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Optimal Algorithm for Shape from Shading and Path Planning

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Abstract. An optimal algorithm for the reconstruction of a surface from its shading image is presented. The algorithm solves the 3D reconstruction from a single shading image problem. The shading image is treated as a penalty function and the height of the reconstructed surface is a weighted distance. A consistent numerical scheme based on Sethian's fast marching method is used to compute the reconstructed surface. The surface is a viscosity solution of an Eikonal equation for the vertical light source case. For the oblique light source case, the reconstructed surface is the viscosity solution to a different partial differential equation. A modification of the fast marching method yields a numerically consistent, computationally optimal, and practically fast algorithm for the classical shape from shading problem. Next, the fast marching method coupled with a back tracking via gradient descent along the reconstructed surface is shown to solve the path planning problem in robot navigation.

Keywords: fast marching, Eikonal equations, shape from shading, navigation

1. Introduction

One of the earliest problems in the field of computer vision is the reconstruction of a three dimensional object from its single gray level image. The problem, for the case of a diffusive reflectance model of the surface, also known as Lambertian reflectance, is recognized as the 'shape from shading problem' [10, 11]. Various numerical schemes were proposed over the years, most of these methods were based on variational principles that require additional regularization terms that introduce second order derivatives into the minimization process. These terms yield over-smoothed reconstructions, see e.g. [12]. Only two early direct models for the shape from shading did not incorporate extra smoothness terms, the first is the characteristic strips expansion method that Horn used when he first introduced the problem [10], the second is Bruckstein's equal height contours tracking model [3]. Unfortunately, the first implementations of these algorithms suffered from numerical instabilities.

New numerical algorithms based on recent results in curve evolution theory, control theory, and the viscosity framework [5], were applied to the shape from shading problem in [8, 15, 16, 24]. In these advanced numerical algorithms the smoothness assumption is embedded within the scheme without the need for an extra smoothness as a penalty.

Recently, Sethian [26, 27] introduced an $O(N \log N)$ computational steps sequential steps algorithm for solving the Eikonal equation on a rectangular grid, where N is the total number of grid points. This algorithm, known as the 'Fast Marching Method,' relies on a systematic causality relationship based on upwinding, coupled with a heap structure for efficiently ordering the updated points. The method was applied to segmentation in 3D in [23] and to edge integration in [4].

An important property of Sethian's Fast Marching Method that distinguishes it from graph search based methods is that it converges to the continuous physical (viscosity) solution as the rectangular numerical grid is refined. Sethian's method is a finite difference scheme, based on upwind monotone schemes, and has been extended to higher order in Sethian SIAM Review article [28]. In [29], Tsitsiklis gives a different, first order Dijkstra-like algorithm which obtains the viscosity solution through a control-theoretic discretization using a causality relationship based on the optimality criterion.

In this note we first modify Sethian's fast marching method to construct a numerical solution for the oblique light source shape from shading problem. Next, the method is used for path planning in robotic navigation with small number of degrees of freedom.

2. Shape from Shading

Let us first review the shading image formation model for a 3D Lambertian object. Assume, that the object we try to reconstruct is given as a function $z(x, y) : R^2 \to R$, whose surface normal at each point is given by $\vec{n}(x, y) : R^2 \to S^2$. Next, let the light source direction be given by $\vec{l} \in S^2$. Then, the intensity image, $I(x, y) : R^2 \to R$, for an orthographic. projection of the object is given by the inner product $I(x, y) = \vec{l} \cdot \vec{n}(x, y)$.

For the simple vertical light source case $\vec{l} = (0, 0, 1)$, in which the light source is located near the viewer, the shading image is given by $I(x, y) = \frac{1}{\sqrt{1+z_x^2+z_y^2}}$. The problem in hand is the reconstruction of $z(x, y)$ from its gradient magnitude at each point that is given by $|\nabla z(x, y)| = \sqrt{(I(x, y))^{-2} - 1}$. This equation is known as the *Eikonal equation.* See [30] for a 'shading from shape' Eikonal based technique.

The fast marching method is an $O(N \log N)$ numerical algorithm for solving the Eikonal equation, e.g. $|\nabla z(x, y)| = f(x, y)$. The first version of the algorithm is based on the following numerical approximation of the Eikonal equation

$$\begin{aligned}&\left(\max\left(D_{ij}^{-x}z,\ -D_{ij}^{+x}z,\ 0\right)\right)^2\\&\quad+\left(\max\left(D_{ij}^{-y}z, -D_{ij}^{+y}z, 0\right)\right)^2 = f_{ij}^2, \qquad (1)\end{aligned}$$

where $z_{ij} = z(i\Delta x,\ j\Delta y)$, and $D_{ij}^{-x}z = (z_{ij} - z_{i-1,j})/\Delta x$ is the standard backwards derivative approximation, $D_{ij}^{+x}z = (z_{i+1,j} - z_{ij})/\Delta x$ is the standard forward derivative approximation in the x direction, and similarly for the y direction. This numerical approximation selects the correct viscosity solution for the shape from shading problem as proven by Rouy and Tourin [24]. One important observation is that information always flow form small to large values of the solution z. Therefore, the surface z may be reconstructed by first setting all z values to ∞, and the correct height value at the local minimum points. In case the height values at these locations are unknown, then a global topology solver can be applied [2, 17].

Assume, for simplicity, that we deal with a single known minimum point. An alternate scanning directions of the numerical grid, while solving the quadratic equation (1) for z_{ij} at each visited grid point, would eventually converge. Yet, the rate of convergence depends on the complexity of the surface z_{ij}. If we reconstruct a connected spirals like surface, then there is a need for $O(N)$ scans in the worst case, that yields a complexity bounded by $O(N^2)$, see e.g. [1, 8]. Note that this is a worst case analysis. That is, for alternating scanning directions based methods the complexity is data dependent and ranges between $O(N)$ and $O(N^2)$. For simple surfaces, convergence can be achieved in few iterations.

Assume without loss of generality that $\Delta x = \Delta y = 1$, and initiate all $z_{ij} = \infty$ besides the minimum point that is set to zero. Then, the update step for z_{ij} can be written as the following simple procedure

- Let $z_1 = \min\{z_{i-1,j}, z_{i+1,j}\}$; and $z_2 = \min\{z_{i,j-1}, z_{i,j+1}\}$;
- If $|z_1 - z_2| < f_{ij}$ then $z_{ij} = \frac{z_1+z_2+\sqrt{2f_{ij}^2-(z_1-z_2)^2}}{2}$; else $z_{ij} = \min\{z_1,\ z_2\} + f_{ij}$;

The fast marching method introduces order to these update steps. Points are updated and accepted by their values from small to large. The selection of the smallest point among the set of candidate points and the update of its neighboring grid points involves an $O(\log N)$ worst case complexity, that yields a total of $O(N \log N)$ worst case computational complexity. The order of updates is similar to that of Dijkstra's graph search algorithm [7, 25], and is based on a heap structure of the points at the front. The main difference from Dijkstra's method is the numerical update step. Actually, one may use the finite numerical accuracy to avoid the ordering and reduce the total complexity to $O(N)$.

Our shading image is usually given on a rectangular pixels grid. Therefore, the fast marching method can be directly applied to solve the shape from shading problem with a vertical light source. However, for the general oblique light source, the model to be solved reads $|\nabla z(x, y)| = f(x, y, z(x, y))$. For this more general case, the right hand side depends on $z(x, y)$. The question is how to include this partial differential equation, which is not an Eikonal equation anymore, within the fast marching framework.

Let us focus on the oblique light source case in which the light source direction is different than that of the viewer. The shading image for this case is

$$I(x, y) = \vec{l} \cdot \vec{n} = (l_1, l_2, l_3).\frac{(-z_x, -z_y, 1)}{\sqrt{1 + z_x^2 + z_y^2}}.$$

W.l.o.g. we choose $l_2 = 0$. The shading image is then given by

$$I(x, y) = (l_1, 0, l_3) \cdot \vec{n}, \tag{2}$$

where $l_1^2 + l_3^2 = 1$. Equation (2) involves the term z_x. It requires some additional thought to construct a monotonic approximation to this term and an appropriate update rule.

If we could have the brightness image in the light source coordinates $\tilde{I}(\tilde{x})$, then the problem would have become the vertical light source case, which is given by the Eikonal equation

$$\tilde{z}_{\tilde{x}}^2 + \tilde{z}_y^2 = \frac{1}{\tilde{I}(\tilde{x}, y)^2} - 1. \tag{3}$$

Lee and Rosenfeld [21] suggested the light source coordinates 'to improve' early shape from shading algorithms. Adopting their suggestion, we view the reflectance map as an 'almost' Eikonal equation which can be solved efficiently. In the light source coordinate system, the right hand side depends on the surface itself via

$$\tilde{I}(\tilde{x}, y) = I(l_3\tilde{x} + l_1\tilde{z}, y). \tag{4}$$

That is, we need to evaluate the value of the surface at a point in order to find the 'brightness' and only then plug it to Eq. (4) and use the fast marching method to solve Eq. (3).

In order to overcome this dependence, we use the directional propagation and 'adopt' the smallest $\tilde{z}$value from all the neighbors of the updated grid point. The new update step then reads

- Let $\tilde{z}_1 = \min\{\tilde{z}_{i-1,j}, \tilde{z}_{i+1,j}\}$ and $\tilde{z}_2 = \min\{\tilde{z}_{i,j-1}, \tilde{z}_{i,j+1}\}$;
- Let $k = l_3 i + l_1 \min\{\tilde{z}_1, \tilde{z}_2\}$;
- If $|\tilde{z}_1 - \tilde{z}_2| < f_{kj}$ then $\tilde{z}_{ij} = \frac{\tilde{z}_1+\tilde{z}_2+\sqrt{2f_{kj}^2-(\tilde{z}_1-\tilde{z}_2)^2}}{2}$; else $\tilde{z}_{ij} = \min\{\tilde{z}_1, \tilde{z}_2\} + f_{kj}$;

Where $\tilde{z}_{ij} = z(i\Delta\tilde{x}, j\Delta y)$, and $f_{kj} = f(k\Delta x, j\Delta y)$. Again, w.l.o.g. we assume $\Delta\tilde{x} = \Delta y = 1$, and $f(x, y) = I(x, y)^{-2} - 1$. The numerical algorithm in this case is still consistent, one pass since the smallest $\tilde{z}$ neighbor will never change its value, and is thus within the fast marching framework. The map between the light source coordinates $(\tilde{x}, y, \tilde{z})$ and the image coordinates (x, y, z) is a rotation given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} l_3 & 0 & l_1 \\ 0 & 1 & 0 \\ -l_1 & 0 & l_3 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ y \\ z \end{pmatrix}.$$

We have thereby extended the fast marching method to the case of $|\nabla z| = F(z)$ relevant to the oblique light source shape from shading problem. A consistent solution can be computed with $O(N \log N)$, where N is the total number of pixels (grid points).

We tested the algorithm on a synthetic shading image, see Fig. 1. Observe that we do not deal here with self casting shadows (see [22]), nor with solving the global topological structure (see [2, 8, 17]).

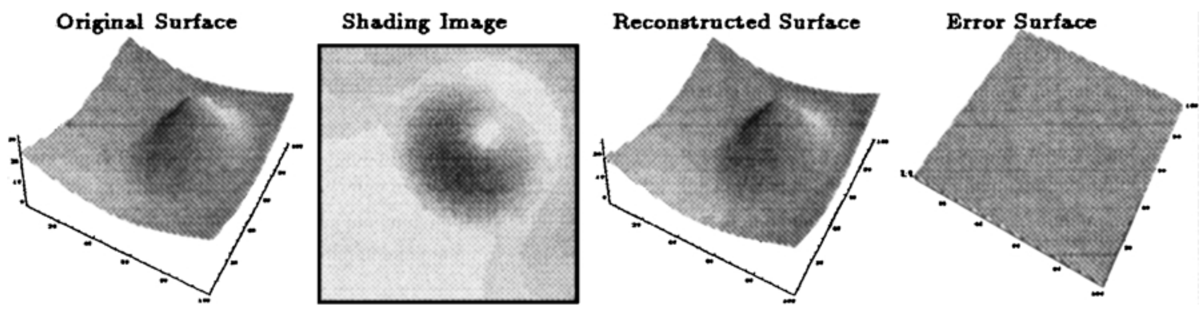

Figure 1. The reconstruction of the surface at the left, from its shading image, with $\vec{l} = (0.2, 0, 0.96)$. Right is the difference between the original surface and its reconstruction.

3. Fast Marching Methods for Path Planning

In this section, we apply the fast marching method to construct optimal paths for navigation problems with relatively few degrees of freedom. We begin with a two-dimensional path planning problem with constraints; and consider a point robot in a uniform domain with obstacles. Here, we treat obstacles by simply setting the possible speed of the robot $g(x, y)$ to be infinity inside the obstacles. Thus, given a starting point A and end point B, the robot is allowed to come arbitrarily close to the obstacles.

The optimal path is defined by the weighted arclength $d\tilde{s}^2 = g(x, y)^2\, ds^2$, where $ds^2 = dx^2 + dy^2$ is the Euclidean arclength, and $g(x, y)$ is the weight over the domain. We search for the path $C(s) = \{x(s), y(s)\}$, with $C(0) = A$ and $C(L) = B$, where L is the total arclength, that minimizes $\min_C \int g(C(s))\, ds$. For an arbitrary parameterization p of the curve, the above geometric functional reads

$$\int_0^1 g(C(p))|C_p(p)|\, dp, \tag{5}$$

for which the Euler-Lagrange geometric equation is

$$\kappa g = \langle \nabla g, \vec{N} \rangle.$$

Where $\kappa = \langle C_{ss}, \vec{N} \rangle$ is the curvature of the curve C, and $\vec{N}$ its normal. In order to link between the shape from shading and the path planning problem we show that the optimal paths are the gradient descent contours of a specific Eikonal equation.

Lemma 1. *The optimal paths between two points A and B, i.e. the curves that minimize Eq. (5) are the gradient descent contours of the function u that satisfies the Eikonal equation*

$$|\nabla u| = g, \quad \text{and} \quad u(A) = 0.$$

Proof: Let us prove that the EL equation $g\kappa = \langle \nabla g, \vec{N} \rangle$ holds for the 'flow lines' of the function u that satisfies $|\nabla u| = g$.

The curve tangent of the flow lines is defined by $\vec{T} = C_s = \frac{\nabla u}{|\nabla u|}$, thus $\vec{N} = \frac{\bar{\nabla} u}{|\nabla u|} = \frac{\{-u_y, u_x\}}{|\nabla u|}$. The curvature is defined as

$$\begin{aligned}
\kappa &= \langle C_{ss}, \vec{N} \rangle = \langle \partial_s \vec{T}, \vec{N} \rangle \\
&= \left\langle \partial_s \frac{\nabla u}{g}, \frac{\bar{\nabla} u}{g} \right\rangle \\
&= \frac{1}{g} \left\langle \left\{ \left\langle \nabla \left(\frac{u_x}{g} \right), C_s \right\rangle, \left\langle \nabla \left(\frac{u_y}{g} \right), C_s \right\rangle \right\}, \{-u_y, u_x\} \right\rangle \\
&= \frac{1}{g^3} \left(u_x u_y (u_{yy} - u_{xx}) + u_{xy} \left(u_x^2 - u_y^2 \right) \right).
\end{aligned}$$

We also have that

$$\begin{aligned}
&\langle \nabla g, \vec{N} \rangle \\
&\quad = \left\langle \nabla g, \frac{\bar{\nabla} u}{g} \right\rangle \\
&\quad = \frac{1}{g} \left\langle \frac{\{u_x u_{xx} + u_y u_{xy}, u_x u_{xy} + u_y u_{yy}\}}{g}, \{-u_y, u_x\} \right\rangle \\
&\quad = \frac{1}{g^2} \left(u_x u_y (u_{yy} - u_{xx}) + u_{xy} \left(u_x^2 - u_y^2 \right) \right). \qquad \square
\end{aligned}$$

We just showed that by solving the Eikonal equation for the function u anchored at the initial location, we can backtrack the optimal path by flowing along the gradient descent of that function from any given 'final' location. Note that for $g = 1$, the function u is the distance map and the flow lines are straight lines, which are indeed the optimal paths between two points in a Euclidean domain. This backtracking idea was used in [4, 6, 14] for finding the optimal paths in 2D and 3D domains.

Next, we add one more dimension to the problem and allow the robot to rotate. Instead of the point robot, we now consider a two-dimensional rectangle with a given width and length. Thus, the initial position A of the robot in the configuration space is specified by the position of the center of the rectangle, plus an orientation angle θ between 0 and 2π. The final configuration B is similarly specified, and the goal is to construct the optimal path from A to B, that minimizes the integration along the arclength $ds^2 = dx^2 + dy^2 + d\psi^2$.

In the absence of obstacles, a completely straightforward application of the fast marching method is possible; one discretizes the configuration space into a three-dimensional grid, that is, we similarly grid both $\mathbb{R}^2$ and θ between 0 and 2π, employing periodic boundary conditions in θ. Then, we solve the Eikonal equation

$$\left[u_x^2 + u_y^2 + u_\theta^2 \right]^{1/2} = 1, \tag{6}$$

with $u(A) = 0$. Back tracking along the gradient descent of u yields the optimal path. In the presence of obstacles, we first apply the classical shrinking approach, where the robot is shrank into a point. For every discretized angle θ_i, we dilate the shape of the obstacles with a 'structuring element' corresponding

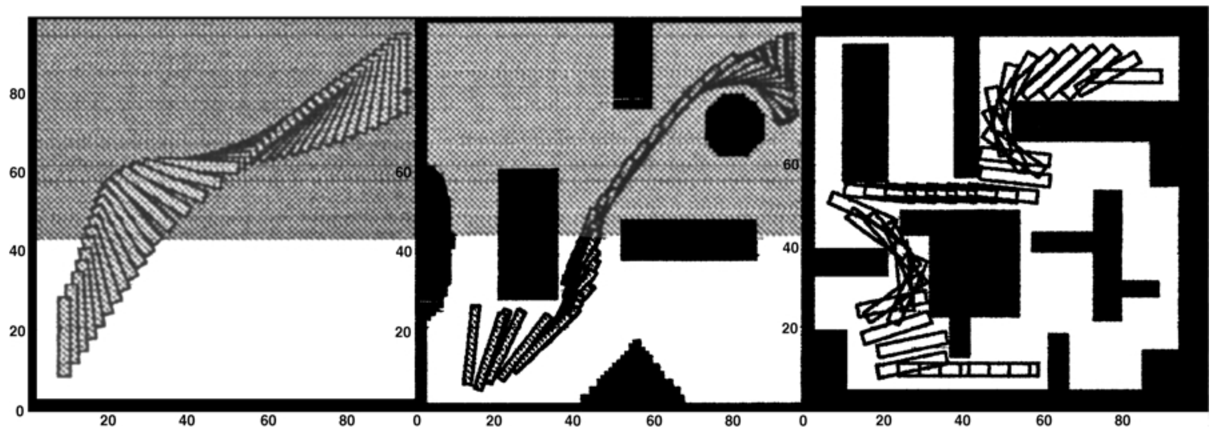

Figure 2. Navigating a rectangle with rotation in 2D (3DOF).

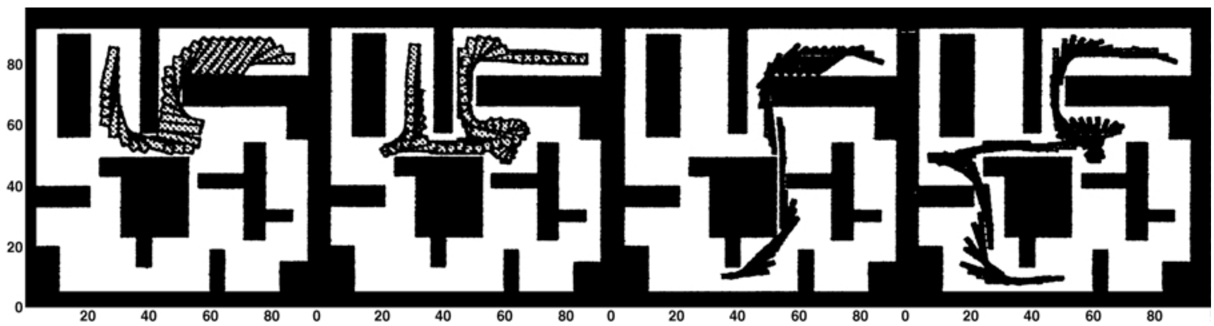

Figure 3. Path planning under different potentials of the free configuration space.

to the robot at the given angle. This 3D configuration space construction can be done in $O(N)$ using the fast morphological operations methods in [9]. In Fig. 2, we show several examples of a two-dimensional robot with rotational angle, navigating in a plane with obstacles.

We now turn to examples in which the goal is to compute the optimal path, and imagine trying to not simply avoid obstacles, but to navigate under a potential function which weights the free work space [13, 20], or even the whole free C-space. Now, the "tuneven" terrain weighting is determined by the distance from the obstacles.

Figure 3 presents the effect of the potential defined in C-space on the course of the optimal path. We have tested several potentials that are functions of the distance map: a constant potential $P_1(x, y, \theta) = 1$, and $P_1(x, y, \theta) = d^8(x, y, \theta)$, where $d(x, y, \theta)$ is the distance from the point (x, y, θ) to the closest obstacle. The solution to the Eikonal equation $|\nabla u(x, y, \theta)| = 1/(1 + P(x, y, \theta))$ via the fast marching method yields the desired input for a back propagation (via an inverse gradient descent) procedure. We note, that in this case, we used the fast marching method itself to determine the value of the distance map $d(x, y, \theta)$ by computing the distance map $d(x, y, \theta)$ in the $R^2 \times S^1$ C-space.

Figure 4 presents the optimal path for a four joints arm robot (4 DOF). The optimal path is defined via the arclength $ds^2 = d\psi_1^2 + d\psi_2^2 + d\psi_3^2 + d\psi_4^2$, where ψ_i is the angle between the stick i, and stick $i - 1$ along the robot arm. In the appendix we derive a useful algorithm for the update stage of the fast marching numerical method for an n dimensional Eikonal equation.

Finally, we present results of an extension of the fast marching method to non-rectangular grids that was presented in [18]. The method extends the numerical monotonic approximation to the Eikonal equation through a geometric interpretation of the update procedure. We applied this method to path planning on curved domains with possible weights, and to

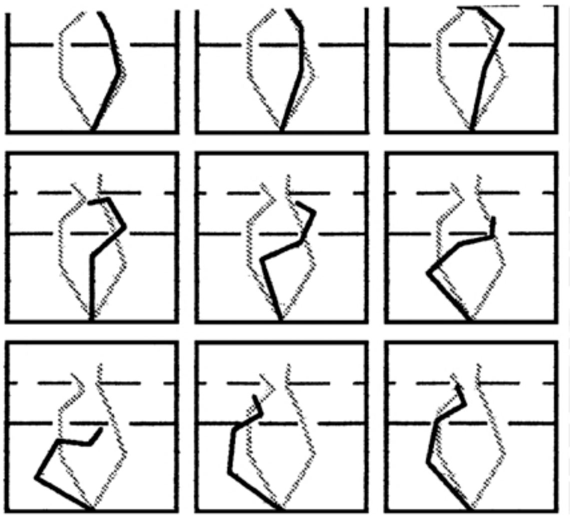

Figure 4. Navigation with 4 DOF: Left to right top to bottom.

computation of Voronoi diagrams, and offset curves on non-flat triangulated domains [19]. Figure 5 presents minimal geodesics as optimal paths, Voronoi diagram and offset curves on a curved triangulated surface. Figure 6 presents minimal geodesics as optimal paths on weighted (right) and unweighted curved surfaces (left). The darker the texture on the left, the lower the cost or weight.

4. Conclusion

We presented an optimal algorithm for surface reconstruction from its shading image. The computational complexity upper bound, of $O(N \log N)$, is data independent. It is the most efficient sequential algorithm for Horn's original formulation of the shape from shading problem and a natural extension and application of the fast marching method. Next, we showed how the fast marching method can be applied to path planning problems as a consistent numerical method for locating optimal paths in configuration spaces with small number of dimensions. These two seemingly unrelated problems and their optimal solutions are samples of the wide range of possible applications for the fast marching method and its extensions.

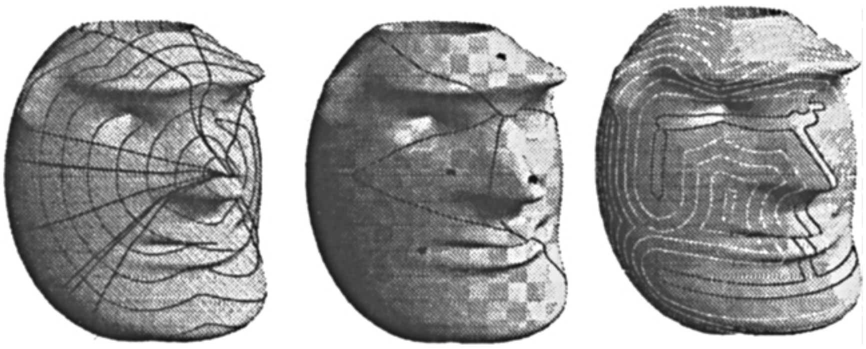

Figure 5. Fast marching on triangulated surfaces: Minimal geodesics, Voronoi diagram, and geodesic offsets.

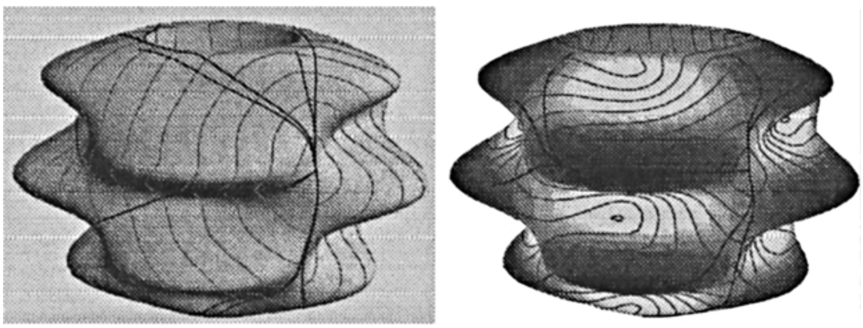

Figure 6. Fast marching on triangulated surfaces: Minimal geodesics, and minimal weighted geodesics.

Appendix

Let us derive a simple and useful procedure for the update stage of the fast marching numerical method for an n dimensional Eikonal equation. Consider the numerical approximation to the Eikonal equation $|\nabla u| - F = 0$ in n dimensions that is given by

$$\sum_{k\in\{1,\dots,n\}} \max(D^{-k}, -D^{+k}, 0)^2 - F^2 = 0.$$

Where D^{-k} and D^{+k} denote the backwards and forward partial derivatives with respect to the k coordinate (e.g. $D^{-1} = u_{i,j} - u_{i-1,j}$ for the 2D case with $\Delta x = 1$).

We need to solve for the largest $u_{1,\dots,n} = t$, that is a root of the above quadratic equation. Let us rewrite the equation, and again search for the largest t that satisfies

$$\sum_{k\in\{1,\dots,n\}} \max(t - \min(u_{k-1}, u_{k+1}), 0)^2 - F^2 = 0,$$

where u_{k-1} indicates the value at the -1 point along the k coordinate (e.g. $u_{1-1} \equiv u_{i-1,j}$ in the 2D case).

Let $a_k = \min(u_{k-1}, u_{k+1})$, where the min is taken along the k coordinate (e.g. $a_2 \equiv \min(u_{i,j-1}, u_{i,j+1})$ in the 2D case). In case there exists a root bigger than all a_k, the solution is the root of the quadratic equation

$$\sum_{k\in\{1,\dots,n\}} (t - a_k)^2 - F^2 = 0,$$

and is given by

$$t = \frac{1}{n}\left(\Sigma_k a_k + \sqrt{nF^2 + \Sigma_k \Sigma_l a_k a_l - n\Sigma_k a_k^2}\right).$$

Let $a_M = \max_k(a_k)$. Inserting the demand $t > a_M$ in the above equation (t should be larger than all the values that generate it) we end up with the following simple condition

$$F^2 > \Sigma_k (a_M - a_k)^2.$$

Then, we readily obtain the following lemma that help us derive a simple selection procedure for the update stage of the fast marching method.

Lemma 2. *Under the condition $F^2 > \Sigma_k(a_M - a_k)^2$ the discriminant of Eq.* (7) *is greater than zero*: $nF^2 + \Sigma_k \Sigma_l a_k a_l - n\Sigma_k a_k^2 > 0$. *I.e. the above t is a legitimate value that satisfies $t > a_M$.*

Proof: Let $F^2 = \Sigma_k(a_M - a_k)^2$. Then the discriminant of the cubic root of Eq. (7) is given by

$$\begin{aligned} &nF^2 + \Sigma_k \Sigma_l a_k a_l - n\Sigma_k a_k^2 \\ &\quad = n\Sigma_k (a_M - a_k)^2 + \Sigma_k \Sigma_l a_k a_l - n\Sigma_k a_k^2 \\ &\quad = n\Sigma_k\left(a_M^2 - 2a_M a_k + a_k^2 - a_k^2\right) + \Sigma_k \Sigma_l a_k a_l \\ &\quad = n\Sigma_k\left(a_M^2 - 2a_M a_k\right) + \Sigma_k \Sigma_l a_k a_l \\ &\quad = n^2 a_M^2 - 2na_M \Sigma_k a_k + \Sigma_k \Sigma_l a_k a_l \\ &\quad = (na_M)^2 - 2(na_M)(\Sigma_k a_k) + (\Sigma_k a_k)^2 \\ &\quad = (\Sigma_k a_k - na_M)^2 > 0 \qquad \square \end{aligned}$$

Based on the above lemma, an efficient way of applying the n dimensional update step of the numerical approximation to Eikonal equation via the fast marching method is given as follows:

1. Compute $\{a_k\}$, and sort them in an increasing order, so that a_1 is the smallest, and a_n is the largest value. Let $m = n$.
2. While $F^2 < \sum_{k\in\{1,\dots,m\}} (a_m - a_k)^2$ do $m = n - 1$.
3. Compute the updated value

$$t = \frac{1}{m}\left(\Sigma_k a_k + \sqrt{mF^2 + \Sigma_k \Sigma_l a_k a_l - m\Sigma_k a_k^2}\right),$$

where the $\sum$ indexes are $\{1, \dots, m\}$. Let the point value be t, if its less than its current value.

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