u_t)(\cdot, t)\| \|v(\cdot, t)\| dt, \end{aligned}$$

which is of the form

$$(3.9) \quad \chi^2(T) + R(T) \leq A(T) + 2 \int_0^T B(t) \chi(t) dt,$$

with

$$\begin{aligned}\chi(T) &= \|(\pi^- u - u_N)(\cdot, T)\|, \\ R(T) &= \int_0^T \|(\pi^+ q - q_N)\|^2 dt + \Theta_{T, \mathcal{T}}(\pi^- u - u_N), \\ A(T) &= \|\pi^- u_0 - u_0\|^2 + \|\pi^+ q - q\|_{Q_T}^2 + \frac{d}{c_{11}(b)} \|(\pi^+ q - q)(b^-, \cdot)\|_{(0, T)}^2, \\ B(t) &= \|(\pi^-(u_t) - u_t)(\cdot, t)\|.\end{aligned}$$

Since inequality (3.9) holds true for all $T > 0$, Lemma 3.1 now follows after a simple application of the following result.

Lemma 3.11. *Suppose that for all $t > 0$ we have*

$$\chi^2(t) + R(t) \leq A(t) + 2 \int_0^t B(s) \chi(s) ds,$$

where R , A , and B are nonnegative functions. Then, for any $T > 0$,

$$\sqrt{\chi^2(T) + R(T)} \leq \sup_{0 \leq t \leq T} A^{1/2}(t) + \int_0^T B(t) dt.$$

Proof. Define $\kappa(t) = 2 \int_0^t B(s) \chi(s) ds$ and fix $T > 0$. Setting $S_T = \sup_{0 \leq t \leq T} A(t)$, the hypothesis implies that for $0 \leq t \leq T$

$$\kappa'(t) = 2B(t)\chi(t) \leq 2B(t)\sqrt{A(t) + \kappa(t)} \leq 2B(t)\sqrt{S_T + \kappa(t)}.$$

Integrating over $(0, T)$ yields

$$\int_0^T \frac{\kappa'(t)}{\sqrt{S_T + \kappa(t)}} dt \leq 2 \int_0^T B(t) dt.$$

Hence,

$$\sqrt{S_T + \kappa(T)} \leq \sqrt{S_T} + \int_0^T B(t) dt.$$

Since $\sqrt{\chi^2(T) + R(T)} \leq \sqrt{S_T + \kappa(T)}$, the assertion in Lemma 3.11 follows. \square

This completes the proof of Lemma 3.1.

3.3. Proof of the hp -approximation results. This section is devoted to the proof of Lemma 3.2 which follows from the subsequent finer approximation results after an application of Stirling's formula.

Let $I = (-1, 1)$ and recall that

$$|u|_{V^s(I)}^2 := \int_I |u^{(s)}(x)|^2 (1+x)^s (1-x)^s dx.$$

We have:

Proposition 3.12. *Let $w' \in V^s(I)$ for $s \in \mathbb{N}_0$. Let $\pi^\pm \in \mathcal{P}^p(I)$ be defined by*

$$(3.10) \quad (\pi^\pm w - w, v)_I = 0 \quad \forall v \in \mathcal{P}^{p-1}(I), \quad \pi^\pm w(\mp 1) = w(\mp 1).$$

Then we have

$$\|\pi^\pm w - w\|_I^2 \leq \left(\frac{6}{(2p+1)^2} \right) \frac{(p-k)!}{(p+k)!} |w'|_{V^k(I)}^2,$$

and

$$|(\pi^\pm w - w)(\pm 1)|^2 \leq \left(\frac{2}{2p+1}\right) \frac{(p-k)!}{(p+k)!} |w'|_{V^k(I)}^2,$$

for any $0 \leq k \leq \min(p, s)$.

Proof. The first estimate was obtained by Schötzau [22] and Schötzau and Schwab [23]. Nevertheless, we present a detailed proof for the sake of completeness. We consider only π^- since the proof for π^+ is similar. We proceed in several steps.

Step 1: First, we derive bounds on the difference $w - \pi^- w$ in terms of the Legendre coefficients of w . To do so, denote by $L_i(x)$, $i \geq 0$, the Legendre polynomial of degree i on I and expand the function w into the series $w = \sum_{i=0}^\infty w_i L_i$ with $w_i = \int_I w(x) L_i(x) dx / \|L_i\|_I^2$. Since $L_i(+1) = 1$, it can be seen from (3.10) that $\pi^- w$ is uniquely given by the series

$$\pi^- w = \sum_{i=0}^{p-1} w_i L_i + \left(\sum_{i=p}^\infty w_i\right) L_p.$$

Hence, the difference $w - \pi^- w$ can be written as

$$w - \pi^- w = \sum_{i=p+1}^\infty w_i L_i - \left(\sum_{i=p+1}^\infty w_i\right) L_p.$$

Let P_p be the L^2 -projection from $L^2(I)$ onto $\mathcal{P}^p(I)$. Since $w - P_p w = \sum_{i=p+1}^\infty w_i L_i$, $\|L_i\|_I^2 = \frac{2}{2i+1}$ and $L_i(\pm 1) = (\pm 1)^i$, we obtain from the definition of π^- the following bounds:

$$(3.11) \quad \|w - \pi^- w\|_I^2 = \|w - P_p w\|_I^2 + \left|\sum_{i=p+1}^\infty w_i\right|^2 \frac{2}{2p+1},$$

$$(3.12) \quad |(w - \pi^- w)(-1)|^2 = 4 \left|\sum_{i=0}^\infty w_{p+1+2i}\right|^2.$$

Step 2: To estimate the sums in the above equalities, we start by expanding w' into the series $w' = \sum_{i=0}^\infty b_i L_i$. Integrating this expression yields

$$w(x) = w(-1) + \sum_{i=0}^\infty b_i \int_{-1}^x L_i(s) ds,$$

and employing for $i \geq 1$ the identity

$$\int_{-1}^x L_i(s) ds = \frac{1}{2i+1} (L_{i+1}(x) - L_{i-1}(x)),$$

and rearranging terms, we obtain

$$w(x) = (w(-1) + b_0) L_0(x) + \sum_{i=1}^\infty \frac{b_{i-1}}{2i-1} L_i(x) - \sum_{i=0}^\infty \frac{b_{i+1}}{2i+3} L_i(x).$$

Comparing coefficients in the Legendre expansions, we conclude that

$$w_i = \frac{b_{i-1}}{2i-1} - \frac{b_{i+1}}{2i+3}, \quad i \geq 1.$$

Hence, after some simple algebraic manipulations,

$$\sum_{i=p+1}^{\infty} w_i = \frac{b_p}{2p+1} + \frac{b_{p+1}}{2p+3},$$

and

$$\sum_{i=0}^{\infty} w_{p+1+2i} = (-1)^{p+1} \frac{1}{2p+1} b_p.$$

As a consequence,

$$\left| \sum_{i=p+1}^{\infty} w_i \right| \leq \frac{1}{\sqrt{2p+1}} \left(\frac{2b_p^2}{2p+1} + \frac{2b_{p+1}^2}{2p+3} \right)^{1/2},$$

$$\left| \sum_{i=0}^{\infty} w_{p+1+2i} \right| = \frac{1}{2p+1} |b_p|,$$

and since $\|w'\|_I^2 = \sum_{i=0}^{\infty} b_i^2 \frac{2}{2i+1}$, we get

$$(3.13) \quad \left| \sum_{i=p+1}^{\infty} w_i \right| \leq \frac{1}{\sqrt{2p+1}} \|w'\|_I,$$

$$(3.14) \quad \left| \sum_{i=0}^{\infty} w_{p+1+2i} \right| = \frac{1}{\sqrt{2(2p+1)}} \|w'\|_I.$$

Step 3: Now, note that after inserting the estimates (3.13) and (3.14) into (3.11) and (3.12), respectively, we get

$$\|w - \pi^- w\|_I^2 \leq \|w - P_p w\|_I^2 + \frac{2}{(2p+1)^2} \|w'\|_I^2,$$

$$|(w - \pi^- w)(-1)|^2 = \frac{2}{2p+1} \|w'\|_I^2.$$

Replacing w in these inequalities by $w - q$, where q is an arbitrary polynomial of degree p , and taking into account that $P_p(q) = q$ and $\pi^-(q) = q$ give

$$(3.15) \quad \|w - \pi^- w\|_I^2 \leq \|w - P_p w\|_I^2 + \frac{2}{(2p+1)^2} \|w' - q'\|_I^2,$$

$$(3.16) \quad |(w - \pi^- w)(-1)|^2 \leq \frac{2}{2p+1} \|w' - q'\|_I^2.$$

Schwab [24] proved that

$$\|w - P_p w\|_I^2 \leq \frac{(p-k)!}{(p+2+k)!} |w'|_{V^k(I)}^2,$$

for any $0 \leq k \leq \min(p, s)$, and the existence of a polynomial $q \in \mathcal{P}^p(I)$ such that

$$\|w' - q'\|_I^2 \leq \frac{(p-k)!}{(p+k)!} |w'|_{V^k(I)}^2,$$

for any $0 \leq k \leq \min(p, s)$. We now simply insert these estimates into (3.15) and (3.16) to conclude that

$$\begin{aligned} \|w - \pi^- w\|_I^2 &\leq \left(\frac{1}{(p+2+k)(p+1+k)} + \frac{2}{(2p+1)^2} \right) \frac{(p-k)!}{(p+k)!} |w'|_{V^k(I)}^2, \\ |(w - \pi^- w)(-1)|^2 &\leq \frac{2}{2p+1} \frac{(p-k)!}{(p+k)!} |w'|_{V^k(I)}^2, \end{aligned}$$

for any $0 \leq k \leq \min(p, s)$.

The corresponding estimates for π^+ are obtained by symmetry. Since

$$(p+2+k)(p+1+k) \geq (2p+1)^2/4,$$

this proves Proposition 3.12. □

From Proposition 3.12 we obtain by standard scaling and interpolation arguments the following hp -approximation properties of π^\pm :

Corollary 3.13. *For each interval $I_j = (x_{j-1}, x_j)$, $j = 1, \dots, M$, we have for $w \in H^{s_j+1}(I_j)$, $s_j \geq 0$ real, the estimates*

$$\|\pi^\pm w - w\|_{I_j}^2 \leq C \left(\frac{h_j}{2} \right)^{2k_j+2} \left(\frac{1}{p_j^2} \right) \frac{\Gamma(p_j - k_j + 1)}{\Gamma(p_j + k_j + 1)} \|w\|_{H^{k_j+1}(I_j)}^2,$$

and

$$\begin{aligned} |(\pi^+ w - w)(x_j)|^2 &\leq C \left(\frac{h_j}{2} \right)^{2k_j+1} \left(\frac{1}{p_j} \right) \frac{\Gamma(p_j - k_j + 1)}{\Gamma(p_j + k_j + 1)} \|w\|_{H^{k_j+1}(I_j)}^2, \\ |(\pi^- w - w)(x_{j-1})|^2 &\leq C \left(\frac{h_j}{2} \right)^{2k_j+1} \left(\frac{1}{p_j} \right) \frac{\Gamma(p_j - k_j + 1)}{\Gamma(p_j + k_j + 1)} \|w\|_{H^{k_j+1}(I_j)}^2, \end{aligned}$$

for any $0 \leq k_j \leq \min(p_j, s_j)$. The constant $C > 0$ is independent of h_j, p_j and k_j .

4. EXTENSIONS

The a priori error estimate of Theorem 3.4 can be easily extended to the case of general boundary conditions and to general numerical fluxes.

4.1. Other boundary conditions. Theorem 3.4 holds unchanged for Neumann, Robin or mixed boundary conditions. To see this, let us consider, for example, the following Neumann boundary conditions:

$$\sqrt{d} u_x(a) = q_N(a), \quad \sqrt{d} u_x(b) = q_N(b).$$

First, we take

$$(u, q)(a^-) = (u(a^+), q_N(a)), \quad (u, q)(b^+) = (u(b^-), q_N(b)).$$

Then we redefine the numerical flux as follows:

$$\hat{\mathbf{h}}(x_j) = \begin{cases} (cu(a^+) - \sqrt{d}q_N(a), -\sqrt{d}u(a^+))^\top & \text{for } j = 0, \\ (cu(x_j^-) - \sqrt{d}q(x_j^+), -\sqrt{d}u(x_j^-))^\top & \text{for } j = 1, \dots, M-1, \\ (cu(b^-) - \sqrt{d}q_N(b), -\sqrt{d}u(b^-))^\top & \text{for } j = M. \end{cases}$$

Note that this flux is obtained by setting $c_{12} \equiv \sqrt{d}/2$ and

$$(c_{11}, c_{12})(x_j) = \begin{cases} -(c/2, \sqrt{d}/2) & \text{for } j = 0, \\ (c/2, \sqrt{d}/2) & \text{for } j = 1, \dots, M. \end{cases}$$

Now, we simply have to go through the proof of Lemma 3.1 in section 3.2 to verify that the basic error estimate of Lemma 3.1 still holds with b as before and

$$A(T) = \|\pi^- u_0 - u_0\|^2 + \|\pi^+ q - q\|_{Q_T}^2.$$

This and the approximation result of Lemma 3.3 imply that the estimate of Theorem 3.4 holds in this case.

4.2. General numerical fluxes. In the case of general numerical fluxes, the optimality of the estimate of Theorem 3.4 is lost in both h and p . Theoretically, the main reason is that now there are new terms in $B(\pi_N \mathbf{w} - \mathbf{w}, \pi_N \mathbf{e})$ which were equal to zero for the flux (2.6).

Indeed, in the purely convective case, $d = 0$, the new term is

$$\begin{aligned} \eta := & \int_0^T \sum_{j=1}^{M-1} (-c/2 + c_{11}(x_j)) (\pi^- u - u)(x_j^+, t) [\pi^-(u_N - u)](x_j^+, t) dt \\ & + \int_0^T (-c/2 + c_{11}(a)) (\pi^- u - u)(a, t) [\pi^-(u_N - u)](a^+, t) dt, \end{aligned}$$

which is estimated as follows:

$$\begin{aligned} |\eta| \leq & \int_0^T \sum_{j=1}^{M-1} \frac{(-c/2 + c_{11}(x_j))^2}{2 c_{11}(x_j)} (\pi^- u - u)^2(x_j^+, t) dt \\ & + \int_0^T \frac{(-c/2 + c_{11}(a))^2}{2 c_{11}(a)} (\pi^- u - u)^2(a^+, t) dt \\ & + \frac{1}{2} \int_0^T \sum_{j=1}^{M-1} c_{11}(x_j) [\pi^-(u_N - u)]^2(x_j^+, t) dt \\ & + \frac{1}{2} \int_0^T c_{11}(a) [\pi^-(u_N - u)]^2(a^+, t) dt. \end{aligned}$$

The last two terms are absorbed by the term $|\pi_N \mathbf{e}|_T^2$ and the first two are bounded, using Lemma 3.3, by

$$C_{11} \Phi(s) \frac{h^{\min\{s,p\}+1/2}}{\max\{1,p\}^{s+1/2}} \|u^{(s+1)}\|_{Q_T},$$

where

$$C_{11} := \max_{0 \leq j \leq M-1} \frac{(-c/2 + c_{11}(x_j))^2}{2 c_{11}(x_j)}.$$

Hence, the error estimate is

$$\|\mathbf{e}\|_{E,T} \leq \Psi(s) \frac{h^{\min\{s,p\}+1/2}}{\max\{1,p\}^{s+1/2}} \left\{ \frac{h^{1/2}}{\max\{1,p\}^{1/2}} \|u^{(s+1)}\|_{\varepsilon,T} + C_{11} \|u^{(s+1)}\|_{Q_T} \right\}.$$

Note the loss of half a power in both h and p .

In the case in which $d \neq 0$, the following additional term appears:

$$\begin{aligned} \zeta &:= \int_0^T \sum_{j=1}^{M-1} (\sqrt{d}/2 - c_{12}(x_j)) (\pi^+ q - q)(x_j^-, t) [\pi^-(u_N - u)](x_j, t) dt \\ &+ \int_0^T \sum_{j=1}^{M-1} (\sqrt{d}/2 - c_{12}(x_j)) (\pi^- u - u)(x_j^+, t) [\pi^+(q_N - q)](x_j, t) dt \\ &+ \int_0^T (\sqrt{d}/2 - c_{12}(a)) (\pi^- u - u)(a^+, t) [\pi^+(q_N - q)](a^+, t) dt. \end{aligned}$$

By using the approximation results of Lemma 3.3, we see that we can bound ζ as follows:

$$|\zeta| \leq C_{12} \Phi(s) \frac{h^{\min\{s,p\}+1/2}}{\max\{1,p\}^{s+1/2}} \int_0^T \Theta(t) dt,$$

where

$$C_{12} := \max_{0 \leq j \leq M-1} |\sqrt{d}/2 - c_{12}(x_j)|,$$

and

$$\begin{aligned} \Theta(t) &:= \sqrt{d} \|u_x^{(s+1)}(t)\| \left\{ \sum_{j=1}^{M-1} [\pi^-(u_N - u)]^2(x_j, t) \right\}^{1/2} \\ &+ \|u^{(s+1)}(t)\| \left\{ \sum_{j=0}^{M-1} [\pi^+(q_N - q)]^2(x_j, t) \right\}^{1/2}. \end{aligned}$$

Next, we use the following inverse inequality:

$$\max\{|w(x_{j-1}^+)|, |w(x_j^-)|\} \leq C_i \frac{p}{\sqrt{h_j}} \|w\|_{I_j},$$

for $w \in V_N$ in order to get

$$\begin{aligned} |\zeta| &\leq C_i C_{12} \Phi(s) \frac{h^{\min\{s,p\}}}{\max\{1,p\}^{s-1/2}} \int_0^T \left\{ \sqrt{d} \|u_x^{(s+1)}(t)\| \|\pi^-(u_N - u)(t)\| \right. \\ &\quad \left. + \|u^{(s+1)}(t)\| \|\pi^+(q_N - q)(t)\| \right\} dt. \end{aligned}$$

Note that this produces an *additional* loss of half power in h and a full power in p . Thus, after a few simple manipulations, we obtain the following estimate for general numerical fluxes:

$$\|\mathbf{e}\|_{E,T} \leq \Lambda(C_{11}, C_{12}, s) \frac{h^{\min\{s,p\}}}{\max\{1,p\}^{s-1/2}} \|u^{(s+1)}\|_{\mathcal{E},T}.$$

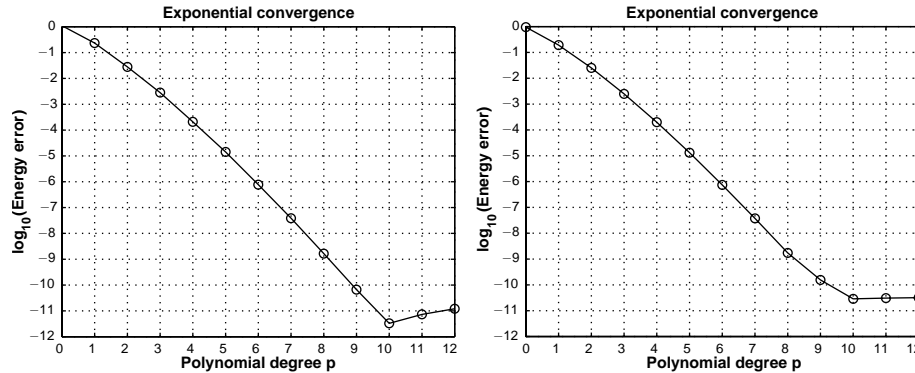


FIGURE 1. Exponential convergence in p for an analytic exact solution. In both examples, the diffusion coefficient is $d = 0.1$. The convection coefficient is $c = 0.1$ (left) and $c = 1.0$ (right).

5. NUMERICAL RESULTS

The purpose of this section is to numerically validate the a priori error estimates given in section 3. In all our experiments, we use a TVD Runge–Kutta time stepping method (see Shu and Osher [26, 27]), with sufficiently small time steps, such that the overall error is governed by the spatial error.

5.1. Exponential convergence. Our first example illustrates the exponential convergence in p for analytic solutions. We solve (2.1) on the space-time domain $Q_T = \Omega \times J = (0, 1) \times (0, 1)$, with exact solution $u(x, t) = \exp(-dt) \sin(2\pi(x - ct))$. We use a fixed grid consisting of a uniform mesh with 4 elements and increase the polynomial degree p . The corresponding errors in the energy norm at time $T = 1$ are shown in Figure 1. The diffusion coefficient is $d = 0.1$ and the convection coefficient is chosen as $c = 0.1$ (left) and $c = 1.0$ (right). The curves clearly show exponential rates of convergence as predicted in (3.4) of section 3. Since the quadrature points and weights used to determine the LDG solution are computed only with an accuracy of 10^{-12} , the curves bottom out for $p \geq 10$.

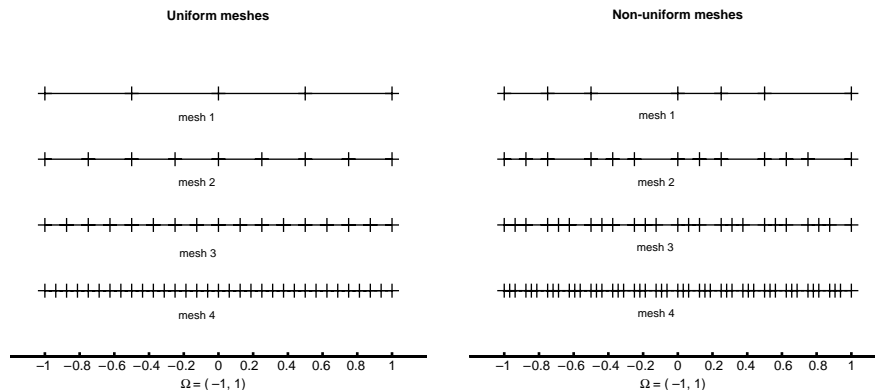


FIGURE 2. Sequence of uniform and nonuniform meshes in $\Omega = (-1, 1)$.

5.2. Optimal order of convergence in h . In these examples, we show that an optimal order of convergence of $p + 1$ is achieved when using the numerical flux in (2.6). For this set of tests, we solve (2.1) on $Q_T = \Omega \times J = (-1, 1) \times (0, 1)$, again with exact solution $u(x, t) = \exp(-dt) \sin(2\pi(x - ct))$. To determine numerically the convergence order we consider the two sequences of successively refined meshes $\{\mathcal{T}_i\}$ shown in Figure 2. Since our analysis is valid for arbitrary meshes, we choose the second sequence to consist of nonuniform meshes whereas the first one contains uniform meshes. Note that in both cases the mesh-size parameter of \mathcal{T}_{i+1} is half of the one of \mathcal{T}_i .

TABLE 1. Orders of convergence for the h -version and a smooth exact solution with $c = 0.1$, $d = 0.01$.

p	Nonuniform grid			Uniform grid		
	r_1	r_2	r_3	r_1	r_2	r_3
0	0.6733	0.6999	0.8801	0.4964	0.7817	0.8728
1	1.5527	2.0295	1.9846	1.8123	1.8739	1.9658
2	2.6972	2.8663	2.9384	2.5580	2.9190	2.9504
3	3.6891	4.1849	3.9948	4.0393	3.9472	3.9840
4	4.8392	4.9388	4.9562	4.7086	4.9489	4.9681
5	5.8042	6.2034	5.9937	6.1268	5.9660	5.9880
6	6.8673	6.9757	6.9365	6.7812	6.9566	6.9652

TABLE 2. Orders of convergence for the h -version and a smooth exact solution with $c = 0.1$, $d = 0.1$.

p	Nonuniform grid			Uniform grid		
	r_1	r_2	r_3	r_1	r_2	r_3
0	0.6425	1.1482	0.9394	0.7124	0.9293	0.9405
1	1.6591	1.7262	2.0137	1.4881	1.9553	1.9914
2	3.0588	3.1997	2.9655	3.1656	2.9458	2.9738
3	3.8242	3.6791	3.9919	3.5969	3.9727	3.9888
4	4.9699	5.2152	4.9785	5.1662	4.9545	4.9840
5	5.8760	5.6978	5.9926	5.6391	5.9782	5.9915
6	6.9644	7.2217	6.9748	7.1813	6.9659	6.9860

TABLE 3. Orders of convergence for the h -version and a smooth exact solution with $c = 0.1$, $d = 1.0$.

p	Nonuniform grid			Uniform grid		
	r_1	r_2	r_3	r_1	r_2	r_3
0	0.7198	1.2008	0.9773	0.9099	0.9330	0.9768
1	1.6489	1.6060	1.9993	1.2495	1.9696	1.9920
2	3.0070	3.2280	2.9858	3.2297	2.9582	2.9875
3	3.5119	3.5757	3.9939	3.1284	3.9728	3.9926
4	5.0340	5.2264	4.9905	5.2742	4.9704	4.9917
5	5.4512	5.5980	5.9949	5.0866	5.9790	5.9944
6	7.0632	7.2251	6.9689	7.3031	6.9774	6.9817

If $e(\mathcal{T}_i)$ denotes the error on the i th mesh in the energy norm, then the numerical rate of convergence r_i is defined as

$$r_i = \log \left(\frac{e(\mathcal{T}_{i+1})}{e(\mathcal{T}_i)} \right) / \log(0.5).$$

In Tables 1, 2 and 3, we present these numerical orders $\{r_i\}$ in the energy norm at $T = 1.0$ for polynomials of degree 0 to 6 on the above two mesh-sequences. In all the experiments we use the same convection coefficient and increase the diffusion coefficient from 0.01 to 1.0. The results show that our estimates are optimal in h not only for convection dominated problems, but also for diffusion dominated problems. In all the cases the numerical orders agree with the theoretical orders of our error estimates in Theorem 3.4.

5.3. Nonsmooth solutions. In this subsection, we present some numerical results to illustrate the performance of the LDG method for a solution that is nonsmooth in space.

We consider first the h -version and start by solving the purely convective ($d = 0$) problem (2.1), on $Q_T = \Omega \times J = (0, 1) \times (0, 1)$ with $c = 0.1$ and with data chosen in such a way that the exact solution is $u(x, t) = x^\pi t$. The corresponding uniform and nonuniform spatial discretizations are similar to those used in section 5.2 (cf., Figure 2). In this purely convective problem an order of convergence of

TABLE 4. Orders of convergence for the h -version and the non-smooth exact solution $x^\pi t$ for the purely convective case $c = 0.1$, $d = 0$.

p	Nonuniform grid			Uniform grid		
	r_1	r_2	r_3	r_1	r_2	r_3
0	0.0932	1.0417	1.0029	0.9751	0.9926	0.9976
1	0.2705	1.8839	1.9163	1.8082	1.8966	1.9524
2	0.4419	2.8961	2.9684	2.8795	2.9708	2.9776
3	1.3895	3.4832	3.5605	3.5243	3.6076	3.6162
4	2.8854	3.6059	3.6209	3.6067	3.6224	3.6310
5	3.5372	3.6250	3.6316	3.6230	3.6252	3.6332
6	3.6194	3.6288	3.6298	3.6252	3.6292	3.6343

TABLE 5. Orders of convergence for the h -version and the non-smooth exact solution $x^\pi t$ for the convection-diffusion case $c = 0.1$, $d = 0.1$.

p	Nonuniform grid			Uniform grid		
	r_1	r_2	r_3	r_1	r_2	r_3
0	0.7546	0.8327	0.9669	0.8405	0.9604	0.9957
1	1.8273	1.9542	1.9992	1.9383	1.9829	1.9956
2	2.9891	3.0066	3.0025	2.9853	2.9811	2.9704
3	3.0040	2.7869	2.6861	2.6750	2.6493	2.6430
4	2.6493	2.6426	2.6413	2.6424	2.6408	2.6409
5	2.6399	2.6403	2.6408	2.6397	2.6403	2.6408
6	2.6393	2.6403	2.6407	2.6393	2.6403	2.6408

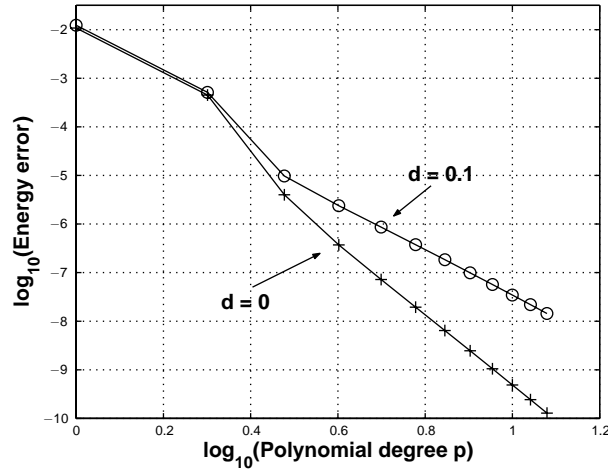


FIGURE 3. The p -version for the nonsmooth exact solution $x^{\pi}t$. The convection coefficient is $c = 0.1$ and the diffusion coefficient is $d = 0.1$ (top curve) and $d = 0$ (bottom curve).

$\min\{\pi + 0.5, p + 1\}$ is expected from our error estimate in section 3. These orders can clearly be seen in Table 4. Again, they are calculated for the energy norm at $T = 1.0$.

Now, we consider the linear problem with diffusion, i.e., with $c = 0.1$ and $d = 0.1$. Again, we choose the data in such a way that the exact solution is $u(x, t) = x^{\pi}t$. The results at $T = 1.0$ are shown in Table 5. From our a priori error estimate, an order of convergence of $\min\{\pi - 0.5, p + 1\}$ is expected. This is what we actually see for all values of p except for $p = 2$. In this case, we observe an order of convergence of 3 instead of the predicted $\pi - 0.5 \approx 2.6416$. Since the order of convergence for $p > 2$ is smaller than 3, we believe that an error cancellation might be taking place which occurs only for $p = 2$.

Since the $x^{\pi}t$ solution is singular at the mesh point $x = 0$, we expect a doubling of the convergence rate in the p -version where p is increased on a fixed mesh \mathcal{T} ; see, e.g., Schwab [24]. This is shown in Figure 3 for the same model problems as above. We can see a convergence rate of $2\pi + 1$ in the purely hyperbolic case ($d = 0$) and of $2\pi - 1$ in the convection-diffusion case, respectively, which corresponds to an exact doubling of the rates. However, in our theoretical results, if we insert the weighted bounds from Lemma 3.2 in the proof of Theorem 3.4, we obtain rates of 2π in the hyperbolic case and of $2\pi - 2$ in the convection-diffusion case, resulting in a loss of one power of p and indicating the suboptimality of Lemma 3.2 with respect to the weighted spaces $|\cdot|_{V^s(I)}$.

5.4. Testing the optimality of the smoothness requirement. To test if the smoothness on the exact solution required by our Theorem 3.4 when $d \neq 0$ is optimal, we consider problem (2.1) with $c = 0.1$, $d = 0.1$, homogeneous Dirichlet boundary conditions and initial data $u_0(x) = x(1 - x)$. The results at $T = 1.0$ are given in Table 6. Theorem 3.4 predicts an order of convergence of $\min\{1.5, p + 1\}$ but we actually see an order of convergence of $\min\{2.5, p + 1\}$. This gives a strong indication that, to obtain optimal orders of convergence at least

TABLE 6. Orders of convergence for the h -version and a non-smooth exact solution corresponding to the initial data $u_0(x) = x(1-x)$ with $c = 0.1$, $d = 0.1$.

p	Nonuniform grid			Uniform grid		
	r_1	r_2	r_3	r_1	r_2	r_3
0	0.8765	0.9020	0.8555	0.9235	0.9546	0.9370
1	1.7059	1.8495	1.8887	1.8491	1.9158	1.9044
2	2.4844	2.4854	2.5047	2.4822	2.4672	2.5089
3	2.4984	2.4953	2.4908	2.4884	2.4979	2.3841
4	2.5001	2.4961	2.4808	2.4893	2.4716	2.0190
5	2.5001	2.5034	2.4875	2.4911	2.4666	2.4900
6	2.5022	2.5022	2.4932	2.4963	2.4949	2.4618

in h , less smoothness of the exact solution than required in Theorem 3.4 for $d \neq 0$ is sufficient. However, obtaining optimality in the smoothness of the exact solution seems to ask for more sophisticated theoretical techniques than the ones available in the current literature and has to be addressed in future work.

5.5. Robust exponential convergence. Our last example shows that robust exponential convergence can be obtained in the presence of a boundary layer, when suitable meshes are used; see Remark 3.8. We solve (2.1) on $(0, 1) \times (0, 1)$ with $c = 0.1$, $d = 0.1$ and right-hand side such that the exact solution is $u(x, t) = t(1 - e^{(1-x)/\varepsilon})$. For small ε , this solution has an exponential boundary layer of strength $O(\varepsilon)$ at the outflow boundary $x = 1$. In Figure 4, we compare the p -version of the LDG method when using uniform and geometric meshes. Both meshes are chosen in such a way that they have the same number of elements; however, the distribution of the grid points is different: In the geometric mesh the size of the first element near $x = 1$ is on the order of the length of the boundary layer, $O(\varepsilon)$,

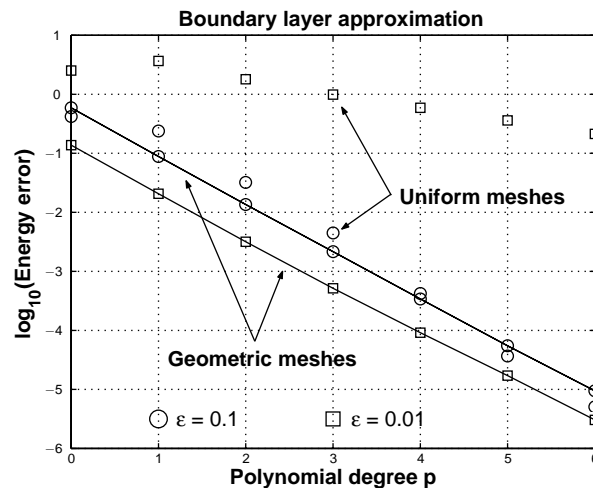


FIGURE 4. Exponential rates of convergence in the presence of a boundary layer on uniform and geometric meshes for $\varepsilon = 0.1$ and $\varepsilon = 0.01$.

the size of the next element is twice the size of the previous and so forth. The errors in the energy norm at $T = 1.0$ for $\varepsilon = 0.1$ and $\varepsilon = 0.01$ are depicted in Figure 4. All the curves show exponential rates of convergence. However, the robustness of the rates for the geometric boundary layer meshes can clearly be observed, whereas the uniform mesh performs orders of magnitude worse for $\varepsilon = 0.01$.

6. CONCLUDING REMARKS

In this paper, we have obtained optimal error estimates for the hp -version of the LDG method for the model problem of the initial boundary value problem for a one-dimensional convection-diffusion equation. We have shown that this is possible by a careful choice of the numerical fluxes and the associated projections π^+ and π^- ; we have also shown how this optimality in h and p is lost, at least theoretically, when general numerical fluxes are used.

Our numerical results confirm the optimality in h of our main result and the exponential convergence that follows when the solution is analytic. These results also indicate that the smoothness requirement on the exact solution is too stringent. The problem of obtaining optimality in the smoothness of the exact solution seems to ask for more sophisticated theoretical techniques than the ones available in the current literature and constitute the subject of ongoing work. Also, extensions of our main result to the more challenging cases of nonconstant coefficients c and d , and to the multidimensional case will be considered elsewhere.

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