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Optimal and instantaneous control of the instationary Navier-Stokes equations

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Contents

CHAPTER 1

Introduction

1. Aims and scope of flow control

A general optimal control problem may be subdivided into the following interrelated parts [**91**]:

(1) **Problem statement**
Definition of the goal, cost function or performance index

(2) **State estimation problem**
Knowledge of the current state of the system

(3) **Modelling and system identification**
Knowledge of how the environment effects the past, present and future of the system

(4) **Optimization**
Determination of the best control policy based on parts 1., 2. and 3.

Of course, for control problems in fluid flow this frame has a mathematical and an engineering component. From the engineering point of view it often is desirable to minimize drag, increase mixing, reduce turbulent kinetic energy and so forth in a channel, say. Engineers measure these quantities using micro-electronical devices (MEMS), heated wires or particle image velocimetry (PIV), and they impose control actions by blowing and suction, movement or heating of walls, application of volume forces by electric fields or variations of the shape, just to mention a few of various possibilities. Moreover, the fluid is characterized as incompressible or non-Newtonian and its behaviour is periodic with respect to time or on (portions of) the boundary, just to count for a small number of possible constellations.

In order to apply mathematical approaches to the control of fluid flow these terminologies have to be translated into mathematical language. The first question to pose is how to model fluid flow mathematically. For this purpose first of all the region of fluid flow has to be specified and to be made accessible to a mathematical description. This is, from the theoretical point of view, easy. Modelling fluid flow clearly depends on the physical properties of the fluid. Most commonly, the fluid is assumed to be incompressible, so that its state can approximately be described by the (instationary) Navier-Stokes equations. Once such a model is available, physical terms like drag and turbulent kinetic energy may be expressed in terms of the model variables and thus can be customized for mathematical performance indexes. The same holds true for control actions. Finally, blowing and suction or movement of walls may be regarded as boundary conditions for the flow variables, external forces applied in the domain of flow as inhomogeneities.

The choice of the control policy depends on the control target and the environment. If the system to be controlled is shielded against external influences, it can be desirable to provide a time dependent control function that, for example steers the system from a given state to a desired one. This would correspond to an optimal (open-loop) control problem (find the best time dependent control function for the specified environment). A more general form of control activity is (optimal) closed-loop or feedback control (find the best feedback control law). It allows for feeding back into the system control information obtained by currently available state information or estimation. This is a much more general concept than optimal control and, at least optimal feedback, also much more complicated to determine.

In the first part this work is concerned with optimal distributed control of the full Navier-Stokes System and with instantaneous control in its second part, which is a suboptimal variant of feedback control.

The distributed control problem considered in the first part of this work is of the form

$$(1)\qquad \begin{cases} \min\ J(y,u) & \\ \text{s.t.} & \\ \frac{\partial y}{\partial t} + (y\cdot\nabla)y - \nu\Delta y + \nabla p = Bu & \text{in } \Omega^T, \\ -\text{div}\, y = 0 & \text{in } \Omega^T, \\ y(t,\cdot) = 0 & \text{on } (\partial\Omega)^T, \\ y(0,\cdot) = y_0 & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ denotes a bounded domain and $\Omega^T := (0,T)\times\Omega$. Since for appropriate controls Bu there is a one-to-one correspondence between state and controls there are in principal two possibilities to tackle this control problem: As unconstrained minimization problem for the functional $\hat{J}(u) = J(y(u),u)$, or as constraint optimization problem, treating the Navier-Stokes equations as explicit constraints. Up to now in most approaches the first possibility has been utilized, see for example [**1, 33, 39**]. In this work the focus is on the second approach. In Chapter 3 and Chapter 4 the results obtained in [**53**] for the optimization problem (1) are extended to a more general analytic frame and the discussion of further second order algorithms is supplemented. In Chapter 3 the concise analytical frame for the constraint minimization approach to (1) is developed. The decoupling of state and control in the problem formulation among other things allows an elegant derivation of first and second order derivatives of the functional $\hat{J}$. The main results of Chapter 3 are contained in Theorem 1.1 (Existence of solutions), Proposition 2.2 (properties of the linearized Navier-Stokes equations and their adjoint) and Theorem 3.1 (Existence of Lagrange multipliers). In Section 4 second order analysis for the functional $\hat{J}$ and the Lagrangian L associated to the above optimization problem is provided. These results are utilized in Chapter 4 to prove local convergence of Newton's method (Theorem 1.1) and of the BFGS method (Theorem 2.1) applied to the unconstraint minimization problem for $\hat{J}$, as well as for the prove of local convergence of the basic SQP method (Theorem

3.1) and its Schur-complement variant (Theorem 4.1), the reduced SQP method (Theorem 5.1) and the reduced SQP-BFGS method (Theorems 6.1, 6.2), when applied to numerically solve the constraint minimization problem. The numerical results for Newton's method applied to the control of the driven cavity flow are presented in Section 7. Moreover, in the same section a numerical comparison between Newton's method and gradient-type algorithms with step size control is provided. It shows that despite its complex inexact implementation Newton's method needs much less cpu-time in this numerical example. However, for more complex flows like unsteady flow around a circular cylinder the numerical solution of the optimal control problem utilizing Newton's method needs an initialization step to provide a suitable initial guess [**2**]. This is clearly due to the local nature of Newton's method and suggests the use of hybrid first-order second-order methods or the development of globalization strategies for second order methods for optimal control of the instationary Navier-Stokes equations.

For many applications computing optimal controls is very time consuming and often far from being achievable with currently available computing facilities. This in particular holds true for the control of realistic three dimensional flows in medium to large sized control horizons and also remains valid for two-dimensional instationary flows such as those around a cylinder or over a backward facing step, say or the selective cooling of steel. In order to circumvent the mentioned bottleneck suboptimal control concepts have been developed. Their aim consists in providing good controls with reasonable effort. For the mentioned applications suboptimal control concepts prove very powerful [**14, 18, 49, 62, 106**].

The suboptimal control method discussed in the second part of this work is called instantaneous control. It is a simplification of model predictive control [**31**], which often is also referred to as receding horizon control [**64, 92, 99**]. Receding horizon control can best be explained at hand of a chess analogy: One computes the optimal next n moves, say with respect to the current situation in the game. Then one applies the first of these moves to continue the game and repeats the procedure. Instantaneous control in this context would compare to a variant of rapid chess: apply an approximation of the first move for $n = 1$.

For the control problem (1) the instantaneous control strategy works as follows. The Navier-Stokes equations are discretized with respect to time. Then, at every time instance a stationary control problem with a instantaneous version of the functional J is approximately solved by applying exactly one gradient step to its solution.

In Chapter 5 the instantaneous control strategy is developed at hand of the linear-quadratic regulator problem for finite dimensional plants of the form

$$\dot{y} + Ay = b(y) + u, \quad y(0) = y_0, \tag{2}$$

see also [**46**]. It turns out that instantaneous control can be interpreted as discrete-in-time control approach to (2), which in turn may be regarded as the stable discretization of a continuous control law $u = Ky$ for the plant (2) (Theorem 4.1). In Section 5 conditional stability of the discrete and the continuous control laws is proved in Theorems 5.1, 5.2, 5.3, 5.4, 5.5 and 5.6. In Chapter 6 the results obtained in [**54**] for the instantaneous control strategy applied to tracking-type control of the instationary Burgers equation are extended to the control of the instationary Navier-Stokes equations. In the solenoidal setting the controlled Navier-Stokes system, similar to the finite dimensional case, then is of the form

$$\dot{y} + Ay = b(y) + K(y), \quad y(0) = y_0, \tag{3}$$

with a nonlinear control operator $K(y)$. Existence of a unique solution to (3) is proved in Theorem 2.1. Moreover, in Theorems 3.1 and 3.2 exponential decay of the state to the desired state in terms of the parameters defining K is proved. The corresponding results for the related discrete-in-time control law are given in Theorems 4.1, 4.2 and Theorem 4.3. In Section 5 these results are numerically validated. In Chapter 7 instantaneous control is investigated from a more practical point of view. For the backward facing step flow serving as model problem the method is utilized to construct a boundary observation based closed-loop feedback boundary control mechanism in order to reduce the re-attachment length of the flow (Algorithms 3.2 and 4.1). In Section 5 the performance of the controller is numerically illustrated. Further details concerning this example can be found in [**19**].

Chapter 2 presents the notations and preliminary results. Theorem 2.1 and Proposition 2.1 are of particular importance for the analysis presented in Chapters 3 and 4. They contain the a-priori estimates for the Navier-Stokes equations and for the linearized Navier-Stokes equations and their adjoints. A proof of the latter is given in Appendix 2. In Appendix 1 the formal derivation of the optimality system for a general optimization problem for the instationary Navier-Stokes equations is provided.

2. A brief review in active flow control

In active flow control optimal and suboptimal control approaches are utilized to compute controls. The following subsections contain a selection of references to the literature for both control branches.

2.1. Optimal control. In [**1**] Abergel and Temam apply optimal control theory to some fluid mechanics problems and derive optimality conditions for various cost functionals. As solution procedure they propose gradient-type methods. A similar approach is taken by Glowinski in [**33**]. Optimal control strategies with focus on turbulent flows are discussed by Bewley, Choi, Temam and Moin in [**13**], compare also [**14, 105**]. In [**16, 17**] Chang and Collis combine large eddy simulation and flow control and obtain numerical results similar to that presented in [**14**], but at significantly lower numerical costs. Together with Ziane, Bewley and Temam extended the optimal control concept for the Navier-Stokes equations to robust control in [**15**]. Gunzburger and Manservisi investigate optimization problems for the instationary Navier-Stokes equations with distributed controls in [**39, 37**], and for boundary controls in [**40**]. See also [**56**] and the PhD [**87**] of the latter author. The long-time behaviour for distributed optimal control of the Navier-Stokes equations is investigated by Hou and Yan in [**59**]. They compare the optimal trajectory to that produced by a simple distributed closed-loop control law, for whose decay can be estimated. In [**12**] Berggren presents a fully discrete boundary control problem for the instationary Navier-Stokes equations and utilizes a limited memory BFGS approach for its numerical solution. Justen investigates control problems for the thermally coupled Navier-Stokes equations in [**67**], and Clerc, Le Tallec, Mallet, Ravachol and Stoufflet optimal control of the parabolized Navier-Stokes system in [**21**]. Joslin, Gunzburger, Nicolaides, Erlebacher and Hussaini in [**66**] present an optimal boundary control strategy for boundary layers which is based on tracking a prescribed normal stress distribution along an observation boundary. Protas and Styczek derive a boundary control problem with state observation on the boundary for the wake flow around a circular cylinder in [**94**]. Kunisch and the author applied a gradient algorithm to compute boundary controls for a backward facing step flow in [**50**] and compared the result to that obtained by the instantaneous control strategy. In [**52**] the same authors discuss Newton's method and present a convergence proof for the distributed optimal control problem. A general analytic frame for the distributed control problem treating the Navier-Stokes equations as explicit constraints is given by Kunisch and the author in [**53**]. The same source also contains convergence proofs for various optimization algorithms applied to the solution of the optimal control problem.

For a more analytical approach to control problems for the instationary Navier-Stokes equations see for example the work [**30**] by Fursikov, Gunzburger & Hou, and that by Fursikov [**29**], Coron [**23**] and Farbre [**27**] concentrating on controllability questions, as well as the references cited therein.

Investigations concerning optimal control of the stationary Navier-Stokes equations are presented by Ghattas and Bark in [**32**], where they apply Newton and quasi-Newton SQP techniques to the solution of optimal control problems for the stationary Navier-Stokes equations in two and three dimensions and compare their results to them obtained by the classical gradient method. Further investigations on control problems for the stationary Navier-Stokes equations are provided by Desai and Ito in [**25**], by Gunzburger, Hou and Svobodny in [**36, 57**], by Málek and Roubiček in [**86**], by Loncarcic for the Stokes equations in [**83**] and by Heinkenschloss in [**43**]. In [**74**] Kunisch and Marduel numerically investigate the optimal control problem with thermal controls for non-isothermal viscoelastic fluids in a suddenly expanding channel. An excellent overview on diverse aspects of optimal control for fluids is given by Gunzburger in [**35**], and also by Sritharan in [**101**].

2.2. Suboptimal control. In the recent past two suboptimal approaches to the control of fluids have been proposed, namely receding horizon control and control by system reduction. The idea of receding horizon control techniques is to replace the open-loop optimal control problem on the full time horizon by a sequence of optimal control problems on short control horizons that move forward in time. Control by system reduction keeps the control horizon fix and replaces the full Navier-Stokes system by a suitable low order model, for which the optimal control problem then is solved exactly. The resulting reduced optimal control is used as control function for the full system.

2.2.1. *Receding horizon control, instantaneous control and other feedback control concepts.* The concept of instantaneous control has been introduced as a suboptimal control procedure for the instationary Burgers equation by Choi, Temam, Moin and Kim in [**20**]. From then onwards this control concept has been utilized in a great number of numerical approaches to the control of fluid flow. In [**18**] Choi successfully applies instantaneous control to unsteady flows around a circular cylinder. The same author together with Min presents boundary control concepts which only utilize state information on the cylinder surface for the same flow configuration in [**88**]. In [**81**] Choi together with Kim and Lee applies instantaneous control to drag reduction for channel flows, and in [**19**] together with Kunisch and the author to reduce the re-attachment length of a laminar backward facing step flow. Moreover, in the same work the interpretation of the method as closed-loop control approach utilizing boundary measurements only is given. Related work of Kunisch and the author can be found in [**51**], and in [**47**] Kauffmann and the author apply instantaneous control to the unsteady flow around a circular cylinder. In [**14**] Bewley, Moin and Temam use receding horizon control to re-laminarize a turbulent channel flow, and Chang and Collis in [**17**] use large eddy simulation combined with receding horizon control to reproduce the numerical results obtained by Bewley et al, but at significantly lower numerical costs. Kunisch and Marduel apply instantaneous control with thermal controls to transient non-isothermal viscoelastic fluids in [**75**].

There are only a few analytical results available for receding horizon type control techniques applied to control fluid flow. In [**58**] Hou and Yan prove stability for the (1,1)-receding horizon approach with distributed controls for a tracking-type problem. Here, (1,1) means that the length of the control horizon coincides with the step size of time integration. For the instantaneous control technique with distributed controls similar results are proved by Volkwein and the author for the instationary Burgers equation in [**54**]. The results presented there are extended to the instationary Navier-Stokes equations in Chapter 6 of this work. An interesting observation is made by Heinkenschloss in [**41**] where he shows that the first step of the Gauß-Seidel method applied to the numerical solution of quadratic optimization problems for discretized parabolic equations is equivalent to the application of instantaneous control to the parabolic equation, compare also [**42**].

A different feedback approach is presented by Ito and Kang in [**60**], where they describe an approach to obtain suboptimal solutions to the optimal feedback control problem for systems governed by the Burgers and the Navier-Stokes equations. Heuristic feedback control approaches to the instationary Navier-Stokes equations with distributed controls are investigated by Gunzburger and Manservisi in [**38**], and by Hou and Yan in [**59**]. Joshi, Speyer and Kim present a feedback control approach to stabilize disturbances in a plane Poiseuille flow in [**65**]. In a series of papers [**71, 72, 73**] Koumoutsakos develops a closed-loop boundary control mechanism for fluid flows which utilizes pressure information on the boundary only and applies it to control the vortex dipole. In [**80**] Lee, Kim, Babcock and Goodmann apply neural networks to turbulence control for drag reduction.

2.2.2. *System reduction and control.* In the system reduction approach to the control of the instationary Navier-Stokes equations a Galerkin approximation for the spatial approximation of the full Navier-Stokes equations is used that utilizes global basis functions that contain characteristics of the expected controlled flow and represent the dynamics of the flow. This is in contrast to finite element based Galerkin schemes where the basis elements are not related to the physical properties of the system that they approximate. Consequently one expects that only a few basis elements will provide a reasonably good description of the controlled flow. The result is a low order model for the Navier-Stokes equations which then is used as subsidiary condition in the optimization process. The control computed for the reduced system is used as suboptimal control for the full equations.

Various reduced system approaches differ in the choice of the basis functions. In the following the approach is sketched for the control problem (1) utilizing basis functions obtained by the snapshot variant of proper orthogonal decomposition (POD) introduced by Sirovich in [**100**]. To begin with denote by $y^1, \dots, y^n$ an ensemble of snapshots of the flow corresponding to the time instances $t_1, \dots, t_n$ and make for the flow the Ansatz

$$y = \bar{y} + \sum_{i=1}^{n} \alpha_i \Phi_i \tag{4}$$

with modes Φ_i and the mean $\bar{y}$ that are obtained as follows:

(1) Compute mean $\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y^i$
(2) Build correlation matrix $K = k_{ij}, k_{ij} = \int_{\Omega} (y^i - \bar{y})(y^j - \bar{y})\, dx$
(3) Compute eigenvalues $\lambda_1, \dots, \lambda_n$ and eigenvectors $v^1, \dots, v^n$ of K
(4) Set $\Phi_i := \sum_{j=1}^{n} v^i_j (y^j - \bar{y})$
(5) Normalize $\Phi_i = \frac{\Phi_i}{||\Phi_i||}$

The functions $\Phi_i, i = 1, \dots, n$ constructed in this way are called modes. They are mutually orthonormal and optimal in terms of their ability to represent the flow kinetic energy [**100**], i.e. they are optimal with respect to the L^2 scalar product in the sense that no other basis of $D :=$ span$\{y_1 - \bar{y}, \dots, y_n - \bar{y}\}$ can contain more energy in fewer elements, compare [**4, 77**]. The modes are used as test functions in the Galerkin formulation of the Navier-Stokes system. Note, that as linear combinations of the snapshots the modes are divergence free. In order to obtain a low-dimensional basis for the Galerkin Ansatz modes corresponding to small eigenvalues are neglected.To make this idea more precise let $D^M :=$ span$\{\Phi_1, \dots, \Phi_M\}$ $(1 \leq M \leq N :=$dim$D)$ and define the relative information content of this basis by

$$\mathrm{I}(M) := \sum_{k=1}^{M} \lambda_k \,/\, \sum_{k=1}^{N} \lambda_k .$$

If a basis is required that contains $\kappa\%$ of the total information contained in the space D, say the dimension M of the subspace D^M is determined by

$$M = \text{argmin}\left\{\mathrm{I}(M);\ \mathrm{I}(M) \geq \frac{\kappa}{100}\right\}.$$

The reduced dynamical system is obtained by plugging in (4) into the Navier-Stokes System and using a subspace D^M containing sufficient information as test space. This results in

$$(y_t, \Phi_j) + \nu(\nabla y, \nabla\Phi_j) + ((y\nabla)y, \Phi_j) = (Bu, \Phi_j) \quad (j = 1, \dots, M).$$

which may be rewritten as

$$\dot{\alpha} + A\alpha = n(\alpha) + \beta + r, \quad \alpha(0) = a_0.$$

Here the initial values y_0 are approximated by $\bar{y} + \sum_{k=1}^{M}(y_0 - \bar{y}, \Phi_k)\Phi_k$. The matrix A is the POD stiffness matrix and the inhomogeneity r results from the contribution of the mean $\bar{y}$ to the Ansatz in (4). The M-vector β results from the contribution of the controls, which are also sought in the space D^M, i.e. for them the Ansatz

$$u = \sum_{i=1}^{M} \beta_i \Phi_i \tag{5}$$

is made. The reduced optimization problem corresponding to (1) is obtained by plugging in (4) and (5) into the cost functional and utilizing the reduced dynamical system as subsidiary condition in the optimization process. Altogether one obtains

$$(\text{ROM}) \begin{cases} \min J(y,u) = J(\alpha, \beta) \\ \text{s.t.} \\ \dot{\alpha} + A\alpha = n(\alpha) + \beta + r, \quad \alpha(0) = a_0, \end{cases}$$

and every optimal control $u^* = \sum_{i=1}^{M} \beta_i^* \Phi_i$ of the reduced optimization problem is considered as suboptimal control for problem (1). Of course, this approach can be extended to obtain suboptimal boundary controls.

There are a few contributions to the control of the instationary Navier-Stokes equations using system reduction. In [**102**] Tang, Graham and Peraire apply the snapshot variant of POD to the control of the unsteady wake flow around a cylinder. In their approach control is introduced into the system via cylinder rotation. Arian, Fahl and Sachs present a sequential POD boundary control approach for fluid flows which successively improves the low order model and apply it to the control of driven cavity flows. Their approach takes up an idea presented by Afanasiev and the author in [**2**] for adaptive distributed control of a wake flow using proper orthogonal decomposition. In [**98**] Ravindran uses the snapshot POD approach for boundary control of the backward-facing step flow.

System reduction and control utilizing the reduced basis method is introduced by Ito and Ravindran in [**63, 62**] where it is applied to the control of the flow over the backward facing step gouverned by stationary Navier-Stokes equations. In [**98**] Ravindran extends the method to boundary control of the instationary Navier-Stokes equations and compares his numerical results for the backward-facing step flow to that obtained by the snapshot variant of the method. In [**49**] Kunisch and the author also present numerical experiments for boundary control of the backward facing step flow utilizing the reduced basis approach.

Further applications of system reduction and control are given by Atwell and King in [**4, 5**], by the author and Kauffmann in [**47, 48**], by Kunisch and Volkwein in [**77, 78**], by Prudhomme and Le Letty in [**95**], by Ly and Tran in [**84, 85**], and by Volkwein in [**107**].

CHAPTER 2

Notation and preliminary results

1. Quasi-Stokes problems

In this work the following results concerning Stokes- and quasi-Stokes problems are frequently used. Parts of the exposition follow the book of Constantin/Foias [**22**].

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $\partial\Omega \in C^2$ with outward unit normal η (some of the results presented below would only require $\partial\Omega$ locally Lipschitz). Let $H^m(\Omega)^k$ $(0 \leq m < \infty)$ for $k \in \mathbb{N}$ denote the Sobolev space of measurable square integrable vector functions with values in $\mathbb{R}^k$, whose derivatives up to order m are measurable and square integrable, equipped with the norm $|\cdot|_{H^m(\Omega)^k}$. The closure of $C_0^\infty(\Omega)^k$ in $H^m(\Omega)^k$ is denoted by $H_0^m(\Omega)^k$, the dual of the latter space is denoted by $H^{-m}(\Omega)^k$. Denote by Δ the (2×2) matrix operator containing the Laplacian in its diagonal, and zeros, else. Let $f \in H^{-1}(\Omega)^2$. In primitive variables (y, p) the quasi-Stokes equations are given by

$$\begin{aligned} y - \nu\Delta\, y\, + \nabla p = f \quad & \text{in } \Omega \\ -\operatorname{div} y = 0 \quad & \text{in } \Omega \\ y = 0 \quad & \text{on } \partial\Omega, \end{aligned} \tag{6}$$

with $\nu = 1/\mathrm{Re}$ denoting the viscosity parameter. In order to pose (6) in a proper functional analytic setting introduce the solenoidal spaces

$$H = \overline{\{v \in C_0^\infty(\Omega)^2 :\ \operatorname{div} v = 0\}}^{|\cdot|_{L^2(\Omega)^2}} \text{ and } V = \overline{\{v \in C_0^\infty(\Omega)^2 :\ \operatorname{div} v = 0\}}^{|\cdot|_{H^1(\Omega)^2}},$$

where the superscripts denote closures in the respective norms. The space H endowed with the usual $L^2(\Omega)^2$-inner product $(\cdot,\cdot)$ is a Hilbert space, and with the scalar product

$$(u, v)_V \ := \int_\Omega \nabla u\, \nabla v\, dx$$

V also becomes a Hilbert space. Then, $V \hookrightarrow\hookrightarrow H \hookrightarrow V^*$ dense.

DEFINITION 1.1. The vector function y is called a weak solution of the quasi-Stokes equations (6) if

$$\begin{cases} y \in V \quad \text{and} \\ (y, v) \ + \ \nu(y, v)_V \ = \ f(v) \quad \text{for all } v \in V. \end{cases} \tag{7}$$

REMARK 1.1. If y is a weak solution then there exists a function $p \in L^2(\Omega)$ such that the couple (y,p) solves (6) in the variational sense and for every variational solution $(y,p) \in H_0^1(\Omega)^2 \times L^2(\Omega)$ to (6) the function y is a weak solution, i.e. satisfies (7) [**22**, Propositon 2.1],[**103**, Ch. I, Theorem 2.1].

THEOREM 1.1. For every $f \in V^*$ there exists a unique weak solution y to the quasi-Stokes equations.

Proof. Follows immediately from the Lax-Milgram theorem. □

From [**103**, Ch. I, Proposition 2.3] one has the following regularity results for solutions to (6).

THEOREM 1.2. Let $m \geq -1$, Ω be of class $C^r(r = \max(m+2,2))$ and f $\in H^m(\Omega)^2$. Then there exists a unique solution $(y,p) \in H^{m+2}(\Omega)^2 \cap V \times H^{m+1}(\Omega) \cap L^2(\Omega)/\mathbb{R}$ to (6). Moreover,

$$|y|_{H^{m+2}(\Omega)^2} + |p|_{H^{m+1}(\Omega)/\mathbb{R}} \leq C\,|f|_{H^m(\Omega)^2}.$$

DEFINITION 1.2. Let $P : L^2(\Omega)^2 \to H$ denote the Leray projector [**22**, Remark 1.10]. Then, the Stokes operator S is given by

$$S : \mathcal{D}(S) \subset H \to H, \quad S := -P\Delta, \quad \mathcal{D}(S) = H^2(\Omega)^2 \cap V. \tag{8}$$

The Stokes operator has the following properties.

THEOREM 1.3. The operator S is selfadjoint and its inverse, S^{-1}, is a compact operator on H. There holds

$$|v|_{H^2(\Omega)^2} \leq C\,|Sv|_{L^2(\Omega)^2} \quad \text{for all } v \in \mathcal{D}(S).$$

Proof. [**22**, Theorem 4.3, 4.4, Proposition 4.7]. □

Utilizing the Stokes operator the quasi-Stokes equations may be formulated as

$$y + \nu S y = f.$$

Although this formulation and, more general the usage of the Stokes operator only makes sense for right-hand-sides $f \in L^2(\Omega)^2$ and functions $y \in \mathcal{D}(S)$ it is for notational reasons also utilized in situations which do not match these requirements. However, it will follow from the specific context in which sense the equation has to be understood.

2. Instationary Navier-Stokes equations in 2d

The Navier-Stokes equations in primitive variables (y,p) are given by

$$\begin{aligned} y_t + (y\nabla)y - \nu\Delta y + \nabla p = f &\quad \text{in } \Omega^T \\ -\operatorname{div} y = 0 &\quad \text{in } \Omega^T \\ y = 0 &\quad \text{on } (\partial\Omega)^T \\ y(0) = y_0 &\quad \text{in } \Omega. \end{aligned} \tag{9}$$

Here, ν denotes the constant kinematic viscosity coefficient and the superscript T denotes the time-space cylinder of the domain built over the interval $(0,T)$. For example $\Omega^T = (0,T)\times\Omega$. The vector function y_0 denotes an initial value. Define for $1 \le p,q \le \infty$

$$W_q^p = \{v \in L^p(V)\colon v_t \in L^q(V^*)\} \quad \text{and} \quad Z := L^2(V)\times H,$$

W_q^p endowed with the norm

$$|v|_{W_q^p} = |v|_{L^p(V)} + |v_t|_{L^q(V^*)},$$
$$H^{2,1}(Q) := \{v \in L^2(V\cap H^2(\Omega)^2); v_t \in L^2(H)\}$$

equipped with the norm

$$|v|^2_{H^{2,1}(Q)} := |v|^2_{L^2(V\cap H^2(\Omega)^2)} + |v_t|^2_{L^2(H)},$$

and set

$$W := W_2^2.$$

Furthermore, let

$$\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{L^2(V^*),L^2(V)}.$$

Here, V^* denotes the dual space of V and $L^p(V)$ is an abbreviation for $L^p(0,T;V)$, and similarly $L^q(V^*) = L^q(0,T;V^*)$. Recall that up to a set of measure zero in $(0,T)$ elements $v \in W_q^2$ $(q\ge 1)$ can be identified with elements in $C([0,T];V^*)$, and even in $C([0,T];H)$ for $q \ge 2$), compare [**24**, p.521], elements $w \in H^{2,1}(Q)$ can be identified with elements in $C([0,T];V)$ and the embedding $W_q^p \hookrightarrow L^p(H)$ $(1<p,q<\infty)$ is compact [**90**, Aubin-Lions, Theorem 1.4], i.e.

$$W_q^2 \hookrightarrow C([0,T];V^*)\ (1\le q<2), \quad W_q^2 \hookrightarrow C([0,T];G)\ (q\ge 2), \quad H^{2,1}(Q) \hookrightarrow C([0,T];V)$$
$$\text{and} \quad W_q^p \hookrightarrow\hookrightarrow L^p(H)\ (1<p,q<\infty). \tag{10}$$

The variational formulation of (9) in the solenoidal setting is given by:

For given $y_0 \in H$ find $y \in W$ with $y(0) = y_0$ in H and

$$\langle y_t, v\rangle + \langle (y\cdot\nabla)y, v\rangle + \nu(\nabla y,\nabla v) = \langle f, v\rangle \ \forall\, v\in L^2(V). \tag{11}$$

For (11) one has

THEOREM 2.1. Let $f \in L^2(V^*)$, $y_0 \in H$. Then (11) admits a unique variational solution $y \in W$ and satisfies the a-priori estimates

i. $|y|_{L^\infty(H)} + |y|_{L^2(V)} \ \le\ C\ \left\{|f|_{L^2(V^*)} + |y_0|_H\right\}$,

ii. $|y_t|_{L^2(V^*)} \ \le\ C\ \left\{|f|_{L^2(V^*)} + |f|^2_{L^2(V^*)} + |v_0|_H + |v_0|^2_H\right\}$,

iii. $|y|_W \ \le\ C\ \left\{|f|_{L^2(V^*)} + |f|^2_{L^2(V^*)} + |v_0|_H + |v_0|^2_H\right\}$.

If $y_0 \in V$ and $f \in L^2(H)$ the function w is an element of $H^{2,1}(Q)$ and satisfies

iv. $|y|_{L^\infty(V)} + |y|_{L^2(H^2(\Omega)^2)} \ \le\ C(|y_0|_H, |f|_{L^2(V^*)})\ \left\{|f|_{L^2(H)} + |y_0|_V\right\}$,

v. $|y_t|_{L^2(H)} \ \le\ C(|y_0|_H, |f|_{L^2(V^*)})\ \left\{|f|_{L^2(H)} + |y_0|_V + |f|^{\frac{3}{2}}_{L^2(H)} + |y_0|^{\frac{3}{2}}_V\right\}$, i.e.

vi. $|y|_{H^{2,1}(Q)} \leq C(|y_0|_H, |f|_{L^2(V^*)}) \left\{|f|_{L^2(H)} + |y_0|_V + |f|^{\frac{3}{2}}_{L^2(H)} + |y_0|^{\frac{3}{2}}_V\right\}$.

Proof. Can be found in [**22, 103**]. □

Now define for $y \in W$ the bounded linear operator $A(y) \in \mathcal{L}(W, Z^*)$ by

$$A(y)v = (v_t + (v \cdot \nabla)y + (y \cdot \nabla)v - \nu\Delta v, v(0)).$$

Then

$$A(y)v = (g, v_0) \text{ in } Z^* \iff \begin{cases} v_t + (v \cdot \nabla)y + (y \cdot \nabla)v - \nu\Delta v = g, \\ v(0) = v_0. \end{cases} \tag{12}$$

where the latter system is understood in the variational sense. There holds

PROPOSITION 2.1. Let $y \in W$, $v_0 \in H$ and $g \in L^2(V^*)$. Then (12) admits a unique variational solution $v \in W$. For every $f \in W^*$ the adjoint equation

$$A(y)^* w = f \quad \text{in } W^*$$

admits a unique variational solution $w = (w^1, w^0) \in Z$. If $f \in L^q(V^*) \cap W^*$ $(1 \leq q \leq \infty)$, then for every $0 \leq \epsilon \leq \min\{q-1, \frac{1}{3}\}$ the function w^1 is an element of $W^2_{1+\epsilon}$ and the variational solution of

$$\begin{cases} -w^1_t + (\nabla y)^t w^1 - (y \cdot \nabla)w^1 - \nu\Delta w^1 = f \\ w^1(T) = 0, \end{cases} \tag{13}$$

and it satisfies $w^1(0) = w^0$. The functions v and w^1 satisfy the a-priori estimates

i. $|v|_{L^\infty(H)} + |v|_{L^2(V)} \leq C(|y|_{L^2(V)}) \left\{|g|_{L^2(V^*)} + |v_0|_H\right\}$,
ii. $|v_t|_{L^2(V^*)} \leq C(|y|_{L^2(V)}, |y|_{L^\infty(H)}) \left\{|g|_{L^2(V^*)} + |v_0|_H\right\}$,
iii. $|w^1|_{L^2(V)} \leq C(|y|_{L^2(V)}, |y|_{L^\infty(H)}) \, |f|_{W^*}$.
iv. $|w^1_t|_{L^{1+\epsilon}(V^*)} \leq C(T^{\frac{1-\epsilon}{2(1+\epsilon)}}, T^{\frac{1-3\epsilon}{4(1+\epsilon)}}, |y|_{L^2(V)}, |y|_{L^\infty(H)}) \left\{|f|_{W^*} + |f|_{L^{1+\epsilon}(V^*)}\right\}$ for all $(0 \leq \epsilon \leq \min\{q-1, \frac{1}{3}\})$.

If in addition $y \in L^\infty(V)$ and $f \in L^2(V^*)$, then $w^1 \in W$ and

v. $|w^1|_{L^2(V)} + |w^1_t|_{L^2(V^*)} \leq C(|y|_{L^\infty(V)}) \, |f|_{L^2(V^*)}$,

For $y \in W \cap L^\infty(V) \cap L^2(H^2(\Omega)^2)$, $v_0 \in V$ and $g, f \in L^2(H)$ the unique solutions v of (12) and w^1 of (13) are elements of $H^{2,1}(Q)$ and satisfy the a-priori estimates

vi. $|v|_{H^{2,1}(Q)} \leq C(|y|_{L^\infty(V)}, |y|_{L^2(H^2(\Omega)^2)}) \left\{|g|_{L^2(H)} + |v_0|_V\right\}$

and

vii. $|w^1|_{H^{2,1}(Q)} \leq C(|y|_{L^\infty(V)}, |y|_{L^2(H^2(\Omega)^2)}) \, |f|_{L^2(H)}$.

Proof. Is given in Appendix 2. □

3. Navier-Stokes numerics

The numerical treatment of the Navier-Stokes equations is illustrated by means of the system

$$
\left\{
\begin{aligned}
y_t + (y\nabla)y - \tfrac{1}{\mathrm{Re}}\Delta y + \nabla p = f &\quad \text{in } \Omega^T \\
-\operatorname{div} y = 0 &\quad \text{in } \Omega^T \\
y = y_d &\quad \text{on } \Gamma_d^T \\
\nu\partial_\eta y - p\eta = 0 &\quad \text{on } \Gamma_n^T \\
y(0) = y_0 &\quad \text{in } \Omega.
\end{aligned}
\right. \tag{14}
$$

As generalization of (9) here the boundary of the spatial domain is decomposed into a Dirichlet part Γ_d and a Neumann part Γ_n with $\partial\Omega = \Gamma_d \cup \Gamma_N$, $\Gamma_d \cap \Gamma_N = \emptyset$, where on the latter natural boundary conditions are prescribed, compare [**44**]. The use of these boundary conditions is motivated by the backward facing step numerics presented in Section 5, compare [**6, 7, 19, 33**].

3.1. Time discretization. Let $n \in \mathbb{N}$. For $\Delta t := T/n > 0$ the time grid on $[0, T]$ is defined by $t_0 := 0$, $t_i := i\Delta t$ $(i = 1, \ldots, n)$. As time discretization schemes the semi-implicit Euler scheme and the fractional step θ-scheme [**96**] as operator splitting [**33**] are used. In the semi-implicit Euler-scheme the nonlinearity is discretized explicitly, so that the resulting scheme for $j \geq 0$ is given by

$$
\left\{
\begin{aligned}
\tfrac{y^{j+1} - y^j}{\Delta t} - \nu\Delta y^{j+1} + \nabla p^{j+1} = f^j - (y^j\nabla y^j) &\quad \text{in } \Omega \\
-\operatorname{div} y^{j+1} = 0 &\quad \text{in } \Omega \\
y^{j+1} = y_d^{j+1} &\quad \text{on } \Gamma_d \\
\nu\partial_\eta y^{j+1} - p^{j+1}\eta = 0 &\quad \text{on } \Gamma_n,
\end{aligned}
\right. \tag{15}
$$

with $y^0 = y(0)$. At every time instance a quasi-Stokes problem has to be solved. As is well known Euler-schemes are strongly A-stable and are easy to implement. On the other hand these schemes are only first-order accurate and quite dissipative.

Let $\theta \in (0, \frac{1}{2}$. The fractional step θ-scheme as an operator splitting for the Navier-Stokes equations (14) proceeds as follows: For $j > 0$ and $y^0 = y(0)$ find $y^{j+\theta}$, $y^{j+1-\theta}$, y^{j+1}, $p^{j+\theta}$ and p^{j+1} such that

$$
(16)\qquad \begin{cases} \frac{y^{j+\theta}-y^j}{\theta\Delta t} - \alpha\nu\Delta y^{j+\theta} + \nabla p^{j+\theta} = & f^{j+\theta} \\ & +\beta\nu\Delta y^j - (y^j\nabla y^j) \quad \text{in } \Omega \\ -\operatorname{div} y^{j+\theta} = & 0 \quad \text{in } \Omega \\ y^{j+\theta} = & y_d^{j+\theta} \quad \text{on } \Gamma_d \\ \alpha\nu\partial_\eta y^{j+\theta} - p^{j+\theta}\eta = & -\beta\nu\partial_\eta y^j \quad \text{on } \Gamma_n, \end{cases}
$$

$$
(17)\qquad \begin{cases} \frac{y^{j+1-\theta}-y^{j+\theta}}{(1-2\theta)\Delta t} - \beta\nu\Delta y^{j+1-\theta} = & f^{j+1-\theta} + \alpha\nu\Delta y^{j+\theta} \\ & -(y^{j+1-\theta}\nabla y^{j+1-\theta}) - \nabla p^{j+\theta} \quad \text{in } \Omega \\ y^{j+1-\theta} = & y_d^{j+1-\theta} \quad \text{on } \Gamma_d \\ \beta\nu\partial_\eta y^{j+1-\theta} = & p^{j+\theta}\eta - \alpha\nu\partial_\eta y^{j+\theta} \quad \text{on } \Gamma_n, \end{cases}
$$

$$
(18)\qquad \begin{cases} \frac{y^{j+1}-y^{j+1-\theta}}{\theta\Delta t} - \alpha\nu\Delta y^{j+1} + \nabla p^{j+1} = & f^{j+1} + \beta\nu\Delta y^{j+1-\theta} \\ & -(y^{j+1-\theta}\nabla y^{j+1-\theta}) \quad \text{in } \Omega \\ -\operatorname{div} y^{j+1} = & 0 \quad \text{in } \Omega \\ y^{j+1} = & y_d^{j+1} \quad \text{on } \Gamma_d \\ \alpha\nu\partial_\eta y^{j+1} - p^{j+1}\eta = & -\beta\nu\partial_\eta y^{j+1-\theta} \quad \text{on } \Gamma_n. \end{cases}
$$

The factors α and β have to satisfy $\alpha + \beta = 1$ and $\alpha > 1/2$ in order to obtain numerical damping. Stability investigations for this scheme can found in [**28, 70, 89**]. Numerical experiments show that this splitting scheme is second order accurate [**8**] and nearly non-dissipative. In numerical practice frequently the choices $\theta = 1 - 1/\sqrt{2}$, $\alpha = (1-2\theta)/(1-\theta)$ and $\beta = \theta(1-\theta)$ are used.

The steps (16)-(18) decouple the treatment of the solenoidal condition and the nonlinearity. In (16) and (18) linear quasi-Stokes problems have to be solved and the nonlinearity is treated explicitly. In (17) a Burgers like system of equations has to be solved, where the divergence free condition is dropped and the pressure gradient is taken from the previous quasi-Stokes computation.

3.2. Spatial discretization. For the spatial discretization the Taylor-Hood finite element with piecewise linear pressure and piecewise quadratic velocity approximations is used, see [**55**] which guarantees a stable discretization for the quasi-Stokes problems (15), (16) and (18).

The discretized quasi-Stokes problems resulting from the spatial discretization of (15), (16) and (18) are solved numerically by a preconditioned conjugate gradient method applied to the Schur-complement [**7, 33**]. To be more precise denote by V_h and P_h the discrete velocity and pressure space, respectively, where h denotes the grid-size of the triangulation. Then, the discrete versions of (15), (16) and (18) are of the form

Find $y_h \in V_h$, $p_h \in P_h$ such that

$$\left\{ \begin{array}{ll} \int\limits_\Omega y_h v_h + \gamma \nabla y_h \nabla v_h - \operatorname{div} v_h p_h \, dx = \int\limits_\Omega r_h v_h \, dx & \text{for all } v_h \in V_h \\ -\int\limits_\Omega \operatorname{div} y_h q_h \, dx = 0 & \text{for all } q_h \in P_h. \end{array} \right. \tag{19}$$

For example in (15) one has $\gamma = \Delta t \nu$ and p_h stands for the approximation of $\Delta t p$. The right-hand-side r_h contains the discretized velocity from the previous time step and the discretized nonlinearity. The action of the Schur-complement operator $A_h : P_h \to P_h$ corresponding to (19) is defined by

$$A_h q_h \;=\; -\mathrm{div}_h \left(Id_{V_h} - \gamma \Delta_h \right)^{-1} \nabla_h q_h, \tag{20}$$

where for given $q_h \in P_h$ one has $w_h = \left(Id_{V_h} - \gamma \Delta_h \right)^{-1} \nabla_h q_h$ iff

$$\int\limits_\Omega w_h v_h + \gamma \nabla w_h \nabla v_h \, dx = \int\limits_\Omega \operatorname{div} v_h q_h \, dx - \int\limits_{\Gamma_n} q_h v_h \eta \, d\Gamma_n \text{ for all } v_h \in V_h.$$

An appropriate preconditioner $S_h : P_h \to P_h$ for the Schur-complement operator (20) is given by $S_h := \big((-\Delta_h)^{-1} + \gamma Id \big)$, i.e.

$$S_h q_h \;=\; \phi_{q_h} + \gamma q_h,$$

with $w_h = \phi_{q_h}$ the solution of

$$\int\limits_\Omega \nabla w_h \nabla v_h \, dx = \int\limits_\Omega q_h v_h \, dx \text{ for all } v_h \in V_h^*,$$

see [**33**], where also more details can be found. Here, V_h^* denotes the 'dual' of the space V_h, i.e. the finite element space containing Ansatz functions with zero boundary conditions on Γ_n and natural boundary conditions on Γ_d.

The numerical solution of the nonlinear Burgers equation (17) is performed with a preconditioned GMRES-algorithm with restart. For that purpose the nonlinearity in every application of the GMRES algorithm is freezed. The result of the iteration is used to update the nonlinearity and GMRES is restarted. This procedure is repeated until convergence is obtained. As preconditioner diagonal scaling is used [**7**].

The same discretization techniques with obvious modifications are used to solve numerically adjoint equations, which in general are (variants of) linearizations of the Navier-Stokes equations.

CHAPTER 3

Optimal control of the instationary Navier-Stokes equations

In the following sections the analytical framework for optimal distributed control of the instationary Navier-Stokes equations in two spatial dimensions is developed. The problem is formulated as constrained optimization problem with the Navier-Stokes equations serving as subsidiary conditions. This is different to the approaches presented for example in [**1, 39**] where the one-one correspondence between state and control is utilized to formulate an unconstrained optimization problem. Section 1 contains the problem formulation, the definitions of appropriate function spaces and introduces the operators for the subsidiary conditions. In Section 2 the first and second derivatives of the cost functional and the subsidiary conditions are derived and investigated with respect to their differentiability properties. In Section 3 the first order optimality system is investigated and existence and uniqueness of Lagrange multipliers is proved. Finally, in Section 4 some results related to the second-order analysis of the optimal control problem are proved.

1. The optimal control problem

The subject of the investigations in the present chapter is given by the distributed control problem

$$
\text{(21)} \qquad \textbf{(P)} \quad \begin{cases} \min\ J(y,u) \text{ over } (y,u) \in W \times U & \\ \text{s.t.} & \\ \frac{\partial y}{\partial t} + (y \cdot \nabla)y - \nu\Delta y + \nabla p = Bu & \text{in } \Omega^T, \\ -\text{div}\, y = 0 & \text{in } \Omega^T, \\ y(t,\cdot) = 0 & \text{on } (\partial\Omega)^T, \\ y(0,\cdot) = y_0 & \text{in } \Omega. \end{cases}
$$

Here U denotes the Hilbert space of controls,

$$X := W \times U \text{ and } x := (y,u) \text{ for } (y,u) \in W \times U.$$

Introducing the nonlinear mapping $e : X \to Z^*$,

$$e(x) = (e^1(x), e^2(x)) = (\tfrac{\partial y}{\partial t} + (y \cdot \nabla)y - \nu\Delta y - Bu, y(0) - y_0)$$

problem **(P)** equivalently can be rewritten as

$$
\text{(22)} \qquad \begin{cases} \min\ J(x) \text{ over } x \in X \\ \text{subject to } e(x) = 0 \quad \text{in } Z^*. \end{cases}
$$

Comparing (21) to (22) it is worth noting that the conservation of mass, as well as the boundary condition are realized in the choice of space W while the dynamics are described by the condition $e(x) = 0$.
The control space U is identified with its dual U^*,

$$B \in \mathcal{L}(U, L^2(V^*))$$

denotes the control extension operator and $y_0 \in H$ an initial value. The cost functional

$$J\colon X \to \mathbb{R}, \quad J(x) = J_1(y) + J_2(u).$$

is assumed to be of separable type, bounded from below, weakly lower semi-continuous, twice Fréchet differentiable with locally Lipschitzean second derivative, and radially unbounded in u, i.e.

$$J(x) \to \infty \quad (|u|_U \to \infty), \text{ for every } y \in W.$$

Due to Theorem 2.1 with respect to existence problem (22) is equivalent to

$$\min\ \hat{J}(u) = J(y(u), u) \ \text{ subject to } \ u \in U, \tag{23}$$

where $y(u) \in W$ satisfies $e(y(u), u) = 0$.

THEOREM 1.1. Problem (22) admits a solution $x^* = (y(u^*), u^*) \in X$.

Proof. Since J is bounded from below there exists a minimizing sequence $\{(y_n, u_n)\} = \{y(u_n), u_n\}$ in X. Due to the radial unboundedness property of J in u the sequence $\{u_n\}$ is bounded in U. Since $B \in \mathcal{L}(U, L^2(V^*))$ Theorem 2.1 iii. implies that the sequence $\{(y_n, u_n)\}$ is bounded in $W \times U$ and hence there exists a subsequence with a weak limit $x^* \in X$. Weak lower semi-continuity of $x \to J(x)$ implies that

$$J(x^*) = \inf\{J(x)\colon x \in W \times U,\ e(x) = 0\},$$

and it remains to show that $y^* = y(u^*)$. Since by (10) (after changing to another subsequence) $\{y_n\}$ converges strongly to y^* in $L^2(H)$ this can be achieved by passing to the limit in (11) with y replaced by $y(u_n)$ and f replaced by Bu_n, see [**1**]. □
For more regular initial values and controls one can obtain more.

COROLLARY 1.1. Let $y_0 \in V$ and $B \in \mathcal{L}(U, L^2(H))$. Then Problem (22) admits a solution $x^* = (y(u^*), u^*) \in H^{2,1}(Q) \times U$.

Proof. Now $\{y_n\}$ converges to y^* weakly in $H^{2,1}(Q)$, and thus (after changing to another subsequence), even strongly in $L^2(V)$ [**103**, Ch. III,Theorem 2.1]. The sequence $\{Bu_n\}$ converges weakly to Bu^* in $L^2(H)$. The claim now follows with Theorem 2.1, vi. as in the proof of the previous theorem. □

EXAMPLE 1.1. The assumptions above are satisfied for cost functionals including tracking type functionals

$$F_1(x) = \frac{1}{2}\int\limits_{\Omega^T} |y-z|^2 dx\, dt + \frac{\gamma}{2}\int\limits_{\Omega} |y(T)-z(T)|^2 dx + \frac{\alpha}{2}\, |u|_U^2 \tag{24}$$

and functionals involving the vorticity of the fluid

$$F_2(x) = \frac{1}{2}\int\limits_{\Omega^T} |\nabla_x \times y(t,\cdot)|^2\, dx\, dt + \frac{\alpha}{2}\, |u|_U^2, \tag{25}$$

where $\alpha, \gamma > 0$ and $z \in W$ are given. Of course, these functionals are even infinitely often differentiable on X.

2. Derivatives

In this section differentiability properties of the constraint e are proved. Furthermore, representations for the first and second derivatives of $\hat{J}$ appropriate for the treatment of (23) by the Newton and quasi–Newton method are derived.

PROPOSITION 2.1. The operator $e = (e^1, e^2)\colon X \to Z^*$ is twice continuously differentiable with Lipschitz continuous second derivative. The action of the first two derivatives of e^1 are given by

$$\langle e^1_x(x)(w,s), \phi\rangle = \langle w_t, \phi\rangle + \langle (w\cdot\nabla)y, \phi\rangle + \langle (y\cdot\nabla)w, \phi\rangle + \nu(\nabla w, \nabla\phi)_{L^2(L^2)} - \langle Bs, \phi\rangle,$$

where $x = (y,u) \in X$, $(w,s) \in X$ and $\phi \in L^2(V)$, and

$$\begin{aligned}\langle e^1_{xx}(x)(w,s)(v,r), \phi\rangle = \langle e^1_{yy}(x)(w,v), \phi\rangle = \\ \langle (w\cdot\nabla)v, \phi\rangle + \langle (v\cdot\nabla)w, \phi\rangle =: \langle H(\phi)w, v\rangle_{W^*,W},\end{aligned} \tag{26}$$

where $(v,r) \in X$.

Proof. Since e^2 is linear it is sufficient to restrict the attention to e^1. Let $b\colon V\times V\times V \to \mathbb{R}$ be defined by

$$b(u,v,\phi) = \langle (u\cdot\nabla)v, \phi\rangle_{V^*,V},$$

and recall that, due to (129)

$$|b(u,v,\phi)|^2 \le 2|u|_H\ |u|_V\ |v|_H\ |v|_V\ |\phi|_V^2, \tag{27}$$

for all $(u, v, \phi) \in V \times V \times V$. To argue local Lipschitz continuity of e, let $x, \tilde{x} \in X$ and $\phi \in L^2(V)$. One finds

$$\begin{aligned}\langle e^1(x) - e^1(\tilde{x}), \phi\rangle &= \langle (y - \tilde{y})_t, \phi\rangle + \langle ((y - \tilde{y}) \cdot \nabla)\tilde{y}, \phi\rangle \\ &\quad + \langle (y \cdot \nabla)(y - \tilde{y}), \phi\rangle + \nu(\nabla(y - \tilde{y}), \nabla\phi) + \langle B(\tilde{u} - u), \phi\rangle \\ &\leq \sqrt{2}\int_0^T |y - \tilde{y}|_H^{1/2}|y - \tilde{y}|_V^{1/2}(|\tilde{y}|_H^{1/2}|\tilde{y}|_V^{1/2} + |y|_H^{1/2}|y|_V^{1/2})|\phi|_V dt \\ &\qquad + C|x - \tilde{x}|_X\, |\phi|_{L^2(V)}.\end{aligned}$$

Here and below the constant C is independent of $x, \tilde{x}$ and ϕ. Due to the continuous embedding of W into $L^\infty(H)$ one has

$$\begin{aligned}\langle e^1(x) - e^1(\tilde{x}), \phi\rangle \leq C \Big[&|x - \tilde{x}|_X\, |\phi|_{L^2(V)} + |y - \tilde{y}|^{1/2}_{L^\infty(H)}(|\tilde{y}|^{1/2}_{L^\infty(H)} + |y|^{1/2}_{L^\infty(H)}) \\ &+ \textstyle\int_0^T |y - \tilde{y}|_V^{1/2}(|\tilde{y}|_V^{1/2} + |y|_V^{1/2})|\phi|_V dt\Big].\end{aligned}$$

Using Hölder's inequality this further implies the estimate

$$\begin{aligned}\langle e(x) - e(\tilde{x}), \phi\rangle \leq C \Big[&|x - \tilde{x}|_X + |y - \tilde{y}|^{1/2}_{L^\infty(H)}(|\tilde{y}|^{1/2}_{L^\infty(H)} + |y|^{1/2}_{L^\infty(H)}) \\ &+ \Big(\textstyle\int_0^T |y - \tilde{y}|_V(|\tilde{y}|_V + |y|_V)dt\Big)^{1/2}\Big]\, |\phi|_{L^2(V)}\end{aligned}$$

and consequently,

$$\langle e^1(x) - e^1(\tilde{x}), \phi\rangle \leq C\, |x - \tilde{x}|_X(|y|_W + |\tilde{y}|_W)|\phi|_{L^2(V)}.$$

This estimate establishes the local Lipschitz continuity of e. To verify that the formula for e_x given above represents the Fréchet - derivative of e one estimates

$$\begin{aligned}&|e^1(\tilde{x}) - e^1(x) - e^1_x(x)(\tilde{x} - x)|_{L^2(V^*)} = \sup_{|\phi|_{L^2(V)}=1} \textstyle\int_0^T |b(y - \tilde{y}, y - \tilde{y}, \phi)|\, dt \\ &\leq \sup_{|\phi|_{L^2(V)}=1} \textstyle\int_0^T |y - \tilde{y}|_H\, |y - \tilde{y}|_V\, |\phi|_V dt \\ &\leq C|y - \tilde{y}|_W \sup_{|\phi|_{L^2(V)}=1} \textstyle\int_0^T |y - \tilde{y}|_V |\phi|_V\, dt \\ &\leq C|y - \tilde{y}|^2_W\end{aligned}$$

and Fréchet - differentiability of e follows. To show Lipschitz continuity of the first derivative let $x, \tilde{x}$ and (v, r) be in X and estimate

$$\begin{aligned}&|(e^1_x(\tilde{x}) - e^1_x(x))(v, r)|_{L^2(V^*)} = \sup_{|\phi|_{L^2(V)}=1} \textstyle\int_0^T |b(y - \tilde{y}, v, \phi) + b(v, y - \tilde{y}, \phi)|\, dt \\ &\leq 2\sqrt{2} \sup_{|\phi|_{L^2(V)}=1} \textstyle\int_0^T |y - \tilde{y}|_H\, |y - \tilde{y}|_V\, |v|_H\, |v|_V\, |\phi|_V dt \\ &\leq C|y - \tilde{y}|_W\, |v|_W.\end{aligned}$$

Clearly, since the nonlinearity of the Navier-Stokes equations is quadratic the second derivative is given by the expression (26). As the second derivative is independent of the point at which it is taken and it is necessarily Lipschitz continuous. $\square$

PROPOSITION 2.2. Let $x \in X$. Then $e_y(x)\colon W \to Z^*$ is a homeomorphism. Moreover, if the inverse of its adjoint $e_y^{-*}(x)\colon W^* \to Z$ is applied to an element $f \in W^* \cap L^\alpha(V^*)$, $\alpha \in [1, 4/3]$ then, setting $(w, w_0) := e_y^{-*}(x)f \in Z$ one has $w_t \in L^\alpha(V^*)$, $w(0) = w_0$ and w is the variational solution to (13).

Proof. Since $e_y(x)$ coincides with $A(y)$ defined in (12) the claim follows as in the proof of Proposition 2.1. $\square$

Next the derivatives of the functional $\hat{J}$ defined in (23) are computed. As a consequence of Propositions 2.1 and 2.2 and the implicit function theorem (see [**111**]) the mapping $u \to y(u)$ is twice Fréchet differentiable in a neighbourhood of u with Lipschitz continuous second derivatives and its first derivative at u in direction δu is given by

$$y'(u)\delta u = -e_y^{-1}(x)e_u(x)\delta u, \tag{28}$$

where $x = (y(u), u)$. By the chain rule one obtains

$$\langle \hat{J}'(u), \delta u\rangle_U = \langle J_u(x) - e_u^*(x)e_y^{-*}(x)J_y(x), \delta u\rangle_U.$$

Introducing the variable

$$\lambda = (\lambda^1, \lambda^0) = -e_y^{-*}(x)J_y(x) \in Z \tag{29}$$

one obtains utilizing Proposition 2.1, iii. with $f = -J_y(x) \in W^*$ the Riesz representation for the first derivative of $u \to \hat{J}(u)$:

$$\hat{J}'(u) = J_u(x) + e_u^*\lambda.$$

If in addition $J_y(x) \in L^q(V^*) \cap W^*$ the same proposition gives $\lambda^0 = \lambda^1(0)$ and $\lambda^1 \in W^2_{1+\epsilon}$ $(0 \le \epsilon \le \min\{q-1, \frac{1}{3}\})$. Moreover, λ^1 can be characterized as the variational solution to

$$\begin{cases} -\lambda^1_t + (\nabla y)^t\lambda^1 - (y \cdot \nabla)\lambda^1 - \nu\Delta\lambda^1 = -J_y(x) \\ \lambda^1(T) = 0. \end{cases} \tag{30}$$

REMARK 2.1. As an immediate consequence of v. and vii. in Proposition 2.1 one obtains $\lambda^1 \in W$ if $y \in L^\infty(V)$ and $J_y(x) \in L^2(V^*)$, and $\lambda^1 \in H^{2,1}(Q)$ if $y \in L^\infty(V) \cap L^2(H^2(\Omega)^2)$ and $J_y(x) \in L^2(H)$.

The computation of the second derivative of $\hat{J}''(u) \in \mathcal{L}(U)$ of $\hat{J}$ is more involved. Let $(\delta u, \delta v) \in U \times U$ and note that the second derivative of $u \to y(u)$ from U to W can be expressed as

$$y''(u)(\delta u, \delta v) = -e_y^{-1}(x)e_{yy}(x)(y'(u)\delta u, y'(u)\delta v).$$

By the chain rule one has

$$
\begin{aligned}
&\langle \hat{J}''(u)\delta u, \delta v\rangle_U = \langle J_{yy}(x)y'(u)\delta u, y'(u)\delta v\rangle_{W^*,W} \\
&+\langle J_y(x), y''(u)(\delta u, \delta v)\rangle_{W^*,W} + \langle J_{uu}(x)\delta u, \delta v\rangle_U \\
&= \langle J_{yy}(x)y'(u)\delta u, y'(u)\delta v\rangle_{W^*,W} + \langle J_y(x), e_y^{-1}e_{yy}(x)(y'(u)\delta u, y'(u)\delta v\rangle_{W^*,W} \\
&+\langle J_{uu}(x)\delta u, \delta v\rangle_U \\
&= \langle J_{yy}(x)y'(u)\delta u, y'(u)\delta v\rangle_{W^*,W} + \langle \lambda^1, e^1_{yy}(x)(y'(u)\delta u, y'(u)\delta v\rangle \\
&+\langle J_{uu}(u)\delta u, \delta v\rangle_U.
\end{aligned}
$$

In order to obtain a more compact representation for $\hat{J}''(u)$ now introduce the Lagrangian $L\colon X\times Z \to \mathbb{R}$

$$L(x,\lambda) = J(x) + \langle e(x), \lambda\rangle_{Z^*Z} \tag{31}$$

and the matrix operator

$$T(x) = \begin{pmatrix} -e_y^{-1}(x)e_u(x) \\ \mathrm{Id}_U \end{pmatrix} \in \mathcal{L}(U,X). \tag{32}$$

The second derivative of L with respect to x can be expressed as

$$L_{xx}(x,\lambda) = \begin{pmatrix} J_{yy}(x)+ & \langle e^1_{yy}(x)(\cdot,\cdot),\, \lambda^1\rangle & 0 \\ & 0 & J_{uu}(x) \end{pmatrix} \in \mathcal{L}(X,X^*).$$

The above computations for $\hat{J}''(u)$ together with (28) imply that

$$\hat{J}''(u) = T^*(x)L_{xx}(x,\lambda)T(x), \tag{33}$$

where $x = (y(u), u)$.

EXAMPLE 2.1. For the cost functional $J = F_1$ in (24) one obtains

$$\hat{J}'(u) \;=\; \alpha\, u \;-\; B^*\lambda^1 \in U$$

and

$$\langle J_y(x), w\rangle_{W^*,W} = \int_0^T (y-z, w)\, dt + \gamma(y(T)-z(T), w(T)),$$

$$\langle J_{yy}(x)w, v\rangle_{W^*,W} = \int_0^T (v,w)\, dt + \gamma(v(T), w(T)), \quad J_{uu}(x) = \alpha\, \mathrm{Id}_U,$$

so that for $r, s \in U$ with $w = -e_y^{-1}(x)Br$ and $v = -e_y^{-1}(x)Bs$

$$(\hat{J}''(u)r, s)_U = \langle J_{yy}(x)w, v\rangle_{W^*,W} + \langle e^1_{yy}(x)(v, w), \lambda^1(x)\rangle + \alpha\,(r, s)_U,$$

with $e_y(x)$ defined by (12) and the action of $e_{yy}(x)$ defined in (26) of Proposition 2.1. Note that for this cost functional the initial condition for λ^1 in (30) becomes $\lambda^1(T) = -\gamma(y(T) - z(T))$.

Similarly for the cost functional $J = F_2$ defined in (25). There holds

$$\langle J_y(x), w\rangle_{W^*,W} = \int_0^T (\nabla_x \times y, \nabla_x \times w)\,dt,$$

$$\langle J_{yy}(x)w, v\rangle_{W^*,W} = \int_0^T (\nabla_x \times v, \nabla_x \times w)\,dt, \quad J_{uu}(x) = \alpha\,\mathrm{Id}_U,$$

and thus, as above

$$(\hat{J}''(u)r, s)_U = \langle J_{yy}(x)w, v\rangle_{W^*,W} + \langle e^1_{yy}(x)(v, w), \lambda^1(x)\rangle + \alpha\,(r, s)_U.$$

3. First order necessary optimality condition

A sufficient condition for the existence of Lagrange multipliers associated to solutions $x^* = (y^*, u^*)$ of the constraint minimization problem **(P)** in (21) is given by the surjectivity of the operator $e_x(x^*)$.

THEOREM 3.1. Let $x \in X$. Then the operator $e_x(x) : X \to Z^*$ is surjective.

Proof. Let $(f, v_0) \in Z^*$ and $\tilde{u} \in U$ arbitrary, but fixed. Due to Proposition 2.1 the equation

$$e_x(x)(v, \tilde{u}) \;=\; (f, v_0) \quad \text{in } Z^* \iff e_y(x)v \;=\; (f + B\tilde{u}, v_0) \quad \text{in } Z^*$$

admits a unique solution $v \in W$ with $v(0) = v_0$. Since $\tilde{u}$ was chosen arbitrary this proves the claim. $\square$

THEOREM 3.2. (Existence and uniqueness of Lagrange multipliers)
Let $x^* = (y^*, u^*) \in X$ be a solution of the optimization problem (21). Then there exists a unique Lagrange multiplier $\lambda^* \in Z$ which together with the optimal solution x^* satisfies the optimality system

$$\left\{\begin{array}{rcll} L_y(x^*, \lambda^*) &=& J_y(x^*) + \langle e_y(x^*)(\cdot)_W, \lambda^*\rangle_{Z^*,Z} = 0 & \text{in } W^* \\ L_u(x^*, \lambda^*) &=& J_u(x^*) + \langle e_u(x^*)(\cdot)_U, \lambda^*\rangle_{Z^*,Z} = 0 & \text{in } U \\ && e(x^*) = 0 & \text{in } Z^*. \end{array}\right. \tag{34}$$

Furthermore, if $J_y(x^*) \in L^q(V^*) \cap W^*$ $(1 \le q \le \infty)$ then there holds $\lambda^* = (\lambda^{*^1}, \lambda^{*^0}) \in W^2_{1+\epsilon} \times H$ for all $(0 \le \epsilon \le \min(q-1, \frac{1}{3}))$ with $\lambda^{*^0} = \lambda^{*^1}(0)$ and λ^{*^1} satisfies the a-priori estimates

L1 $|\lambda^{*^1}|_{L^2(V)} \;\le\; C(|y^*|_{L^2(V)}, |y^*|_{L^\infty(H)})\,|J_y(x^*)|_{W^*}$, L2 $|\lambda_t^{*^1}|_{L^{1+\epsilon}(V^*)} \leq C(T^{\frac{1-\epsilon}{2(1+\epsilon)}}, T^{\frac{1-3\epsilon}{4(1+\epsilon)}}, |y^*|_{L^2(V)}, |y^*|_{L^\infty(H)})\, |J_y(x^*)|_{L^{1+\epsilon}(V^*)}$ $(0 \leq \epsilon \leq \min(q-1, \frac{1}{3}))$.

Moreover, λ^{*^1} is the variational solution of (30).

Proof. Since $e_y(x^*) = A(y^*)$, $J_y(x^*) \in W^*$ and $y^* \in W$ the first part of the claim follows from Proposition 2.1 with w replaced by λ^* and f replaced by $-J_y(x^*)$. The second part follows from the same proposition, iii., since now $J_y(x^*) \in L^q(V^*) \cap W^*$. □

For more regular states, controls and cost functionals one obtains more regular Lagrange multipliers.

THEOREM 3.3. (Regularity for Lagrange multipliers)
Let $x^* = (y^*, u^*) \in X$ be a solution of the optimization problem (21) such that $y^* \in L^\infty(V)$. Furthermore, let $J_y(x^*) \in L^2(V^*)$. Then the unique Lagrange multiplier λ^* from Theorem 3.2 is an element of $X = W \times H$ and satisfies the a-priori estimates

L3 $|\lambda^{*^1}|_{L^2(V)} \leq C(|y^*|_{L^2(V)})\, |J_y(x^*)|_{L^2(V^*)}$,

L4 $|\lambda_t^{*^1}|_{L^2(V^*)} \leq C(|y^*|_{L^\infty(V)})\, |J_y(x^*)|_{L^2(V^*)}$.

If $y_0 \in V$, $B \in \mathcal{L}(U, L^2(H))$ and $J_y(x^*) \in L^2(H)$ the variable λ^{*^1} is an element of $H^{2,1}(Q)$ and satisfies the a-priori estimate

L5 $|\lambda^{*^1}|_{H^{2,1}(Q)} \leq C(|y^*|_{L^\infty(V)}, |y^*|_{L^2(H^2(\Omega)^2)})\, |J_y(x^*)|_{L^2(H)}$.

Proof. The first part follows from Proposition 2.1, v.. The second part from the same proposition, vii., since the assumptions on the data by Corollary 1.1 imply $y^* \in H^{2,1}(Q)$. □

EXAMPLE 3.1. For the cost functional F_1 from (27) the optimality system (34) in primitive variables (y, p) reads

$$
\begin{aligned}
y_t^* + (y^*\nabla)y^* - \nu\Delta y^* + \nabla p^* &= Bu && \text{in } \Omega^T\\
-\operatorname{div} y^* &= 0 && \text{in } \Omega^T\\
y^* &= 0 && \text{on } (\partial\Omega)^T\\
y^*(0) &= y_0^* && \text{in } \Omega
\end{aligned}
$$

$$
\begin{aligned}
-\lambda_t^{*^1} - \nu\Delta\lambda^{*^1} - (y^*\nabla)\lambda^{*^1} + (\nabla y^*)^t\lambda^{*^1} + \nabla\xi^* &= -(y^* - z) && \text{in } \Omega^T\\
-\operatorname{div}\lambda^{*^1} &= 0 && \text{in } \Omega^T\\
\lambda^{*^1} &= 0 && \text{on } (\partial\Omega)^T\\
\lambda^{*^1}(T) &= -\gamma(y^*(T) - z(T)) && \text{in } \Omega
\end{aligned}
$$

$$
\alpha u - B^*\lambda^{*^1} = 0 \qquad \text{in } U.
$$

For the derivation of the terminal condition for λ^{*^1} see Appendix 1 or follow the lines of the proof of Proposition 2.1 with f replaced by $-F_{1_y}(x)$. Since $F_{1_y}(x) = f_1 + f_2 \in L^2(H) + C(H)^*$ with f_2 vanishing on $\{v \in W, v(T) = 0\}$ all conclusions of this proof remain valid and the terminal condition follows by integration by parts with respect to time in (133).

The optimality system for F_2 given by (25) differs in the right-hand side of the adjoint equations, where the term $-(y^* - z)$ has to be replaced by $-\nabla \times \nabla \times y^*$ and $\lambda^{*^1}(T) = 0$.

4. Second order conditions

In order to provide convergence analysis for second order methods it will be essential to derive conditions that ensure positive definiteness of $\hat{J}''(u^*)$ and a sufficient second order optimality condition for the Lagrangian $L(x, \lambda)$. The key to these conditions are the a-priori estimates of Proposition 2.1. One shall also prove that the difference $\hat{J}''(u^*) - J_{uu}(x^*)$ is compact. This property is required for the rate of convergence analysis of quasi-Newton methods. In the first result we assert positive definiteness of the Hessian provided that $J_y(x)$ is sufficiently small, a condition which is applicable to tracking-type problems.

LEMMA 4.1. (Positive definiteness of Hessian)
Let $u \in U$ and assume that $J_{yy}(x) \in \mathcal{L}(W, W^*)$ be positive semi-definite and $J_{uu}(x) \in \mathcal{L}(U)$ be positive definite, where $x = (y(u), u)$. Finally, let $B \in \mathcal{L}(U, L^2(V^*))$. Then, the Hessian $\hat{J}''(u)$ is positive definite provided that $|J_y(x)|_{W^*}$ is sufficiently small.

Proof: Recall that by (33)

$$\hat{J}''(u) = T^*(x) L_{xx}(x, \lambda) T(x),$$

where $\lambda = \lambda(x)$ is defined in (29). For $s \in U$ let $w := -e_y^{-1}(x) e_u(x) s$. Then $w \in W$ by Proposition 2.1. Since

$$\begin{aligned} \hat{J}''(u) = {} & e_u^*(x) e_y^{-*}(x) J_{yy}(x) e_y^{-1}(x) e_u(x) \cdot + \\ & e_u^*(x) e_y^{-*}(x) \langle e_{yy}^1(x)(e_y^{-1}(x) e_u(x) \cdot, \cdot), \lambda^1(x) \rangle + J_{uu}(x) \cdot \end{aligned} \tag{35}$$

one has

$$(\hat{J}''(u)s, s)_U \; = \; \langle J_{yy}(x) w, w \rangle_{W^*, W} + \big\langle \langle e_{yy}^1(x) w, \cdot), \lambda^1(x) \rangle, w \big\rangle_{W^*, W} + (J_{uu}(x) s, s)_U .$$

The third addend in this equation is bounded from below by a constant times $|u|_U^2$ and, since $J_{yy}(x)$ is positive semi-definite the first addend is non-negativ. In order to tackle the second addend define

$$\mathcal{R} := e_u^*(x) e_y^{-*}(x) \langle e_{yy}^1(x)(e_y^{-1}(x) e_u(x) \cdot, \cdot), \lambda^1(x) \rangle \; \in \mathcal{L}(U) \tag{36}$$

and recall that for $g, h \in W$

$$\langle e_{yy}^1(x)(g, h), \lambda^1(x) \rangle \; = \; \int_0^T \int_\Omega (g \cdot \nabla) h \lambda^1 \; + \; (h \cdot \nabla) g \lambda^1 \, dx dt.$$

Using (27), the continuity of the embedding $W \hookrightarrow L^\infty(H)$ and Proposition 2.1 one estimates

$$|\langle e_{yy}^1(x)(w, w), \lambda^1(x) \rangle| \; \leq \; C \, |w|_W^2 \, |\lambda^1|_{L^2(V)} \leq C \, |s|_U^2 \, |J_y(x)|_{W^*}$$

with a constant independent of w. Therefore,

$$(\mathcal{R}s, s)_U \geq -C\, |s|_U^2\, |J_y(x)|_{W^*}.$$

Now let $\delta > 0$ such that $(J_{uu}(x)s, s)_U \geq \delta\, |s|_U^2$. Then by the definition of w and the above estimate for $\mathcal{R}$

$$(\hat{J}''(u)s, s)_U \geq (\delta - C|J_y(x)|_{W^*})\; |s|_U^2,$$

which gives the claim. □

LEMMA 4.2. Let $x \in X$ and denote by $\lambda = \lambda(x) \in Z$ the solution of (29) associated to x. Let $J_{yy}(x) \in \mathcal{L}(W, W^*)$ be positive semi-definite, let $J_{uu}(x) \in \mathcal{L}(U)$ be positive definite and let $|J_y(x)|_{W^*}$ be sufficiently small. Then, $L_{xx}(x, \lambda)$ is positive definite on the kernel of $e_x(x)$, i.e.

$$\langle L_{xx}(x, \lambda)\tilde{x}, \tilde{x}\rangle_{X^*X} \geq c\, |\tilde{x}|_X^2 \quad \text{for all } \tilde{x} \in \mathcal{N}(e_x(x)) \tag{37}$$

with a positive constant c independent of x.

Proof. Let $(v, \tilde{u}) \in \mathcal{N}(e_x(x))$. Then v solves (12) with $v_0 = 0$ and $g = B\tilde{u}$. Due to Proposition 2.1, $v \in W$ and satisfies

$$|v|_W \leq C(|y|_{L^2(V)}, |y|_{L^\infty(H)}, \|B\|_{\mathcal{L}(U, L^2(V^*))})\, |u|_U. \tag{38}$$

Let $\delta > 0$ be chosen such that $J_{uu}(x)(\tilde{u}, \tilde{u}) \geq \delta |\tilde{u}|_U^2$. One finds

$$\begin{aligned}\langle L_{xx}(x, \lambda)(v, \tilde{u}), (v, \tilde{u})\rangle_{X^*,X} &= J_{yy}(x)(v, v) + \langle e^1_{yy}(x)(v, v), \lambda^1\rangle + J_{uu}(x)(\tilde{u}, \tilde{u}) \\ &\geq \delta\, |\tilde{u}|_U^2 - 2\sqrt{2}\int_0^T |v|_H |v|_V |\lambda^1|_V\, dt \geq \delta\, |\tilde{u}|_U^2 - C|\tilde{u}|_U^2\, |\lambda^1|_{L^2(V)}.\end{aligned}$$

Here and below C denotes a generic constant independent of $(v, \tilde{u})$. As in the proof of Lemma 4.1

$$\langle L_{xx}(x, \lambda)(v, \tilde{u}), (v, \tilde{u})\rangle_{X^*,X} \geq (\delta - C|J_y(x)|_{W^*})\; |\tilde{u}|_U^2,$$

so that the claim follows. □

REMARK 4.1. It follows from Example 2.1 that the assumptions of Lemma 4.1 and Lemma 4.2 are satisfied for both, the functional F_1 defined in (24) and the functional F_2 in (25).

In order to prove super-linear convergence for quasi-Newton methods in infinite dimensions a compactness property for the difference between the initial approximation of the Hessian and the Hessian at the local solution is needed [**34, 79**]. To ensure this property for the optimization problem under consideration the differentiability properties of the functional J and the regularity properties for the controls have to be enhanced.

LEMMA 4.3. Let $u \in U$, let $y_0 \in H$ and assume $B \in \mathcal{L}(U, L^2(H))$. Furthermore, let $J_1 : W \cup L^p(V) \to \mathbb{R}$ be twice continuously Fréchet differentiable for some $1 \leq p < \infty$ Then, the difference

$$\hat{J}''(u) - J_{uu}(x)$$

is compact, where $x = (y(u), u)$ and $y(u)$ denotes the uniquely defined state corresponding to the control u.

Proof. By Theorem 2.1 $y_0 \in H$ implies $y(u) \in W$. Utilizing (36) one may rewrite

$$\hat{J}''(u) - J_{uu}(x) = e_u^*(x)e_y^{-*}(x)J_{yy}(x)e_y^{-1}(x)e_u(x) \cdot + \mathcal{R}.$$

It will be shown that both addenda in this equation define compact operators on U. To begin with let $\mathcal{U}$ be a bounded subset of U. Since $B \in \mathcal{L}(U, L^2(H))$ the set $\{B\delta u : \delta u \in \mathcal{U}\}$ is bounded in $L^2(H) \subset L^2(V^*)$, and from Proposition 2.1, i. and ii. it follows that

$$S = \{e_y^{-1}(x)e_u(x)\delta u : \delta u \in \mathcal{U}\}$$

is bounded in W. By the differentiability assumption on J at x the second derivative $J_{yy}(x)$ can be identified with an element of $\mathcal{L}(W \cup L^p(V), W^* \cap L^q(V^*))$ ($q = \frac{p}{p-1}$ for $p > 1$, $q = \infty$ for $p = 1$). As a consequence, $J_{yy}(x)(S)$ is a bounded subset of $W^* \cap L^q(V^*)$. Since $y \in W$ Proposition 2.1 shows that

$$\tilde{S} = \{e_y^{-*}(x)J_{yy}(x)(z) : z \in S\} \subset W^2_{1+\epsilon} \times H \text{ for } (0 < \epsilon \leq \min\{q-1, \frac{1}{3}\}). \tag{39}$$

Since $W^2_{1+\epsilon} \hookrightarrow\hookrightarrow L^2(H)$ and $e_u^*(x)z = -B^*z^1$ for $z = (z^1, z^2) \in Z$ the set $e_u^*(x)\tilde{S}$ is pre-compact in U.
To cope with the second addend let $z \in S$. Since $\lambda^1 \in L^2(V)$ (even in $W^2_{1+\min\{q-1,\frac{1}{3}\}}$), straightforward estimation yields that the element $\langle e^1_{yy}(x)(z, \cdot), \lambda^1\rangle = H(\lambda^1)z$ of W^* is in fact in $W^* \cap L^{4/3}(V^*)$. A further application of Proposition 2.1 gives $\{e_y^{-*}H(\lambda^1)z; z \in S\}$ bounded in $W^2_{\frac{4}{3}} \times H$. Since $W^2_{\frac{4}{3}} \hookrightarrow\hookrightarrow L^2(H)$ it follows as above that the set

$$\{e_u^*(x)e_y^{-*}(x)\langle e^1_{yy}(x)(z, \cdot), \lambda^1\rangle; z \in S\}$$

is pre-compact in U. $\square$

CHAPTER 4

Convergence for second order methods

Up to now in most approaches to optimal control problems involving the instationary Navier-Stokes equations gradient-type methods have been proposed as numerical solution algorithms [**1, 14, 20, 39, 50**]. Clearly, the drawback of gradient-type algorithms is their slow, in general linear, convergence which is accompanied by a huge number of function and gradient evaluations. Numerically evaluating cost functions and computing gradient information necessitates the numerical solution of the Navier-Stokes equations and of a backward-in-time linear system of convection-diffusion equations with the computed Navier-Stokes solution as coefficient function. Therefore, slow convergence of the iterative solver corresponds to a large number of system solutions and thus, to very high computational costs, even in two-dimensional problems [**50, 87**]. In any case these facts justify the investigation of solution approaches with better convergence properties, compare [**43**].

Ghattas and Bark [**32**] applied Newton and quasi-Newton SQP techniques to the solution of stationary optimal control problems in two and three dimensions and compared them to the classical gradient method. Their investigations insinuate the potential of second order methods in the field of flow control. For a numerical application of a quasi-Newton method to the control of Navier-Stokes equations see [**12**].

This chapter contains a description and a comparison of second order methods to solve to optimization problem (22) and (23). In this context there has to be distinguished between methods incorporating the Navier-Stokes equations as explicit constraints and methods utilizing the one-to-one correspondence state-control which is a consequence of Theorem 2.1.

The first six Sections in this chapter contain a discussion with respect to complexity and local convergence properties of the following algorithms.

i.) Newton method in the control space (Section 1)
ii.) BFGS-method (Section 2)
iii.) Classical SQP method (Section 3)
iv.) Schur-complement SQP method (Section 4)
v.) Reduced SQP method (Section 5)
vi.) Reduced SQP-BFGS method (Section 6)

In Section 7 numerical results for Newton's method applied to a tracking-type control problem are presented. Among other things the presentation contains a numerical comparison of the method to the gradient and conjugate gradient methods with step-size control.

Unless otherwise specified throughout this chapter u^* denotes a solution to (23), x^* a solution to (22) and λ^* the corresponding Lagrange multiplier (which exists and is unique due to Theorem 3.1). $N(u^*)$ is the neighborhood of u^* determined by the use of the implicit function theorem as in Section 2.

1. Newton's algorithm

Newton's algorithm applied to the solution of the optimization problem (23) is given by

ALGORITHM 1.1. (Newton Algorithm).

(1) Choose $u^0 \in N(u^*)$, set $k = 0$.
(2) Do until convergence
   i) solve $\hat{J}''(u^k)\delta u^k = -\hat{J}'(u^k)$,
   ii) update $u^{k+1} = u^k + \delta u^k$,
   iii) set $k = k + 1$.

The dimension of the linear system in 2. i) is that of the control space U. From the characterization (35) of the Hessian $\hat{J}''(u^k)$ one concludes that its evaluation requires as many solutions of the linearized Navier–Stokes equations (28) and the adjoint equations (29) with appropriate right hand sides as the dimension of U. If U is infinite dimensional an appropriate discretization must be carried out. Now assume that the dimension of U is large so that direct evaluation of $\hat{J}''(u^k)$ is not feasible. In this case 2. i) must be solved iteratively, e. g. by a conjugate gradient technique. Then 2. i) is refered to as the "inner " loop as opposed to the do–loop in 2. which is the "outer" loop of the Newton algorithm. The inner loop at iteration level k of the outer loop requires to

(i) evaluate $\hat{J}'(u^k)$, i. e. given u^k compute $y(u^k)$ from (11) and λ^1 from (29) with $x = (y(u^k), u^k)$,
(ii) iteratively evaluate the action of $\hat{J}''(u^k)$ on δ_j^k, the j–th iterate of the inner loop on the k–th level of the outer loop.

More detailed, the iterate $q = \hat{J}''(u^k)\delta_j^k$ is obtained by successively applying the steps

a) solve in $L^2(V^*)$ for $v \in W$

$$\begin{aligned} v_t + (v \cdot \nabla)y + (y \cdot \nabla)v - \nu\Delta v &= B\delta_j^k \\ v(0) &= 0, \end{aligned}$$

where $y = y(u^k)$,

b) evaluate $J_{yy}(x)v + \langle e_{yy}^1(x)(v, \cdot), \lambda^1\rangle$,

c) solve in W^* for $w \in L^2(V)$

$$e_y(x)^*w = J_{yy}(x)v + \langle e_{yy}^1(x)(v, \cdot), \lambda^1\rangle$$

d) and finally set $q := J_{uu}\delta u + B^*w$.

Recall that for $s \in W$

$$\langle e^1_{yy}(x)(v,s), \lambda^1\rangle = \int_0^T \int_\Omega ((v\cdot\nabla)s\lambda^1 + (s\cdot\nabla)v\lambda^1)dx\,dt,$$

so that the second addend on the right-hand side of c) is even an element of $L^{4/3}(V^*) \cap W^*$. If now $J_{yy}(x)v \in L^q(V^*) \cap W^*$ Proposition 2.1 implies $w \in W^2_{1+\epsilon}$ for $0 \le \epsilon \le \min\{q-1, \frac{1}{3}\}$. Moreover, w is the variational solution of

$$\begin{aligned} -w_t + (\nabla y)^t w - (y\cdot\nabla)w - \nu\Delta w &= J_{yy}(x)v + \langle e^1_{yy}(x)(v,\cdot), \lambda^1\rangle \\ w(T) &= 0. \end{aligned}$$

Summarizing, for the outer iteration of the Newton method one Navier–Stokes solve for $y(u^k)$ and one linearized Navier–Stokes solve for $\lambda(u^k)$ are required. For the inner loop one forward (–in time) as well as one backwards linearized Navier–Stokes solve per iteration are necessary.

REMARK 1.1. Concerning initialization note that if initial guesses $(y_0, u_0) \in W \times U$ are available (with y_0 not necessarily $y(u_0)$) then alternatively to the initialization in Algorithm 1.1 this information can be used advantageously to compute the adjoint variable λ^1 required for the initial guess for the right hand side of the linear system as well as to carry out steps a) - c) for the evaluation of the Hessian. There is no necessity to recompute $y(u_0)$ from u_0.

For Algorithm 1.1 one has the following local convergence result for its application to problem (23).

THEOREM 1.1. (Local convergence of Newton's method)
Let u^* denote a solution to problem (23). Assume that the corresponding state $y^* = y(u^*)$ has its initial condition y_0 in H. Furthermore, let $J_y(x^*) \in W^*$ be sufficiently small, $J_{yy}(x^*) \in \mathcal{L}(W, W^*)$ be positive semi-definite, $J_{uu}(x^*) \in \mathcal{L}(U)$ be positive definite, where $x^* = (y(u^*), u^*)$, and let $B \in \mathcal{L}(U, L^2(V^*))$. Then there exist a neighbourhood $\mathcal{U}(u^*)$ such that u^* is the only solution to problem (23) in $\mathcal{U}(u^*)$ and for every starting value $u^0 \in \mathcal{U}(u^*)$ the iterates $\{u^n\}_{n\in\mathbb{N}}$ of Newton's Algorithm 1.1 converge quadratically to u^*.

Proof. Since $y_0 \in H$ and $B \in \mathcal{L}(U, L^2(V^*))$, Theorem 2.1 implies $y^* \in W$. This in turn together with Lemma 4.1 yields positive definiteness of the Hessian $\hat{J}''(u^*)$. Therefore, u^* is a local solution on U. Moreover, by a continuity argument, there exists a neighbourhood $\mathcal{U}(u^*) \subset U$ such that $\hat{J}''(u)$ is Lipschitz continuous for all elements $u \in \mathcal{U}(u^*)$. Therefore, the suppositions of both, the Newton-Mysovskii and the Newton-Kantorovitch Theorems [**26**, Theorem 1.1,Theorem 1.2] can be satisfied, so that the claim follows. □

REMARK 1.2. Newton's method is locally second order convergent for the minimization problem (23) with both, $\hat{J}$ defined through F_1 from (24) and F_2 from (25).

2. The BFGS method

To avoid the difficulties of evaluating the action of the exact Hessian in Algorithm 1.1 one can resort to quasi–Newton algorithms. Here, one of the most prominent candidates, the BFGS–method is considered. In order to formulate the algorithm define for w and z in U the rank–one operator $w \otimes z \in \mathcal{L}(U)$, whose action is given by

$$(w \otimes z)(v) = \langle z, v \rangle_U \, w.$$

In the BFGS–method the Hessian $\hat{J}''$ at u^* is approximated by a sequence of operators H^k.

ALGORITHM 2.1. (BFGS–Algorithm)

(1) Choose $u^0 \in N(u^*)$, $H^0 \in \mathcal{L}(U)$ symmetric, set $k = 0$.
(2) Do until convergence
  i) solve $H^k \delta u^k = -\hat{J}'(u^k)$,
  ii) update $u^{k+1} = u^k + \delta u^k$,
  iii) compute $\hat{J}'(u^{k+1})$,
  iv) set $s^k = u^{k+1} - u^k$, $d^k = \hat{J}'(u^{k+1}) - \hat{J}'(u^k)$,
  v) update $H^{k+1} = H^k + \frac{d^k \otimes d^k}{\langle d^k, s^k \rangle_U} - \frac{H^k s^k \otimes H^k s^k}{\langle H^k s^k, s^k \rangle_U}$,
  vi) set $k = k + 1$.

Note that the BFGS–algorithm requires no more system solves than the gradient algorithm applied to (22), which is one forward solution of the nonlinear equation to obtain $y(u^k)$ and one backward solve of the linearized equation (30) obtain the adjoint variable $\lambda(u^k)$. For the BFGS algorithm applied to the solution of problem (23) the following convergence result holds true.

THEOREM 2.1. (Local convergence of the BFGS method)
Let u^* denote a solution to problem (23). Assume that the corresponding state $y^* = y(u^*)$ has its initial condition y_0 in H. Furthermore, let $J_y(x^*) \in W^*$ be sufficiently small, $J_{yy}(x^*) \in \mathcal{L}(W, W^*)$ be positive semi-definite and $J_{uu}(x^*) \in \mathcal{L}(U)$ be positive definite, where $x^* = (y(u^*), u^*)$. Then there exist a $\delta > 0$ and an $\epsilon > 0$ such that for all u^0 and all symmetric and positive definite operators $H^0 \in L(U)$ with

$$|H^0 - \hat{J}''(u^*)|_{\mathcal{L}(U)} < \delta \quad \text{and} \quad |u^0 - u^*|_U < \epsilon$$

the BFGS method in Algorithm 2.1 converges linearly to u^* and u^* is a local solution.
If in addition $J_1 : W \cup L^p(V) \to \mathbb{R}$ is twice continuously Fréchet differentiable for some $1 \leq p < \infty$, $B \in \mathcal{L}(U, L^2(H))$ and $H^0 := J_{uu}(x^*)$, the convergence is super-linear.

Proof: By Lemma 4.1 the Hessian $\hat{J}''(u^*)$ is self-adjoint and positive definite. This proves the first part of the theorem, i.e. local uniqueness of u^* and linear convergence of the iterates $\{u^k\}_{k \in \mathbb{N}}$ to u^* in a neighbourhood of u^*, see [**34**, Theorem 5.1]. With the additional assumptions made for the second part the claim follows from Lemma 4.3 since for the choice $H^0 := J_{uu}(x^*)$ the difference $H^0 - \hat{J}''(u^*)$ is compact, so that the compactness assumption in [**34**] is satisfied, see also [**69**, Theorem 2.3]. $\square$

REMARK 2.1. The differentiability and compactness assumptions for super-linear convergence of the iterates in the BFGS method are satisfied for the cost functionals F_1 in (24) and F_2 in (25). In both cases $J_{uu}(x) = \alpha\, \mathrm{Id}_U$, which is positive definite, $J_{yy}(x)$ is positive semi-definite, F_1 is infinitely often differentiable on W and its first derivative can be decomposed as $F_{1_y} = f_1 + f_2 \in L^2(H) + C(H)^*$, where f_2 vanishes on the set $\{v \in W; v(T) = 0\}$, see the end of Section 3. This gives the desired regularity properties of the solution w^1 to (13) with right hand side $f = f_2$, see the proof of Lemma 4.3 and of Proposition 2.1. The required differentiability properties are satisfied for F_2 if $p = 2$ since $F_{2_y} \in L^2(V^*)$.

3. Basic SQP–method

Instead of the reduced minimization problem (23) now consider (22) as a minimization problem for the functional J over the space X with the explicit constraint $e(x) = 0$.

The basic SQP-algorithm consists in applying Newton's method to the first order optimality system (34), which for briefety is rewritten as

$$\begin{aligned} L_x(x,\lambda) &= 0 \quad \text{in } X^* \\ L_\lambda(x,\lambda) &= 0 \quad \text{in } Z^*, \end{aligned} \tag{40}$$

where the Lagrangian L is defined in (31). In algorithmical form the method can be formulated as follows.

ALGORITHM 3.1. (SQP–algorithm)

(1) Choose $(x^0, \lambda^0) \in B((x^*, \lambda^*))$, set $k = 0$.
(2) Do until convergence
   i) solve

$$\begin{bmatrix} L_{xx}(x^k,\lambda^k) & e_x^*(x^k) \\ e_x(x^k) & 0 \end{bmatrix} \begin{bmatrix} \delta x^k \\ \delta\lambda^k \end{bmatrix} = -\begin{bmatrix} J_x(x^k) & + e_x^*(x^k)\lambda^k \\ & e(x^k) \end{bmatrix} \tag{41}$$

   ii) update $(x^{k+1}, \lambda^{k+1}) = (x^k, \lambda^k) + (\delta x^k, \delta\lambda^k)$,
   iii) set $k = k + 1$.

Since $e_x(x^*)$ is surjective by Theorem 3.1, due to Theorem 3.2 there exists a unique Lagrange multiplier $\lambda^* \in Z$ such that (40) holds with $(x, \lambda) = (x^*, \lambda^*)$. The SQP–method will be well defined and locally second order convergent, if in addition to the surjectivity of $e_x(x^*)$ a second order optimality condition (37) holds for $x = x^*$. Moreover, due to the regularity properties of e there exists a neighborhood $S((x^*, \lambda^*))$ such that $L_{xx}(x, \lambda)$ is uniformly positive definite on $\ker(e_x(x))$ for every $(x, \lambda) \in S((x^*, \lambda^*))$.

THEOREM 3.1. (Local convergence of the SQP method)
Let $x^* = (y^*, u^*)$ denote a solution to problem (22). Assume that the corresponding state y^* has its

initial condition y_0 in H. Furthermore, let $J_y(x^*) \in W^*$ be sufficiently small, $J_{yy}(x^*) \in \mathcal{L}(W, W^*)$ be positive semi-definite and $J_{uu}(x^*) \in \mathcal{L}(U)$ be positive definite, where $x^* = (y^*, u^*)$. Finally, let λ^* denote the Lagrange-multiplier associated to x^*. Then there exist a neighbourhood $S(x^*, \lambda^*) \subset X \times Z$ such that for all $(x^0, \lambda^0) \in S(x^*, \lambda^*)$ the SQP-algorithm 3.1 is well defined and its iterates $\{(x^n, \lambda^n)\}_{n \in \mathbb{N}}$ converge quadratically to (x^*, λ^*) in $W \times U \times L^2(V) \times H$.

Proof: Since J and e are twice Fréchet differentiable, $e_x(x^*)$ is surjective by Proposition 2.2 and condition (37) is satisfied due to Lemma 4.2 the claim follows for instance from the convergence results obtained in [**61**] for the Augmented Lagrangian-SQP method, which includes the basic SQP-method as special case. $\square$

REMARK 3.1. With the assumptions of Theorem 3.1 the convergence of the Augmented Lagrangian-SQP Algorithms 1 and 2 in [**61**] for the control problem (22) can easily be established.

Just as for Newton's method step 2. i.) is the difficult one. While in contrast to Newton's method neither the Navier–Stokes equation nor its linearization need to be solved, the dimension of the system matrix which is twice the dimension of the state plus the dimension of the control space is formidable for applications in fluid mechanics. In addition from experience with Algorithm 3.1 for other optimal control problems, see [**68, 108**] for example, it is well known that preconditioning techniques must be applied to solve (41) efficiently. As a preconditioner one might consider the (action of the) operator $P\colon X^* \times Z^* \to X \times Z$ given by

$$P = \begin{bmatrix} 0 & 0 & R \\ 0 & J_{uu}(x^k)^{-1} & 0 \\ R^* & 0 & 0 \end{bmatrix},$$

where $R\colon Z^* \to W$ is the inverse to the (discretized) instationary Stokes operator or the (discretized) linearization of the Navier–Stokes equation at the state y^k, either one with homogenous boundary conditions.

One iteration of the preconditioned version of Algorithm 3.1 therefore requires two linear parabolic solves, one forward and one backwards in time. As a consequence, even with the application of preconditioning techniques, the numerical expense counted in number of parabolic system solves is less for the SQP–method than for Newton's method. However, the number of iterations of iterative methods applied to solve the system equations in Algorithms 1.1 and 3.1 strongly depends on the system dimension, which is much larger for Algorithm 3.1 than for Algorithm 1.1.

3.1. SQP compared to Newton. Newton's method applied to the reduced problem 23 can be characterized as a version of the SQP-algorithm which conserves feasibility of its iterates with respect to the subsidiary condition $e(x) = 0$. Infact, one can rewrite the update step 2. i) in Algorithm 1.1 and bring it into the form of the SQP-step (41). To begin with observe that the right hand side in the update step in Newton's algorithm can be written with the help of the adjoint variable λ from (29) and the operator $T(x)$ defined in (32) as

$$-\hat{J}_u(u) = -J_u(x) - e_u^*(x)\lambda = -T^*(x)\begin{bmatrix} 0 \\ J_u(x) + e_u^*(x)\lambda \end{bmatrix},$$

where the iteration indices are dropped. As a consequence, with $\delta y = y'(u)\delta u$ from (28) the update can be written as

$$T^*(x)L_{xx}(x,\lambda)\begin{bmatrix} \delta y \\ \delta u \end{bmatrix} = -T^*(x)\begin{bmatrix} 0 \\ J_u(x) + e_u^*(x)\lambda \end{bmatrix},$$

so that

$$L_{xx}(x,\lambda)\begin{bmatrix} \delta y \\ \delta u \end{bmatrix} + \begin{bmatrix} 0 \\ J_u(x) + e_u^*(x)\lambda \end{bmatrix} \in \mathcal{N}(T^*(x))$$

holds. Since $e_x(x) \in \mathcal{L}(X, Z^*)$ and $\mathcal{N}(e_x(x)) = \mathcal{R}(T) \subseteq X$ it follows that $\mathcal{R}(T)$ is closed and one has the sequence of identities

$$\mathcal{N}(T^*(x)) = \mathcal{R}(T(x))^\perp = \mathcal{N}(e_x(x))^\perp = \mathcal{R}(e_x^*(x)).$$

Thus, there exists $\delta\lambda \in Z$ such that

$$-e_x^*(x)\delta\lambda = L_{xx}(x,\lambda)\begin{bmatrix} \delta y \\ \delta u \end{bmatrix} + \begin{bmatrix} 0 \\ J_u(x) + e_u^*(x)\lambda \end{bmatrix}.$$

Using this equation together with the definition of δy, Newton's update may be rewritten as

$$\begin{bmatrix} L_{xx}(x,\lambda^k) & e_{x^*}(x) \\ e_x(x) & 0 \end{bmatrix}\begin{bmatrix} \delta y \\ \delta u \\ \delta\lambda \end{bmatrix} = -\begin{bmatrix} 0 \\ J_u(x) + e_u^*(x)\lambda \\ 0 \end{bmatrix}. \tag{42}$$

To further compare the structure of the Newton and the SQP–methods assume for an instance that x is feasible for the primal equation, i.e. $e(x) = 0$ and (x,λ) is feasible for the adjoint equation (29), i.e. $e_y^*(x)\lambda = -J_y(x)$. Then the right hand side of (41) has the form

$$-\begin{bmatrix} 0 \\ J_u(x) + e_u^*\lambda \\ 0 \end{bmatrix}$$

and comparing to (42) one observes that the linear systems describing the Newton and the SQP–methods coincide. In general the nonlinear primal and the linearized adjoint equation will not be satisfied by the iterates of the SQP–method. Therefore, one may refer to the SQP–method as an outer or unfeasible method, while the Newton method is a feasible one.

4. Schur–complement SQP–method

This and the following sections are devoted to a discussion of the basic SQP–method with the goal of reducing the size of the system matrix in 2. i.) of Algorithm 3.1. In [**68**] it is proposed to use the Schur complement in step 2. i) of Algorithm 3.1. To formulate the Schur–complement SQP–method one shall require that

(H3) $$L_{xx}^{-1}(x,\lambda) \in \mathcal{L}(X^*, X) \quad \text{for all } (x,\lambda) \in S((x^*,\lambda^*)).$$

Here, $S((x^*,\lambda^*))$ denotes a neighborhood of (x^*,λ^*). For $(x,\lambda) \in S((x^*,\lambda^*))$ define

$$\mathcal{O}(x,\lambda) \colon = e_x(x) L_{xx}(x,\lambda)^{-1} e_x^*(x).$$

Then, the operator $\mathcal{O}(x,\lambda)$ is an element of $\mathcal{L}(Z,Z^*)$ for all $(x,\lambda) \in S(x^*,\lambda^*))$.

The Schur–complement algorithm results from Algorithm 3.1 by first eliminating δx in favour of the variable $\delta\lambda$ and then solving for δx.

ALGORITHM 4.1. (Schur–complement SQP–algorithm).
This is Algorithm 3.1 with 2. i.) replaced by

i) Solve

α) $\mathcal{O}(x^k,\lambda^k)\delta\lambda^k = e(x^k) - e_x(x^k) L_{xx}(x^k,\lambda^k)^{-1}\left\{J_x(x^k) + e_x^*(x^k)\lambda^k\right\}$,

β) $L_{xx}(x^k,\lambda^k)\delta x^k = -\left\{J_x(x^k) + e_x^*(x^k)\lambda^k\right\} - e_x^*(x^k)\delta\lambda^k$.

The right-hand sides in α) and β) are elements of Z^* and X^*, respectively. When solving for the increments in α) and β) the operator L_{xx}^{-1} has to be applied repeatedly. To consider it more closely recall that J is of separable type, i.e. $J(y,u) = J_1(y) + J_2(u)$. Then

$$L_{xx}(x,\lambda)\begin{bmatrix}\delta y\\ \delta u\end{bmatrix} = \begin{bmatrix} r_y \\ r_u\end{bmatrix} \text{ in } X^*$$

results in

$$J_{uu}(x)\delta u = r_u \text{ in } U$$

and

$$J_{yy}(x)\delta y + (\nabla\delta y)^t\lambda^1 - (\delta y\nabla)\cdot\lambda^1 = r_y \text{ in } W^*. \tag{43}$$

For the later equation in a neighborhood of (x^*,λ^*) one can prove

LEMMA 4.1. Let x^* denote a solution to (22) and let λ^* be the corresponding Lagrange multiplier. Assume that $J_{uu}(x^*)$ admits a bounded inverse and that $J_{yy}(x^*)$ is positive definite. Assume further that $J_y(x^*) \in W^*$ is sufficiently small. Then, there exists a neighborhood $S(x^*,\lambda^*) \subset X \times Z$ such that for all $(x,\lambda) \in S(x^*,\lambda^*)$ equation (43) admits a unique solution $\delta y \in W$ for every $r_y \in W^*$. Moreover, the operator $\mathcal{O}(x,\lambda)$ admits a bounded inverse in $S(x^*,\lambda^*)$.

Proof. It can be shown with the techniques applied in the proof of Lemma 4.2 that

$$a : W \times W \to \mathbb{R}, \quad a(u,v) := \langle u, J_{yy}(x)v\rangle_{W,W^*} + \langle u, H(\lambda^1)v\rangle_{W,W^*}$$

with $H(\lambda^1)$ defined in (26) due to the positive definiteness of $J_{yy}(x^*)$ defines a continuous coercive form on $W \times W$ for all (x,λ) in a sufficiently small neighborhood $S(x^*,\lambda^*)$ of (x^*,λ^*). The first claim now follows from the Lax-Milgram lemma [**3**, Satz 4.7]. The second claim follows with the assumptions on $J_{uu}(x^*)$ and $J_{yy}(x^*)$, the properties of the subsidiary condition $e(x^*)$ and its linearization, and a continuity argument. □

LEMMA 4.2. Let $x \in X$ and $\lambda(x)$ be given by (29). If $J_{yy}(x) \in \mathcal{L}(W,W^*)$ and $J_{uu}(x) \in \mathcal{L}(U)$ are invertible, then **(H3)** holds with (x^*,λ^*) replaced by (x,λ) provided $|J_y(x)|_{W^*}$ is sufficiently small.

Proof. Since $(x,\lambda) \mapsto L_{xx}(x,\lambda)$ from $W \times L^2(V)$ to $\mathcal{L}(X^*,X)$ is continuous it suffices to argue that $L_{xx}(x,\lambda)$ is continuously invertible. Due to the invertibility assumptions on $J_{yy}(x)$ and $J_{uu}(x)$, and the structure of $L_{xx}(x,\lambda)$ it suffices to assert that $\langle e^1_{yy}(x)(\cdot,\cdot),\lambda^1\rangle$ is sufficiently small in $\mathcal{L}(W,W^*)$. This can be achieved by making $|J_y(x)|_{W^*}$ small. □

Theorem 3.1 together with Lemma 4.1 (or together with Lemma 4.2) gives local convergence of the Schur-complement SQP-algorithm 4.1.

THEOREM 4.1. (Local convergence of the Schur-complement SQP-method)
With the notations and suppositions of Theorem 3.1, let the assumptions of Lemma 4.1 (Lemma 4.2) be satisfied. Then there exist a neighbourhood $S(x^*,\lambda^*) \subset X \times Z$ such that for all $(x^0,\lambda^0) \in S(x^*,\lambda^*)$ the Schur-complement SQP-algorithm 4.1 is well defined and its iterates $\{(x^n,\lambda^n)\}_{n\in\mathbb{N}}$ converge quadratically to (x^*,λ^*).

Note that (43) is a transport equation if J_1 is of tracking type as is the case for F_1 defined in (24). Transport problems of this kind are investigated in [**10**]. Comparing the number of necessary equation solves for the inner loop of Algorithm 4.1 to those of Algorithm 1.1 and keeping in mind that the number of necessary iterations is determined by the dimension of the linear system the Schur–complement SQP–method does not appear to be an efficient competitor to Newton's method even if the dimension of (the discretization of) U happens to be large. The difference in favour of Newton's methods is further enforced by the experience that the Schur–complement SQP–method requires preconditioning to be efficient. This experience has admittedly be made for a different class of optimal control problems [**68**] (where the nonlinearity is of exponential type and L_{xx} can easily be inverted) but one expects that preconditioning would also be required for optimal control for the Navier–Stokes equation. Preconditioners, however, typically require further linear system solves.

5. Reduced SQP–method

The idea of the reduced SQP–method is to replace (41) (for $x = x^k$) with an equation in $\ker e_x(x)$, so that the reduced system is of smaller dimension than the original one. The following development of the reduced system follows the lines of [**76**]. To begin with recall the definition of $T(x): U \to X$ and define $A(x): Z^* \to X$ by

$$A(x) = \begin{bmatrix} e_y^{-1}(x) \\ 0 \end{bmatrix}. \tag{44}$$

Then, A is a right–inverse to $e_x(x)$. In fact, one has

i) $\ker e_x(x) = \mathcal{R}(T(x)) = \left\{ \begin{bmatrix} -e_y^{-1}(x)e_u(x)v \\ v \end{bmatrix} : v \in U \right\},$

ii) $e_x(x)T(x) = 0$ in Z^*,

iii) $e_x(x)A(x) = I_{Z^*}$.

By Proposition 2.2 and $B \in \mathcal{L}(U, L^2(V^*))$ the operator $T(x)$ is an isomorphism from U to $\ker e_x(x)$ and hence the second equality in (41) given by

$$e_x(x)\delta x = -e(x)$$

can be expressed as

$$\delta x = T(x)\delta u - A(x)e(x). \tag{45}$$

Using this in the first equality of (41) one finds

$$L_{xx}(x,\lambda)T(x)\delta u - L_{xx}(x,\lambda)A(x)e(x) + e_x^*(x)\delta\lambda = -(J_x(x) + e_x^*(x)\lambda).$$

Application of $T^*(x)$ to this last equation and ii) from above imply that if δu is a solution coordinate of (41) then it also satisfies

$$T^*(x)L_{xx}(x,\lambda)T(x)\delta u = T^*(x)L_{xx}(x,\lambda)A(x)e(x) - T^*(x)J_x(x). \tag{46}$$

Once δu is computed from (46) then δy and $\delta\lambda$ can be obtained from (45) (which requires one forward linear parabolic solve) and the first equation in (41) (another backwards linear parabolic solve).

Note that if x is feasible then the first term on the right hand side of (46) is zero and (46) is identical to step 2. i) in Newton's Algorithm 1.1.

This again reflects the fact that Newton's method can be viewed as an SQP–method that obeys the feasibility constraint $e(x) = 0$. It also points at the fact that the amount of work (measured in equation solves) for the inner loop coincides for both the Newton and the reduced SQP–methods. The significant difference between the two methods lies in the outer iteration. To make this evident next the reduced SQP–algorithm is specified.

ALGORITHM 5.1. (Reduced SQP–algorithm with exact second order information).

(1) Choose $x^0 \in B(x^*)$, set $k = 0$.
(2) Do until convergence
   i) Lagrange multiplier update: solve
$$e_y^*(x^k)\lambda^k = -J_y(x^k)$$
   ii) Solve
      α)
$$\begin{aligned} T^*(x^k)L_{xx}(x^k,\lambda^k)T(x^k)\delta u^k &= T^*(x^k)L_{xx}(x^k,\lambda^k)A(x^k)e(x^k) \\ &\quad -T^*(x^k)J_x(x^k) \end{aligned}$$
      β)
$$e_y(x^k)\delta y^k = -e(x^k) - e_u(x^k)\delta u^k,$$
   iii) update
$$x^{k+1} = x^k + (\delta y^k, \delta u^k),$$
   iv) set $k = k + 1$.

Note that in the algorithm specified the update of the Lagrange variable is different to that derived in the procedure outlined above. In fact for reduced SQP–methods there is no "optimal" update strategy for λ. The two choices described above are natural and frequently used [**76, 79**]. To implement Algorithm 5.1 two linear parabolic systems have to be solved in steps 2. i) and 2. ii) β) and, in addition two linear parabolic system solves are necessary to evaluate the term involving the operator A on the right hand side of 2. ii) α). In applications this term is often neglected since it vanishes at x^*.

The differences between the reduced SQP–method and Newton's method are stated next.

(1) Most significantly the velocity field is updated by means of the nonlinear equation in Newton's method and via the linearized equation in the reduced SQP–method.
(2) The right hand sides of the linear systems differ due to the appearance of the term involving the operator A. As mentioned above this term is frequently not implemented.
(3) Formally there is a difference in the initialization procedure in that y^0 is chosen independently from u^0 in the reduced SQP–method and $y^0 = y(u^0)$ in Newton's method. However, if a good initial guess y^0 independent from $y(u^0)$ is available, it can be utilized in Newton's method as well.

Thus, the methods turn out to be very similar. The preliminaries for the proof local convergence for the reduced SQP algorithm are given in the following two lemmas.

LEMMA 5.1. Let $x \in X$. Then, the right-inverse $A(x)$ defined in (44) is an element of $\mathcal{L}(Z^*, X)$, the operator $T(x) \in \mathcal{L}(U, X)$ and the mappings $x \mapsto R(x)$ from X into $\mathcal{L}(Z^*, X)$ and $x \mapsto T(x)$ from X into $\mathcal{L}(U, X)$ are Fréchet differentiable with Lipschitz continuous derivatives.

Proof. An immediate consequence of i., ii. in Proposition 2.1 and the identities ii) and iii) above together with the differentiability properties of the mapping $x \mapsto e_x(x)$. $\square$

LEMMA 5.2. The mapping $x \mapsto \lambda(x)$ from $X \to Z$ defined by (29) is locally Lipschitz continuous.

Proof. Let $\tilde{x} \in X$ and let $\mathcal{U}(\tilde{x}) \subset X$ be a (bounded) neighbourhood of $\tilde{x}$. For $x, \bar{x} \in \mathcal{U}(\tilde{x})$ set

$$\bar{\lambda} = \lambda(\bar{x}) = -e_y^{-*}(\bar{x})J_y(\bar{x}), \quad \lambda = \lambda(x) = -e_y^{-*}(x)J_y(x).$$

Then

$$\bar{\lambda} - \lambda = -\left[e_y^{-*}(\bar{x})(J_y(\bar{x}) - J_y(x))\right] + \left[e_y^{-*}(\bar{x}) - e_y^{-*}(x)\right] J_y(x),$$

so that straightforward estimation, utilizing $J_{yu} \equiv 0$, gives

$$|\bar{\lambda} - \lambda|_Z \leq \|e_y^{-*}(\bar{x})\|_{\mathcal{L}(W^*,Z)} \int_0^1 \|J_{yy}(y + s(\bar{y} - y), \bar{u})\|_{\mathcal{L}(W,W^*)}\, ds\, |\bar{y} - y|_W + \left| \left[e_y^{-*}(\bar{x}) - e_y^{-*}(x)\right] J_y(x)\right|_Z.$$

With $\mu := -e_y^{-*}(\bar{x})J_y(x)$ the last addend can be rewritten as

$$\left| \left[e_y^{-*}(\bar{x}) - e_y^{-*}(x)\right] J_y(x)\right|_Z = |\lambda^1 - \mu^1|_{L^2(V)} + |\lambda^0 - \mu^0|_H.$$

Furthermore, $w := \lambda^1 - \mu^1$ solves

$$-w_t - \nu\Delta w - (\bar{y}\nabla)w + (\nabla\bar{y})^t w = ((y - \bar{y})\nabla)\lambda^1 - (\nabla(y - \bar{y}))^t\lambda^1 =: F \quad \text{in } W^*, \quad w(T) = 0.$$

With the techniques of the proof of Lemma 4.2 and Proposition 2.1, iii.and iv. one gets $F \in W^* \cap L^\alpha(V^*)$ for $1 \leq \alpha \leq 4/3$, so that $w(T)$ is well defined and

$$|F|_{W^*} + |F|_{L^\alpha(V^*)} \leq C|y - \bar{y}|_W\, |\lambda^1|_{L^2(V)}$$

with a positive constant C depending on $|\bar{y}|_{L^2(V)}$ and $|\bar{y}|_{L^\infty(H)}$. This together with a further inspection of the proof of the Proposition 2.1 also gives

$$|\lambda^0 - \mu^0|_H + |\lambda^1 - \mu^1|_{L^2(V)} + |\lambda^1_t - \mu^1_t|_{L^\alpha(V^*)} \leq C|y - \bar{y}|_W\, |\lambda^1|_{L^2(V)}.$$

Finally, by Proposition 2.1, iii. there holds

$$|\lambda^1|_{L^2(V)} \leq C(|\bar{y}|_{L^\infty(H)}, |\bar{y}|_{L^2(V)})\, |J_y(x)|_{W^*}.$$

Altogether

$$|\bar{\lambda} - \lambda|_Z \leq C\big(\sup_{x \in \mathcal{U}(\tilde{x})} \{|y|_{L^\infty(H)}, |y|_{L^2(V)}\}\big)\big(\sup_{x \in \mathcal{U}(\tilde{x})} \|e_y^{-*}(x)\|_{\mathcal{L}(W^*,Z)} \sup_{x \in \mathcal{U}(\tilde{x})} \|J_{yy}(x))\|_{\mathcal{L}(W,W^*)} + \sup_{x \in \mathcal{U}(\tilde{x})} |J_y(x))|_{W^*}\big)\, |\bar{x} - x|_X,$$

which is the claim. $\square$

The convergence result for Algorithm 5.1 is stated in

THEOREM 5.1. Let λ^* denote the Lagrange multiplier associated to x^*. Assume that $J_{yy}(x) \in \mathcal{L}(W, W^*)$ is positive semi-definite, let $J_{uu}(x) \in \mathcal{L}(U)$ be positive definite and let $|J_y(x)|_{W^*}$ be sufficiently small. Then there exist a neighbourhood $\mathcal{U}(x^*) \subset X$ such that for all $x^0 \in \mathcal{U}(x^*)$ the reduced SQP-algorithm 5.1 is well defined and its iterates $\{x^n\}_{n\in\mathbb{N}}$ converge two-step quadratically to x^*, i.e.

$$|x^{k+1} - x^*|_X \;\leq\; C\,|x^{k-1} - x^*|_X^2$$

for some positive constant C and all $k \in \mathbb{N}$.

Proof: First note that the conclusions of Lemma 4.1 remain valid for the reduced Hessian

$$H_R(x^*, \lambda^*) \;= T(x^*)^* L_{xx}(x^*, \lambda^*) T(x^*) \;\in \mathcal{L}(U). \tag{47}$$

Furthermore, a continuity argument implies positive definiteness of $H_R(x, \lambda)$ in a neighbourhood $\tilde{\mathcal{U}}(x^*)$ of x^*. By Lemma 5.1 the mappings $x \mapsto T(x)$ and $x \mapsto A(x)$ are Fréchet differentiable with Lipschitz continuous derivatives. Furthermore, by Lemma 5.2 the mapping $x \mapsto \lambda(x)$ is locally Lipschitz continuous. This in particular implies for the Lagrange multiplier updates λ^k the estimate

$$|\lambda^k - \lambda|_Z \;\leq\; C\,|x^k - x^*|_X, \quad x^k \in \tilde{\mathcal{U}}(x^*),$$

with a positive constant C. Altogether, the assumptions for Corollary 3.6 in [**79**] are met and there exists a neighbourhood $\hat{\mathcal{U}}(x^*)$ such that for all $x^0 \in \mathcal{U}(x^*) := \hat{\mathcal{U}}(x^*) \cap \tilde{\mathcal{U}}(x^*)$ the claim follows. $\square$

6. Reduced SQP-BFGS method

Except for the BFGS method, common to all methods presented up to now is the presence of an inner iteration. Even if the dimension of the discretized reduced Hessian (47) is small (as it may the case in boundary control problems) its structure makes its numerical computation infeasible. A good compromise between feasible quasi-Newton methods and unfeasible SQP methods might be the reduced SQP-BFGS method. This method utilizes a positive definite approximation of the reduced Hessian in step 2.ii.)α.) of Algorithm 5.1 on the expense of two more linear parabolic systems solves. The following algorithm is taken from [**76**, Algorithm A1].

ALGORITHM 6.1. (Basic reduced SQP-BFGS algorithm)
Let x^* be a solution of problem (22).

(1) Choose $x^0 \in B(x^*)$, $H^0 \in L(U)$ positive definite, set $k = 0$
(2) Do until convergence
   i.) Lagrange multiplier update, compute
      $e_y^*(x^k)\lambda^k = -J_y(x^k)$
   ii.) Solve
      α.) $H^k \delta u^k = -T^*(x^k)\left\{J_x(x^k) - L_{xx}(x^k, \lambda^k) A(x^k) x(x^k)\right\}$

β.) $e_y(x^k)\delta y^k = -e(x^k) - e_u(x^k)\delta u^k$

iii.) Update

$$x^{k+1} = x^k + \begin{bmatrix} \delta y^k \\ \delta u^k \end{bmatrix}$$

iv.) Update H^k

$$v = T(x^k + T(x^k \delta u^k))^* J_x(x^k + T(x^k \delta u^k)) - T(x^k)^* J_x(x^k)$$
$$H^{k+1} = h^k + \frac{v \otimes v}{\langle v, \delta u^k \rangle_{U^* U}} - \frac{(H^k \delta u^k) \otimes H^k \delta u^k)}{\langle (H^k \delta u^k), \delta u^k \rangle_{U^* U}}$$

v.) $k = k+1$

enddo

(3) stop

The additional amount of numerical work in comparison to the reduced SQP method is given by the rank-2 update in step 2.iv.). Nevertheless, in the case of small system dimension of the discretized control space an inner iteration procedure can be neglected. Recall again, that every step the inner iteration amounts to the solution of two linear parabolic problems. As a conclusion the reduced SQP-BFGS method is the most promising competitor of Newton's Algorithm 1.1. There holds

THEOREM 6.1. (Local linear convergence of the reduced SQP-method)
Let $x^* = (y^*, u^*)$ denote a solution to problem (22). Assume that the corresponding state y^* has its initial condition y_0 in H. Furthermore, let $J_y(x^*) \in W^*$ be sufficiently small, $J_{yy}(x^*) \in \mathcal{L}(W, W^*)$ be positive semi-definite and $J_{uu}(x^*) \in \mathcal{L}(U)$ be positive definite. Finally, let λ^* denote the unique Lagrange-multiplier associated to x^* and let $B \in \mathcal{L}(U, L^2(V^*))$. Then there exist a $\delta > 0$ and an $\epsilon > 0$ such that for all x^0 and all positive definite $H^0 \in L(U)$ with

$$\|H^0 - H(X^*, \lambda^*)\|_{L(U,U^*)} \leq \delta \quad \text{and} \quad \|x^0 - x^*\|_X \leq \epsilon$$

the iterates $\{x^n\}_{n \in \mathbb{N}}$ of the reduced SQP-BFGS-algorithm 6.1 converge linearly to x^*.

Proof: With the same argumentation as in the proof of Lemma 4.1 it can be shown that the reduced Hessian $H_R(x^*, \lambda^*)$ defined in (47) is positive definite. It follows from Lemma 5.1 and the identities

$$e_x(x)T(x) = 0 \text{ in } Z^* \quad \text{and} \quad e_x(x)A(x) = I_{Z^*}, \quad x \in X$$

that the mappings $x \mapsto T(x)$ from X into $\mathcal{L}(U, X)$ and $x \mapsto A(x)$ from X into $\mathcal{L}(Z^*, X)$ are Fréchet differentiable with Lipschitz continuous derivatives. In particular, this implies positive definiteness of $H_R(x, \lambda)$ in a neighbourhood $\tilde{S}(x^*)$ of x^*, where λ is associated to x via i). in Algorithm 6.1. Moreover, by Theorem 3.1 $e_x(x^*)$ is surjective and from Lemma 4.2 one deduces the validity of the second order sufficiency condition (37). Altogether, we meet the assumptions 2.1 to 2.4 and the suppositions of Theorem 2.3 for Algorithm A1 in [**76**] (see also [**79**, (A1),...,(A3) and Algorithm 2.1]), i.e. there exist positive constants δ, η such that if

$$|x^0 - x^*|_W \leq \delta \quad \text{and} \quad \|H_R(x^0, \lambda^0) - H_R(x^*, \lambda^*)\|_{\mathcal{L}(U)} \leq \eta,$$

the sequence $\{x^n\}_{n\in\mathbb{N}}$ converges linearly to x^*. By a continuity argument one may choose a neighbourhood $S(x^*) \subset X$ such that both of these requirements are satisfied for starting values $x^0 \in S(x^*) \cap \tilde{S}(x^*)$, which is the desired result. □

THEOREM 6.2. (Local super-linear convergence of the reduced SQP-method)
Let all assumptions of the previous theorem be satisfied. In addition assume that $J\colon W \cup L^p(V) \to \mathbb{R}$ is twice continuously Fréchet differentiable for some $1 \leq p < \infty$, $B \in \mathcal{L}(U, L^2(H))$ and

$$H^0 := J_{uu}(x^*).$$

Then, the convergence of the iterates $\{x^n\}_{n\in\mathbb{N}}$ is super-linear.

Proof: Due to [**76**, Theorem 2.5] it is sufficient to show that the difference

$$H^0 \;-\; H_R(x^*, \lambda^*)$$

is compact. Since for the reduced Hessian $H_R(x, \lambda)$ the decomposition (35) into a compact part plus $J_{uu}(x)$ remains valid, the claim follows with Lemma 4.3. □

7. Newton's method for driven-cavity control

The numerical examples in this section should first of all demonstrate the feasibility of utilizing Newton's method for optimal control of the two-dimensional instationary Navier-Stokes equations in a workstation environment despite the formidable size of the optimization problem. The total number of unknowns (primal-, adjoint-, and control variables) in Example 7.1 below, for instance, is of order $2.2*10^6$. The control problem is given by (21) with cost function J defined by

$$J(y, u) := \frac{1}{2} \int_{Q_o} |y - z|^2 \, dxdt + \frac{\alpha}{2} \int_{Q_c} |u|^2 \, dxdt, \tag{48}$$

where $Q_c := \Omega_c \times (0, T)$ and $Q_o := \Omega_o \times (0, T)$, with Ω_c and Ω_o subsets of $\Omega = (0, 1)^2$ denoting the control and observation volumes, respectively. The flow is driven by stationary tangential blowing at the upper part of the spatial domain, i.e. the homogeneous boundary conditions in (21) are replaced by $y_1(t, \cdot) = 1$ on $(\partial\Omega)^T$. In the following examples $T = 1$, $U := L^2(Q_c)$, $\nu = 1/400$ and B is the indicator function of Q_c. The results for Newton's method will be compared to those of the gradient algorithm, which for the sake of convenience is formulated next.

ALGORITHM 7.1. **Gradient Algorithm**

(1) Set $k = 0$ and choose u^0,
(2) Set $d := -\hat{J}'(u^k)$ and compute

$$\rho^* = \text{arg min}_{\rho>0} I(\rho) := \hat{J}(u^k + \rho d)$$

(3) Set

$$u^{k+1} = u^k + \rho^* d$$

(4) Set $k = k + 1$ and goto 2.

Given a control u the evaluation of the gradient of J at a point u amounts to solving the state equations in (21) for y and (30) for the adjoint variable λ. Implementing a step size rule to determine an approximation of ρ^* is numerically expensive as every evaluation of the functional $\hat{J}$ at a control u requires solving the instationary Navier-Stokes equations with right hand side Bu.

In order to provide approximations to the optimal step size ρ^* consider for $u \in U$ and search direction $d \in U$ the solutions $v \in W$ and $w \in L^2(V)$ of the systems

(49)
$$\begin{aligned} v_t - \nu \Delta v + (y \cdot \nabla) v + (v \cdot \nabla) y &= Bd \\ v(0) &= 0 \end{aligned}$$

and

(50)
$$\begin{aligned} -w_t - \nu \Delta w - (y \cdot \nabla) w + (\nabla y)^t w &= J_{yy}(x) v - (v \cdot \nabla) \lambda^1 + (\nabla v)^t \lambda^1 \\ w(T) &= 0, \end{aligned}$$

where $y = y(u)$ and $\lambda \in Z$ defined through (29) is the associated adjoint variable.

(1) For a given search direction $d \in U$ interpolate the function $I(\rho)$ by a quadratic polynomial using the values $I(0), I'(0)$ and $I''(0)$, i.e.
$$I(\rho) \doteq I(0) + I'(0)\rho + \frac{1}{2} I''(0) \rho^2$$
and use the unique zero
$$\rho_1^* = \frac{-\langle \hat{J}'(u), d \rangle_U}{\alpha |d|_U^2 + \langle B^* w, d \rangle_U}$$
of the equation $I'(\rho) = 0$ as approximation of ρ^*, with w given by (50).

(2) Use the linearization of the mapping $\rho \mapsto y(u + \rho d)$ at $\rho = 0$,
$$y(u + \rho d) \doteq y(u) + \rho y'(u) d,$$
in the cost functional J. This results in the quadratic approximation
$$I_2(\rho) := J(y(u) + \rho y'(u) d, u + \rho d)$$
of the functional $I(\rho)$. Now use the unique root

(51)
$$\rho_2^* = \frac{-\langle \hat{J}'(u), d \rangle_U}{\alpha |d|_U^2 + |v|_{L^2(Q_o)}^2}$$

of the equation $I_2'(\rho) = 0$ as approximation of ρ^*, with v given in (49).

The denominator of ρ_1^* equals $I''(0) = \langle \hat{J}''(u) d, d \rangle_U$. Utilizing (35) it can be concluded as in the proof of Lemma 4.1 that the denominator is positive, provided that the state $y(u)$ is sufficiently close to z in $L^2(H)$.

Note that the computation of ρ_1^* requires the solution of linearized Navier-Stokes equations forward and backward in time, whereas that of ρ_2^* only requires one solve of the linearized Navier-Stokes equations. In addition, a numerical comparison shows that the step-size guess ρ_2^* performs better than ρ_1^*, both with respect to the number of iterations in the gradient method and with respect to computational time. Therefore, for the numerical results presented below the step size proposal ρ_2^* is used. Thus, every iteration of the gradient algorithm amounts to solving the nonlinear Navier-Stokes equations forward in time and the associated adjoint equations backward in time for the computation of the gradient, and to solving linearized Navier-Stokes equations forward in time for the step size proposal.

The inner iteration of Newton's method is performed by the conjugate gradient method, the choice of which is justified in a neighbourhood of a local solution u^* of the optimal control problem by the positive definiteness of $\hat{J}''(u^*)$, provided the desired state z is sufficiently close to the optimal state $y(u^*)$.

For the numerical tests the target flow z is given by the Stokes flow with boundary condition $z_1 = 1$ in tangential direction, see Fig. 1. The termination criterion for the j-th iterate δu_j^k in the conjugate gradient method is chosen as

$$\frac{|\hat{J}''(u^k)\delta u_j^k + \hat{J}'(u^k)|}{|\hat{J}'(u^0)|} \leq \min\left\{ \left(\frac{|\hat{J}'(u^k)|}{|\hat{J}'(u^0)|}\right)^{\frac{3}{2}}, 10^{-2}\frac{|\hat{J}'(u^k)|}{|\hat{J}'(u^0)|} \right\} \text{ or } j \geq 50.$$

Since the system equation 2.i) in Algorithm 1.1 is only solved approximately at best the super-linear convergence rate of the in-exact Newton method can be expected for the numerical results to be presented. The initialization for Newton's method is $u^0 := 0$.

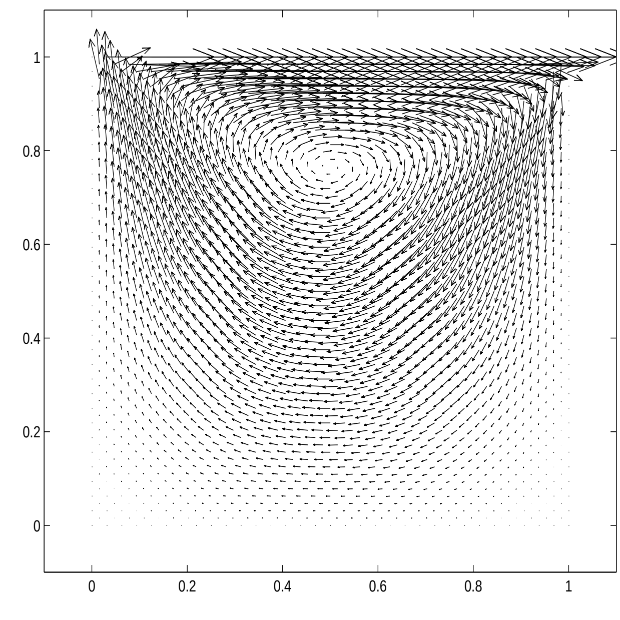

FIGURE 1. Control target, Stokes flow in the cavity

The discretization of the Navier-Stokes equations, its linearization and adjoint is carried out by using parts of the code developed by Bänsch in [**6**], which is based on Taylor-Hood finite elements for spatial discretization. As time step size $\delta t = .00625$ is taken, which results in 160 grid points for the time grid. The grid for the spatial discretization contains 545 pressure and 2113 velocity nodes. All computations are performed on a DEC-ALPHA^{TM}station 500.

Iteration	CG-steps	$\frac{\lvert\hat{J}'(u)\rvert}{\lvert\hat{J}'(u^0)\rvert}$	$\frac{\lvert\delta u^k\rvert_U}{\lvert\delta u^{k-1}\rvert_U}$	$\hat{J}(u^k)$
1	-	1.e0	-	1.196202e-2
2	13	3.358825e-1	1.	3.226486e-3
3	11	5.058497e-2	0.492	1.617913e-3
4	18	8.249029e-3	0.422	1.482032e-3
5	17	1.409278e-4	0.079	1.480533e-3
6	19	4.686819e-6	0.032	1.480534e-3

TABLE 1. Performance of Newton's method for Example 7.1

EXAMPLE 7.1. Here the results for $\Omega_c = \Omega_o = (0,1)^2$ and $\alpha = 10^{-2}$ are presented. Table 1 confirms super-linear convergence of the in-exact Newton method. To achieve the the same accuracy as Newton's method the gradient algorithm requires 96 iterations. The computing time with Newton's method is approximately 45 minutes whereas the gradient method requires 110 minutes. This demonstrates the superiority of Newton's method over the gradient algorithm for this example. For larger values of α and coarser time and space grids the difference in computing time is less drastic. In fact this difference increases with decreasing α and increasing mesh refinement. As expected a significant amount of computing time is spent for read-write actions of the variables to the hard-disc in the sub-problems, especially when there is no local hard-disc available.

In Figures 2, 3, 4 the evolution of the cost functional, the difference to the Stokes flow and the control as a function of time are documented. It can be observed that Newton's method tends to over-estimate the control in the first iteration step, whereas the gradient algorithm approximates the optimal control from below, see Figure 4. Graphically there is no significant change after the second iteration for Newton's method. These comments hold for quite a wide range of values for α.

In Fig. 5 the uncontrolled flow together with the controlled flow and the control action at the end of the time interval are presented.

In the previous example the observation volume Ω_o and the control volume Ω_c each cover the whole spatial domain. From the practical point of view this is not feasible. However, from the

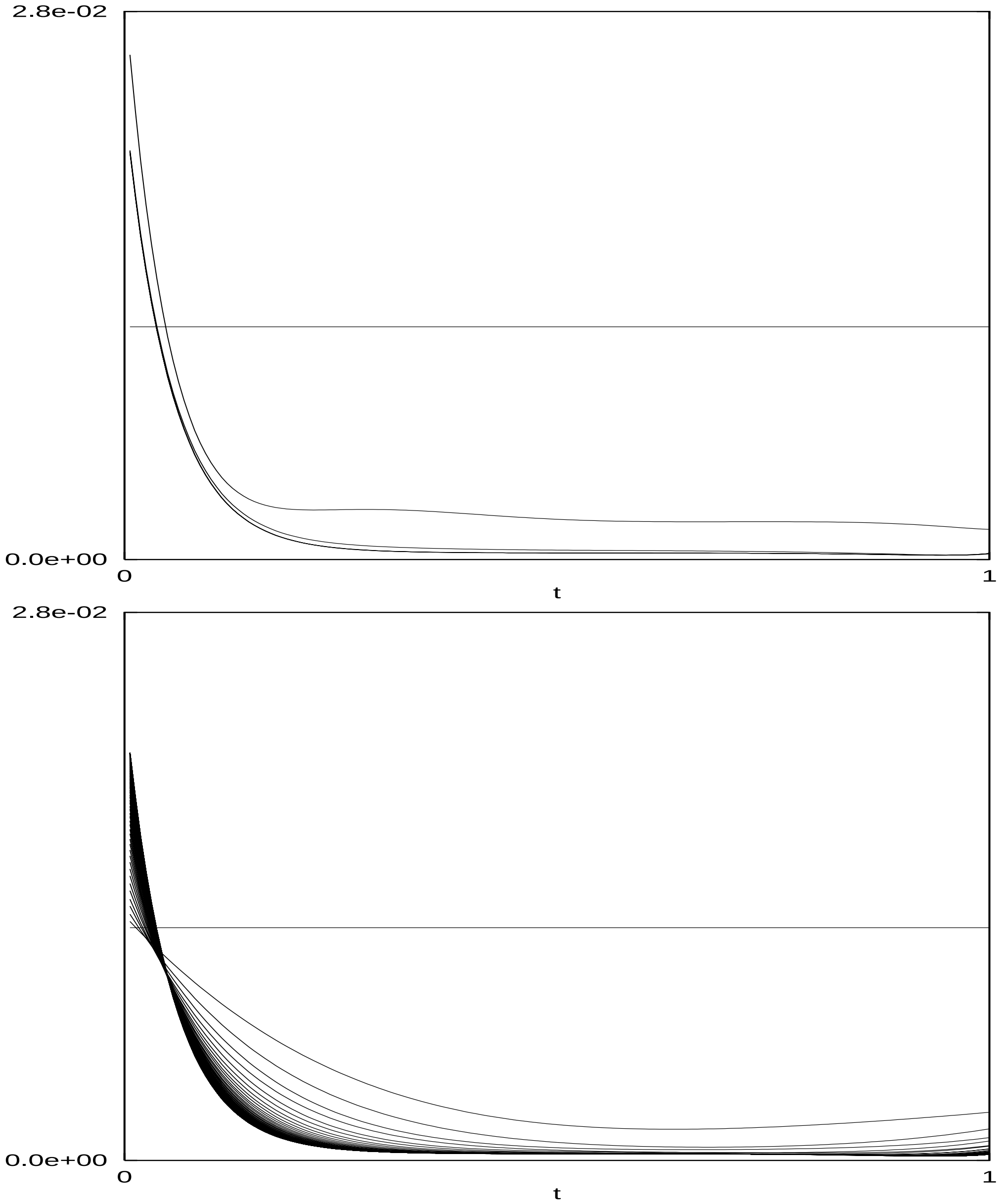

FIGURE 2. Newton's method (6 Iterations) (top) versus Gradient algorithm (96 Iterations), Re=400, $\alpha = 10^{-2}$: Evolution of cost functional for relative accuracy = 1.d-5

numerical standpoint this is a complicated situation, since the inhomogeneities in the primal and adjoint equations are large.

The next two numerical examples presented deal with different observation and control volumes. This results in smaller control and observation volumes than in Example 7.1, and thus the primal and adjoint equations are numerically simpler to solve.

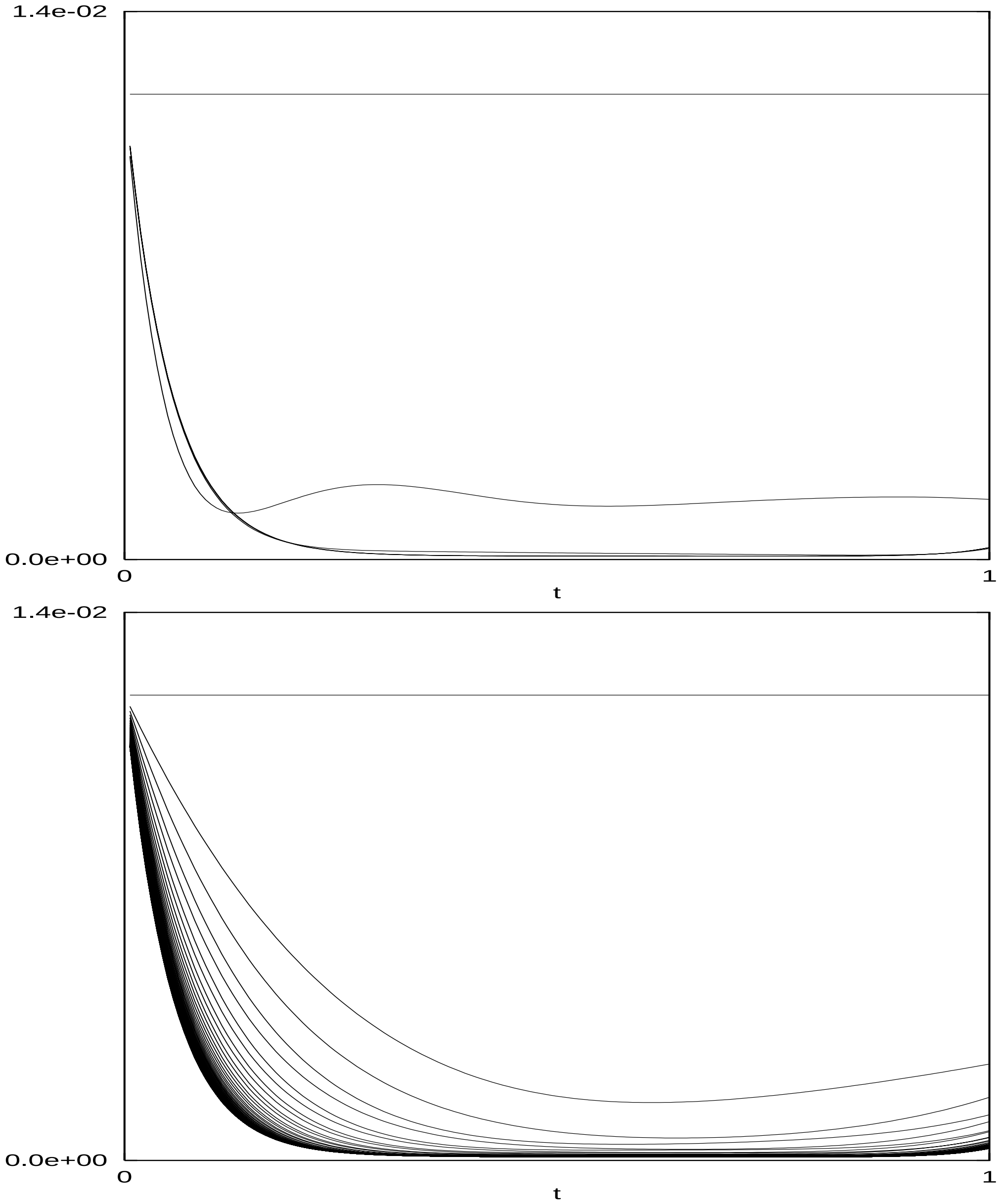

FIGURE 3. Newton's method (6 Iterations) (top) versus Gradient algorithm (96 Iterations), Re=400, $\alpha = 10^{-2}$: Evolution of difference to Stokes flow for relative accuracy = 1.d-5

EXAMPLE 7.2. Here $\Omega_o = (0., 1.) \times (0.85, 0.95)$ and $\Omega_c = (0., 1.) \times (0.9, 1.)$. The spatial and temporal discretizations as well as the parameter α are the same as in Example 7.1. Newton's method takes 15 minutes cpu-time and its convergence statistics are presented in Tab. 2. The gradient algorithm needs 25 iterations and 26 minutes cpu to reduce the value of the cost functional from $\hat{J}(u^0) = 1.25 \times 10^{-3}$ to $\hat{J}(u^*) = 6.3 \times 10^{-4}$.

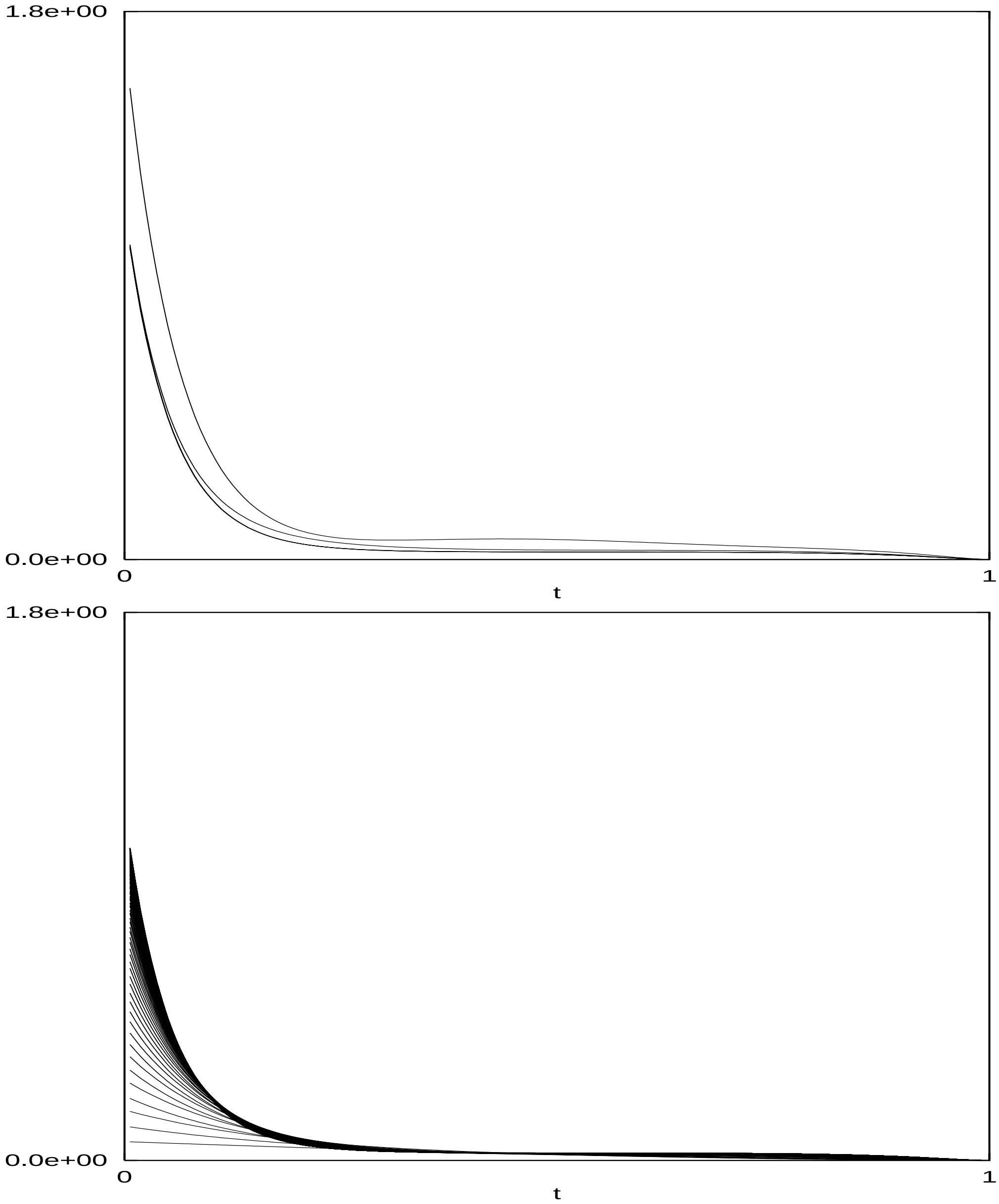

FIGURE 4. Newton's method (6 Iterations) (top) versus Gradient algorithm (96 Iterations), Re=400, $\alpha = 10^{-2}$: Evolution of control for relative accuracy = 1.d-5

EXAMPLE 7.3. Here $\Omega_o = (0.,1.) \times (0.2, 0.5)$ and $\Omega_c = (0.,1.) \times (0.4, 0.7)$. Again, the discretization of the spatial and the time domain as well as the parameter α are the same as in Example 7.2. The gradient algorithm needs 38 iterations to reduce the value of the cost functional from $\hat{J}(u^0) = 3.25 \times 10^{-3}$ to $\hat{J}(u^*) = 8.11 \times 10^{-4}$. It takes about 80 minutes cpu-time. The Polak-Ribiére variant of the conjugate gradient algorithm converges after 37 iterations and yields a slightly better reduction of the residual. The amount of of cpu-time needed is nearly equal to that taken by the gradient

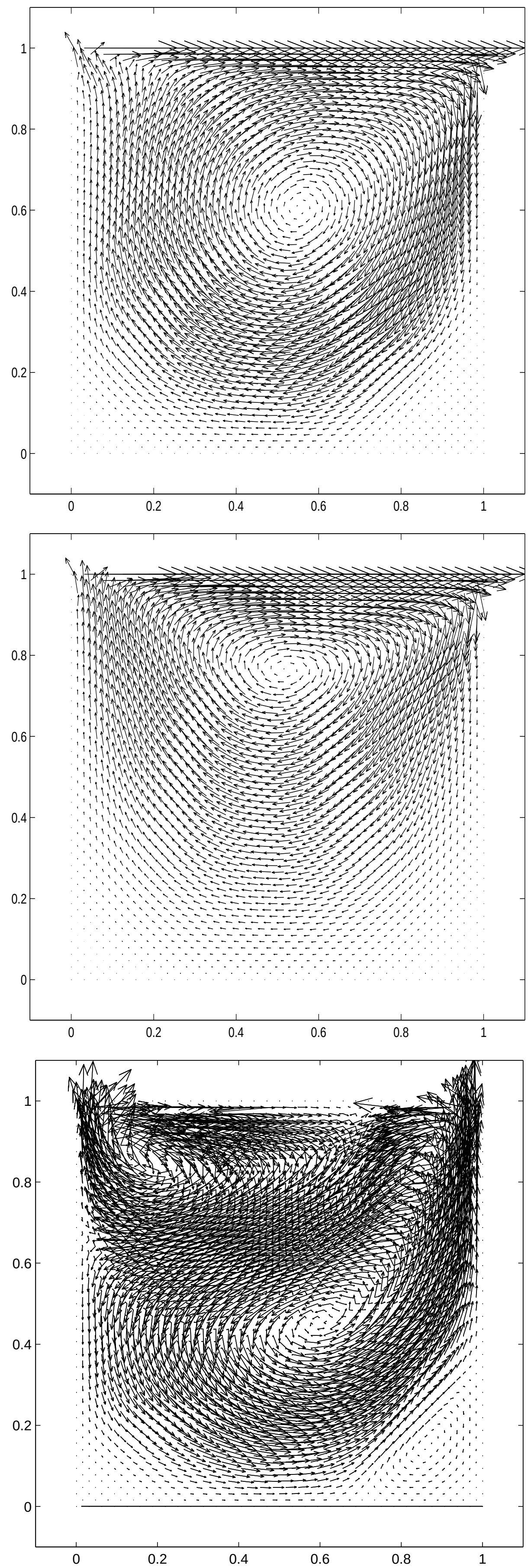

FIGURE 5. Results for $\alpha = 10^{-2}$, from top to bottom: uncontrolled flow, controlled flow at $t = 1$, and control force at $t = 0.75$

Iteration	CG-steps	$\frac{\|\hat{J}'(u)\|}{\|\hat{J}'(u^0)\|}$	$\frac{\|\delta u^k\|_U}{\|\delta u^{k-1}\|_U}$	$\hat{J}(u^k)$
1	-	1.e0	-	1.254304e-3
2	7	9.266545e-3	1.	6.438873e-4
3	8	9.958369e-4	0.16	6.270625e-4
4	8	5.919598e-5	0.054	6.269869e-4
5	5	2.027455e-5	0.015	6.269868e-4

TABLE 2. Performance of Newton's method for Example 7.2

algorithm. Newton's method is faster. It converges within 7 iterations to the approximate solution and needs 65 minutes cpu-time. The average cpu-time for the inner iteration loop is 7.5 minutes. As in the previous examples the average cost of a conjugate gradient iteration in the inner loop decreases with decreasing residual of the outer-iteration loop. The results are depicted in Tab. 3. Again super-linear convergence of the method is confirmed.

Iteration	CG-steps	$\frac{\|\hat{J}'(u)\|}{\|\hat{J}'(u^0)\|}$	$\frac{\|\delta u^k\|_U}{\|\delta u^{k-1}\|_U}$	$\hat{J}(u^k)$
1	-	1.e0	-	3.253402e-3
2	5	3.824511e-1	1.	1.198001e-3
3	8	2.270326e-1	0.600	9.829474e-4
4	8	5.762604e-2	0.619	8.333535e-4
5	12	9.527054e-3	0.404	8.108330e-4
6	11	1.920783e-4	0.083	8.105986e-4
7	18	9.283444e-6	0.029	8.106008e-4

TABLE 3. Performance of Newton's method for Example 7.3

CHAPTER 5

Instantaneous control for finite dimensional systems

This chapter presents the development of the instantaneous control method with distributed controls for linear and non-linear dynamical systems. In Section 1 a brief discussion of the scope of the method is given. Section 2 contains a brief discussion of the linear quadratic regulator problem, which is chosen as model problem for the investigations. In Section 3 the application of the instantaneous control approach to the model problem is described. Section 4 deals with the interpretation of the instantaneous method as closed-loop control law. Stability properties and convergence of the corresponding discrete and the continuous controller for tracking-type problems are proven in Section 5. Finally, a comparison between the controller obtained by the instantaneous control method and the optimal closed-loop Riccati controller for a two-dimensional example is presented in Section 6.

1. Scope of the method

Various restrictions apply to the optimal control approach for the instationary Navier-Stokes equations. They are caused by limits in available computing power and storage facilities as well as by the limited applicability of the resulting open-loop control policies in practical applications.

Instantaneous control provides a powerful tool for coping with the mentioned bottlenecks while providing reasonable controls [**50, 81, 88**]. Furthermore, instantaneous control can be interpreted as closed-loop feedback control approach which, applied to the Navier-Stokes equations in a natural way leads to practically implementable feedback strategies for fluid flows [**18, 19, 20, 62, 102**]. The nature of the approach is completely different from that of the Hamilton-Jacobi-Bellman method since it does not rely on the dynamical programming principle of Bellman [**11**]. Rather, it successively determines approximations of the objective while marching forward in time. The method is closely related to the model predictive control approach [**31**], which often is also refered to as receding horizon control, see [**92, 99, 91**]. It has been shown in several numerical studies that the instantaneous control approach is implementable in workstation environments and is very effective in reducing costs [**50, 54**]. Moreover, certain variants of the instantaneous method lead to practically relevant controllers for fluid flows, since they can be implemented as to determine boundary controls based on boundary information only [**19, 81, 88**].

The method works as follows. The dynamical system is discretized in time. Then, at every time instance a stationary control problem with an instantaneous version of the feedback gain as cost function is solved approximately by applying one step of the gradient method. The control is used to steer the system to the next time slice. As is shown this discrete approach allows for the interpretation as a semi-discretization of a closed-loop control law. Provided the parameters involved in the method, such as step size in time integration and the descent parameter in the gradient algorithm are adjusted in a proper way the discrete control algorithms and the related continuous closed-loop control laws are proved to be stable.

2. The model problem

In this section the derivation of control laws ultilizing the linear-quadratic regulator problem (LQR problem) in its simplest form is outlined. To make the ideas of the approach as transparent as possible questions of controllability and observability are not investigated. These topics in connection with the approach presented here are discussed in [**110**]. The model problem can be formulated as follows: Given a desired state vector $z \in \mathbb{R}^n$ and an initial state $x_0 \in \mathbb{R}^n$, find a control vector $u(t) \in \mathbb{R}^n$ such that the objective function

$$J(x,u) = \hat{J}(u) = \frac{1}{2}\int_0^T \gamma|u(t)|^2 + |x(t)-z|^2\,dt \tag{52}$$

is minimal compared to the value of J at all input vectors $v(t) \in \mathbb{R}^n$ satisfying

$$\begin{aligned} \dot{x}(t) + Ax(t) &= b(t) + v(t), \\ x(0) &= x_0. \end{aligned} \tag{53}$$

Here, $\gamma > 0$ weighs the costs for the control input, the matrix $A \in \mathbb{R}^{n\times n}$ is assumed to be regular and the inhomogeneity is denoted by b. It is well known that this problem admits a unique solution

$$u^* = \frac{1}{\gamma}\mu, \tag{54}$$

where the adjoint state $\mu \in \mathbb{R}^n$ is connected to the state x by the linear forward-backward in time Hamilton equations

$$\begin{aligned} \dot{x} + Ax &= b + \tfrac{1}{\gamma}\mu, \\ -\dot{\mu} + A^*\mu &= -(x-z), \\ x(0) &= x_0, \\ \mu(T) &= 0. \end{aligned}$$

A short computation shows that the control u^* in (54) for z independent of time may be computed with the help of the solution of the two-point boundary value problem

$$\begin{aligned} \ddot{\mu} - (A^* - A)\,\dot{\mu} - \left(AA^* + \tfrac{1}{\gamma}I\right)\mu &= b - Az \\ A^*\mu(0) - \dot{\mu}(0) &= z - x_0 \\ \mu(T) &= 0. \end{aligned}$$

Choosing for the control the Ansatz

$$u(t) = \frac{1}{\gamma}\{-P(t)(x-z) + p(t)\} \tag{55}$$

leads to the time-variant feedback matrix P and a time-dependent vector field p satisfying the Riccati matrix differential equations

$$\begin{aligned} \dot{P} - A^*P - PA + I - \tfrac{1}{\gamma}P^2 &= 0 \\ \dot{p} - \tfrac{1}{\gamma}Pp - A^*p &= P(b - Az) \\ P(T) &= 0 \\ p(T) &= 0. \end{aligned} \tag{56}$$

In practical applications the use of an numerical approximation of the optimal control law (55) amounts to the storage of an $n \times n$-matrix and an n-vector at every time instance. Especially for large time intervals $T >> 0$ or/and large system dimension the application of this law is not workable, not to mention the enormous amount of computational work necessary to provide the numerical solution of (56).

For infinite time horizon $T = \infty$ the optimal control law is also given by (55), where now the feedback matrix P and the vector p are the suitable time-invariant stationary solutions of (56), i.e. P is positive definite and the couple and (P, p) satisfies the quadratic system

$$\begin{aligned} -A^*P - PA + I - \tfrac{1}{\gamma}P^2 &= 0 \\ -\tfrac{1}{\gamma}Pp - A^*p &= P(b - Az). \end{aligned} \tag{57}$$

Investigations on algorithms for the numerical solution of this problem are discussed, for example, by Rauter and Sachs in [**97**]. However, for large system dimensions as they arise from the discretization of parabolic equations, say the effort to solve Eqs. 57 numerically is enormous. It is one goal of this work to present an approach which provides numerically computable control laws even for systems of very large dimension.

3. The instantaneous control strategy

The approach taken is based on a time discretization of equation (53). For this purpose let $0 = t_0 < t_1 < \cdots < t_m = T$ denote an equidistant grid on the time interval $[0, T]$ with step size $h = \frac{T}{m}$. At each discrete time level t_i a stationary control problem is solved for an approximate optimal control u_i^* and this control is used to steer the system from t_i to t_{i+1}, where a new approximate optimal control is determined. Unless otherwise specified from now onwards z denotes a sufficiently smooth bounded time dependent function.

As time discretization method the implicit Euler method is chosen, and as cost function an instantaneous version of (52) is taken. The optimization problem in every time step then has the form

$$(P_i)\left\{\begin{array}{l} \min J(x^{j+1},u^{j+1}) = \hat{J}(u^{j+1}) = \frac{\gamma}{2}|u^{j+1}|^2 \;+\; \frac{1}{2}|x^{j+1}-z^j|^2 \\ \text{s.t.} \\ \;(I+hA)x^{j+1} \;\;=\;\; x^j + hb^j + u^{j+1}, \end{array}\right.$$

which, due to the quadratic character of the cost function, admits a unique solution (x^{j+1}, u^{j+1}). This tuple together with the uniquely determined Lagrange multiplier μ^{j+1} solves the corresponding first order optimality conditions given by

$$\begin{array}{rcl} (I+hA)x^{j+1} & = & x^j + hb^j + u^{j+1} \\ (I+hA^*)\mu^{j+1} & = & -(x^{j+1}-z^j) \\ \gamma u^{j+1} - \mu^{j+1} & = & 0. \end{array}$$

This system has the solution

$$\begin{array}{l} x^{j+1} = (I+hA)^{-1}\left(x^j + hb^j + \frac{1}{\gamma}S\left(h(Az^j - b) - (x^j - z^j)\right)\right) \\ \mu^{j+1} = -S(x^j - z^j) - hS(b^j - Az^j), \end{array}$$

where

$$S := \left((I+hA)(I+hA^*) + \frac{1}{\gamma}I\right)^{-1}.$$

The gradient of the functional $\hat{J}$ at u in direction v is given by

$$\nabla\hat{J}(u)(v) \;=\; (\gamma u - \mu, v),$$

where $(\cdot,\cdot)$ denotes the usual inner product in $\mathbb{R}^n$.

In order to obtain a stable control law for the dynamical system (53), at every time level t_i exactly one step of the gradient algorithm with a suitable step size $\rho > 0$ is applied to solve (P_i). This approach from now on will be referred to as **instantaneous control**. It may be interpreted as an in-exact variant of finite-horizon model predictive control [**31, 92, 93, 99**] and may be written in algorithmical form as follows.

ALGORITHM 3.1. (Instantaneous control)

(1) Given initial values x^0, set $j = 0$, $t_0 = 0$

(2) Given u_0^j, solve

$$\begin{array}{rcl} (I+hA)x & = & x^j + hb^j + u_0^j \\ (I+hA^*)\mu & = & -(x - z^j) \end{array}$$

(3) Set $\nabla\hat{J}(u_0^j) = \gamma u_0^j - \mu$

(4) Given ρ, set $u^{j+1} = u_0^j - \rho\nabla\hat{J}(u_0^j)$

(5) Solve

$$(I+hA)x^{j+1} = x^j + hb^j + u^{j+1}$$

(6) Set $t_{j+1} = t_j + h$, $j = j + 1$
(7) If $t_j < T$ goto 2.

Note, that the optimal step size $\rho = \rho^*$ in step 4. of Algorithm 3.1 can be computed exactly and is given by

$$\rho^* = -\frac{\nabla \hat{J}(u)d}{\gamma d^2 - \mu(d)d}, \tag{58}$$

where for given direction d, $\mu(d)$ is the adjoint state computed from

$$\begin{cases} (I + hA)x &= x_o + hb + d \\ (I + hA^*)\mu &= -(x - z). \end{cases}$$

Here, x_o denotes the state at the previous time slice. The denominator in (58) is equal to $D^2\hat{J}(u)(d,d)$ and therefore, due to the quadratic nature of the optimization problem (P_i) is always positive. It is worth noting that even for the nonlinear problems investigated later the step size guess (58) performs very reliably.

In the next section Algorithm 3.1 is interpreted as semi-implicit discretization scheme of a dynamical system related to the one given in Eq. (53).

4. Discrete and continuous output control laws

The instantaneous control approach presented in Algorithm 3.1 allows for an interpretation as semi-implicit time integration scheme of a dynamical system. In order to derive the corresponding differential equation abbreviate

$$B := (I + hA)^{-1}, \quad B^* := (I + hA^*)^{-1}, \tag{59}$$

set $f := Az$ and require

$$h \max\{\|A\|, \|A^*\|\} =: h\mathbf{M} < 1, \tag{60}$$

so that the matrices B, and thus B^*, are well defined. With these preparations one can prove

THEOREM 4.1. Let $u_0^j := 0$ in Algorithm 3.1. Then Algorithm 3.1 is equivalent to the semi-implicit time discretization

$$(I + hA)x^{j+1} = x^j + hb^j - \rho B^*B(x^j - z^j) - h\rho B^*B(b^j - f^j), \quad x^0 := x_0, \tag{61}$$

of the dynamical system

$$\dot{x} + Ax = b - \frac{\rho}{h}B^*B(x - z) - \rho B^*B(b - f), \quad x(0) = x_0. \tag{62}$$

Proof. For a proof it remains to show that for u^{j+1} in step 4. of Algorithm 3.1

$$u^{j+1} = -\rho B^*B(x^j - z^j) - h\rho B^*B(b^j - f^j)$$

holds true. For this purpose note, that $f^j = Az^j$ implies $B^*z^j = B^*Bz^j + hB^*Bf^j$. Using this in steps 2.-5. of Algorithm 3.1 yields the desired result. □

REMARK 4.1. Note, that in (62) h and ρ may now be regarded as parameters, although they stem from the time step size in the discretization process and the step size in the gradient step, respectively.

Now let for a moment b depend nonlinearly on the state. In order to track the desired state z with the control law (62) z by necessity has to satisfy the homogeneous equation

$$g_z := \dot{z} + Az - b(z) + \rho B^*B(b(z) - Az) = 0, \quad z(0) = z_0.$$

This restriction for the desired state z can be avoided by replacing (61) by the modified discrete control law

$$(I + hA)x^{j+1} = x^j + hb(x^j) + g_z^j - \rho B^*B(x^j - z^j) - h\rho B^*B(b(x^j) - Az^j), \quad x^0 := x_0, \tag{63}$$

where

$$g_z^j := z^{j+1} - z^j + hAz^{j+1} - hb(z^j) + h\rho B^*B(b(z^j) - Az^j),$$

and (62) accordingly by

$$\dot{x} + Ax = b(x) + g_z - \frac{\rho}{h}B^*B(x - z) - \rho B^*B(b(x) - f), \quad x(0) = x_0, \tag{64}$$

REMARK 4.2. With u_0^j defined to be the solution of

$$\left(\mathrm{Id} - \frac{\rho}{1-\gamma\rho}B^*B\right)u_0^j = \frac{1}{1-\gamma\rho}g_z^j$$

in Algorithm 3.1 the control laws (63) and (64) can be derived from Algorithm 3.1 in the same way as the laws (61) and (62) in the proof of Theorem 4.1.

In the next section the stability properties of the differential equation (64) and of its discretization (63) are characterized in terms of the parameters involved in their formulation.

5. Convergence of the control laws

This section shows that provided the system parameters ρ and h are chosen suitably instantaneous control steers the system towards the desired state z.

To begin with first note, that in the linear case $b(x) = b(z) = b$ holds, so that the difference scheme (63) can equivalently be rewritten as

$$(I + hA)\Phi^{j+1} = \Phi^j - \rho B^*B\Phi^j,$$

which in turn is semi-implicit time discretization of the the differential equation

$$\dot{\Phi} + A\Phi = -\frac{\rho}{h} B^* B \Phi,$$

where $\Phi := x - z$ and h serves as discretization parameter.

5.1. Stability of the discrete schemes. To derive stability conditions for the scheme (63) it is sufficient to ensure that the matrix

$$C_I := (I + hA)^{-1}(I - \rho B^* B) \tag{65}$$

has spectral radius lower than or equal to 1. With M defined in (60) a sufficient condition for this to hold is given in

THEOREM 5.1. Let the parameters $h > 0$ and $0 < \rho < 2$ satisfy the relation

$$\frac{1}{h} \geq \frac{(14\rho + 2)\, M}{1 - |1 - \rho|}. \tag{66}$$

Then the discretization (63) of (64) is stable.

Proof. The matrix C_I may be written as

$$C_I = I \underbrace{-\rho B^* B - hA}_{=:hC_A} + \rho h A B^* B + \left(\sum_{k=2}^{\infty} (-h)^k A^k \right) (I - \rho B^* B).$$

For the matrix hC_A write

$$\begin{aligned} hC_A = \quad & -\rho B^* B - hA = \\ & -\rho \left[I + \sum_{k=1}^{\infty} (-h)^k A^k + \sum_{k=1}^{\infty} (-h)^k A^{*k} + \right. \\ & \left. + \left(\sum_{k=1}^{\infty} (-h)^k A^k \right) \left(\sum_{k=1}^{\infty} (-h)^k A^{*k} \right) \right] - hA. \end{aligned}$$

Using this representation and the fact that h satisfies the estimate $2hM < 1$ straight forward estimation gives

$$\|I + hC_A\| \;\leq\; |1 - \rho| + hM \{6\rho + 1\}.$$

The estimate

$$\|B^* B\| \;\leq\; \frac{1}{(1 - hM)^2} \;\leq\; 4$$

together with the above decomposition of the matrix C_I yields the desired condition. ☐

5.2. Stability of the continuous systems. As an immediate consequence of the proof of the previous theorem one obtains a condition for exponential stability of the differential equations (62) and (64).

THEOREM 5.2. Let the assumptions of Theorem 5.1 be satisfied. Furthermore, let condition (66) hold with strict inequality sign. Then, the differential equation (62) (and thus (64)) is exponentially stable.

Proof. The matrix C_A is the system matrix of the differential equation (62). Following the lines of the proof of the previous theorem one concludes that the spectral radius r_σ of the matrix $I + hC_A$ satisfies

$$r_\sigma(I + hC_A) \;\le\; \|I + hC_A\| < \;1 \quad \text{(for all } h \text{ satisfying (66))}.$$

This implies Re($\lambda(C_A)) < 0$ for all eigenvalues $\lambda(C_A)$ of C_A. □

5.3. Convergence on the infinite time horizon. To prove convergence of the iterates of the discrete controller (63) and of the state of the continuously controlled system (64) to the desired state z for large times is now a minor step.

THEOREM 5.3. Let the assumptions of Theorems 5.1 and 5.2 be satisfied. Then, for every $\epsilon > 0$ there exists $j_0(h, \rho, x^0)$ such that

$$\sup_{j\in\mathbb{N}, j\ge j_0} |x^j - z^j| \;\le\; \epsilon, \qquad \sup_{j\in\mathbb{N}, j\ge j_0} |u^j| \;\le\; \epsilon$$

holds. For the continuous controllers, similarly, for every $\epsilon > 0$ there exists a time t_0 such that

$$\sup_{t\ge t_0} |x(t) - z(t)| \;\le \epsilon \tag{67}$$

holds true.

Proof. Since $\Phi : x - z$ solves the stable differential equation

$$\dot{\Phi} \;=\; \underbrace{-(A + \frac{\rho}{h}B^*B)}_{=:C_A}\,\Phi,$$

which admits the unique solution

$$\Phi(t) \;=\; e^{C_A t}\Phi(0),$$

(67) follows from exponential stability of the system (64).

In order to prove the first claim for the discrete controller (63) recall that

$$(I + hA)\Phi^{j+1} \;=\; \Phi^j - \rho B^*B\Phi^j$$

holds for the iterates of (63). Since strict inequality in condition (66) implies $\|C_I\| < 1$ one concludes

$$|\Phi^j| \;\le\; \|C_I\|^j\,|x^0 - z^0|.$$

This is the desired result for the iterates x^j. For the controls u^j the result immediately follows from the representation

$$u^{j+1} = -\rho B^* B(x^j - z^j)$$

and the estimate for the iterates x^j. □

5.4. The nonlinear case. Denote the by $C_b^m([0,\infty),\mathbb{R}^n)$ the set of m-times continuously differentiable functions with bounded derivatives up to order m on the infinite time interval $[0,\infty)$ and values in $\mathbb{R}^n$.

The results of Theorem 5.3 are now extended to nonlinear dynamical systems. For this purpose it is sufficient to suitably modify the requirements for stability in Theorem 5.1 for right-hand sides b which now depend nonlinear on the state x. In order to investigate Algorithm 3.1 with u_0^j given as in Remark 4.2 for the nonlinear case it is sufficient to consider the differential equation

$$\dot\Phi(t) + A\Phi(t) = -\frac{\rho}{h}B^* B\Phi(t) - (\rho B^* B - I)(b(t,x(t)) - b(t,z(t))),\ \Phi(0) = x_0 - z(0), \tag{68}$$

together with its semi-implicit discretization

$$(I + hA)\Phi^{j+1} = \Phi^j - \rho B^* B\Phi^j - h(\rho B^* B - I)(b(t_j,x^j) - b(t_j,z^j)),\ \Phi^0 = \Phi(0), \tag{69}$$

where $b^j := b(t_j,x^j)$. On the nonlinearity b the assumption

ASSUMPTION 5.1. $b = b(t,q) \in C_b^1([0,\infty)\times\mathbb{R}^n,\mathbb{R}^n)$

is imposed. In particular, this assumption implies the Lipschitz-continuity of the nonlinearity b with Lipschitz constant $L_b = \|Db\|_\infty$.

To derive sufficient conditions for the stability of the discrete scheme (69) note that by the mean value theorem

$$b(t,x) - b(t,z) = \underbrace{\int_0^1 b_q(t, z + s(x-z))ds}_{=:b_q(t,\xi)}\ \Phi.$$

Using this identity, the discrete scheme (69) may be written as

$$\Phi^{j+1} = \underbrace{C_I\left(I + hb_q(t_j,\xi)\right)}_{=:\tilde C_I}\Phi^j, \quad \Phi^0 = \Phi(0),$$

where C_I denotes the matrix given in (65). As an immediate consequence of this identity one obtains with M defined in (60)

THEOREM 5.4. Let the parameters $h > 0$ and $0 < \rho < 2$ satisfy the relation

$$\frac{1}{h} \geq \frac{(14\rho + 2)\left(M + \frac{1}{2}L_b\right) + |1-\rho|L_b}{1 - |1-\rho|}. \tag{70}$$

Then the discretization (69) of (68) is stable.

Proof. Similar to that of Theorem 5.1. □

As a corollary one also obtains the stability of the continuous system. The proof of the following theorem is the same as that of Theorem 5.2.

THEOREM 5.5. Let the assumptions of Theorem 5.4 be satisfied. Furthermore, let condition (70) hold with strict inequality sign. Then, the differential equation (68) is exponentially stable.

Similarly to Theorem 5.3 there holds a convergence result in the nonlinear case.

THEOREM 5.6. Let the assumptions of Theorems 5.5 be satisfied. Then, for every $\epsilon > 0$ there exists a $j_0(h, \rho, x^0)$ such that

$$\sup_{j\in\mathbb{N}, j\geq j_0} |x^j - z^j| \leq \epsilon, \qquad \sup_{j\in\mathbb{N}, j\geq j_0} |u^j| \leq \epsilon$$

holds. For the continuous controllers, similarly, for every $\epsilon > 0$ there exists a time $t_0 > 0$ such that

$$\sup_{t\geq t_0} |x(t) - z(t)| \leq \epsilon$$

holds true.

REMARK 5.1. The results of this chapter are generalized by Wunder in [**110**] where he investigates plants of the form

$$x^{j+1} = Bx^j + Eu^j, \quad j = 0, 1, 2, \ldots \quad \text{with } x^0 \text{ given.}$$

Here, $x^j \in \mathbb{R}^n$, $u^j \in \mathbb{R}^m$, $m \leq n$, $B = (I + hA)^{-1}$ and $E\colon \mathbb{R}^m \to \mathbb{R}^n$ denotes a control extension operator. Among other things Wunder proves in [**110**, Satz 4.7] that instantaneous control is stabilizing if h is sufficiently small and A is positive definite on $\mathcal{N}(E^t)$.

6. An example

As example investigate the linearization of the equation

$$\ddot{\varphi} + \sin\varphi = 0$$

at the unstable stationary solution $\varphi = \pi, \dot{\varphi} = 0$:

$$\dot{x} + \underbrace{\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}}_{=:A} x = \underbrace{\begin{bmatrix} 0 \\ -\pi \end{bmatrix}}_{=:f}.$$

Here $x = [\varphi, \dot{\varphi}]^t$ and the state z to be tracked is given by $z = [\pi, 0]^t$. Furthermore, let $b = [0, -\pi]$, so that $Az = b$ holds.

The stationary Riccati solution is computed in order to compare it to the results obtained by the control law given by Algorithm 3.1. Using the stationary Riccati solutions P and p from (57) to stabilize the linearized equations leads to the feedback law

$$\dot{x} \;=\; -\left(A + \tfrac{1}{\gamma}P\right)x + \left(f - \tfrac{1}{\gamma}Pz\right) = -\left(A + \tfrac{1}{\gamma}P\right)(x - z),$$

where the positive definite matrix P and the vector p are given by

$$P = \begin{bmatrix} \sqrt{\gamma^2+\gamma} & \gamma \\ \gamma & \sqrt{\gamma^2+\gamma} \end{bmatrix}, \quad p = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Using this, a short calculation gives the optimal state for the infinite horizon problem,

$$x = \begin{bmatrix} \pi \\ 0 \end{bmatrix} + e^{-t\cdot\sqrt{1+\frac{1}{\gamma}}} \begin{bmatrix} \alpha \\ \beta \end{bmatrix},$$

where α and β depend on the initial state of the differential equation. The control has the form $u = -\frac{1}{\gamma}P(x-z)$ and is given by

$$u = -\tfrac{1}{\gamma} \begin{bmatrix} \alpha\sqrt{\gamma^2+\gamma} + \beta\gamma \\ \alpha\gamma + \beta\sqrt{\gamma^2+\gamma} \end{bmatrix} e^{-t\cdot\sqrt{1+\frac{1}{\gamma}}}.$$

Two more lines of computation finally yield that the cost $J = \displaystyle\int_0^\infty \|x-z\|^2 + \gamma\|u\|^2\,dt$ is given by

$$J = \sqrt{\gamma^2+\gamma}\left(\alpha^2 + 2\alpha\beta\sqrt{\tfrac{\gamma}{\gamma+1}} + \beta^2\right) \;=\; O(\gamma)\;(\gamma \to 0).$$

Now proceed similarly for the instantaneous control law. To begin with, let $\frac{1}{h} > \|A\| = 1$. Then, $B^* = B = Id - hA + O(h^2)\;(h \to 0)$ holds and $B^*B = 1 - 2hA + O(h^2)\;(h \to 0)$. For the matrix $C_A = -(A + \frac{\rho}{h}B^*B)$ one obtains the approximation

$$\tilde{C}_A \;\doteq\; \begin{bmatrix} -\frac{\rho}{h} & 1-2\rho \\ 1-2\rho & -\frac{\rho}{h} \end{bmatrix},$$

which is of order $O(h)\;(h \to 0)$. Using this approximations it is sufficient to investigate the feedback control for $w = x - z$ given by

$$\dot{w} = \tilde{C}_A w$$

the exact solution of which is given by

$$x \;=\; \alpha e^{\lambda_1 t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta e^{\lambda_2 t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

and deviates from that of the control law (64) only by a term of size $O(h)\;(h \to 0)$. The eigenvalues $\lambda_{1/2}$ of the matrix $\tilde{C}_A$ are given by

$$\lambda_{1/2} \;=\; -\frac{\rho}{h} \pm \text{sign}(1-2\rho)|1-2\rho|,$$

so that negative eigenvalues are guaranteed, provided the parameter h is sufficiently small. Finally, the control and the costs are computed. For the control $u = -\frac{\rho}{h}B^*Bw + Az - b(z)$ one obtains

$$u = -\left(\alpha e^{\lambda_1 t}\left(\tfrac{\rho}{h} + 2\rho\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta e^{\lambda_2 t}\left(\tfrac{\rho}{h} + 2\rho\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) \longrightarrow 0 \ (t \ \to \ \infty),$$

the value of the costs J are given by

$$J = \frac{\alpha^2}{\lambda_1}\left(1 + \gamma\left(\frac{\rho}{h} + 2\rho\right)^2\right) + \frac{\beta^2}{\lambda_2}\left(1 + \gamma\left(\frac{\rho}{h} + 2\rho\right)^2\right) \leq C\,\left(\frac{h}{\rho} + \frac{\gamma\rho}{h}\right)$$

with some positive constant C independent of ρ, γ and h. As a result one obtains that the optimal Riccati controller and the controller obtained by the instantaneous control Algorithm 3.1 have the same qualitative behaviour.

CHAPTER 6

Instantaneous control for the instationary Navier-Stokes equations

This chapter investigates the stability properties of the instantaneous control method developed in Chapter 5 applied to the control of the instationary Navier-Stokes equations with distributed controls. In Section 1 the instantaneous control strategy is adapted to the Navier-Stokes equations and the corresponding continuous and discrete controllers are introduced. Section 2 contains an existence and uniqueness proof for the controlled Navier-Stokes equations with desired state equal to zero. In Section 3 these results are extended to tracking of sufficiently smooth states and exponential decay of the controlled state to the desired state is proved. In Section 4 similar results are provided for the corresponding discrete-in-time control procedure. Finally the results are numerically justified in Section 5.

1. Framework

The instantaneous stabilization method (61) developed in Chapter 5 is applied to construct a distributed feedback control policy for tracking of the instationary Navier-Stokes equations. As in the finite dimensional case the control gain is to track a given state z which for example can be chosen as the Stokes flow in an observation cylinder $\Omega_s^T \subseteq \Omega^T$, say. In order to put this problem into the framework developed in Chapter 5 let

$$A \;:=\; \nu S,$$

where S denotes the Stokes operator introduced in (8) and denote the nonlinearity by

$$b(y) \;=\; b(y, \nabla y) \;:=\; -P\left[(y\nabla)y\right]$$

where P denotes the Leray projector. Note that this definition only makes sense for $(y\nabla)y \in L^2(\Omega)^2$, compare [**82**]. However, it is utilized for notational reasons also for functions not satisfying these regularity requirements and it will follow from the specific context how this term has to be understood. In this setting the Navier-Stokes equations (9) for $f = 0$ in variational formulation may be rewritten as Burgers equation in the space V,

$$\begin{aligned} y_t + Ay &= b(y), \\ y(0) &= y_0, \end{aligned}$$

which has to be understood in the sense of (11). The analogue to the operator B in (59) here is defined through

$$v \;=\; Bf \quad \text{for } f \in V^* \quad \Longleftrightarrow \quad v + hAv \;=\; f \text{ in } V^*,$$

where for $f \in V^*$ the equation is understood in the weak sense and due to Theorem 1.1 admits a unique solution. It follows from Theorem 1.3 that B is linear, bounded, selfadjoint and compact in H. With these preparations the continuous control law (64) is of the form

$$y_t + Ay - b(y) = K_h y \quad \text{and} \quad y(0) = y_0, \tag{71}$$

where the controller K_h is given by

$$K_h y = -\frac{\rho}{h} BB(y - z) - \rho BB(b(y) - b(z)) + z_t + Az - b(z). \tag{72}$$

In this context the analogue to (63) is given by

$$\frac{y^{j+1} - y^j}{h} + Ay^{j+1} - b(y^j) = K_h^D y^j, \quad j = 0, 1, \dots \quad \text{and } y^0 = y_0, \tag{73}$$

where for $j \geq 0$

$$K_h^D y = -\frac{\rho}{h} BB(y^j - z^j) - \rho BB(b(y^j) - b(z^j)) + \frac{z^{j+1} - z^j}{h} + Az^{j+1} - b(z^j). \tag{74}$$

Recall that the system parameters h and ρ, respectively stem from a time discretization parameter and from a step size in the gradient algorithm, respectively. Unless otherwise stipulated, throughout this chapter

ASSUMPTION 1.1. $0 \neq y_0 \in V$ and $z \in H^{2,1}(Q)$

is asssumed. Note that this assumption on the desired state z in particular implies that $z(0)$ is meaningful. Moreover, $z(0) \in V$.

2. Existence and uniqueness of solutions

In this section existence and uniqueness of a solution to (71) is proven and its regularity is investigated. Since the desired state z in this context plays the role of an inhomogeneity whose regularity properties could be adapted to the necessities of the proof, for the moment $z \equiv 0$ is assumed. Moreover, $\partial\Omega$ is assumed to be as smooth as required for the applications of Theorem 1.2.

THEOREM 2.1. Let $y_0 \in H$ be a given initial state and let $h > 0$ be fix. Then, there exits a threshold $\rho_0 = \rho_0(|y_0|_H, \nu) > 0$ such that for every $0 < \rho \leq \rho_0$ the equation

$$\begin{cases} y_t + Ay - b(y) = -\frac{\rho}{h} BBy - \rho BB(b(y)) \\ \qquad\qquad y(0) = y_0 \end{cases} \tag{75}$$

for every $T > 0$ admits a unique weak solution $y \in W$ which satisfies the a-priori estimates

$$\begin{cases} |y|_H^2 \leq 2\,\mathrm{e}^{-\frac{\rho}{h}t}\,|y_0|_H^2 & \text{for all } t \in [0, T] \\ |y|_{L^\infty(H)}^2 \leq 2\,|y_0|_H^2, & \\ |y|_{L^2(V)}^2 \leq \frac{4}{\nu}\,|y_0|_H^2 & \text{and} \\ |y_t|_{L^2(V^*)}^2 \leq C(\nu)\,\left\{1 + \frac{\rho}{h}\right\}\,|y_0|_H^2. & \end{cases} \tag{76}$$

Proof. The proof follows the lines of the existence proof for the time-dependent Navier-Stokes equations given in [**103**, Chap. III] and is divided into two parts, namely the derivation of a-priori estimates and the passage to the limit. It is worth noting that in addition to the well understood analytical handling of the nonlinearity of the Navier-Stokes equations one in essence has to cope with the technical difficulties arising from non-local solution operator BB applied to the nonlinearity of the Navier-Stokes equations. However, as will be demonstrated the smoothing properties of this operator and its sign enhance the stability properties of the Navier-Stokes equations.

Let ρ_0 be given by

$$\rho_0 = \frac{\nu^2}{8\nu^2 + 4|y_0 - z(0)|_H^2}. \tag{77}$$

2.1. Part 1: A-priori estimates via Galerkin-ansatz. Let $\psi_1, \ldots, \psi_m, \ldots$ denote the eigenfunctions of the operator S^{-1} (which exist as elements of $\mathcal{D}(S)$ due to Theorem 1.3 and are total in V). For $m \in \mathbb{N}$ define the discrete Ansatz space by $V_m = \mathrm{span}\,\{\psi_1, \ldots, \psi_m\} \subset V$ and choose for $m \in \mathbb{N}$ a vector $(y_1^0, \ldots, y_m^0) \in \mathbb{R}^m$ such that

$$y_{m_0} := \sum_{i=1}^m y_i^0 \chi_i \longrightarrow y_0 \quad (m \longrightarrow \infty) \text{ in } V.$$

Set

$$y_m(t,x) := \sum_{i=1}^m y_i^m(t)\psi_i(x)$$

and note that by the linearity of the operator B

$$\begin{aligned} (By_m)(t,x) &= \sum_{i=1}^m y_i^m(t)\,(B\psi_i)(x) \text{ and} \\ (B((y_m\nabla)y_m))(t,x) &= \sum_{i,l=1}^m y_i^m(t)y_l^m(t)\,(B((\psi_i\nabla)\psi_l))(x) \end{aligned}$$

holds. Inserting y_m into the state equation and using ψ_j, $j = 1, \ldots, m$, as test functions leads to the system of nonlinear differential equations for the vector $Y(t) := (y_1^m, \ldots, y_m^m)^t$ given next.

$$\begin{aligned} M\dot{Y} + \nu DY + \left[\sum_{i,l=1}^m y_i^m(t)y_l^m(t)\int_\Omega (\psi_i\nabla)\psi_l\psi_j\,dx\right]_{j=1}^m \\ + \frac{\rho}{h}\tilde{M}Y + \rho\left[\sum_{i,l=1}^m y_i^m(t)y_l^m(t)\int_\Omega B((\psi_i\nabla)\psi_l)B(\psi_j)\,dx\right]_{j=1}^m &= 0, \\ Y(0) = Y_0 = (y_1^0, \ldots, y_m^0)^t. \end{aligned} \tag{78}$$

Here, the matrix M denotes the mass matrix, D denotes the stiffness matrix and the matrix $\tilde{M}$ is defined by

$$\tilde{M} = (\tilde{m}_{il})_{i,l=1}^m, \quad \tilde{m}_{il} = \int_\Omega B\psi_i B\psi_l\,dx.$$

Since the mass matrix is regular, the nonlinear system (78) admits a solution in some maximal time interval $[0, t_m)$. It will follow from the a-priori estimates of the solution that this interval may be chosen as large as desired.

To obtain a-priori estimates for y_m scalar-multiply (78) with Y. Application of Young's inequality then leads to the differential inequality

$$\frac{1}{2}\frac{d}{dt}\,|y_m|_H^2 + \nu\,|y_m|_V^2 + \frac{\rho}{h}|By_m|_H^2 \leq \frac{\rho h}{2}|B((y_m\nabla)y_m)|_H^2 + \frac{\rho}{2h}\,|By_m|_H^2, \tag{79}$$

for the function $t \mapsto |y_m(t)|_H^2$. Note, that $w_m := B((y_m\nabla)y_m)$ exists in V and is unique, since $(y_m\nabla)y_m$ is an element of V^*. Next, estimate $|w_m|_H^2$. By the definition of B the function w_m satisfies

$$\nu h\,|w_m|_V^2 + |w_m|_H^2 = \int_\Omega (y_m\nabla)y_m w_m\,dx \leq \frac{1}{4\nu h}|(y_m\nabla)y_m|_{V^*}^2 + \nu h|w_m|_V^2.$$

Estimation of the term $|(y_m\nabla)y_m|_{V^*}^2$ utilizing the first estimate in (129) leads to

$$|w_m|_H^2 \leq \frac{1}{2\nu h}|y_m|_H^2|y_m|_V^2. \tag{80}$$

Note that in the first estimate in (129) $C = \sqrt{2}$ holds. Estimate (80) together with the differential inequality (79) leads to

$$\frac{1}{2}\frac{d}{dt}|y_m|_H^2 + \left(\nu - \frac{\rho}{4\nu}|y_m|_H^2\right)|y_m|_V^2 + \frac{\rho}{2h}|By_m|_H^2 \leq 0. \tag{81}$$

The outermost addend on the left-hand-side can be estimated with

LEMMA 2.1. For $y \in V$ let $w := By$. Then $w \in V \cap H^3(\Omega)^2$ and $Sw \in V$. Moreover,

$$|w|_H^2 \;\geq\; |y|_H^2 \;-\; 2\nu h|y|_V^2.$$

Proof of Lemma 2.1. By the definition of the operator B the regularity claim for w follows from Theorem 1.2. Thus, $Sw \in H^1(\Omega)^2$ [**22**, Remark 1.10] and since $y \in V$ even an element of V. Therefore, Sw can be utilized as test function in the equation

$$w \,+\, hAw \;=\; y.$$

This gives

$$|w|_V^2 \,+ \nu h|Sw|_H^2 \;=\; \int_\Omega \nabla w\nabla y\,dx \;\Rightarrow\; |w|_V^2 + 2\nu h|Sw|_H^2 \;\leq\; |y|_V^2.$$

Furthermore, using y as test function the latter estimate leads to

$$|y|_H^2 \;=\; \int_\Omega yw \,+\, \nu h\nabla y\nabla w\,dx \;\leq\; \frac{1}{2}|y|_H^2 + \frac{1}{2}|w|_H^2 + h\nu|y|_V|w|_V \leq \frac{1}{2}|y|_H^2 + \frac{1}{2}|w|_H^2 + h\nu|y|_V^2,$$

which gives the claim. □

From (81) it now follows that

$$\frac{1}{2}\frac{d}{dt}|y_m|_H^2 + \left(\nu(1-\rho) - \frac{\rho}{4\nu}|y_m|_H^2\right)|y_m|_V^2 + \frac{\rho}{2h}|y_m|_H^2 \leq 0. \tag{82}$$

Since $y_m(0) \to y_0$ in H one may fix d_H and some $m_0 \in \mathbb{N}$ such that

$$|y_m(0)|_H^2 \;\leq\; 2|y_0|_H^2 = d_H \quad \text{for all } m \geq m_0.$$

Then, for all $0 < \rho \leq \rho_0$ with ρ_0 from (77) $\nu(1-\rho) - \frac{\rho}{4\nu} d_H > \frac{\nu}{2}$ holds true.

Since (78) describes a nonlinear system of ordinary differential equations there exists some t_m such that a solution exists and is continuous in the interval $[0, t_m)$. It shall be shown that $t_m = T$. To begin with assume that $|ym(t)|_H \to \infty (t \to t_m)$. Since

$$\nu(1-\rho) - \frac{\rho}{4\nu}|y_m(0)|_H^2 \;\geq\; \nu(1-\rho) - \frac{\rho}{4\nu} d_H \;\geq\; \frac{7\nu}{8} > \;\frac{\nu}{2},$$

there exists a time $t_m^* \leq t_m$ and $t_m^* > 0$ such that

$$\nu(1-\rho) - \frac{\rho}{4\nu}|y_m(t_m^*)|_H^2 \;=\; \frac{\nu}{2}$$

and

$$\nu(1-\rho) - \frac{\rho}{4\nu}|y_m(t)|_H^2 \;>\; \frac{\nu}{2} \quad \text{for all } t \in [0, t_m^*).$$

Now (82) implies

$$|y_m(t)|_H^2 \;\leq\; \mathrm{e}^{-\frac{\rho}{h}t}\, |y_m(0)|_H^2 \;\leq\; \mathrm{e}^{-\frac{\rho}{h}t}\, d_H \quad \text{for all } t \in [0, t_m^*],$$

and this in turn together with a continuity argument gives

$$\frac{\nu}{2} \;=\; \nu(1-\rho) - \frac{\rho}{4\nu}|y_m(t_m^*)|_H^2 \;\geq\; \nu(1-\rho) - \frac{\rho}{4\nu} d_H\, \mathrm{e}^{-\frac{\rho}{h}t_m^*} \;>\; \frac{\nu}{2},$$

which is a contradiction. Therefore, $t_m = t_m^* = T$ and

$$|y_m(t)|_H^2 \;\leq\; \mathrm{e}^{-\frac{\rho}{h}t}\, d_H \quad \text{for all } t \in [0, T].$$

Integration of (82) with respect to time then yields

$$|y_m|_{L^2(V)}^2 \;\leq\; \frac{2}{\nu}\, d_H \quad \text{and} \quad |y_m|_{L^2(H)}^2 \;\leq\; \frac{2h}{\rho}\, d_H.$$

Utilizing these estimates one obtains for the time derivative the estimate

$$|y_{m_t}|_{L^2(V^*)}^2 \;\leq\; C(\nu)\, \left\{1 + \frac{\rho}{h}\right\}\, d_H.$$

Altogether,

a) The sequences $\{y_m\}_{m\in\mathbb{N}} \subset L^\infty(H)$ and $\{y_m\}_{m\in\mathbb{N}} \subset L^2(V)$, are bounded independent of the parameter h and the dimension m of the Ansatz space, for $T > 0$ arbitrary, but fixed .
b) The sequence $\{y_{m_t}\}_{m\in\mathbb{N}} \subset L^2(V^*)$ is bounded by a constant of size $1 + \frac{\rho}{h}$ and independent of the dimension m of the Ansatz space, for $T > 0$ arbitrary, but fixed.

2.2. Part 2: Passage to the limit. Now a) and b) from above are equivalent to $\{y_m\}_{m\in\mathbb{N}} \subset W$ bounded and since $W \hookrightarrow\hookrightarrow L^2(H)$ and $W \hookrightarrow C([0,T];H)$ one concludes convergence of a subsequence, again denoted by $\{y_m\}_{m\in\mathbb{N}}$, weakly in W, weak* in $L^\infty(H)$ and strong in $L^2(H)$ to a function $y \in W$. Thus, passing to the limit in (78) in the linear terms is straightforward, as is for the non-linear term of the Navier-Stokes equations due to the strong convergence of the subsequence in $L^2(H)$, see [**103**, Lemma 3.2]. It finally remains to establish

$$\int_0^T \int_\Omega B(y_m - y)Bw_j\chi(t)\,dxdt \to 0 \quad (m\to\infty)$$

and

$$\int_0^T \int_\Omega B((y_m\nabla)y_m - (y\nabla)y)Bw_j\chi(t)\,dxdt$$
$$= \int_0^T \int_\Omega ((y_m\nabla)y_m - (y\nabla)y)BBw_j\chi(t)\,dxdt \to 0 \quad (m\to\infty),$$

where $\chi \in C_0^\infty(0,T)$. For the first term the boundedness of B and the strong convergence of $\{y_m\}_{m\in\mathbb{N}}$ in $L^2(H)$ gives the result. Since for $\chi \in C_0^\infty(0,T)$ the functions $BB\chi w_j \in C^1(\bar{Q})$ a further application of [**103**, Lemma 3.2] gives the convergence in the second term. Uniqueness of the solution is an immediate consequence of inequality (82). This completes the proof of Theorem 2.1. $\square$

REMARK 2.1. In the proof of Theorem 2.1 only $y_0 \in H$ is required.

The decay rate for the $L^\infty(H)$-estimate in (76) holds for all $h > 0$ provided $0 < \rho \le \rho_0$. This means that the controller (72) enhances the stability properties of the system up to exponential decay of order $\frac{1}{h}$. As is shown next a similar result holds for the $L^\infty(V)$ norm of the solution y.

THEOREM 2.2. Let ρ_0 be given by (77) and let $0 < \rho \le \rho_0$. Then the solution y of problem (75) is an element of $H^{2,1}(Q)$ and satisfies the a-priori estimates

$$\begin{cases} |y|_V^2 & \le C\,\mathrm{e}^{-\frac{\rho}{h}t}\,|y_0|_V^2 \qquad \text{for all } t\in[0,T], \\ |y|^2_{L^\infty(V)} & \le C(\nu)\,|y_0|_V^2, \\ |y|^2_{L^2(H^2(\Omega)^2\cap V)} & \le C(\nu)\,|y_0|_V^2 \qquad \text{and} \\ |y_t|^2_{L^2(H)} & \le C(\nu)\,\left\{1+\frac{\rho}{h}\right\}\,|y_0|_V^2. \end{cases} \tag{83}$$

Proof. Choose the sequence $\{y_m\}_{m\in\mathbb{N}}$ such that $|y_m(0)|_V^2 \le 2|y(0)|_V^2$. This is possible since $y_m(0) \to y(0)$ in V. Proceeding as in the proof of Theorem 2.1 it is now sufficient to establish a-priori estimates for the solution of (75) in appropriate norms. To begin with denote by $y = y_m$ the solution of (78).

LEMMA 2.2. Let $\kappa := By$ and $\tau := B\kappa$. Then

$$\int_\Omega By\,BSy\,dx \;=\; |y|_V^2 \;-\; \nu h\int_\Omega (S\kappa \,+\, S\tau)\,Sy\,dx$$

and

$$|S\tau|_H^2,\ |S\kappa|_H^2 \ \le\ \frac{1}{4\nu h}\,|y|_V^2. \tag{84}$$

Proof of Lemma 2.2. The definition of κ and τ implies $\kappa \in H^3(\Omega)^2 \cap V$, $\tau \in H^5(\Omega)^2 \cap V$. Moreover, $S\kappa$ and $S\tau$ are elements of V. Integration by parts gives the first part of the claim. To obtain the second claim test the equation for κ with $S\kappa$. This gives

$$|\kappa|_V^2 \ \le |y|_V^2 \quad \text{and} \quad |S\kappa|_H^2 \ \le\ \frac{1}{4\nu h}\,|y|_V^2.$$

Since the same estimate holds with κ replaced by τ and y replaced by κ the lemma is proved. □

Now test (78) with Sy to obtain

$$\frac{1}{2}\frac{d}{dt}\,|y|_V^2 + \nu\,|Sy|_H^2 + \int_\Omega (y\nabla)y\,Sy\,dx = -\frac{\rho}{h}\int_\Omega ByBSy\,dx - \rho\int_\Omega B((y\nabla)y)BSy\,dx.$$

Straightforward estimation utilizing the fourth inequality in (129), (76), (84), Young's inequality and the estimate

$$\rho\int_\Omega B((y\nabla)y)BSy\,dx \ \le\ \frac{\rho^2}{2\nu^2 h}|y|_H^2|y|_V^2 \ +\ \frac{\nu}{4}|Sy|_H^2$$

leads to

$$\frac{1}{2}\frac{d}{dt}|y|_V^2 + \frac{\nu}{4}|Sy|_H^2 + \frac{\rho}{h}\left(1-\rho-\frac{\rho}{2\nu^2}d_H\right)|y|_V^2 \ \le\ C_\nu\,|y|_V^4|y|_H^2. \tag{85}$$

By (77) $1-4\rho-\frac{\rho}{2\nu^2}d_H > \frac{1}{2}$ so that Gronwall's lemma (2.1) applied to the function $v(t) = \mathrm{e}^{\frac{\rho}{h}t}|y|_V^2$ gives

$$|y|_V^2 \ \le\ 2|y_0|_V^2\,\mathrm{e}^{-\frac{\rho}{h}t}\,\mathrm{e}^{2C_\nu|y|^2_{L^\infty(H)}|y|^2_{L^2(V)}} \ \overset{(76)}{\le}\ 2C(\nu)\,\mathrm{e}^{-\frac{\rho}{h}t}|y_0|_V^2.$$

This is the first inequality in (83) and it implies the second one. The third one follows by integrating (85). Finally the fourth inequality in (83) is obtained by testing (78) with y_t and straightforward estimation, utilizing (129). As a consequence of the a-priori estimates for $\{y_m\}_{m\in\mathbb{N}}$ the unique solution of (75) is an element of $H^{2,1}(Q)$, which completes the proof of Theorem 2.2. □

3. Stability of the continuous controller

In this section the stability properties of the control law (71) are investigated. It follows from the a-priori estimates to be derived that this feedback law admits a unique solution provided Assumption 1.1 is satisfied. Moreover, the H and V norms of difference $w = y - z$ decay exponentially with rate $-\frac{\rho}{h}$, which coincides with the results for $z = 0$. To achieve these results the value of ρ_0 in (77) has to be decreased depending on the size of $|z|^2_{L^\infty(H)}$. From now onwards let $d_H = 2|y_0 - z(0)|_H^2$.

Using w as test function in the variational formulation of (71) leads to

$$\frac{\mathrm{d}}{\mathrm{dt}}|w_t|_H^2 + \nu|w|_V^2 + \frac{\rho}{h}|Bw|_H^2 = \int_\Omega \left((w\nabla)w + (z\nabla)w + (w\nabla)z\right)\, w\, dx - \rho\int_\Omega \left(B((w\nabla)w + (z\nabla)w + (w\nabla)z)\right) Bw\, dx \;=\; (1) + \cdots + (6).$$

Since $(1) = (2) = 0$, and addend (4) can be estimated according to (79) and (80) for w replaced by y, it remains to estimate the terms $(3), (5)$ and (6) to derive a differential inequality for $y - z$ which is similar to that given in (82) for y. To begin with, estimate

$$(3) \;\le\; \frac{\nu}{2}|w|_V^2 \;+\; \frac{2}{\nu}|z|_V^2|w|_H^2,$$

$$(4) \;\le\; \rho h|B((w\nabla)w)|_H^2 + \frac{\rho}{4h}|Bw|_H^2 \;\le\; \frac{\rho}{2\nu}|w|_H^2|w|_V^2 + \frac{\rho}{4h}|Bw|_H^2,$$

$$\begin{aligned}(5) + (6) \;&\le\; \rho h\left\{|B((w\nabla)z)|_H^2 + |B((z\nabla)w)|_H^2\right\} + \frac{\rho}{4h}\,|Bw|_H^2 \\ &\le\; \frac{\rho}{4h}\,|Bw|_H^2 \;+\; \frac{\rho}{2\nu}\left\{|w|_V^2|z|_H^2 + |z|_V^2|w|_H^2\right\}\end{aligned}$$

and proceed as for the derivation of (82). This leads to

$$\frac{1}{2}\frac{d}{dt}|w|_H^2 + \left(\nu\left(\frac{1}{2} - \rho\right) - \frac{\rho}{2\nu}\left(|w|_H^2 + |z|_{L^\infty(H)}^2\right)\right)|w|_V^2 + \frac{\rho}{2h}|w|_H^2 \;\le\; \frac{4+\rho}{2\nu}|z|_V^2|w|_H^2.$$

Now, with ρ_0 from (77) let the range of ρ be implicitly defined by the relation

$$0 \;<\; \rho \;\le \min\left(\rho_0, \frac{\nu^2}{2\nu^2 + e^{\frac{4+\rho}{\nu}|z|_{L^2(V)}^2}d_H + |z|_{L^\infty(H)}^2}\right) \tag{86}$$

so that $\nu\left(\frac{1}{2} - \rho\right) - \frac{\rho}{2\nu}\left(e^{\frac{4+\rho}{\nu}|z|_{L^2(V)}^2}d_H + |z|_{L^\infty(H)}^2\right) > 0$. For ρ in this range arguments similar to those applied to (82) give

$$\frac{1}{2}\frac{d}{dt}|w|_H^2 + \left(\nu\left(\frac{1}{2} - \rho\right) - \frac{\rho}{2\nu}\left(e^{\frac{4+\rho}{\nu}|z|_{L^2(V)}^2}d_H + |z|_{L^\infty(H)}^2\right)\right)|w|_V^2 + \frac{\rho}{2h}|w|_H^2 \;\le\; \frac{4+\rho}{2\nu}|z|_V^2|w|_H^2. \tag{87}$$

Since the right-hand-side in (87) is integrable a further application of Gronwall's Lemma (2.1) finally proves

THEOREM 3.1. Let ρ satisfy (86) and let y be the unique solution of (71). Then $w = y - z$ satisfies

$$\left\{\begin{array}{lll} |w|_H^2 & \leq C\left(\nu, |z|_W\right) \mathrm{e}^{-\frac{\rho}{h}t} & \text{for all } t \in [0,T], \\ |w|_{L^\infty(H)}^2 & \leq C\left(\nu, |z|_W\right), & \\ |w|_{L^2(V)}^2 & \leq C\left(\nu, |z|_W\right) & \text{and} \\ |w_t|_{L^2(V^*)}^2 & \leq C\left(\nu, |z|_W\right) \left\{1 + \frac{\rho}{h}\right\}, & \end{array}\right. \tag{88}$$

where $C\left(\nu, |z|_W\right)$ is a positive constant independent of ρ and h.

Note that the estimate for w_t is a direct consequence of the second and third estimate in (88).

REMARK 3.1. For the result of the previous theorem only $y_0 \in H$ and $z \in W$ is required.

To prove decay also for the V norm of w use Sw as test function in (71). This leads to

$$\frac{1}{2}\frac{d}{dt}|w|_V^2 + \nu\,|Sw|_H^2 + \int\limits_\Omega ((y\nabla)y - (z\nabla)z)\; Sw\,dx = \\ -\frac{\rho}{h}\int\limits_\Omega BwBSw\,dx - \rho\int\limits_\Omega B((y\nabla)y - (z\nabla)z)BSw\,dx.$$

Now require for the range of ρ

$$0 \;<\; \rho \;\leq \min\left(\rho_0, \frac{1}{2}\frac{\nu^2}{2\nu^2 + 3e^{\frac{4+\rho}{\nu}|z|_{L^2(V)}^2}d_H + 3|z|_{L^\infty(H)}^2}\right) \tag{89}$$

so that $1-\rho-\frac{3\rho}{\nu^2}\left(e^{\frac{4+\rho}{\nu}|z|_{L^2(V)}^2}d_H + |z|_{L^\infty(H)}^2\right) \geq \frac{1}{2}$. Restricting ρ to this range, similar to the derivation of (87) after several applications of (129), (84), Lemma 2.2 and Young's inequality one ends up with the inequality

$$\frac{1}{2}\frac{d}{dt}|w|_V^2 + \frac{\nu}{2}|Sw|_H^2 + \frac{\rho}{h}\left(1 - \rho - \frac{3\rho}{\nu^2}\left(e^{\frac{4+\rho}{\nu}|z|_{L^2(V)}^2}d_H + |z|_{L^\infty(H)}^2\right)\right)|w|_V^2 \\ \leq\; C_\nu\,\left\{|w|_H^2|w|_V^2 + |z|_H|Sz|_H + |z|_V^2 + |z|_V^4\right\}|w|_V^2. \tag{90}$$

Since the right-hand-side in (90) is integrable a further application of Lemma (2.1) finally proves

THEOREM 3.2. Let ρ satisfy (89) and let y be the unique solution of (71). Then $w = y - z$ satisfies

$$\begin{array}{lll} |w|_V^2 & \leq C\left(\nu, |z|_{H^{2,1}(Q)}\right) \mathrm{e}^{-\frac{\rho}{h}t} & \text{for all } t \in [0,T], \\ |w|_{L^\infty(V)}^2 & \leq C\left(\nu, |z|_{H^{2,1}(Q)}\right), & \\ |w|_{L^2(H^2(\Omega)^2\cap V)}^2 & \leq C\left(\nu, |z|_{H^{2,1}(Q)}\right) & \text{and} \\ |w_t|_{L^2(H)}^2 & \leq C\left(\nu, |z|_{H^{2,1}(Q)}\right)\left\{1 + \frac{\rho}{h}\right\}, & \end{array}$$

where $C\left(\nu, |z|_{H^{2,1}(Q)}\right)$ is a positive constant independent of ρ and h.

4. Stability of discrete controllers

Here, first stability properties of the instantaneous control procedure (73) are investigated. As will be shown stability for a certain parameter range of h and ρ can only be ensured by requiring additionally either largeness of the viscosity parameter ν or smallness of the initial defect $y_0 - z(0)$ between the state and desired state. As will be sketched the latter restriction can be dropped when the procedure is applied to the instationary Burgers equation, see also [**54**]. Finally, a slightly modified version of the control operator (74) is applied to stabilize the fully implicit Euler-discretization of the Navier-Stokes equations and it turns out that stability can be obtained for a considerable parameter range, without requiring smallness of the data or largeness of the viscosity parameter. Fully implicit discretization of the state is a realistic situation, since the discrete controller is applied to stabilize a physical system which is described by the Navier-Stokes equations. Therfore, the choice of the discretization procedure for the uncontrolled state need not be linked to the discrete controller.

Throughout this section it is assumed that

ASSUMPTION 4.1. $0 \neq y_0 \in V$ and $z \in C([0,T]; H^{1,\infty}(\Omega)^2 \cap V)$.

To begin the investigation for the instantaneous control procedure (73) set $w = y^{j+1} - z^{j+1}$, $v = y^j - z^j$, abbreviate $z = z^j$, $j \in \mathbb{N}$, and use w as test function in the variational formulation of (73). This gives

$$
\begin{aligned}
(91)\quad & \frac{1}{2}|w|_H^2 - \frac{1}{2}|v|_H^2 + \frac{1}{2}|w-v|_H^2 + \nu h |w|_V^2 = \\
& \qquad h \int_\Omega \left((v\nabla)v + (z^j\nabla)v + (v\nabla)z^j\right) w \; dx - \rho \int_\Omega BvBw\,dx \\
& \qquad\qquad - \rho h \int_\Omega B\left((v\nabla)v + (z^j\nabla)v + (v\nabla)z^j\right) Bw\,dx = (1) + \cdots + (7).
\end{aligned}
$$

Estimating the odds using (129), (80) and Young's inequality one obtains

$$
\begin{aligned}
(1) &\le \tfrac{\nu h}{4}|w|_V^2 + \tfrac{2h}{\nu}|v|_H^2|v|_V^2, \\
(2) &\le \tfrac{\nu h}{4}|w|_V^2 + \tfrac{|z|_\infty^2 h}{\nu}|v|_H^2, \\
(3) &\le \tfrac{h}{2}|w|_H^2 + \tfrac{|z|_{1,\infty}^2 h}{2}|v|_H^2, \\
(4) &= -\rho \int_\Omega |Bw|^2\,dx - \rho\int_\Omega B(v-w)Bw \le \tfrac{\rho}{2}(\rho-1)|Bw|_H^2 + \tfrac{1}{2}|v-w|_H^2, \\
(5) &= -\rho h \int_\Omega B\left((v\nabla)v\right) Bw\,dx \le \tfrac{3\rho h}{2\nu(1-\rho)}|v|_H^2|v|_V^2 + \tfrac{\rho}{12}(1-\rho)\,|Bw|_H^2,
\end{aligned}
$$

and finally

$$
\begin{aligned}
(6) + (7) \;&=\; -\rho h \int\limits_{\Omega} B\left((v\nabla)z^j + (z^j\nabla)v\right)\, Bw\, dx \\
&\leq\; \left\{\frac{3\rho h^2 |z|_{1,\infty}^2}{1-\rho} + \frac{3\rho h |z|_\infty^2}{4(1-\rho)\nu}\right\} |v|_H^2 + \frac{\rho(1-\rho)}{6}|Bw|_H^2.
\end{aligned}
$$

Now introduce

$$
c_1(\rho, h) \;:=\; \frac{1}{2} + \frac{\rho}{4}(1-\rho) \;-\; \frac{h}{2},
$$

and

$$
\begin{aligned}
c_2^j(\rho, h, z) \;:=\; &\frac{1}{2} + \left[\frac{2h}{\nu} + \frac{3h\rho}{2\nu(1-\rho)}\right] |v|_V^2 \\
&+ \underbrace{\left[\frac{|z|_{1,\infty}^2 h}{2} + \frac{|z|_\infty^2 h}{\nu} + \frac{3\rho h^2 |z|_{1,\infty}^2}{1-\rho} + \frac{3\rho h |z|_\infty^2}{(1-\rho)4\nu}\right]}_{=:\tilde{c}_2(h,\rho,z)}.
\end{aligned}
$$

With this notation the estimates above and Lemma 2.1 together with (91) give

$$
c_1(\rho, h)\, |w|_H^2 \;-\; c_2^j(\rho, h, z)\, |v|_H^2 \;+\; \frac{\nu h}{2}(1-\rho+\rho^2)\, |w|_V^2 \;\leq\; 0. \tag{92}
$$

From now onwards let $w^j := y^j - z^j$, where y^j denote the iterates obtained by (73).

THEOREM 4.1. (Conditional H-norm stability of instantaneous control)

$$
{}^\forall \rho \in (0,1)\; {}^\exists h^*(\rho),\; 0 < \kappa < 1\; {}^\forall j \in \mathbb{N} : \; |w^{j+1}|_H^2 \;\leq\; \kappa^j\, |w^0|_H^2,
$$

provided $0 < h \leq h^*(\rho)$ and

$$
crit \;:=\; \frac{4-\rho}{\nu^2(1-\rho)(1-\rho+\rho^2)}|w^0|_H^2
$$

is sufficiently small.

Proof. Fix $\rho \in (0,1)$, define

$$
\hat{c}_2(\rho, h, z) \;:=\; \frac{1}{2} + crit + \tilde{c}_2(\rho, h, z)
$$

and argue by induction as follows.

(1) Set $j = 0$, choose $h_0 = h_0(\rho)$ and $crit$ so small that for all $0 < h \leq h_0$
   (a) $\hat{c}_2(\rho, h, z) \leq 1$ and $c_2^0(\rho, h, z) \leq 1$,
   (b) $\dfrac{\hat{c}_2(\rho, h, z)}{c_1(\rho, h)} = \kappa_1 < 1$ and
   (c) $\dfrac{c_2^0(\rho, h, z)}{c_1(\rho, h)} = \kappa_2 < 1$

holds. This is possible, since the for $\rho \in (0,1)$ the term $\frac{\rho}{4}(1-\rho)$ in the definition of $c_1(\rho, h)$ is positive. Define $\kappa := \max(\kappa_1, \kappa_2)$. Then (92) implies

$$|w^1|_H^2 \leq \kappa |w^0|_H^2$$

and

$$|w^1|_V^2 \leq \frac{2}{\nu h(1-\rho+\rho^2)}\,|w^0|_H^2 \;=\; \frac{2}{\nu h(1-\rho+\rho^2)}\,\kappa^0 |w^0|_H^2.$$

(2) Now assume that for $j \in \mathbb{N}$
   (a) $|w^j|_H^2 \leq \kappa^j |w^0|_H^2$ and
   (b) $|w^j|_V^2 \leq \dfrac{2}{\nu h(1-\rho+\rho^2)}\,\kappa^{j-1} |w^0|_H^2$

   holds true.

(3) Then conclude from (92)

$$\begin{aligned} c_2^j(\rho, h, z) &= \tfrac{1}{2} + \left[\tfrac{2h}{\nu} + \tfrac{3h\rho}{2\nu(1-\rho)}\right] |w^j|_V^2 + \tilde{c}_2(\rho, h, z) \\ &\leq \tfrac{1}{2} + \left[\tfrac{2h}{\nu} + \tfrac{3h\rho}{2\nu(1-\rho)}\right] \tfrac{2}{\nu h(1-\rho+\rho^2)}\,\kappa^{j-1} |w^0|_H^2 + \tilde{c}_2(\rho, h, z) \\ &\leq \tfrac{1}{2} + crit + \tilde{c}_2(\rho, h, z) = \hat{c}_2(\rho, h, z) \end{aligned}$$

Thus, a further application of (92) implies

$$|w^{j+1}|_H^2 \;\leq\; \frac{c_2^j(\rho, h, z)}{c_1(\rho, h)} |w^j|_H^2 \;\leq\; \frac{\hat{c}_2(\rho, h, z)}{c_1(\rho, h)} |w^j|_H^2 \;\leq\; \kappa^{j+1}\,|w^0|_H^2$$

and

$$|w^{j+1}|_V^2 \;\leq\; \frac{2}{\nu h(1-\rho+\rho^2)}\, c_2^j(\rho, h, z) |w^j|_H^2 \;\leq\; \frac{2}{\nu h(1-\rho+\rho^2)}\,\kappa^j\,|w^0|_H^2,$$

which completes the proof of Theorem 4.1. □

From the previous proof also stability with respect to the V norm can be inferred.

THEOREM 4.2. (Conditional V-norm stability of instantaneous control)

$$\forall_{\rho \in (0,1)}\ \exists_{h^*(\rho),\ 0<\kappa<1}\ \forall_{j \in \mathbb{N},\ 0<h\leq h^*}:\ |w^j|_V \;\leq\; \frac{2}{\nu h(1-\rho+\rho^2)}\kappa^{j-1}\,|w^0|_H,$$

provided

$$crit \;:=\; \frac{4-\rho}{\nu^2(1-\rho)(1-\rho+\rho^2)}|w^0|_H^2$$

is sufficiently small.

REMARK 4.1. The smallness of $crit$ in Theorems 4.1 and 4.2 is a condition either on the smallness of the initial difference between state and desired state or on the smallness on the Reynolds number of the fluid. It has to be required since there are no better estimates available for the term (1) above. The term (5) could be estimated in a slightly different way to obtain a ρ^2 in front of $|v|_V^2$, see the proof of the next Theorem, and therefore could be reduced by decreasing ρ. However, for (1) there is no further knob to fix its size. For the Burgers equation the situation is much more comfortable at this stage. Due to the continuous embedding $H^1 \hookrightarrow L^\infty$ and the well known L^2-H^1 interpolation estimate for L^∞ functions in one spatial dimension one has

$$h \int_\Omega vv'w \, dx \le \frac{\nu h}{4}|w|_V^2 + h^{1-2\alpha}|w|_H^2 + h^{1+2\alpha}\frac{1}{2\sqrt{\nu}}|w|_V^2|w|_H^2 \quad \text{for all } \alpha \in (0,1).$$

Following the lines of the proof of Theorem 4.1 one can now conclude that the smallness requirement on $crit$ may be dropped provided ρ is sufficiently small, since the power of h in the last addend on the right-hand-side of this estimate is larger than one. For more details see [**54**].

Consider now the discrete controller

$$K_h^D y = -\frac{\rho}{h}BB(y^j - z^j) - \rho BB(b(y^j) - b(z^j)) + \frac{z^{j+1} - z^j}{h} + Az^{j+1} - b(z^{j+1}), \tag{93}$$

which differs from the controller defined in (73) only in the last addend on the right-hand-side. In the next theorem the uncontrolled Navier-Stokes equations are discretized with time step size h using the fully implicit Euler scheme and the discrete controller (93) is applied to stabilize the discretized Navier-Stokes system. The modified controller stabilizes the Navier-Stokes equations for the parameters ρ and h in a considerable range, without any smallness or largeness requirements on the data of the Navier-Stokes system. The controlled equation has the form

$$\frac{y^{j+1} - y^j}{h} + Ay^{j+1} - b(y^{j+1}) = K_h^D y^j, \quad j = 0, 1, \dots \quad \text{and } y^0 = y_0, \tag{94}$$

with K_h^D from (93).

THEOREM 4.3. (H- and V-norm stability of (93))
There exists some $\rho^* \in (0,1)$ such that for every $0 < \rho \le \rho^*$ there exists a $h^*(\rho) > 0$ and a positive $\kappa < 1$ such that for all $j \in \mathbb{N}$

$$\begin{aligned} |w^j|_H^2 &\le \kappa^j \, |w^0|_H^2 \quad \text{and} \\ |w^j|_V^2 &\le \tfrac{2}{\nu h(1 - \frac{2}{3}\rho + \frac{2}{3}\rho^2)} \, \kappa^{j-1} \, |w^0|_H^2, \end{aligned}$$

provided $0 < h \le h^*(\rho)$.

Proof. Test (94) with w. Then one obtains

$$\begin{aligned} &\frac{1}{2}|w|_H^2 - \frac{1}{2}|v|_H^2 + \frac{1}{2}|w - v|_H^2 + h\nu|w|_V^2 = \\ &\qquad h \int_\Omega \left((w\nabla)w + (z^j\nabla)w + (w\nabla)z^j\right) w \, dx - \rho \int_\Omega BvBw \, dx \\ &- \rho h \int_\Omega B\left((v\nabla)v + (z^j\nabla)v + (v\nabla)z^j\right) Bw \, dx = (1)' + (2)' + (3)' + (4) + (5)' + (6) + (7). \end{aligned} \tag{95}$$

There holds $(1)' = (2)' = 0$ and $(3)'$ can be estimated as

$$(3)' \;\leq\; \frac{\nu h}{2}|w|_V^2 \;+\; \frac{h}{\nu}|z|_V^2|w|_H^2.$$

Utilizing the estimate $|BBw|_V \leq |w|_V$ one obtains

$$(5)' \;\leq\; \frac{\nu h}{3}|w|_V^2 \;+\; \frac{3\rho^2 h}{2\nu}|v|_H^2|v|_V^2$$

The remaining addenda can be estimated as above. Now introduce

$$c_1(\rho,h,z) \;:=\; \frac{1}{2}+\frac{\rho}{3}(1-\rho) \;-\; \frac{2h}{\nu}|z|^2_{L^\infty(V)} \;-\; \frac{h}{2},$$

and

$$c_2^j(\rho,h,z) \;:=\; \frac{1}{2}+\frac{3\rho^2 h}{2\nu}|v|_V^2 \;+\; \underbrace{\left[\frac{3\rho h^2|z|^2_{1,\infty}}{1-\rho}+\frac{3\rho h|z|^2_\infty}{4(1-\rho)\nu}\right]}_{=:\tilde{c}_2(h,\rho,z)}.$$

With this notation and the estimates above one concludes from (95)

$$c_1(\rho,h,z)\,|w|_H^2 \;-\; c_2^j(\rho,h,z)\,|v|_H^2 \;+\; \frac{\nu h}{2}(1-\frac{2}{3}\rho+\frac{2}{3}\rho^2)\,|w|_V^2 \;\leq\; 0. \tag{96}$$

Now define

$$\hat{c}_2(\rho,h,z) \;:=\; \frac{1}{2}+\frac{3\rho^2}{\nu^2(1-\frac{2}{3}\rho+\frac{2}{3}\rho^2)}|w^0|_H^2+\;\tilde{c}_2(\rho,h,z)$$

and proceed as follows.

(1) Choose $\rho^* \in (0,1)$ such that

$$\frac{3\rho}{\nu^2(1-\frac{2}{3}\rho+\frac{2}{3}\rho^2)}|w^0|_H^2 \;\leq\; \frac{1}{6}(1-\rho) \quad \text{for all } \rho\in(0,\rho^*]. \tag{97}$$

(2) Fix $\rho \in (0,\rho^*]$ and choose $h^* = h^*(\rho) > 0$ such that
   - (a) $\tilde{c}_2(h,\rho,z) \;\leq\; \frac{1}{4}$
   - (b) $\frac{3\rho^2 h}{2\nu}|w^0|_V^2 \;\leq\; \frac{1}{4}$
   - (c) $\frac{h}{2}+\frac{2h}{\nu}|z|^2_{L^\infty(V)}+\tilde{c}_2(h,\rho,z) < \frac{\rho}{6}(1-\rho)$
   - (d) $\frac{c_2^0(\rho,h,z)}{c_1(\rho,h,z)} \;\leq\; \kappa_1 \;<\; 1$

   holds for all h in the interval $(0,h^*]$. Let $h\in(0,h^*]$.

(3) Now conclude from (97) and $\rho\in(0,1)$ that

$$\frac{3\rho^2}{\nu^2(1-\frac{2}{3}\rho+\frac{2}{3}\rho^2)}|w^0|_H^2 \;\leq\; \frac{\rho}{6}(1-\rho) \;<\; \frac{1}{4},$$

which together with (c) implies that

$$\hat{c}_2(\rho,h,z) \;<\; 1 \quad \text{and} \quad \frac{\hat{c}_2(\rho,h,z)}{c_1(\rho,h,z)} \;\leq\; \kappa_2 < 1.$$

Furthermore, (a) and (b) give $c_2^0(\rho,h,z) \leq 1$, so that with (96)

$$|w^1|_V^2 \leq \frac{2c_2^0(\rho,h,z)}{\nu h(1-\frac{2}{3}\rho+\frac{2}{3}\rho^2)}\,|w^0|_H^2 \;\leq\; \frac{2}{\nu h(1-\frac{2}{3}\rho+\frac{2}{3}\rho^2)}\,\kappa^0|w^0|^2.$$

(4) Now assume that for $j \in \mathbb{N}$
  (a) $|w^j|_H^2 \leq \kappa^j |w^0|_H^2$ and
  (b) $|w^j|_V^2 \leq \dfrac{2}{\nu h(1-\frac{2}{3}\rho+\frac{2}{3}\rho^2)}\,\kappa^{j-1}|w^0|_H^2$

  holds true, where $\kappa :=$max$(\kappa_1,\kappa_2) < 1$.

(5) Then conclude from (b)

$$\begin{aligned} c_2^j(\rho,h,z) &= \tfrac{1}{2} + \tfrac{3h\rho^2}{2\nu}|w^j|_V^2 + \tilde{c}_2(\rho,h,z) \\ &\leq \tfrac{1}{2} + \tfrac{3h\rho^2}{2\nu}\tfrac{2}{\nu h(1-\frac{2}{3}\rho+\frac{2}{3}\rho^2)}\,\kappa^{j-1}|w^0|_H^2 + \tilde{c}_2(\rho,h,z) \\ &\leq \hat{c}_2(\rho,h,z). \end{aligned}$$

  Thus, utilizing (96) one more time gives

$$|w^{j+1}|_H^2 \;\leq\; \frac{c_2^j(\rho,h,z)}{c_1(\rho,h,z)}|w^j|_H^2 \;\leq\; \frac{\hat{c}_2(\rho,h,z)}{c_1(\rho,h,z)}|w^j|_H^2 \;\leq\; \kappa^{j+1}\,|w^0|_H^2$$

  and

$$|w^{j+1}|_V^2 \;\leq\; \frac{2}{\nu h(1-\frac{2}{3}\rho+\frac{2}{3}\rho^2)}\,c_2^j(\rho,h,z)|w^j|_H^2 \;\leq\; \frac{2}{\nu h(1-\frac{2}{3}\rho+\frac{2}{3}\rho^2)}\,\kappa^j\,|w^0|_H^2,$$

which completes the proof of Theorem 4.3. □

5. Numerical validation

Here the results obtained in the previous sections are numerically validated. In order to value the performance of the control laws (72) and (74) the numerical example is taken from [**58**]. As is demonstrated below the instantaneous controller presented here steers the H-norm and the V-norm of the difference $y - z$ to zero with exponential decay. This is similar to the numerical results for the (1,1)-receding horizon controller (i.e. control horizon length coincides with time step size) implemented in [**58**]. The instantaneous controls are compared to the optimal control and it turns out that instantaneous controls give a much better reduction of the control gain, but at significant higher overall costs.

The control problem considered here is of tracking type and is given by (21) with cost functional

$$J(y,u) := \frac{1}{2}\int\limits_{\Omega^T} |y-z|^2\,dxdt + \frac{\alpha}{2}\int\limits_{\Omega^T} |u|^2\,dxdt$$

and control space $U := L^2(Q)^2$, with B denoting the injection from U into $L^2(V^*)$. The initial value of the uncontrolled flow is chosen as

$$y(x,0) \;=\; e\left[\begin{array}{c}(\cos 2\pi x_1 - 1)\sin 2\pi x_2\\ -(\cos 2\pi x_2 - 1)\sin 2\pi x_1\end{array}\right]$$

with e denoting the Euler number, and the desired state is time dependent and given by

$$z(t,x) = \left[\begin{array}{c}\varphi_{x_2}(t,x_1,x_2)\\ -\varphi_{x_1}(t,x_1,x_2)\end{array}\right],$$

where φ is defined through the stream function

$$\varphi(t,x_1,x_2) = \theta(t,x_1)\theta(t,x_2)$$

with

$$\theta(t,y) = (1-y)^2(1-\cos 2k\pi t), \quad y \in [0,1].$$

For the results presented $\alpha = 1.e{-}2$, $k = 1$ and the time interval is chosen as $[0,2]$, i.e. $T = 2$. For the discretization in time a equidistant grid with width $\delta t = 0.01$ is used, for the spatial discretization the Taylor-Hood finite element [**55**] is used on a grid containing 1024 triangles with 2113 velocity and 545 pressure nodes. The number of unknowns in the discretized control problem therefore has the magnitude 1.65×10^6, including the primal, adjoint and control variables.

In Fig. 1 the desired flow at $T = 2$ is shown. It forms four cells with opposite flow directions near the cell boarders.

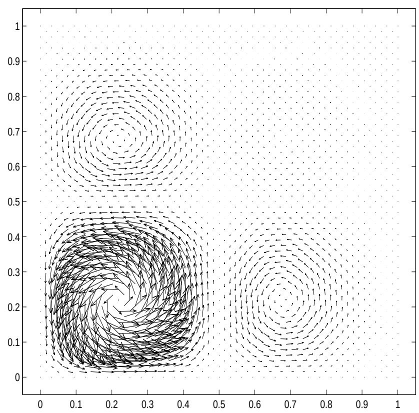

FIGURE 1. Desired flow at $T = 2$

In Figs. 2, 3 the evolution of the optimally controlled flow computed with Newton's method and the instantaneously controlled flow is illustrated a selected time instances. The costs are compared in Fig. 4. For the instantaneous control strategy they become larger with increasing time. This is due to the increasing dynamics of the desired state. As is expected the optimal control strategy equidistributes the costs over the time horizon. The performance of Newton's method applied to solve for an optimal solution is illustrated in Table 1. As one can deduce from the same table and Fig. 5 the numerical solution computed by Newton's algorithm seems to be only a local one. For $\nu = 1/\text{Re} = 1/10$ and $\gamma = 1.e-2$ the numerical computation of the optimal control takes about 45 minutes cpu-time on a DEC-ALPHA^{TM} station 500, which is one half of that needed by the gradient algorithm with step-size rule (51) for the same problem. The instantaneous feedback controller takes about 2 minutes to compute a control function on the time horizon $[0, 2]$.

Fig. 6 shows a comparison of the control actions at $t = 0.1$ and $t = 1$ between the instantaneous and the optimal controls. In Fig. 7 the evolution of the L^2-cost for the instantaneous control law is shown for $\rho = 0.1$ and different values of h. In Fig. 8 $h = 0.1$ is fix and the evolution of the control gain in the L^2- and the H^1-norm for different values of ρ are shown. Exponential decay is observed and thus, the theoretical results of Theorems 3.1, 3.2, 4.1 and 4.2 are confirmed.

Iteration	CG-steps	$\frac{\lvert\hat{J}'(u)\rvert}{\lvert\hat{J}'(u^0)\rvert}$	$\frac{\lvert\delta u^k\rvert_U}{\lvert\delta u^{k-1}\rvert_U}$	$\hat{J}(u^k)$
1	-	1.e0	-	2.385417
2	6	1.388822e-1	1.	2.583791
3	7	2.671826e-3	6.53e-2	2.539879
4	8	5.783690e-6	2.84e-2	2.538508

TABLE 1. Performance of Newton's method for Example 1

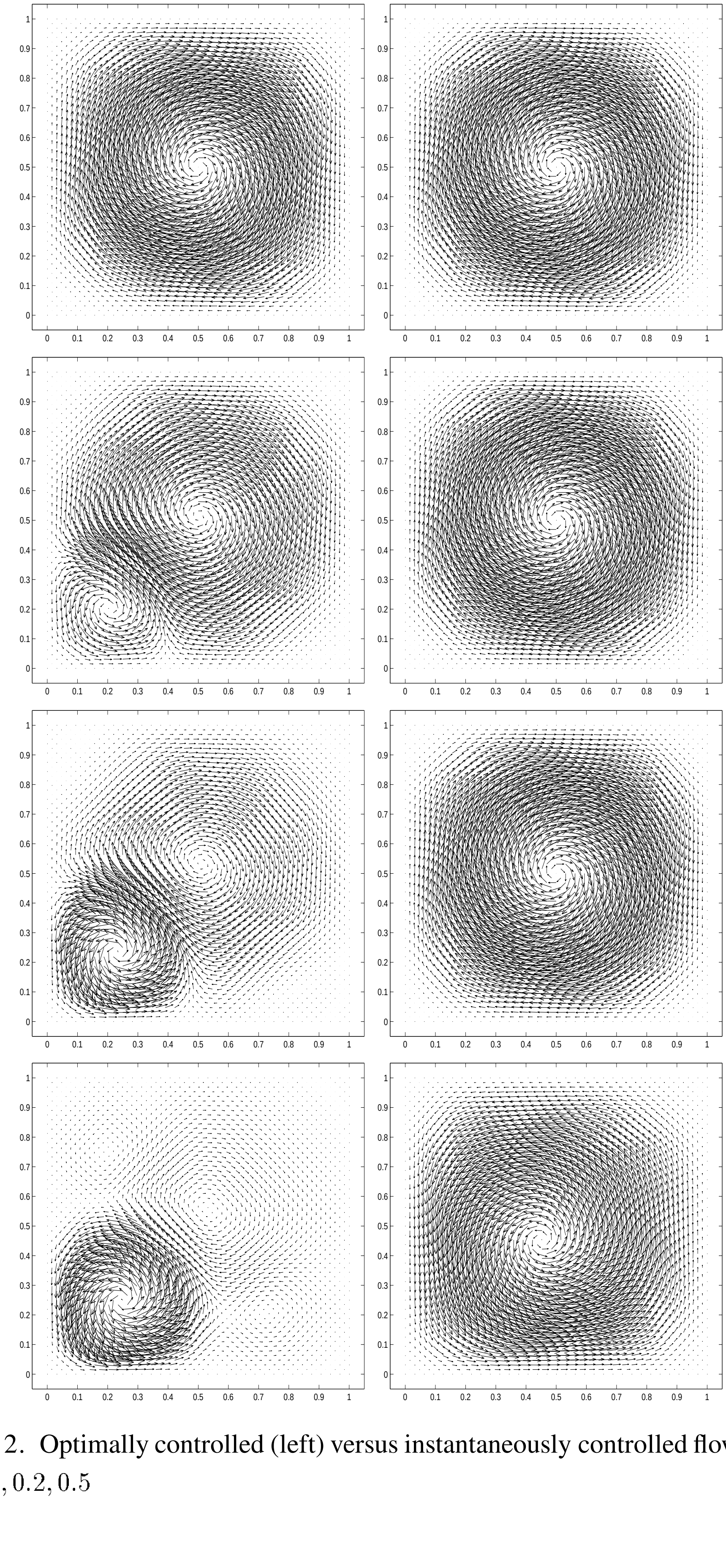

FIGURE 2. Optimally controlled (left) versus instantaneously controlled flows for $t = 0.01, 0.1, 0.2, 0.5$

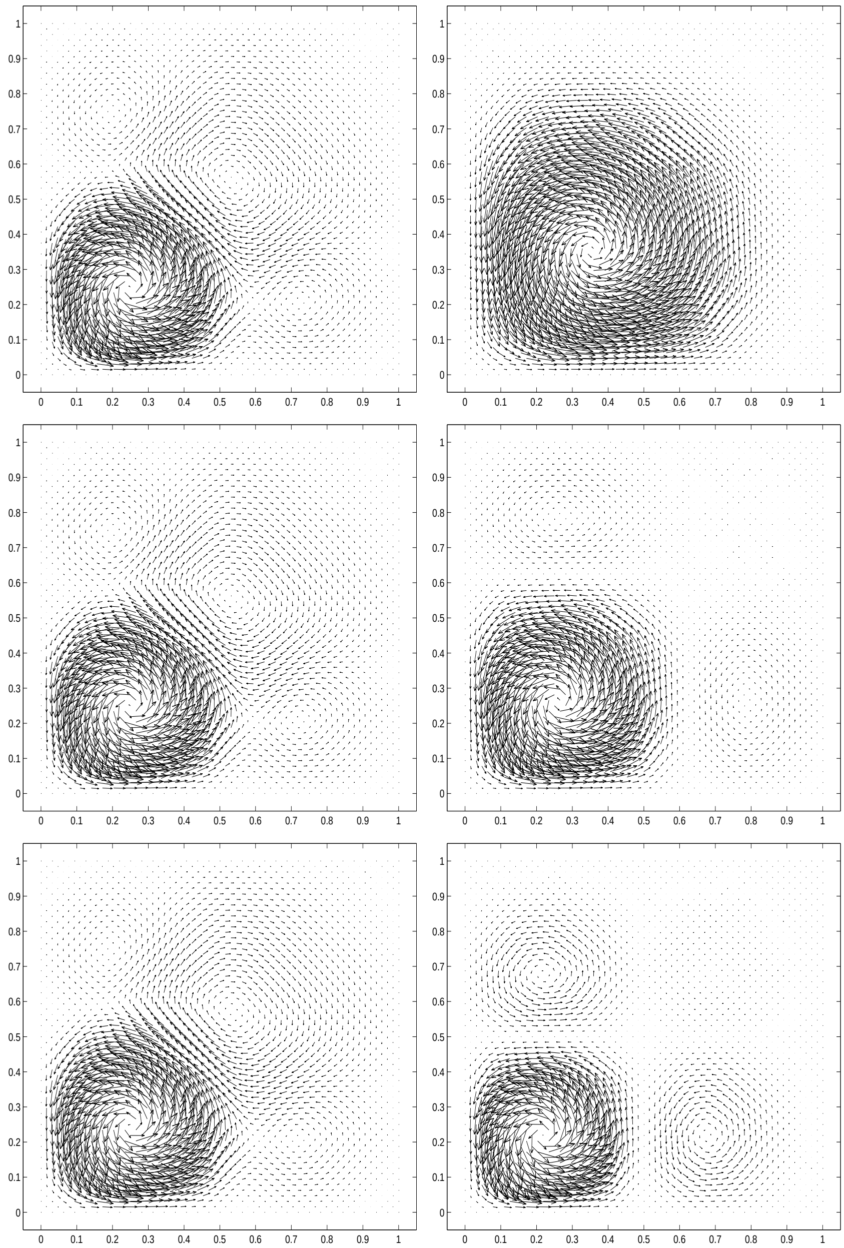

FIGURE 3. Optimally controlled (left) versus instantaneously controlled flows for $t = 1., 1.6, 2.$

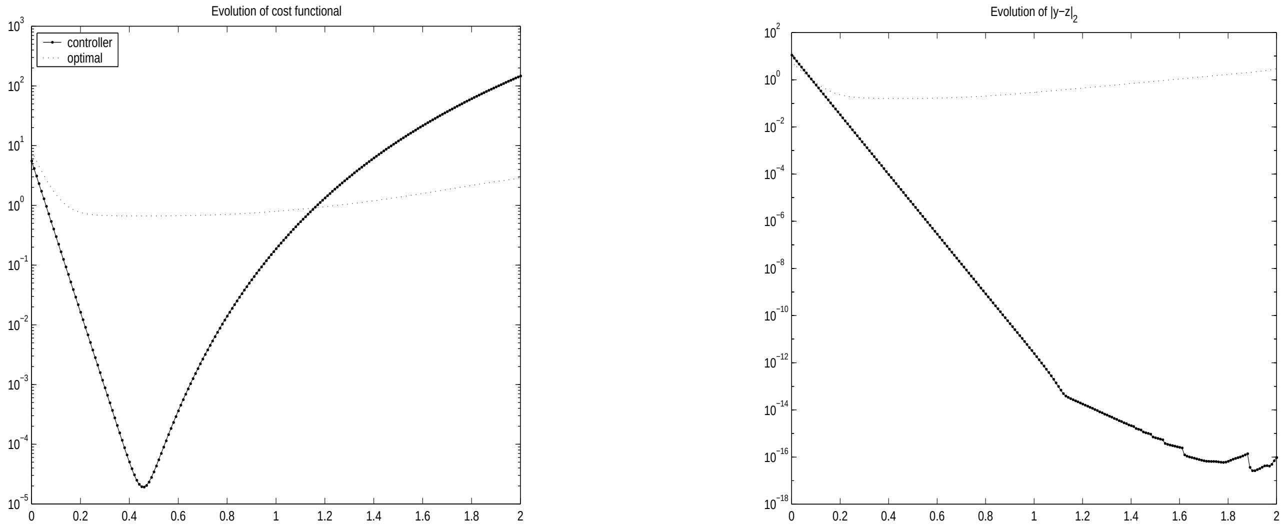

FIGURE 4. Evolution of cost (left) and control gain for $h = 0.01$ and $\rho = 0.1$

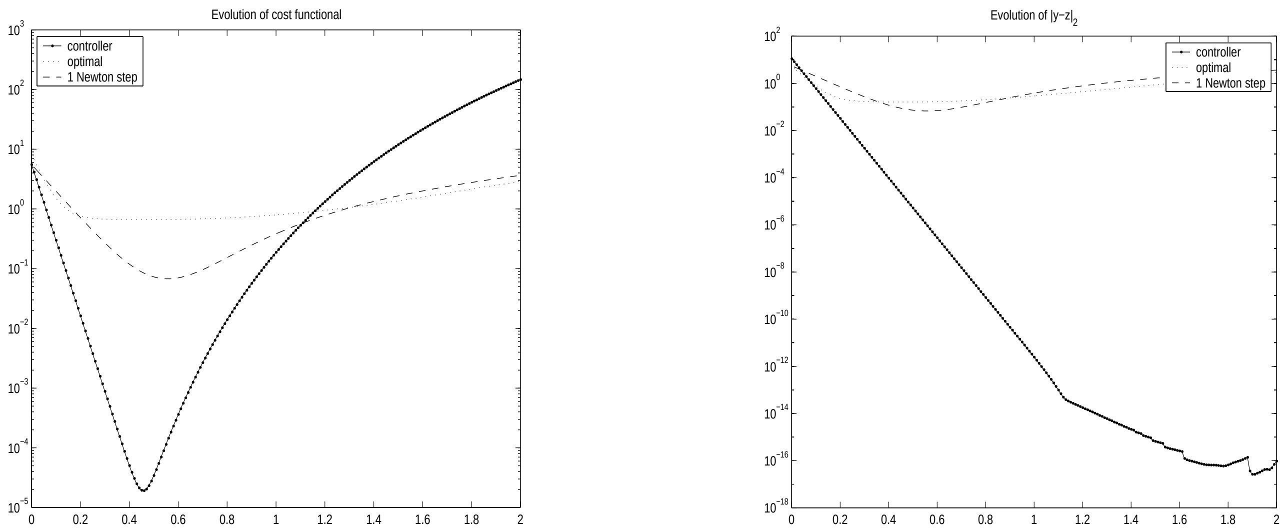

FIGURE 5. Evolution of cost (left) and control gain for $h = 0.01$ and $\rho = 0.1$

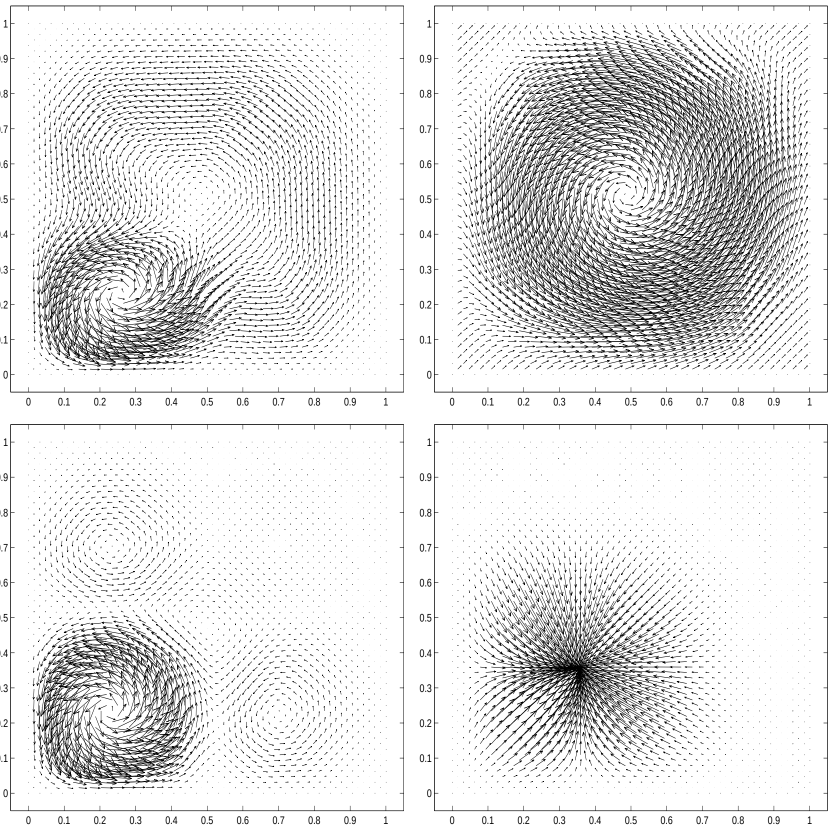

FIGURE 6. Optimal controls (left) and instantaneous controls at $t = 0.1$ (top) and $t = 1$

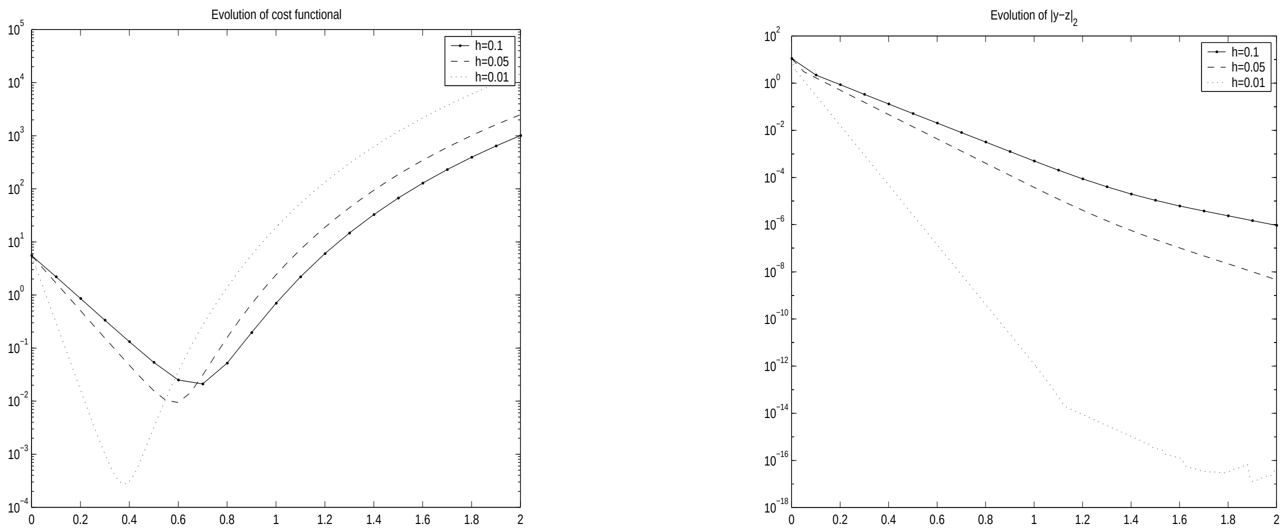

FIGURE 7. Evolution of cost (left) and control gain for $\rho = 0.1$ and $h = 0.01, 0.05, 0.1$

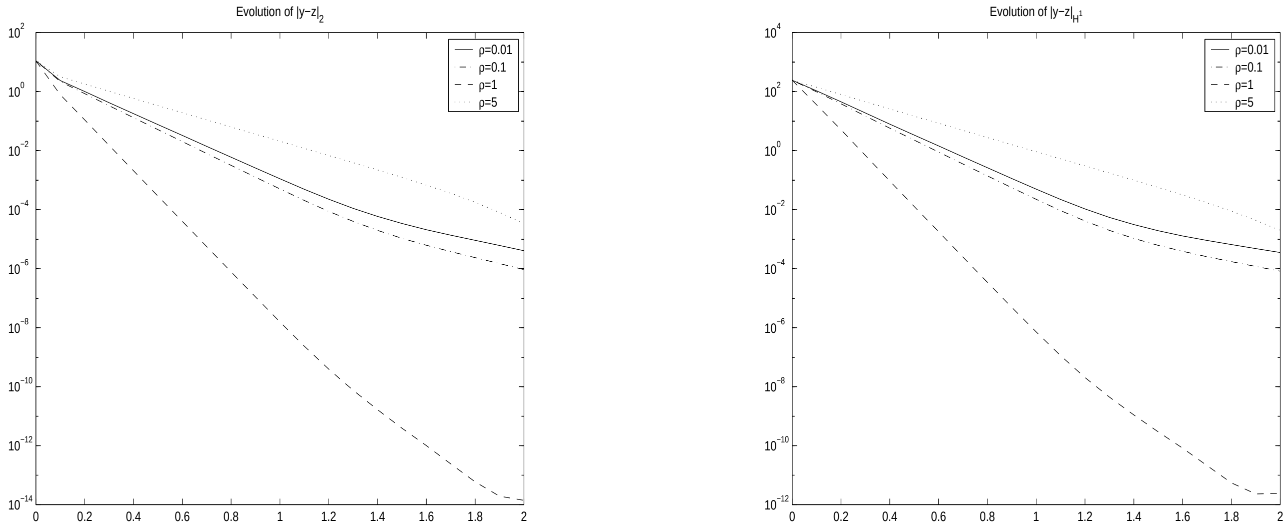

FIGURE 8. Evolution of control gain in L^2-norm (left) and H^1-norm for $h = 0.1$ and $\rho = 0.01, 0.1, 1., 5.$

CHAPTER 7

Instantaneous feedback control for backward facing step flows

In this chapter a more applied control setting is considered. Are distributed control problems with distributed observations the topic of investigation in the previous chapters now the focus is on a rather general boundary control problem with boundary observations. From the mathematical point of view the exposition is formal. However, the corresponding numerical results speak for their own. In Section 1 the model problem is introduced whose optimality system is derived in Section 2. The instantaneous control approach is described in detail in Section 3. Section 4 is dedicated to the step-size estimation in the gradient step. Finally, Section 5 presents some numerical examples.

1. Problem formulation

As model problem consider the control of a backward facing step flow. The control objective is the reduction of the recirculation bubble behind the step, and thus reducing the re-attachment length of a backward-facing-step flow, by controlling the flow at the boundary near the edge of the step. For the following presentation it will be convenient to refer to Fig. 1 which depicts the spatial domain Ω which in this chapter is assumed to be a bounded subset of $\mathbb{R}^2$, that constitutes the flow region and the subsets of the boundary that shall be refered to. The optimal control problem to be investigated in

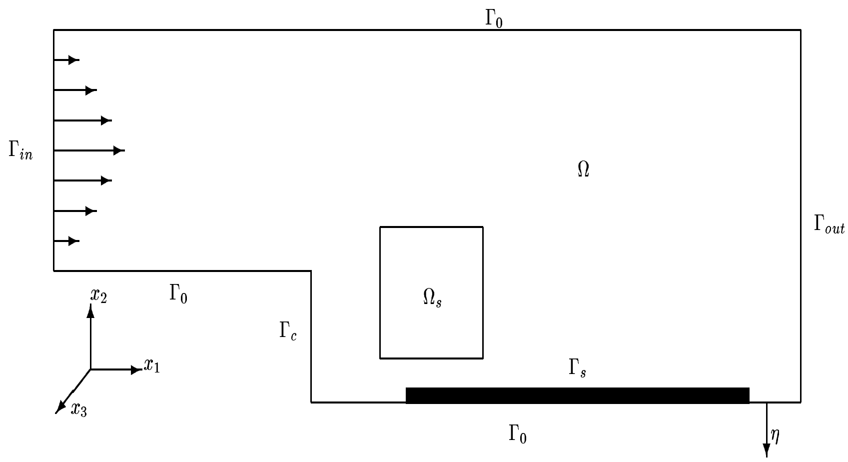

FIGURE 1. Flow region

this chapter is given by

$$
(98) \qquad (P)\begin{cases} \min \hat{J}(g) = J(y,g) \\ \text{s.t.} \\ y_t - \frac{1}{Re}\Delta y + (y\cdot\nabla)y + \nabla p = 0 & \text{in } \Omega^T \\ -\operatorname{div} y = 0 & \text{in } \Omega^T \\ y = y_d & \text{on } \Gamma_d^T \\ y = g & \text{on } \Gamma_c^T \\ \frac{1}{Re}\partial_\eta y = p\eta & \text{on } \Gamma_{out}^T \\ y(0) = y_0. \end{cases}
$$

Here $y = (y_1, y_2)$ denotes the velocity of the fluid in the directions (x_1, x_2) and p denote its pressure. Recall that the space-time cylinder is denoted by $\Omega^T = \Omega \times (0,T)$, $T > 0$. In the case of the backward facing step flow the Reynolds number Re is determined by the bulk velocity of the inlet profile U, the step hight H and the kinematic viscosity ν and is given by $Re = \frac{UH}{\nu}$. As usual y_0 denotes the initial condition of the flow at $t = 0$. The Dirichlet boundary consists of the inflow part Γ_{in}, the control boundary Γ_c and the homogenous part Γ_0. The latter one together with Γ_{in} forms Γ_d. Typical for channel flow configurations is an outflow boundary on which boundary conditions of a different type than Dirichlet boundary conditions may be described in order to model physical flows. Here the so called do-nothing boundary conditions

$$
\nu\partial_\eta y = p\eta \quad \text{on } \Gamma_{out},
$$

are used, where η denotes the unit normal to $\partial\Omega$ in outward direction. This boundary condition is used as a downstream condition in [**33, 44**] and it has been observed numerically in [**9**], e.g. that this choice of boundary condition has little artificial influence on practically observed flow patterns. Control is applied along the boundary Γ_c in form of blowing and suction modelled by the function g which has to be chosen in such a manner that an appropriately defined performance index is minimized.

REMARK 1.1. To prove existence for solutions to the instationary Navier-Stokes equations with do-nothing boundary conditions is a non-trivial task, even in two space dimensions [**8, 44**]. Moreover, regularity results for solutions have not yet been proved. Thus, at the moment a proper functional analytic setting ensuring existence of solutions to problem (98) is not available.

The choice of the cost functional appropriate to capture the goal of influencing and reducing the recirculation bubble is certainly a delicate one. Here, the control methods to be developed relate to functionals of the form

$$
(99) \qquad \hat{J}(g) = J(y,g) = \int_0^T \left[\int_{\Gamma_s} F(\partial_\eta y)\, d\Gamma_s + \frac{\gamma}{2}\int_{\Gamma_c} g^2\, d\Gamma_c \right] dt,
$$

where $\gamma > 0$ and $F : \mathbb{R}^2 \to \mathbb{R}$ is a given, sufficiently smooth function. In the numerical part of this chapter also the functional

$$\hat{J}_1(g) = \frac{1}{2}\int_0^T \left[\int_{\Omega_s} |y - y_{st}|^2\, dx \;+\; \gamma \int_{\Gamma_c} g^2\, d\Gamma_c\right] dt, \tag{100}$$

is investigated. Here $\Omega_s \subset \Omega$ denotes an observation volume inside Ω, and y_{st} denotes the Stokes flow in Ω. Although $\hat{J}_1$ relies on volume observations which is not practical from the applications point of view it serves as validation for the boundary observation-based instantaneous control strategy.

Referring to Fig. 1 one has $\eta = (0, -1)^t$ on Γ_s and hence a special case of (99) is given by

$$J(y, g) = \int_0^T \left[\int_{\Gamma_s} \frac{1}{2}\frac{\partial y_1}{\partial x_2}\left(\frac{\partial y_1}{\partial x_2} - \left|\frac{\partial y_1}{\partial x_2}\right|\right) d\Gamma_s \;+\; \frac{\gamma}{2}\int_{\Gamma_c} g^2\, d\Gamma_c\right] dt. \tag{101}$$

The first part of the cost penalizes a negative wall velocity gradient in the sensor area Γ_s which is supposed to contain the re-attachment point. The second addend weighs the costs for the control acting along Γ_c. For a discussion of further functionals of type (99) see [**19**]. Methods utilizing boundary control with boundary observation are discussed for instance in [**45, 66, 81, 88, 94, 105**].

2. The optimality system for (P)

The optimality system for problem (98) can formally be derived as in Appendix 1 with $R(a, b, c)$ replaced by $F(a)$ and $S(d, e, f) := \frac{1}{2}|d|^2$. In order to state it let g^* denote a solution to (98) (which is assumed to exist) and let $(y^*, p^*) = (y(g^*), p(g^*))$ denote the corresponding optimal velocity field and pressure, respectively satisfying the Navier-Stokes system. Then (y^*, p^*, g^*) satisfies the following

system of coupled equations in the 'primal variables' (y, p, g) and the 'adjoint variables' (ξ, π, μ):
(102)

$$\left\{\begin{array}{rcl}
y_t - \frac{1}{\mathrm{Re}}\Delta y + (y\cdot\nabla)y + \nabla p &=& 0 \quad \text{in } \Omega^T\\
-\operatorname{div} y &=& 0 \quad \text{in } \Omega^T\\
y &=& \left\{\begin{array}{ll} y_d & \text{on } \Gamma_d^T\\ g & \text{on } \Gamma_c^T\end{array}\right.\\
\frac{1}{\mathrm{Re}}\partial_\eta y &=& p\eta \quad \text{on } \Gamma_{out}^T\\
y(0) &=& y_0\\
& & \\
-\xi_t - \frac{1}{\mathrm{Re}}\Delta\xi + (\nabla y)^t\xi - (y\cdot\nabla)\xi + \nabla\pi &=& 0 \quad \text{in } \Omega^T\\
-\operatorname{div}\xi &=& 0 \quad \text{in } \Omega^T\\
\xi &=& \left\{\begin{array}{ll} 0 & \text{on } (\partial\Omega\setminus(\Gamma_{out}\cup\Gamma_s))^T\\ \mathrm{Re}\nabla F(\partial_\eta y) & \text{on } \Gamma_s^T\end{array}\right.\\
\xi(T) &=& 0\\
\frac{1}{\mathrm{Re}}\partial_\eta\xi + (y\cdot\eta)\xi &=& \pi\eta \quad \text{on } \Gamma_{out}^T\\
& & \\
\gamma g + \frac{1}{\mathrm{Re}}\partial_\eta\xi &=& \pi\eta \quad \text{on } \Gamma_c^T.
\end{array}\right.$$

Here $((\nabla y)^t\xi)_i = \sum_j(\partial_{x_i}y_j)\xi_j$. The gradient of $\hat{J}$ at g in direction χ is given by

$$\hat{J}'(g)\chi = \int_0^T\int_{\Gamma_c}(\gamma g + \frac{1}{\mathrm{Re}}\partial_\eta\xi - \pi\eta)\,\chi\, d\Gamma_c dt,$$

where (ξ, π) are computed from (102) with (y^*, p^*, g^*) replaced by (y, p, g). The primal and dual variables are strongly coupled in system (102). The coupling in the coefficient of the state equation for ξ and the boundary condition on Γ_{out} for ξ is due to the nonlinear character of the Navier-Stokes equations. The coupling between g and (ξ, π) and the fact that the primal equation is solved forward whereas the adjoint equation for (ξ, π) is solved backwards in time are characteristic for optimal control problems gouverned by time-dependent partial differential equations.

Note that for the cost functional in (99) the compatibility condition (124) for F (or g) is equivalent to

(103) $$\nabla F(\partial_\eta y)\cdot\eta = 0 \quad \text{on } \Gamma_s^T.$$

For the special case of J given in (101) and the geometry of Fig. 1 one has (note that $\eta = (0, -1)^t$)

$$F(z_1, z_2) = \frac{1}{2}z_1(z_1 + |z_1|)$$

and

$$\nabla F(\partial_\eta y) = \begin{bmatrix} -\frac{\partial y_1}{\partial x_2} + \left|\frac{\partial y_1}{\partial x_2}\right| \\ 0 \end{bmatrix},$$

so that (103) is trivially satisfied.

In principal (98) or (102) can numerically be solved with the methods developed in Chapter 4, at least for two-dimensional problems with laminar flow regime. However, even in the latter case the numerical costs are expected to be very high since already for distributed control with distributed observations the computing time on modern workstation environments is in the range between hours and days. As shall be shown in Section 5 the instantaneous control method introduced in Section 3 computes good controls at low computational costs. Moreover, as already worked out in Chapter 6 it allows for an interpretation of nonlinear discrete-in-time feedback control law which in the case of boundary observation and boundary control relies on observable state information alone.

3. Instantaneous control for boundary controls and boundary observations

Recall that instantaneous control is based on a time discretization of (98). At each discrete time level t_i a stationary control problem is approximately solved by applying one step of the gradient algorithm and the resulting control g_i^* is used to steer the system from t_i to t_{i+1}, where a new control is determined. Recall further that it cannot be claimed that the controls obtained in this manner approximate the optimal control for (98) as the discretization parameter tends to zero. However, this procedure is justified by the effectiveness that it exhibits for numerical examples and the interpretation that it allows for suboptimal feedback controls.

To commence, let $m > 1$ be fixed and set $\delta t = \frac{T}{m}$, $t_i = i\delta t, i = 0, \ldots, m$. As an intermediate step in the derivation consider the case where the Navier-Stokes equations in (102) are approximated by a Crank-Nicolson scheme. At the i-th level of the Crank-Nicolson based suboptimal strategy one approximately solves the following stationary optimal control problem (compare also [**20**]), where the variables (y, p, g) correspond to $(y(t_i), p(t_i), g(t_i))$.

$$\text{(104)} \qquad \begin{cases} \min \hat{J}(g) = J(y,g) = \int_{\Gamma_s} F(\partial_\eta y)\, d\Gamma_s + \frac{\gamma}{2} \int_{\Gamma_c} g^2\, d\Gamma_c \\ \text{s.t.} \\ e_1((y,p),g) \; = \; 0, \end{cases}$$

where

$$\text{(105)} \qquad e_1((y,p),g) \; = \; 0 \iff \begin{cases} \frac{1}{\delta t} y + \frac{1}{2}(y\cdot\nabla)y - \frac{c_1}{2}\Delta y + \nabla p & = & \tilde{R}(y(t_{i-1})) & \text{in } \Omega \\ -\operatorname{div} y & = & 0 & \text{in } \Omega \\ y & = & y_d & \text{on } \Gamma_d \\ \frac{c_1}{2}\partial_\eta y - p\eta & = & -\frac{c_1}{2}\partial_\eta y(t_{i-1}) & \text{on } \Gamma_{out} \\ y & = & g & \text{on } \Gamma_c. \end{cases}$$

Here $\tilde{R}(y(t_{i-1})) = \frac{1}{\delta t} y(t_{i-1}) + \frac{1}{2\mathrm{Re}}\Delta y(t_{i-1}) - \frac{1}{2}(y(t_{i-1})\cdot\nabla)y(t_{i-1})$ is a known inhomogeneity at time t_i and $c_1 := 1/\mathrm{Re}$.

For a given control g the gradient of the cost functional $\hat{J}$ in (104) at point g is given by

$$\text{(106)} \qquad \hat{J}'(g) = (\gamma g + c_1 \partial_\eta \xi - \pi\eta)_{|\Gamma_c}.$$

Here (ξ,π) is a solution to $e^*_{1(y,p)}((y,p),g)(\xi,\pi) = -J_y(y,g)$, i.e.

$$
(107) \qquad \left\{ \begin{array}{rcll}
\frac{1}{\delta t}\xi - \frac{1}{2}(y\cdot\nabla)\xi + \frac{1}{2}(\nabla y)^t\xi & = & \frac{c_1}{2}\Delta\xi - \nabla\pi & \text{in } \Omega \\
-\operatorname{div}\xi & = & 0 & \text{in } \Omega \\
\xi & = & \left\{ \begin{array}{ll} 0 & \text{on } \partial\Omega\setminus(\Gamma_{out}\cup\Gamma_s) \\ \frac{2}{c_1}\nabla F(\partial_\eta y) & \text{on } \Gamma_s \end{array}\right. & \\
\xi\cdot\eta & = & 0 & \text{on } \Gamma_s \\
\frac{c_1}{2}\partial_\eta\xi + \frac{1}{2}(y\cdot\eta)\xi & = & \pi\eta & \text{on } \Gamma_{out}
\end{array}\right.
$$

with (y,p) a solution of the primal equations (105) with boundary values g on Γ_c. The instantaneous control Algorithm 3.1 for the control problem (98) and Crank-Nicolson time integration can be reformulated as

ALGORITHM 3.1. (Instantaneous control with Crank-Nicolson)

(1) Given an initial state y_0, set $j=0$, $t_0=0$.
(2) Given g_j^0, solve
 i. Solve for (y,p) in (105) with $g = g_j^0$.
 ii. Solve for (ξ,π) in (107).
(3) Set $\hat{J}'(g_j^0) = (\gamma g_j^0 + c_1\partial_\eta\xi_j - \pi_j\eta)_{|\Gamma_c}$.
(4) Given $\rho > 0$, set $g_{j+1} = g_j^0 - \rho\,\hat{J}'(g_j^0)$.
(5) Solve for (y_{j+1},p_{j+1}) in (105) with $g = g_{j+1}$ and $\tilde{R}(y_j)$.
(6) Set $t_{j+1} = t_j + h$, $j = j+1$.
(7) Goto 2.

Note that (107) requires information of the velocity inside the domain as well as on its boundary. For computations this requires recalculation of the system matrix for the adjoint equation whenever the primal velocity field y changes. Even more importantly from the feedback-control point of view, one aims for replacing y by observable quantities of the system and computing g from (107) and (106). This would require knowledge of the velocity field y inside Ω which is impractical. To circumvent these drawbacks the discretization is further simplified by replacing the implicit Crank-Nicolson scheme for the nonlinear equation by a semi-implicit time discretization scheme (see also [**20**]). In what follows the Euler method implicit in the viscous term and explicit in the nonlinear convective term is chosen, but different discretization schemes treating the non-linear term explicitly would also work. As is shown below in this case only measurements of the wall shear stress are necessary to compute the adjoint, which in turn determine the control.

Consider the minimization problem

$$
(108) \qquad (P_i)\left\{ \begin{array}{l}
\min \hat{J}(g) = J(y,g) = \int_{\Gamma_s} F(\partial_\eta y)\,d\Gamma_s + \frac{\gamma}{2}\int_{\Gamma_c} g^2\,d\Gamma_c \\
\text{s.t.} \\
e((y,p),g) \;=\; 0,
\end{array}\right.
$$

where

$$e((y,p),g) = 0 \iff \left\{ \begin{array}{rcll} y - c\Delta y + \delta t \nabla p &=& R(y(t_{i-1})) & \text{in } \Omega \\ -\operatorname{div} y &=& 0 & \text{in } \Omega \\ y &=& y_d & \text{on } \Gamma_d \\ c\partial_\eta y &=& \delta t \, p\eta & \text{on } \Gamma_{out} \\ y &=& g & \text{on } \Gamma_c. \end{array} \right. \tag{109}$$

Here, $R(y(t_{i-1})) = y(t_{i-1}) - \delta t(y(t_{i-1}) \cdot \nabla)y(t_{i-1})$ and $c := \frac{\delta t}{\mathrm{Re}}$. Note that the information from time level t_{i-1} to t_i is passed solely through the inhomogeneity R. Again for a given control g the gradient of the cost functional $\hat{J}$ in (104) at point g is given by (106), where now (ξ, π) is a solution to $e^*_{(y,p)}((y,p),g)(\xi,\pi) = -J_y(y,g)$, i.e.

$$\left\{ \begin{array}{rcll} \xi - c\Delta\xi + \nabla\pi &=& 0 & \text{in } \Omega \\ -\delta t \operatorname{div} \xi &=& 0 & \text{in } \Omega \\ \xi &=& \left\{ \begin{array}{ll} 0 & \text{on } \partial\Omega \setminus (\Gamma_{out} \cup \Gamma_s) \\ \frac{1}{c}\nabla F(\partial_\eta y) & \text{on } \Gamma_s \end{array} \right. & \\ \xi \cdot \eta &=& 0 & \text{on } \Gamma_s \\ c\partial_\eta \xi &=& \pi\eta & \text{on } \Gamma_{out} \end{array} \right. \tag{110}$$

with (y,p) the solution of the primal equations $e((y,p),g) = 0$ in (109). Hence, only state information on Γ_s enters into the computation of gradient information. Consequently the instantaneous control Algorithm 3.1 for the control problem (98) can be interpreted as feedback algorithm. The following feedback algorithm may be applied to control arbitrary fluid flows governed by the Navier-Stokes equations.

ALGORITHM 3.2. (Instantaneous feedback control)

(1) Measure $\partial_\eta y$ on Γ_s.
(2) Solve for (ξ, π) in (110).
(3) Given g^0, set $\hat{J}'(g^0) = (\gamma g^0 + c\partial_\eta \xi - \pi\eta)_{|\Gamma_c}$.
(4) Given $\rho > 0$, set $g = g^0 - \rho\, \hat{J}'(g^0)$.
(5) Apply blowing/suction determined by g on Γ_c to the flow.
(6) Goto 1.

In a practical applications Algorithm 3.2 may be applied whenever measurements on Γ_s are available. Time stepping in this algorithm is contained implicitly with step size determined by the CPU-time necessary to solve for the adjoint variables in (110) together with the time span necessary to take the measurements.

4. Line search

In Algorithms 3.1 and 3.2 the descent parameter ρ is a degree of freedom which due to the non-linear dependence of the state on the observation in (104) and (108) in general can not be determined explicitly as is the case in a fully linear setting, compare (58). Now concentrate on Algorithm 3.2. In order to maximize the descent in step 4. it is natural to seek a minimizer ρ^* of the function

$$H(\rho) \; = \; \hat{J}(g - \rho\, s),$$

where $s = \hat{J}'(g)$. Applying the usual linearization technique to the equation $H'(\rho^*) = 0$ leads to the approximation

$$\rho^*_{app} \; = \; \frac{|\hat{J}'(g)|^2_{\mathcal{U}}}{\langle \hat{J}''(g)s, s\rangle_{\mathcal{U}}}$$

of the optimal value ρ^*, where the second derivative $\hat{J}''(g)$ is given by (33) with u replaced by g. The space $\mathcal{U}$ denotes the Hilbert space of the boundary controls, which is identified with its dual. Since the state equations (109) are linear a close inspection of (33) shows that

$$\rho^*_{app} = \frac{|\hat{J}'(g)|^2_{\mathcal{U}}}{\gamma|\hat{J}'(g)|^2_{\mathcal{U}} \; + \; \int_{\Gamma_c}(c\partial_\eta\xi(s) - \pi(s))s\, d\Gamma_c}, \tag{111}$$

where $(\xi(s), \pi(s))$ is obtained by solving first $e_{(y,p)}((y,p),g)((y(s),p(s)) = -e_g((y,p),g)s$, i.e.

$$\left\{ \begin{array}{rcl} y(s) - c\Delta y(s) + \delta t \nabla p(s) & = & 0 \quad \text{in } \Omega \\ -\text{div}\, y(s) & = & 0 \quad \text{in } \Omega \\ y(s) & = & \left\{ \begin{array}{l} 0 \text{ on } \Gamma_d \\ s \text{ on } \Gamma_c \end{array} \right. \\ c\partial_\eta y(s) & = & \delta t\, p(s)\eta \quad \text{on } \Gamma_{out}. \end{array} \right. \tag{112}$$

and then utilizing $(y(s), p(s))$ to solve for $(\xi(s), \pi(s))$ in $e^*_{(y,p)}((y,p),g)(\xi(s), \pi(s)) = -J_{yy}(y,g)(y(s))$, i.e.

$$\left\{ \begin{array}{rcl} \xi(s) - c\Delta\xi(s) + \nabla\pi(s) & = & 0 \quad \text{in } \Omega \\ -\delta t \text{div}\, \xi(s) & = & 0 \quad \text{in } \Omega \\ \xi(s) & = & \left\{ \begin{array}{ll} 0 & \text{on } \Gamma_d \\ \frac{1}{c}\nabla^2 F(\partial_\eta y)\partial_\eta y(s) & \text{on } \Gamma_s \end{array} \right. \\ \xi \cdot \eta & = & 0 \quad \text{on } \Gamma_s \\ c\partial_\eta \xi(s) & = & \pi(s)\eta \quad \text{on } \Gamma_{out}. \end{array} \right. \tag{113}$$

Here, $y = y(g)$ satisfies the linear state equation (109), i.e. $e((y,p),g) = 0$.

The linearization procedure which leads to the approximate value ρ^*_{app} can be made more transparent by applying a Lagrange approach to the minimization of the function $H(\rho)$ similar to that sketched in Appendix 1. To begin with consider for $s = J'(g)$ the constraint minimization problem

$$(L)\begin{cases} \min H(\rho) = \int_{\Gamma_s} F(\partial_\eta y)\, d\Gamma_s + \frac{\gamma}{2}\int_{\Gamma_c}(g-\rho s)^2\, d\Gamma_c \\ \text{s.t. } (y,p) = (y(\rho), p(\rho)) \text{ a solution to} \\ e((y,p), g-\rho s) = 0. \end{cases}$$

The corresponding Lagrangian is given by

$$\begin{aligned} G(y,p,\rho,\xi,\pi,\mu) = \int_{\Gamma_s} F(\partial_\eta y)\, d\Gamma_s + \frac{\gamma}{2}\int_{\Gamma_c}(g-\rho s)^2\, d\Gamma_c + \int_{\Gamma_c}(y-g+\rho s)\mu\, d\Gamma_c \\ + \int_\Omega [(y - c\Delta y + \delta t\nabla p - R)\xi - \pi \mathrm{div} y]\, d\Omega, \end{aligned}$$

and the constraints on Γ_d and Γ_{out} are understood as explicit constraints.

The first order optimality system for (L) is given by

$$G_y(w) + G_p(q) = 0, \text{ and } \quad G_\rho = 0 \tag{114}$$

for all (w,q) satisfying $w = 0$ on Γ_d and $\frac{1}{Re}\partial_\eta w = q\eta$ on Γ_{out}. Condition (114) is equivalent to

$$\begin{aligned} \int_{\Gamma_s} \nabla F(\partial_\eta y)\partial_\eta w\, d\Gamma_s + \int_\Omega (w - c\Delta w)\xi\, dx + \delta t \int_\Omega \xi \nabla q\, dx \\ - \int_\Omega \pi \mathrm{div} w\, dx + \int_{\Gamma_c} w\mu\, d\Gamma_c = 0 \end{aligned} \tag{115}$$

and

$$\int_{\Gamma_c} [\gamma(\rho s - g) + \mu] s\, d\Gamma_c = 0. \tag{116}$$

From (115) it follows that

$$e^*_{(y,p)}((y,p), g-\rho s)(\xi,\pi) \;=\; -J_y(y, g-\rho s), \tag{117}$$

compare (110), and

$$c\partial_\eta \xi + \mu = \pi\eta \quad \text{on } \Gamma_c. \tag{118}$$

From (116) one has

$$\rho^* = \frac{1}{\gamma |s|^2_{L^2(\Gamma_c)}} \int_{\Gamma_c} (\gamma g - \mu) s\, d\Gamma_c$$

and hence with (118)

$$\rho^* = \frac{1}{\gamma |s|^2_{L^2(\Gamma_c)}} \int_{\Gamma_c} (\gamma g + c\partial_\eta\xi - \pi\eta) s\, d\Gamma_c \tag{119}$$

where (ξ,π) is given by (117) with (y,p) determined by the state equations in (L). Since (y,p) and consequently (ξ,π) depend on ρ, (119) is a nonlinear equation for ρ. An approximation to ρ^* can now be derived based on a linearization of ∇F at $\partial_\eta y(g)$. To begin with note first that the solution

$(y,p) = (y(\rho), p(\rho))$ of the equality constraints in (L) is linear with respect to the inhomogeneity in the boundary condition on Γ_c and hence

$$y(\rho) = y(g) - \rho y(s), \quad p(\rho) = p(g) - \rho p(s),$$

where $(y(s), p(s))$ is the solution to (112) and $(y(g), p(g))$ denotes the solution of $e((y,p),g) = 0$. The boundary condition for the adjoint variable along Γ_s is linearized according to

$$\nabla F(\partial_\eta(y(g) - \rho y(s))) \approx \nabla F(\partial_\eta y(g)) - \rho \nabla^2 F(\partial_\eta y(g)) \partial_\eta y(g), \tag{120}$$

where $\nabla^2 F$ is the Hessian of $F : \mathbb{R}^2 \to \mathbb{R}$. Accordingly the adjoint variables are approximated by

$$\xi(\rho) \approx \xi(g) - \rho \xi(s), \quad \pi(\rho) \approx \pi(g) - \rho \pi(s),$$

where $(\xi(s), \pi(s))$ is the solution to (113) and $(\xi(g), \pi(g))$ solves

$$e^*_{(y,p)}((y,p),g)(\xi(g), \pi(g)) \;=\; -J_y(y(g), g), \tag{121}$$

which is equivalent to (110). From (116) and (117) with $(\xi(\rho), \pi(\rho))$ linearized according to (120) an approximation to the optimal step size is formed,

$$\rho^*_{app} = \frac{\int_{\Gamma_c} (\gamma g + c\partial_\eta \xi(g) - \pi(g)\eta) s \, d\Gamma_c}{\int_{\Gamma_c} (\gamma s + c\partial_\eta \xi(s) - \pi(s)) s \, d\Gamma_c}$$

which coincides with that presented in (111).

To summarize, every iteration of the gradient algorithm with step size estimate ρ^*_{app} requires the solution of four quasi-Stokes problems: (110) and (121) for the evaluation of the gradient and two additional ones given by (112) and (113) for the computation of ρ^*_{app}. Note that in the case that F is quadratic (120) is exact and hence ρ^*_{app} gives the optimal step length for (L). Using ρ^*_{app} a more sophisticated feedback control law than that given in Algorithm 3.2 is then given by

ALGORITHM 4.1. (Instantaneous feedback control extended)

(1) Measure $\partial_\eta y$ on Γ_s.
(2) Solve for (ξ, π) in (110).
(3) Given g^0, set $\hat{J}'(g^0) = (\gamma g^0 + c\partial_\eta \xi - \pi\eta)_{|\Gamma_c}$.
(4) Solve (112) for $(y(s), p(s))$ with $s = \hat{J}'(g^0)$
(5) Evaluate $\partial_\eta y(s)$ on Γ_s and solve (113) for $(\xi(s), \pi(s))$.
(6) Compute ρ^*_{app} from (111).
(7) Set $g = g^0 - \rho^*_{app} \hat{J}'(g^0)$.
(8) Apply blowing/suction determined by g on Γ_c to the flow.
(9) Goto 1.

Note, that for the solution of the auxiliary problems (112) and (113) only observations on the boundary are needed. Furthermore, in Algorithms 3.2, 4.1 it is assumed that the state measurements depend on the control g^0, i.e. the current state is determined by the control g^0.

For the cost in (101) the elements of the Hessian must be interpreted as directional derivatives. Recalling $\eta = (0, -1)^t$, for the cost functional in (101) there holds

$$\nabla F(\partial_\eta y) = \begin{bmatrix} -\frac{\partial y_1}{\partial x_2} + \left|\frac{\partial y_1}{\partial x_2}\right| \\ 0 \end{bmatrix}.$$

The Hessian of F exists as directional derivative as long as $\frac{\partial y_1}{\partial x_2} \neq 0$. At $\frac{\partial y_1}{\partial x_2} = 0$ the functional F allows no second derivative. In practice

$$\nabla^2 F(\partial_\eta y)\partial_\eta w = \begin{bmatrix} \left[-1 + \text{sign}\left(\frac{\partial y_1}{\partial x_2}\right)\right] \frac{\partial w_1}{\partial x_2} \\ 0 \end{bmatrix},$$

is taken, where

$$\text{sign}(s) \ := \begin{cases} 1, & s \geq 0 \\ 0, & s < 0. \end{cases}$$

5. Backward-facing-step numerics

In this section numerical results related to the approaches developed in the previous sections are presented. The focus here is two-fold: to numerically prove the reliability of the feedback procedures developed in Algorithms 3.2, 4.1, and to show that instantaneous feedback control applied to laminar backward facing step flows yields remarkable reduction of the value of the cost functional at considerable numerical costs.

The computational domain is the backward-facing-step shown in Fig. 1. The inflow boundary is located at a distance of two step heights, i.e. $2h$ before the step, and an expansion ratio of 2 is used. The outflow boundary is located at a distance of $8h$ behind the step. The Reynolds number is based on the step height h and the bulk velocity U at the inlet. The boundary conditions are no-slip at the top and the bottom walls and *do-nothing* at the outflow boundary.

The optimization goal is the reduction of the recirculation bubble behind the step and thus, of the re-attachment length of the two-dimensional laminar Navier-Stokes flow at Reynolds numbers up to Re$= 400$. For this purpose instantaneous control is applied to problem (98) involving cost functionals of tracking type (100) and of boundary observation type (101). Among other things dependencies on parameters determining the problem formulation as well as on the step size in the gradient algorithm are investigated.

For the discretization of the quasi-Stokes problems the Taylor-Hood finite element is used on a grid containing 1152 triangles, 633 pressure and 2417 velocity nodes. For the numerical solution of the quasi-Stokes problems the least-squares approach developed in [**33**] is used, see also [**6**]. The control strategies applied here are based on first-order semi-implicit time discretization (explicit Euler for the nonlinear term, implicit Euler for the viscous term) with time stepping $\delta t = 0.01$. The final time is $T = 100$. This means that the total number of unknowns in the optimization problem (98) is of order 6×10^8 which makes the problem far too large to be handled with standard optimization software. It is worth noting that the computational time needed for the solution of the optimal tracking type control problem discussed in the next subsection utilizing a gradient method lies in the range between two days and one week on a ORIGIN^{TM} 200, the time span depending on the required accuracy. This is due to the long control horizon. The instantaneous control strategy takes about 6 hours of cpu-time on the same workstation to compute the suboptimal control.

In all figures presented the label *Cost* refers to the evolution of the cost functional $\hat{J}$, *Control Cost* refers to the evolution of the square of the L^2-norm of the control input taken over the control boundary Γ_c, *Difference to Stokes Flow* refers to the evolution of the square of the L^2-norm in the whole of Ω of the difference between the controlled flow and the Stokes flow.

5.1. Volume observations. Here the numerical results for the case of the reconstruction of the Stokes flow in the observation volume $\Omega_s = [4,6] \times [-1,0] \subset \Omega$ behind the step of the flow domain are presented. The functional to be minimized at each time instance is related to the functional $\hat{J}_1$ in (100) and is given by

$$\hat{J}(g) = \frac{1}{2}\int_{\Omega_s} |y - y_{st}|^2 \, dx + \frac{\gamma}{2}\int_{\Gamma_c} g^2 \, d\Gamma_c,$$

where g denotes the boundary control applied at the upper part of the back wall $[2,2]\times[-0.5,0] \subset \Gamma_c$. In all numerical computations presented Re$=400$ and a hybrid version of Algorithms 3.2 and 4.1 is used, see Tab. 2. Note that for this example the adjoint equations (110) and (113) have to be slightly modified. The boundary conditions on Γ_s are now homogeneous, the right-hand-side in (110) is now equal to $-(y - y_{st})\chi_{\Omega_s}$ and the right-hand-side in (113) is equal to $-y(s)\chi_{\Omega_s}$, where χ_{Ω_s} denotes the indicator function on Ω_s. Furthermore, in both algorithms 1. has to be replaced by

1.' Evaluate $y - y_{st}$ in Ω_s.

and in Algorithm 4.1 in addition 5. by

5.' Evaluate $y(s)$ in Ω_s and solve (113) for $(\xi(s), \pi(s))$.

The time history of control and the gradient descent parameter is illustrated in Tab. 5.1.

Time	$0 \le t \le 2$	$2 \le t \le 20$	$20 \le t \le 100$
ρ	no control	1	ρ^*_{app}

TABLE 1. Time history of control and the gradient descent parameters ρ

The influence of the varying penalty parameter γ on the controls is illustrated in Tab. 2. It can be observed that the descent parameter ρ^*_{app} remains nearly constant over the whole time integration interval. This may be due to the quadratic nature of linearly constraint tracking-type control problems. Fig. 2(a) shows the evolution of the costs. For $\gamma = 1$ one observes increasing costs for $t \in [20, 30]$.

γ	1.e0	1.e-1	1.e-2	1.e-3
ρ^*_{app}	0.82	3.13	4.35	4.53

TABLE 2. Optimal descent parameter ρ^*_{app} with the corresponding penalty parameter γ at Re= 400

This is due to the suboptimal character of the approach. Decreasing γ yields decreased costs at time instances where the controls are stationary. The oscillations in the costs for small γ are due to the instantaneous character of the feedback control mechanism. As Fig. 3 shows, these effects can be damped by performing further gradient steps in the instantaneous control problems, see also [**19**]. Moreover, the robustness of the control mechanism increases with the number of gradient steps. To reduce the oscillatory effects the relative amount of changes in the control from one time step to the next should be bounded.

In the evolution of the control costs in Fig. 2(b) one also observes oscillations of the control after the inset of the use of ρ^*. In the long term these oscillations disappear with time and the control costs converge to steady states. This means that the feedback control mechanism is stable. The same holds true for the evolution of the L^2-difference of the flow to the Stokes flow in Fig 2(c). The feedback controller presented here yields a significant reduction of the costs of all targets under consideration. The controlled flows for different penalty parameters are shown in Fig.4. The controls as expected adjust the re-attachment point in a neighbourhood of $x = 4$ (note that the flow is observed in $[4, 6] \times [-1, 0]$).

In Figs. 5 and 6 a numerical comparison between optimal and instantaneous control is presented. Here $\gamma = 10^{-1}$ and Re=300. The optimal control is computed with the gradient Algorithm 7.1, where in step 2. $\rho = 0.2$ is used as descent parameter. The iteration of the algorithm is stopped if the L^2-difference of two successive iterates is smaller than 10^{-3}. Instantaneous control is performed with step size ρ^*_{app}.

Fig. 5, top shows the uncontrolled Navier-Stokes flow with re-attachment point at 8.75. The application of the feedback mechanism up to $T = 70$ yields the flow presented in Fig. 5, bottom with re-attachment point located at $x = 4$, which is expected in view of the location of the observation volume Ω_s. Fig. 5, middle shows the optimal controlled flow. As one can see, the blowing direction on the control boundary is not longer perpendicular to Γ_c. For both approaches the downstream recirculation vanishes nearly completely.

Fig. 6 shows costs for the instantaneous control approach in Algorithm 4.1 (dotted) compared to those for the optimal solution (dash-dotted), compare [**50**]. The numerical amount of instantaneous control corresponds approximately to that for one gradient step in the optimal control problem (2$\times$ Navier-Stokes, 1$\times$ linearized Navier-Stokes). As is expected, the optimal control approach yields smaller overall cost than the instantaneous control strategy. Nevertheless, the suboptimal control strategy proposed here also achieves a remarkable reduction of the costs.

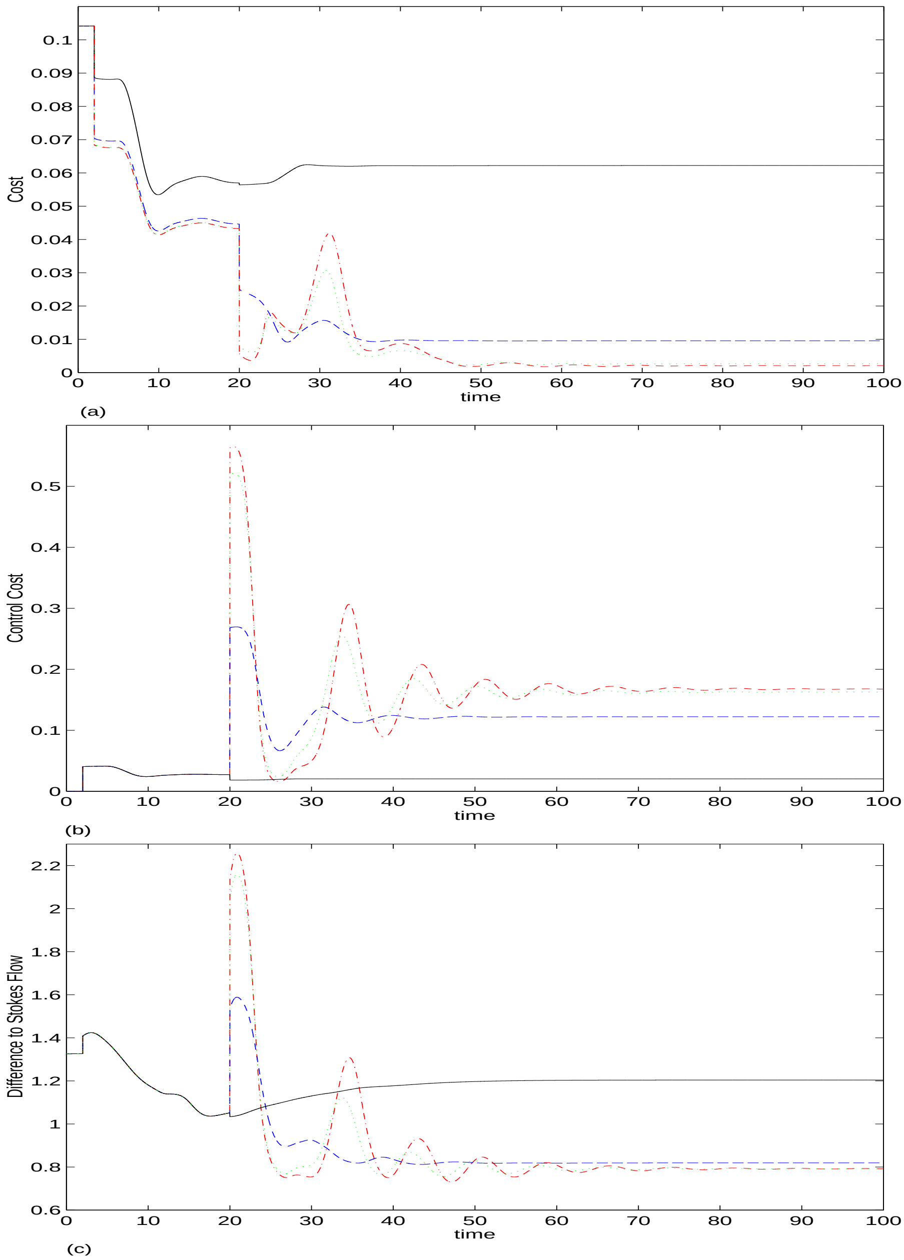

FIGURE 2. Boundary control at Re=400: γ =1(solid), 1.e-1(dashed), 1.e-2(dotted), 1.e-3(dash-dotted). (a) Cost; (b) Control Cost; (c) Difference to Stokes Flow

Further numerical examples involving different cost functionals are presented in [**19**]. Related approaches to the control of fluid flow can be found in [**12, 35, 49, 56, 62, 81, 87, 102**] and the references cited therein.

5.2. Boundary observations. In this section numerical investigations on the influence of sensor area Γ_s on the feedback control mechanism are presented. The functional to be minimized at each

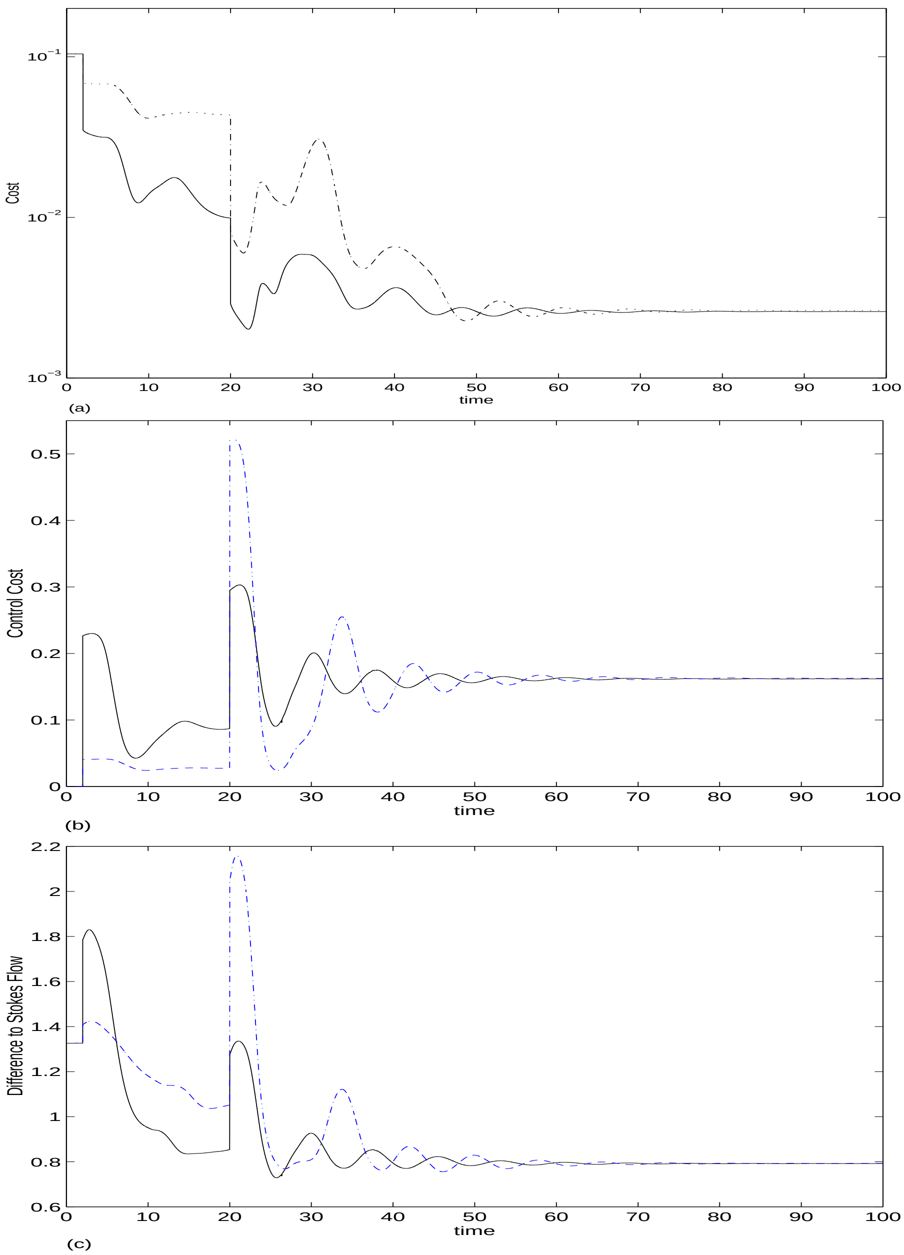

FIGURE 3. Boundary control at Re=400, $\gamma = 1.e-2$: 3 gradient steps(solid), 1 gradient step(dash-dotted). (a) Cost; (b) Control Cost; (c) Difference to Stokes Flow

time instance is related to the functional $\hat{J}$ in (101) and is given by

$$J(g) = \frac{1}{2} \int_{\Gamma_s} \frac{\partial y_1}{\partial x_2} \left(\frac{\partial y_1}{\partial x_2} - \left| \frac{\partial y_1}{\partial x_2} \right| \right) d\Gamma_s + \frac{\gamma}{2} \int_{\Gamma_c} g^2 \, d\Gamma_c,$$

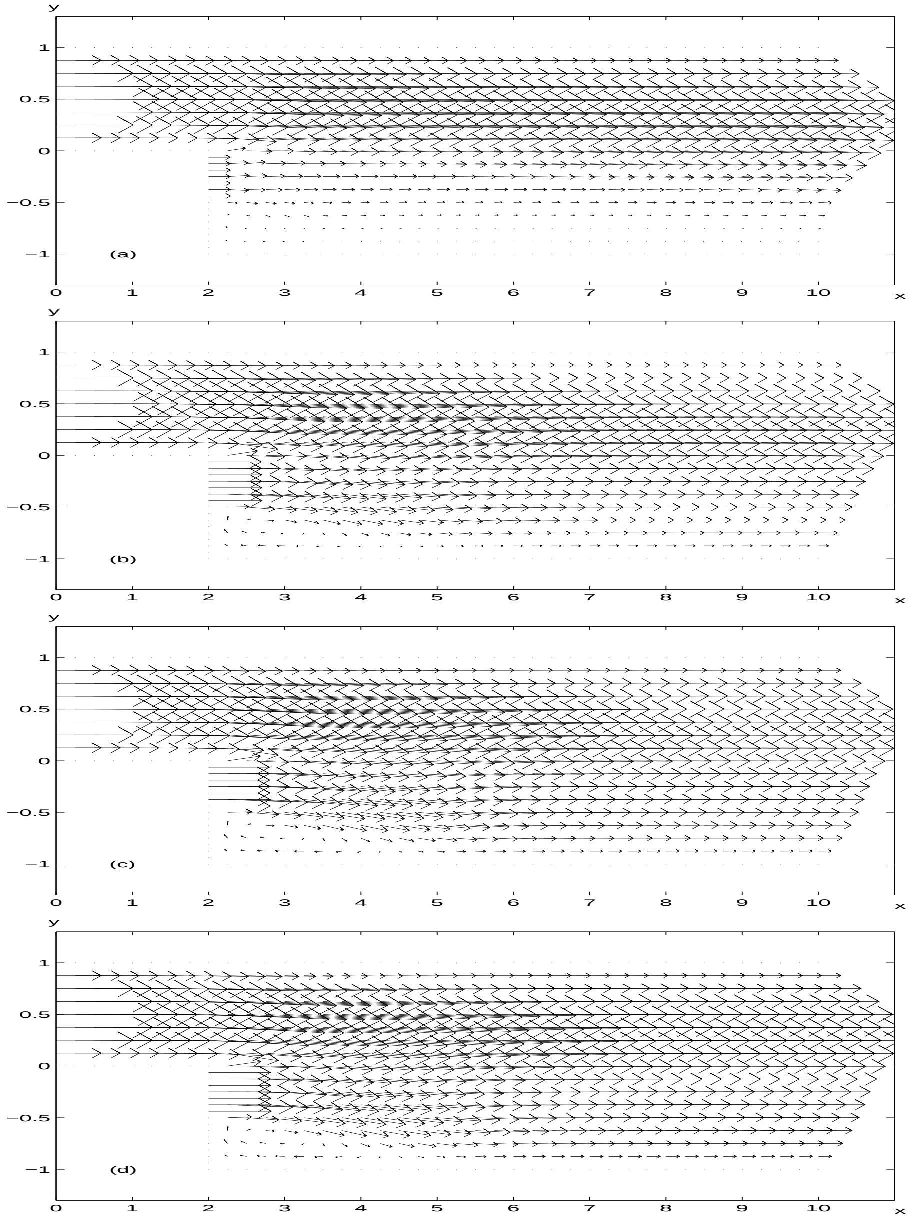

FIGURE 4. Boundary control at Re= 400: Controlled flows at $t = 100$. (a) γ =1; (b) γ =1.e-1; (c) γ =1.e-2; (d) γ =1.e-3

where again g denotes the boundary control applied at the upper part of the back wall Γ_c and y_1 denotes the stream-wise velocity. Consequently, the first part in the cost functional vanishes when there is no recirculation region at Γ_s.

In Fig. 7 the flows corresponding to different sizes of Γ_s are shown. It can be observed that the larger the distance between the control boundary Γ_c and the sensor area Γ_s, the more the control

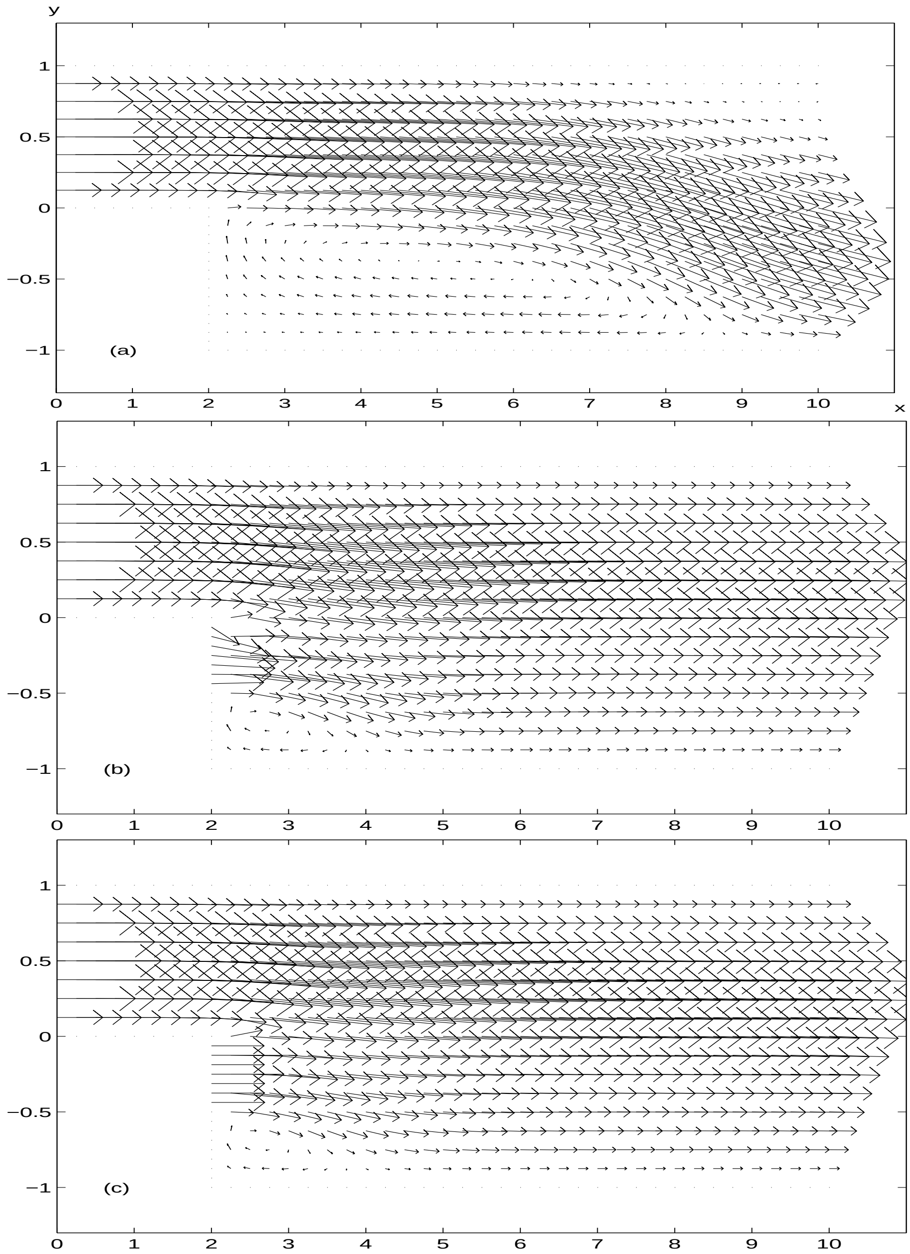

FIGURE 5. (a) Navier-Stokes flow at Re=300, (b) Optimal controlled flow at t= 50, (c) Flow obtained by instantaneous control

mechanism becomes periodic. This is due to the fact that without control the flow evolves to the uncontrolled Navier-Stokes flow and the time span lapsing until the evolving flow affects the sensor area is larger for larger distances between Γ_c and Γ_s. This can also be observed for the evolution of the costs in Fig. 8(a) and the costs of the boundary control in Fig. 8(b). Of course, the dead time in the system

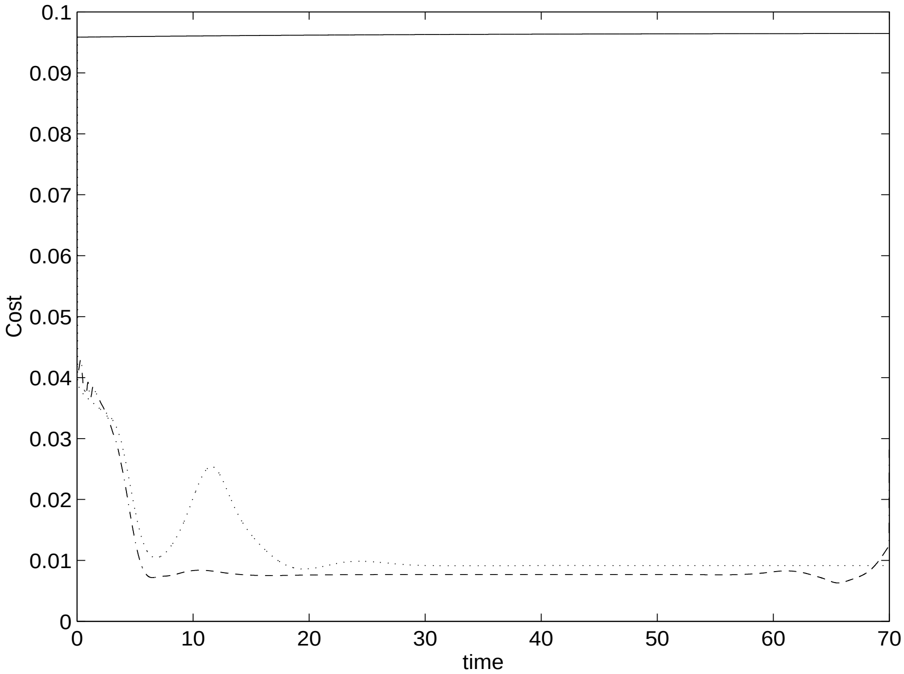

FIGURE 6. Optimal control versus instantaneous control, evolution of cost: uncontrolled (solid), optimal control (dash-dotted), instantaneous control (dotted)

for transport dominated flow is proportional to the distance between Γ_s and Γ_c. Moreover, the controlled flow also depends on the size of Γ_s, but numerical experiments show that the latter dependence plays a less important role. Several videos demonstrating the performance of the instantaneous feedback Algorithms 3.2, 4.1 can be found on the author's homepage at **http://www.math.tu-berlin.de/˜hinze**.

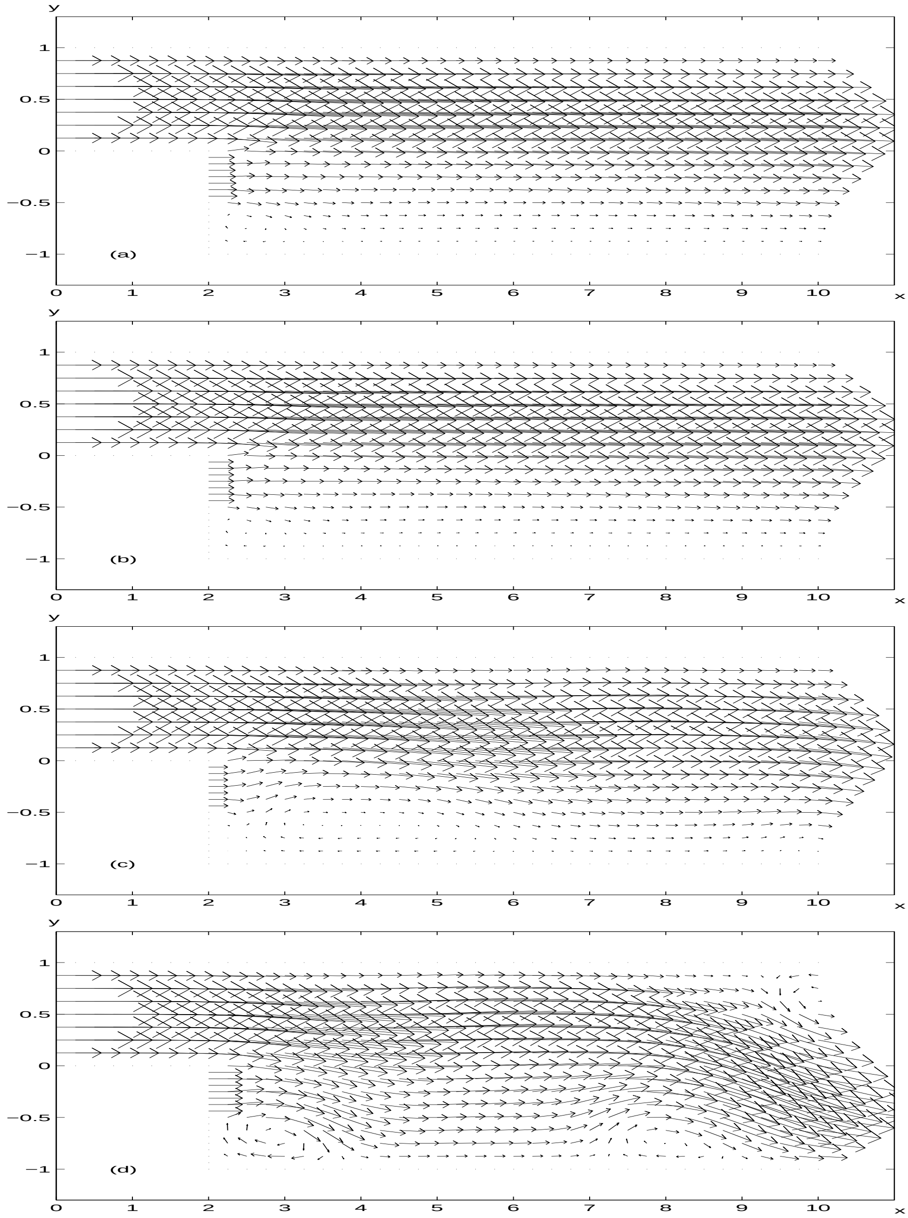

FIGURE 7. Boundary control based on boundary observation at Re= 300: Controlled flows for (a) $\Gamma_s = [2,9]$; (b) $\Gamma_s = [3,9]$; (c) $\Gamma_s = [4,9]$; (d) $\Gamma_s = [5,9]$

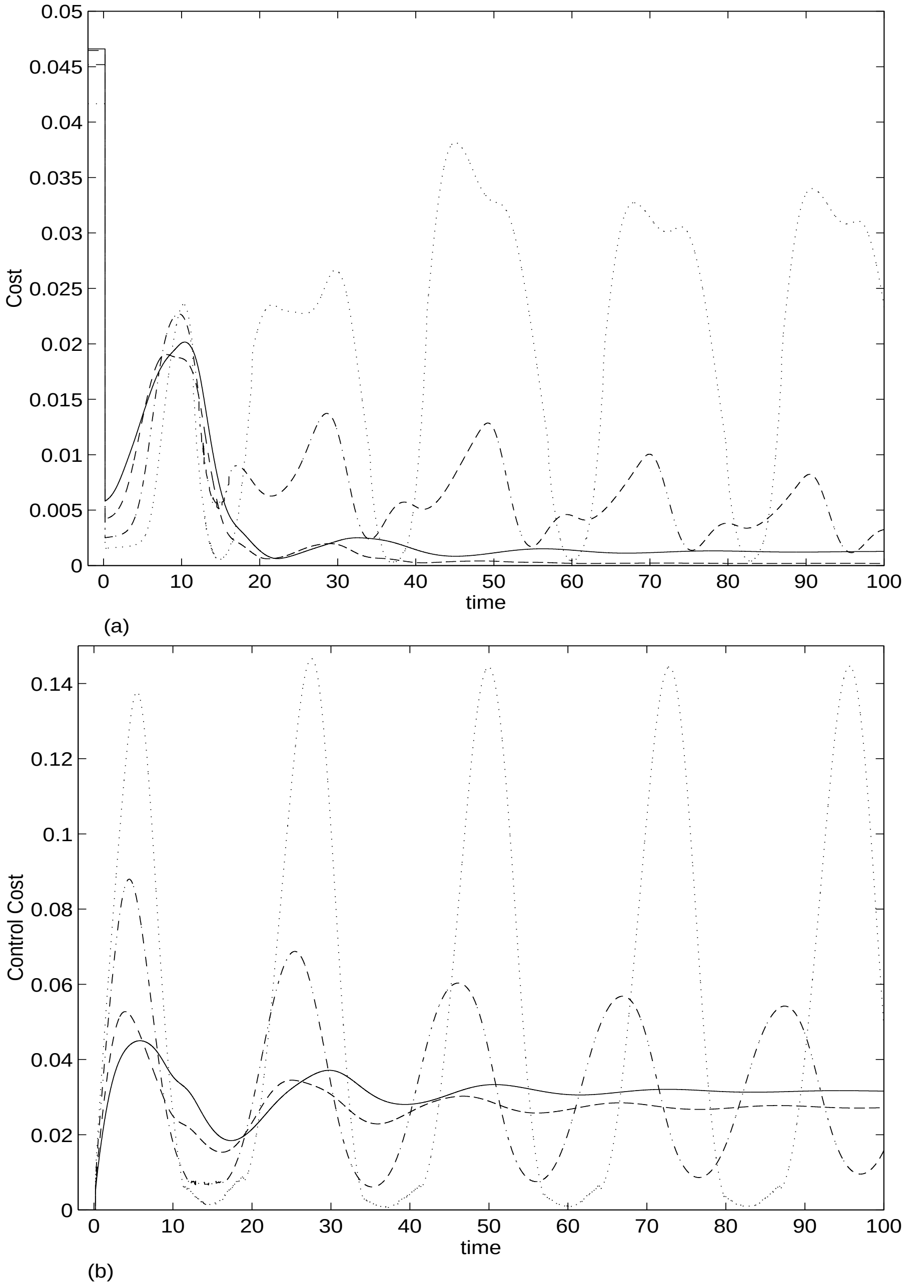

FIGURE 8. Boundary observation at Re=300: $\Gamma_s = [2,9]$(solid), $[3,9]$(dashed), $[4,9]$(dash-dotted), $[5,9]$(dotted). (a) Cost; (b) Control Cost

APPENDIX A

A general optimal control problem, lemmata and proofs

1. A general control problem

Consider the control problem

$$
(122) \qquad (P) \begin{cases} \min J(y,p,u,g) \\ \text{s.t.} \\ \begin{array}{rcll} y_t - \nu\Delta y + (y\cdot\nabla)y + \nabla p &=& \beta Bu & \text{in } Q := (0,T)\times\Omega \\ -\operatorname{div} y &=& 0 & \text{in } Q \\ y &=& \delta Eg & \text{on } \Gamma_d^T \\ \nu\partial_\eta y &=& p\eta & \text{on } \Gamma_{out}^T \\ y(0) &=& y_0, & \end{array} \end{cases}
$$

with the cost functional defined by

$$
\begin{aligned}
J(y,p,u,g) = {} & \alpha \int_0^T \int_\Omega V(y,\nabla y,p)\,dx dt + \beta \int_0^T \int_{\Omega_c} C(u,\nabla u,u_t)\,dx dt \\
& + \gamma \int_0^T \int_{\Gamma_d} R(\partial_\eta y, p, \nabla_s p)\,d\Gamma dt + \delta \int_0^T \int_{\Gamma_c} S(g,\nabla_s g, g_t)\,d\Gamma dt \\
& + \epsilon \int_\Omega G(y(T),\nabla y(T),p(T))\,dx.
\end{aligned}
$$

Here ∇_s denotes the surface gradient and the operators B and E are control extension operators which extend controls defined over the control domains Ω_c and Γ_c, respectively to functions defined over Ω

and Γ_d. Formally the optimality system for problem (122) is given by

$$
\left\{
\begin{array}{rcll}
y_t - \nu\Delta y + (y\cdot\nabla)y + \nabla p &=& \beta\, Bu & \text{in } \Omega^T \\
-\operatorname{div} y &=& 0 & \text{in } \Omega^T \\
y &=& \delta\, Eg & \text{on } \Gamma_d^T \\
\nu\partial_\eta y - p\eta &=& 0 & \text{on } \Gamma_{out}^T \\
y(0) &=& y_0 & \text{in } \Omega \\
&&& \\
-\mu_t - \nu\Delta\mu - (y\cdot\nabla)\mu + (\nabla y)^t\mu + \nabla\xi &=& -\alpha\,(V_1 - \operatorname{div} V_2) & \text{in } \Omega^T \\
-\operatorname{div}\mu &=& -\alpha V_3 & \text{in } \Omega^T \\
\mu &=& \gamma\frac{1}{\nu}R_1 & \text{on } \Gamma_d^T \\
\mu\eta &=& -\gamma\,(R_2 - \operatorname{div}_s R_3) & \text{on } \Gamma_d^T \\
\nu\partial_\eta\mu - \xi\eta + (y\eta)\mu &=& -\alpha V_2\eta & \text{on } \Gamma_{out}^T \\
\mu(T) &=& -\epsilon\,(G_1 - \operatorname{div} G_2(T)) & \text{in } \Omega \\
\epsilon\, G_2(T)\eta &=& 0 & \text{on } \partial\Omega \\
\gamma\, R_3\eta_{\Gamma_d} &=& 0 & \text{on } (\partial\Gamma_d)^T \\
\epsilon G_3(T) &=& 0 & \text{in } \Omega \\
&&& \\
\kappa + \nu\partial_\eta\mu - \xi\eta + (y\eta)\mu &=& -\alpha V_2\eta & \text{on } \Gamma_d^T \\
&&& \\
\beta\left(C_1 - \operatorname{div} C_2 - \frac{d}{dt}C_3\right) &=& \beta\, B^*\mu & \text{in } \Omega_c^T \\
\beta\, C_2\eta &=& 0 & \text{on } \partial\Omega_c^T \\
\beta\, C_3(T) &=& 0 & \text{in } \Omega_c \\
\beta\, C_3(0) &=& 0 & \text{in } \Omega_c \\
&&& \\
\delta\left(S_1 - \operatorname{div}_s S_2 - \frac{d}{dt}S_3\right) &=& \delta\, E^*\kappa & \text{on } \Gamma_c^T \\
\delta\, S_2\eta_{\Gamma_c} &=& 0 & \text{on } \partial\Gamma_c^T \\
\delta\, S_3(T) &=& 0 & \text{on } \Gamma_c \\
\delta\, S_3(0) &=& 0 & \text{on } \Gamma_c.
\end{array}
\right. \tag{123}
$$

Here, S_i denotes the i-th partial derivative of the function $S_i(g(t,x),\nabla_x g(t,x),g_t(t,x))$, the notation $S_i(T)$ is an abbreviation for $S_i(g(T,x),\nabla_x g(T,x),g_t(T,x))$ and $S_i(0) = S_i(g(0,x),\nabla_x g(0,x),g_t(0,x))$, similarly for V_i, C_i, R_i.

Note, that the boundary conditions for μ on Γ_d imply a compatibility condition for the function R (or for the optimal state y and the optimal pressure p) on Γ_d, namely

$$\frac{1}{\nu}\,R_1\eta \;=\; -\,(R_2 - \operatorname{div}_s R_3) \quad \text{on } \Gamma_d^T \tag{124}$$

Proof. Keeping

$$y(0,\cdot) = y_0 \text{ in } \Omega \quad \text{and} \quad \nu\partial_\eta y = p\eta \text{ on } \Gamma_{out}^T \tag{125}$$

as explicit constraints, the Lagrangian associated to the optimization problem (122) is given by

$$L(y,p,u,g,\mu,\xi,\kappa) = J(y,p,u,g) + \int_0^T\int_\Omega (y_t - \nu\Delta y + (y\nabla)y + \nabla p - \beta\, Bu)\,\mu\, dxdt - \int_0^T\int_\Omega \mathrm{div} y\,\xi\, dxdt + \int_0^T\int_{\Gamma_d} (y - \delta\, Eg)\,\kappa\, d\Gamma dt. \tag{126}$$

Every solution of (122) by necessity satisfies the variational equation

$$DL(y,p,u,g,\mu,\xi,\kappa)(v,q,s,h,\tilde\mu,\tilde\xi,\tilde\kappa) \;=\; 0 \tag{127}$$

for all admissible test vectors $(v,q,s,h,\tilde\mu,\tilde\xi,\tilde\kappa)$. Due to the natural boundary condition which has to be satisfied by (y,p) on Γ_{out} the vectors v and the functions p are related by $\nu\partial_\eta y = p\eta$ on Γ_{out}.

First exploit

$$L_y(y,p,u,g,\mu,\xi,\kappa)v + L_p(y,p,u,g,\mu,\xi,\kappa)q = 0 \tag{128}$$

for all (v,q) satisfying

$$v(0,\cdot) = 0 \text{ in } \Omega \quad \text{and} \quad \nu\partial_\eta v = q\eta \text{ on } \Gamma^T_{out}.$$

Integrating by parts with respect to time and space (128) can equivalently be rewritten as

$$\begin{aligned}
&\int_0^T\int_\Omega \big(-\mu_t - \nu\Delta\mu - (y\cdot\nabla)\mu + (\nabla y)^t\mu + \nabla\xi + \alpha\,(V_1 - \mathrm{div}\, V_2)\big)\; v\, dxdt\\
&\quad + \int_0^T\int_{\partial\Omega} (\chi_{\Gamma_d}\kappa + \nu\partial_\eta\mu - \xi\eta + (y\eta)\mu + \alpha V_2\eta)\; v\, d\Gamma dt + \int_0^T\int_{\partial\Omega} (\chi_{\Gamma_d}\gamma R_1 - \nu\mu)\;\partial_\eta v\, d\Gamma dt\\
&\quad\quad + \int_\Omega (\mu(T) + \epsilon\,(G_1 - \mathrm{div}\, G_2))\; v(T)\, dx - \int_\Omega \mu(0)v(0)\, dx + \epsilon\int_{\partial\Omega} G_2(T)\eta v(T)\, d\Gamma\\
&+ \int_0^T\int_\Omega (-\mathrm{div}\,\mu + \alpha V_3)\; q\, dxdt + \int_0^T\int_{\partial\Omega} (\chi_{\Gamma_d}\gamma\,(R_2 - \mathrm{div}_s\, R_3) + \mu\eta)\; q\, d\Gamma dt + \epsilon\int_\Omega G_3(T)q(T)\, dx\\
&\quad\quad + \gamma\int_0^T\int_{\partial\Gamma_d} R_3\eta_{\Gamma_d}\, q\, d(\partial\Gamma)dt = 0 \quad \text{for all } (v,q) \text{ satisfying } v(0) = 0 \text{ and } \nu\partial_\eta v = q\eta \text{ on } \Gamma^T_{out}.
\end{aligned}$$

(1) Choose $q \equiv 0$ and
   - v with compact support in Ω^T. This gives

$$\int_0^T\int_\Omega \big(-\mu_t - \nu\Delta\mu - (y\cdot\nabla)\mu + (\nabla y)^t\mu + \nabla\xi + \alpha\,(V_1 - \mathrm{div}\, V_2)\big)\; v\, dxdt = 0 \quad \forall v$$

and thus

$$-\mu_t - \nu\Delta\mu - (y\cdot\nabla)\mu + (\nabla y)^t\mu + \nabla\xi + \alpha\,(V_1 - \mathrm{div}\,V_2) = 0 \quad \text{in } \Omega^T.$$

- $v(T,\cdot)) \equiv 0$, $\partial_\eta v \equiv 0$, $v = 0$ on Γ_{out} and v arbitrary on Γ_d. Then

$$\int_0^T\int_{\Gamma_d} \left(\kappa + \nu\partial_\eta\mu - \xi\eta + (y\eta)\mu + \alpha V_2\eta\right)\, v\, d\Gamma dt = 0 \quad \forall\, v,$$

  which implies

$$\kappa + \nu\partial_\eta\mu - \xi\eta + (y\eta)\mu = \alpha V_2\eta \quad \text{on } \Gamma_d^T.$$

- $v(T,\cdot)) \equiv 0$, $\partial_\eta v \equiv 0$, $v = 0$ on Γ_d and v arbitrary on Γ_{out}. This implies

$$\int_0^T\int_{\Gamma_{out}} \left(\nu\partial_\eta\mu - \xi\eta + (y\eta)\mu + \alpha V_2\eta\right)\, v\, d\Gamma dt = 0 \quad \forall\, v,$$

  hence,

$$\nu\partial_\eta\mu - \xi\eta + (y\eta)\mu = \alpha V_2\eta \quad \text{on } \Gamma_{out}^T.$$

- $v(T,\cdot)) \equiv 0$, , $v \equiv 0$ on $\partial\Omega$, $\partial_\eta v = 0$ on Γ_{out} and $\partial_\eta v$ arbitrary on Γ_d. Consequently,

$$\int_0^T\int_{\Gamma_d} \left(\gamma R_1 - \nu\mu\right)\, \partial_\eta v\, d\Gamma dt = 0 \quad \forall\, v,$$

  which leads to

$$\gamma R_1 - \nu\mu = 0 \quad \text{on } \Gamma_d^T.$$

- $\partial_\eta v \equiv 0$ on $\partial\Omega$ and $v(T,\cdot)$ with compact support on Ω. Then,

$$\int_\Omega \left(\mu(T) + \epsilon\,(G_1 - \mathrm{div}\,G_2)\right)\, v(T)\, dx = 0 \quad \forall\, v,$$

  which gives the initial values for the co-state μ,

$$\mu(T) + \epsilon\,(G_1 - \mathrm{div}\,G_2) = 0 \quad \text{in } \Omega.$$

- $\partial_\eta v \equiv 0$ on $\partial\Omega$ and $v(T,\cdot)$ arbitrary on $\partial\Omega$. Thus

$$\epsilon\int_{\partial\Omega} G_2(T)\eta v(T)\, d\Gamma = 0 \quad \forall\, v,$$

  which is equivalent to

$$G_2(T)\eta = 0 \quad \text{on } \partial\Omega.$$

(2) Choose $v \equiv 0$ and

- q with compact support in Ω. Then

$$\int_0^T \int_\Omega \left(-\mathrm{div}\,\mu + \alpha V_3\right) q\,dxdt = 0 \quad \forall\, q,$$

which gives the mass balance for the co-state,

$$-\mathrm{div}\,\mu + \alpha V_3 = 0 \quad \text{in } \Omega^T.$$

- q with values on Γ_d, $q_{|\partial\Gamma_d} = 0$ and $q = 0$ on Γ_{out}. This gives

$$\int_0^T \int_{\Gamma_d} \left(\gamma\left(R_2 - \mathrm{div}_s\, R_3\right) + \mu\eta\right) q\,d\Gamma dt = 0 \quad \forall\, q,$$

equivalently,

$$\mu\eta = -\gamma\left(R_2 - \mathrm{div}_s\, R_3\right) \quad \text{on } \Gamma_d^T.$$

- q with arbitrary values on Γ_d and $q = 0$ on Γ_{out}. Then

$$\gamma \int_0^T \int_{\partial\Gamma_d} R_3 \eta_\Gamma\, q\,d\partial\Gamma dt = 0 \quad \forall\, q,$$

which gives the condition

$$R_3 \eta_{\Gamma_d} = 0 \quad \text{on } \left(\partial\Gamma_d\right)^T,$$

- $$\int_\Omega G_3(T) q(T)\,dx = 0 \quad \forall\, q$$

finally gives the condition

$$G_3(T) = 0 \quad \text{in } \Omega.$$

For the remaining term there holds

$$\int_0^T \int_{\Gamma_{out}} \mu\left(-\nu\partial_\eta v + q\eta\right) d\Gamma dt = 0.$$

The optimality conditions for the controls u and g in (123) are obtained with the same technique. Informations for the volume controls are obtained from the identity

$$L_u(y, p, u, g, \mu, \xi, \kappa)s = 0 \text{ for all } s \text{ with values on } \Omega_c^T.$$

Recall, that B is an operator which extends controls defined over Ω_c to control functions defined over the whole of the spatial domain Ω. Integration by parts with respect to time and space this identity can equivalently be rewritten as

$$\beta \int_0^T \int_{\Omega_c} \left(\left(C_1 - \operatorname{div} C_2 - \frac{d}{dt} C_3 \right) - B^* \mu \right) s \, dx dt + \beta \int_0^T \int_{\partial\Omega_c} C_2 \eta \, s \, d\Gamma dt$$

$$+ \beta \int_{\Omega_c} C_3(T)s(T) - C_3(0)s(0) \, dx \;=\; 0 \quad \text{for all admissible test functions } s.$$

Similar as above, choose

- s with compact support in Ω_c^T. This implies

  $$\beta \int_0^T \int_{\Omega_c} \left(\left(C_1 - \operatorname{div} C_2 - \frac{d}{dt} C_3 \right) - B^* \mu \right) s \, dx dt = 0 \quad \forall\, s,$$

  which means

  $$\beta \left(\left(C_1 - \operatorname{div} C_2 - \frac{d}{dt} C_3 \right) - B^* \mu \right) = 0 \quad \text{in } \Omega_c^T.$$

- s with $s(T) = s(0) = 0$ and s arbitrary on $\partial\Omega_c$. Then

  $$\beta \int_0^T \int_{\partial\Omega_c} C_2 \eta \, s \, d\Gamma dt = 0 \quad \forall\, s,$$

  which implies the condition

  $$\beta C_2 \eta = 0 \quad \text{on } (\partial\Omega_c)^T.$$

- s with arbitrary values on Ω_c^T, but $s(T) = 0$. Then

  $$\beta \int_{\Omega_c} -C_3(0)s(0) \, dx \;=\; 0 \quad \forall\, s,$$

  which gives the first boundary condition for the volume controls with respect to time,

  $$\beta C_3(0) \;=\; 0 \quad \text{in } \Omega_c.$$

- s with arbitrary values on Ω_c^T, but $s(0) = 0$. Then

  $$\beta \int_{\Omega_c} C_3(T)s(T) \, dx \;=\; 0 \quad \forall\, s,$$

  which gives the second boundary condition for the volume controls with respect to time,

  $$\beta C_3(T) \;=\; 0 \quad \text{in } \Omega_c.$$

To obtain informations for the boundary controls utilize the identity

$$L_g(y, p, u, g, \mu, \xi, \kappa)h = 0$$

for all h with values on Γ_c^T. Recall, that E is an operator which extends boundary controls defined over Γ_c to control functions defined over the whole of Γ_d.Integration by parts with respect to time and space this identity can equivalently be rewritten as

$$\delta \int_0^T \int_{\Gamma_c} \left(\left(S_1 - \operatorname{div} S_2 - \frac{d}{dt} S_3 \right) - E^* \kappa \right) h \, dx dt + \delta \int_0^T \int_{\partial \Gamma_c} S_2 \eta \, h \, d\Gamma dt$$
$$+ \delta \int_{\Gamma_c} S_3(T)h(T) - S_3(0)h(0) \, dx \;=\; 0 \quad \text{for all admissible test functions } h.$$

Similar as above, choose

- h with compact support in Γ_c^T. This implies
$$\delta \int_0^T \int_{\Gamma_c} \left(\left(S_1 - \operatorname{div}_{\Gamma_c} S_2 - \frac{d}{dt} S_3 \right) - E^* \kappa \right) h \, dx dt = 0 \quad \forall \, h,$$
which means
$$\delta \left(\left(S_1 - \operatorname{div} S_2 - \frac{d}{dt} S_3 \right) - E^* \kappa \right) = 0 \quad \text{in } \Gamma_c^T.$$
- h with $h(T) = h(0) = 0$ and h arbitrary on $\partial \Gamma_c$. Then
$$\delta \int_0^T \int_{\partial \Gamma_c} S_2 \eta \, h \, d\Gamma dt = 0 \quad \forall \, h,$$
which implies the condition
$$\delta S_2 \eta_{\Gamma_c} = 0 \quad \text{on } (\partial \Gamma_c)^T.$$
- h with arbitrary values on Γ_c^T, but $h(T) = 0$. Then
$$\delta \int_{\Gamma_c} -S_3(0)h(0) \, dx \;=\; 0 \quad \forall \, h,$$
which gives the first boundary condition for the volume controls with respect to time,
$$\delta S_3(0) \;=\; 0 \quad \text{in } \Gamma_c.$$
- h with arbitrary values on Γ_c^T, but $h(0) = 0$. Then
$$\delta \int_{\Gamma_c} S_3(T)h(T) \, dx \;=\; 0 \quad \forall \, h,$$

which gives the second boundary condition for the volume controls with respect to time,

$$\delta S_3(T) \;=\; 0 \quad \text{in } \Gamma_c.$$

As is well known, the variations with respect to $(\tilde{\mu}, \tilde{\xi}, \tilde{\kappa})$ reproduce the state equations for (y, p) in (123). Note that for $\Gamma_{out} = \emptyset$ an additional Lagrange multiplier for the mass conservation constraint $\int_{\partial\Omega} Eg\,dx = 0$ need to be introduced.

2. Proof of Proposition 2.1

In the proof of Proposition 2.1 the following Lemmata are frequently used.

LEMMA 2.1. **(Gronwall)** Let v, s, u be three non-negative locally integrable functions on $\mathbb{R}_+$ satisfying

$$v(t) \;\le\; s(t) + \int_0^t u(\tau)v(\tau)\,d\tau \quad \forall\, t \ge 0.$$

Then,

$$v(t) \;\le\; s(0)e^{\int_0^t u(\tau)\,d\tau} + \int_0^t s'(\tau)e^{\int_\tau^t u(\sigma)\,d\sigma}\,d\tau.$$

Proof. In [**109**]. □

LEMMA 2.2. Let

$$b(u, v, w) \;:=\; \int_\Omega (u\cdot\nabla)v\,w\,dx.$$

Then,

$$b(u,v,w) \;\le\; C \begin{cases} |u|_H^{\frac12}|u|_V^{\frac12}|v|_H^{\frac12}|v|_V^{\frac12}|w|_V & \forall\, u,v,w \in V \\ |u|_H^{\frac12}|u|_V^{\frac12}|v|_V^{\frac12}|Sv|_H^{\frac12}|w|_H & \forall\, u \in V, v \in V\cap H^2(\Omega)^2, w \in H \\ |u|_H|v|_V|w|_H^{\frac12}|Sw|_H^{\frac12} & \forall\, u \in H, v \in V, w \in V \cap H^2(\Omega)^2 \\ |u|_H^{\frac12}|Su|_H^{\frac12}|v|_V|w|_H & \forall\, u \in V\cap H^2(\Omega)^2, v \in V, w \in H, \end{cases} \tag{129}$$

with a positive constant C. The estimates are also valid for H replaced by $L^2(\Omega)^2$, V replaced by $H^1(\Omega)^2$ and $|Su|_H$ replaced by $|u|_{H^2(\Omega)^2}$.

Proof. In [**104**]. □

LEMMA 2.3. There exists a positive constant C such that

$$|(\nabla u)^t v|_{L^\alpha(V^*)} + |(u\cdot\nabla)v|_{L^\alpha(V^*)} \\ \le\; C\,T^{\frac{4-3\alpha}{4\alpha}}\,|u|_{L^2(V)}^{\frac12}\,|u|_{L^\infty(H)}^{\frac12}\,|v|_{L^2(V)} \quad \text{for all } u \in W, v \in L^2(V) \text{ and } \alpha \in [1, 4/3].$$

Proof. Let $\xi \in L^{\alpha'}(V)$, α' denoting the dual exponent to α. Utilizing the first estimate in (129) and Hölder's inequality gives

$$\int_0^T \int_\Omega (\xi\nabla)uv\,dxdt \le C\,|u|^{\frac{1}{2}}_{L^\infty(H)} \left(\int_0^T |\xi|_V^{\alpha'}\,dt\right)^{\frac{1}{\alpha'}} \left(\int_0^T |u|_V^{\frac{\alpha}{2}} |v|_V^{\alpha}\,dt\right)^{\frac{1}{\alpha}}$$
$$\le C\,T^{\frac{4-3\alpha}{4\alpha}}\,|\xi|_{L^{\alpha'}(V)}\,|u|^{\frac{1}{2}}_{L^2(V)}\,|u|^{\frac{1}{2}}_{L^\infty(H)}\,|v|_{L^2(V)}.$$

This gives the claim for the first addend. Estimation for the second addend is similar. □.
Note that the power $4/3$ in the previous estimate cannot be improved by requiring $v \in W$. Sufficient conditions for $(\nabla u)^t v + (u\nabla)v \in L^2(V^*)$ are given by requiring in addition that u or $v \in L^\infty(V)$.

Proof of Proposition 2.1. Existence and uniqueness of a solution to (12) can be shown following the lines of the existence and uniqueness proof for the instationary two-dimensional Navier-Stokes equations in [**103**, Chap.III]. In the following the derivation of the necessary a-priori estimates is sketched.

i. Test (12) with $v \in V$ pointwise with respect to time, use $b(u,v,v) = 0$, $b(u,u,v) = b(u,v,u)$ and estimate, utilizing Young's inequality and the first estimate in (129). This results in

$$\frac{1}{2}\frac{d}{dt}|v|_H^2 + \nu|v|_V^2 \;\le\; C_\nu\left\{|g|_{V^*}^2 + |y|_V^2|v|_H^2\right\} + \frac{\nu}{2}|v|_V^2. \tag{130}$$

After integration from 0 to t Gronwall's inequality (2.1) gives

$$|v|_{L^\infty(H)} \;\le\; C\exp\left(2C_\nu|y|_{L^2(V)}\right)\left\{|g|_{L^2(V^*)} + |v_0|_H\right\}. \tag{131}$$

Using (131) in (130), the Cauchy-Schwarz inequality yields

$$|v|_{L^2(V)} \;\le\; C(|y|_{L^2(V)})\left\{|g|_{L^2(V^*)} + |v_0|_H\right\}. \tag{132}$$

Combining (131) and (132) yields the first claim.

ii. Test (12) with $\varphi \in V$ pointwise in time and estimate using the Cauchy-Schwarz inequality and the first estimate in (129). This gives

$$\int_\Omega v_t\varphi\,dx \;\le\; \left\{|g|_{V^*} + \nu|v|_V + C|y|_H^{\frac{1}{2}}|y|_V^{\frac{1}{2}}|v|_H^{\frac{1}{2}}|v|_V^{\frac{1}{2}}\right\}|\varphi|_V,$$

which implies

$$|v_t|^2_{L^2(V^*)} \;\le\; C\left\{|g|^2_{L^2(V^*)} + |v|^2_{L^2(V)} + |y|_{L^\infty(H)}|v|_{L^\infty(H)}|y|_{L^2(V)}|v|_{L^2(V)}\right\}.$$

This, together with $y \in W \subset L^\infty(H)$, and the estimates (131), (132) gives ii. Combining i. and ii. implies

$$|v|_W \;\le\; C(|y|_{L^2(V)}, |y|_{L^\infty(H)})\left\{|g|_{L^2(V^*)} + |v_0|_H\right\}.$$

iii. For $y \in W$ the linear operator $A(y)$ coincides with $e_y(x)$ of Section 2. Due to i.,ii. this operator admits a continuous inverse $A(y)^{-1} \in \mathcal{L}(Z^*, W)$. For the adjoint $A(y)^* \in \mathcal{L}(Z, W^*)$ one finds

$$\begin{aligned}\langle A(y)v, (w^1, w^0)\rangle_{Z^*,Z} &= \langle v, A(y)^*(w^1, w^0)\rangle_{W,W^*} \\ &= \langle v_t, w^1\rangle + \langle v, (\nabla y)^t w^1 - (y\cdot\nabla)w^1 - \nu\Delta w^1\rangle_{W,W^*} + (v(0), w^0)_H\end{aligned}$$

for $v \in W$ and $w^1 \in L^2(V)$. Since $A(y)^{-*} \in \mathcal{L}(W^*, Z)$ for every $f \in W^*$ there exists a unique solution $(w^1, w^0) \in Z$ to

$$\begin{aligned}\langle v_t, w^1\rangle + \langle v, (\nabla y)^t w^1 - (y\cdot\nabla)w^1 - \nu\Delta w^1\rangle_{W,W^*} + (v(0), w^0)_H \\ = \langle v, f\rangle_{W,W^*} \quad \text{for all } v \in W.\end{aligned} \tag{133}$$

From i. and ii. together with the fact that

$$\|A(y)^{-1}\|_{\mathcal{L}(Z^*,W)} = \|A(y)^{-*}\|_{\mathcal{L}(W^*,Z)}$$

one has

$$|w^1|_{L^2(V)} \;\le\; C\left(|y|_{L^2(V)}, |y|_{L^\infty(H)}\right)\, |f|_{W^*}, \tag{134}$$

which is iii.

iv. By Lemma 2.3 and the assumption that $f \in L^q(V^*)$ the mapping

$$t \mapsto \left(-\nu\Delta w^1 + (y\cdot\nabla)w^1 - (\nabla y)^t w^1 + f\right)(t)$$

is an element of $L^{1+\epsilon}(V^*)$, with $\epsilon \in \min\{q-1, 1/3\}$. Now denote by w^1_t the distributional derivative of w^1 and recall that the scalar product $(\cdot,\cdot)_H$ can be continuously extended to $V^* \times V$. It shall be shown that $w^1_t \in L^{1+\epsilon}(V^*) \cap W^*$. Utilizing (133) a short calculation gives

$$\begin{aligned}-\left(\int_0^T w^1_t \chi\, dt, h\right)_H &= \\ &\left(\int_0^T \left\{-\nu\Delta w^1 - (y\nabla)w^1 + (\nabla y)^t w^1 - f\right\}\chi\, dt, h\right)_H \qquad \forall \chi \in C_0^\infty(0,T), h \in V,\end{aligned}$$

so that a density argument together with (134) yields $w^1_t \in L^{1+\epsilon}(V^*) \cap W^*$. Together with $w^1 \in L^2(V)$ this implies $w^1 \in W^2_{1+\epsilon}$ $(0 \le \epsilon \le \min\{q-1, 1/3\})$. In particular, $w \in C([0,T]; V^*)$, compare [**24**, p.521]. From (133) one deduces that the first equation in (13) is well defined in $L^{1+\epsilon}(V^*)$. Choosing appropriate test functions in (133) and utilizing the fact that $w^1(T)$ is well defined in V^* it follows that $w^1(T) = 0$ and $w^0 = w^1(0)$. By Lemma 2.3 (with $\alpha = 1 + \epsilon$) there exists a constant C such that

$$|w_t^1|_{L^\alpha(V^*)} \;\leq\; C\,\left\{T^{\frac{1-3\epsilon}{4(1+\epsilon)}}\,|y|_{L^2(V)}^{\frac{1}{2}}\,|y|_{L^\infty(H)}^{\frac{1}{2}} + T^{\frac{1-\epsilon}{2(1+\epsilon)}}\right\}\,|w^1|_{L^2(V)} \;+\; |f|_{L^\alpha(V^*)}.$$

Combining this estimate with (134) implies the estimate in iv.

v. If $y \in L^\infty(V)$ and $f \in L^2(V^*)$ then $(y \cdot \nabla)w^1 - (\nabla y)^t w^1 \in L^2(V^*)$ and, utilizing (133) one finds that $w_t^1 \in L^2(V^*)$. Moreover, by (129) we have

$$|w_t^1|_{L^2(V^*)} \;\leq\; C\,\left(|y|_{L^\infty(V)}\right)\left\{|w^1|_{L^2(V)} + |f|_{L^2(V^*)}\right\}.$$

Together with (134) this gives the desired estimate in v.

vi. Test (12) with (the Leray-projection of) Δv pointwise in time and utilize Young's inequality and the last estimate in (129) to obtain

$$\frac{1}{2}\frac{d}{dt}|v|_V^2 + \nu|\Delta v|_H^2 \;\leq C_\nu\left\{|g|_H^2 + |y|_V^4|v|_H^2 + |y|_H|\Delta y|_H|v|_V^2\right\} + \frac{\nu}{2}|\Delta v|_H^2. \tag{135}$$

Integration from 0 to t together with (132) results in

$$|v|_V^2 \;\leq\; C(|y|_{L^2(V)}, |y|_{L^\infty(V)})\left\{|g|_{L^2(H)}^2 + |v_0|_V^2\right\} + C_\nu\int_0^t |y|_H|\Delta y|_H|v|_V^2\,dt,$$

so that Gronwall's inequality gives

$$|v|_{L^\infty(V)} \;\leq C(|y|_{L^2(V)}, |y|_{L^\infty(V)}, |y|_{L^\infty(H)}, |y|_{L^2(H^2(\Omega)^2)})\left\{|g|_{L^2(H)} + |v_0|_V\right\}. \tag{136}$$

Using this in (135) yields

$$|v|_{L^2(H^2(\Omega)^2)} \;\leq C(|y|_{L^2(V)}, |y|_{L^\infty(V)}, |y|_{L^\infty(H)}, |y|_{L^2(H^2(\Omega)^2)})\left\{|g|_{L^2(H)} + |v_0|_V\right\}. \tag{137}$$

To estimate $|v_t|_{L^2(H)}$ test (12) with $\varphi \in V$ and use the last estimate in (129). This gives

$$\int_\Omega v_t\varphi\,dx \;\leq \left\{|g|_H + \nu|\Delta v|_H + C|y|_H^{\frac{1}{2}}|\Delta y|_H^{\frac{1}{2}}|v|_V + C|v|_H^{\frac{1}{2}}|\Delta v|_H^{\frac{1}{2}}|y|_V\right\}|\varphi|_H,$$

so that $v \in L^\infty(V) \cap L^2(H^2(\Omega)^2)$ together with (136) and (137) implies

$$|v_t|_{L^2(H)} \;\leq C(|y|_{L^2(V)}, |y|_{L^\infty(V)}, |y|_{L^\infty(H)}, |y|_{L^2(H^2(\Omega)^2)})\left\{|g|_{L^2(H)} + |v_0|_V\right\}.$$

Therefore,

$$|v|_{H^{2,1}(Q)} \;\leq C(|y|_{L^2(V)}, |y|_{L^\infty(V)}, |y|_{L^\infty(H)}, |y|_{L^2(H^2(\Omega)^2)})\left\{|g|_{L^2(H)} + |v_0|_V\right\},$$

which is vi. The estimation for $|w^1|_{L^\infty(V)\cap L^2(H^2(\Omega)^2)}$ is similar to that for $|v|_{L^\infty(V)\cap L^2(H^2(\Omega)^2)}$. In order to cope with $b(\varphi, y, w^1)$ in the estimation of $|w_t^1|_{L^2(H)}$ one utilizes the third estimate in (129) to obtain the estimate vii. $\square$

Bibliography

[1] **Abergel, F.** & **Temam, R.** On some Control Problems in Fluid Mechanics. *Theoret. Comput. Fluid Dynamics*, 1:303–325, 1990.

[2] **Afanasiev, K.** & **Hinze, M.** Adaptive control of a wake flow using proper orthogonal decomposition. In *Shape Optimization & Optimal Design*, Lecture Notes in Pure and Applied Mathematics 216. Marcel Dekker, 2001. see also Preprint No. 648/1999, Fachbereich Mathematik, Technische Universität Berlin.

[3] **Alt, H.W.** *Lineare Funktionalanalysis*. Springer, 1985. 2. Auflage.

[4] **Atwell, J.A.** & **King, B.B.** Proper Orthogonal Decomposition for reduced basis feedback controllers for parabolic equations. ICAM Report 99-01-01, 1999. Interdisciplinary Center for Applied Mathematics, Virginia Polytechnic Institute and State University Blacksburg, Virginia.

[5] **Atwell, J.A.** & **King, B.B.** Reduced Order Controllers for Spatially Distributed Systems via Proper Orthogonal Decomposition. ICAM Report 99-07-01, 1999. Interdisciplinary Center for Applied Mathematics, Virginia Polytechnic Institute and State University Blacksburg, Virginia.

[6] **Bänsch, E.** An adaptive Finite-Element-Strategy for the three-dimensional time-dependent Navier-Stokes-Equations. *J. Comp. Math.*, 36:3–28, 1991.

[7] **Bänsch, E.** Numerical methods for the instationary Navier-Stokes equations with a free capillary surface. Habilitationsschrift, 1998. Mathematische Fakultät der Universität Freiburg.

[8] **Bänsch, E.** Private Communication, 1999.

[9] **Bärwolff, G.; Hinze, M.** & **Koster, F.** Comparison of outflow boundary conditons for channel flows. Preprint in preparation.

[10] **Beirão da Veiga, H.** On a stationary transport equation. *Ann.Univ. Ferrara - Sez. VII - Sc. Mat.*, XXXII:79–91, 1986.

[11] **Bellman, R.** *Dynamic Programming*. Princton University Press, 1957. Princeton.

[12] **Berggren, M.** Numerical solution of a flow control problem: Vorticity reduction by dynamic boundary action. *Siam J. Sci. Comput.*, Vol. 19(No. 3):829–860, 1998.

[13] **Bewley, T.; Choi, H.; Temam, R.** & **Moin, P.** Optimal feedback control of turbulent channel flow. CTR Annual Research Briefs, 1993. Center for Turbulence Research, Stanford University/NASA Ames Research Center, 3-14.

[14] **Bewley, T.R.; Moin, P.** & **Temam, R.** DNS-based predictive control of turbulence: an optimal benchmark for feedback algorithms. *J. Fluid Mech.*, 447:179–225, 2001.

[15] **Bewley, T.; Temam, R.** & **Ziane, M.** A general framework for robust control in fluid mechanics. *Physica D*, 138, 2000.

[16] **Chang, Y.** & **Collis, S.** Computer simulation of active control in complex turbulent flows. Preprint No. 571/1997, 1998.

[17] **Chang, Y.** & **Collis, S.** Active control of turbulent channel flows by large eddy simulation. In *Proceedings of the FEDSM99*. ASME, 1999.

[18] **Choi, H.** Suboptimal Control of Turbulent Flow Using Control Theory. In *Proceedings of the International Symposium. on Mathematical Modelling of Turbulent Flows*, 1995. Tokyo, Japan.

[19] **Choi, H.; Hinze, M.** & **Kunisch, K.** Instantaneous control of backward-facing-step flows. *Applied Numerical Mathematics*, 31:133–158, 1999.

[20] **Choi, H; Temam, R.; Moin, P.** & **Kim, J.** Feedback control for unsteady flow and its application to the stochastic Burgers equation. *J. Fluid Mech.*, 253:509–543, 1993.

[21] **Clerc, M.; Le Tallec, P.; Mallet, M.; Ravachol, M.** & **Stoufflet, B.** Optimal control for the parabolized Navier-Stokes system. In *Computational Fluid Dynamics '96*, pages 139–145. John Wiley & Sons, 1996.

[22] **Constantin, P.** & **Foias, C.** *Navier-Stokes Equations*. The University of Chicago Press, 1988.

[23] **Coron, J.M.** On the controllability of the 2-d incompressible Navier-Stokes equations with the Navier slip boundary condition. *ESAIM: Control, Optimization and Calculus of Variations*, 1:35–75, 1996.

[24] **Dautray, R.** & **Lions, J.L.** *Mathematical Analysis and Numerical Methods for Science and Technology*, volume 5. Springer Verlag, 1992. Evolution Problems I.

[25] **Desai, M.C.** & **Ito, K.** Optimal control of Navier–Stokes equations. *Siam J. Control and Optimization*, 32:1428–1446, 1994.

[26] **Deuflhard, P.** Newton techniques for highly nonlinear problems - theory, algorithms, codes. Skriptum, 1985. Fachbereich Mathematik, Freie Universität Berlin.

[27] **Fabre, C.** Uniqueness results for the Stokes equations and their consequences in linear and nonlinear control problems. *ESAIM: Control, Optimization and Calculus of Variations*, 1:267–302, 1996.

[28] **Fernandez-Cara, E.** & **Beltran, E.M.M.** The convergence of two numerical schemes for the Navier-Stokes equations. *Numer. Math.*, 55:33–60, 1989.

[29] **Fursikov, A.V.** Exact boundary zero controllability of three-dimensional Navier-Stokes equations. *J. Dynamical Control and Systems*, 1:325–350, 1995.

[30] **Fursikov, A.V.; Gunzburger, M.D.** & **Hou, L.S**. Boundary value problems and optimal boundary control for the Navier-Stokes system: the two-dimensional case. *SIAM J. Control and Optimization*, 36:852–894, 1998.

[31] **García, C.E.; Prett, D.M.** & **Morari, M.** Model predictive control: Theory and practice - a survey. *Automatica*, 25(3):335–348, 1989.

[32] **Ghattas, O.** & **Bark, J.** Optimal control of two- and three-dimensional incompressible Navier-Stokes flows. *Journal of Computational Physics*, 136:231–244, 1997.

[33] **Glowinski, R.** Finite element methods for the numerical simulation of incompressible viscous flow; Introduction to the Control of the Navier-Stokes Equations. *Lectures in Applied Mathematics*, 28, 1991.

[34] **Griewank, A.** The local convergence of Broyden-like methods in Lipschitzian problems in Hilbert spaces. *SIAM J. Numer. Anal.*, 24:684–705, 1987.

[35] **Gunzburger, M.D.** *Flow Control*. IMA. Springer, 1995.

[36] **Gunzburger, M.D.; Hou, L.S.** & **Svobodny, T.P.** Analysis and finite element approximation of optimal control problems for the stationary Navier-Stokes equations with Dirichlet boundary controls. M^2AN, 25(6):711–748, 1991.

[37] **Gunzburger, M.D.** & **Manservisi, S.** The velocity tracking problem for Navier-Stokes flows with bounded distributed controls. Siam J. Contr. Optim., 1999. to appear.

[38] **Gunzburger, M.D.** & **Manservisi, S.** Analysis and approximation for linear feedback control for tracking the velocity in Navier-Stokes flows. *Comput. Methods Appl. Mech. Eng.*, 189:803–823, 2000.

[39] **Gunzburger, M.D.** & **Manservisi, S.** Analysis and approximation of the velocity tracking problem for Navier-Stokes flows with distributed control. *Siam J. Numer. Anal.*, 37:1481–1512, 2000.

[40] **Gunzburger, M.D.** & **Manservisi, S.** The velocity tracking problem for Navier-Stokes flows with boundary controls. *Siam J. Control and Optimization*, 39:594–634, 2000.

[41] **Heinkenschloss, M.** Private Communication. Trier, 1999.

[42] **Heinkenschloss, M.** Time-domain decomposition iterative methods for the solution of discretized linear quadratic optimal control problems. Draft, Department of Computational and Applied Mathematics, Rice University, Houston, USA.

[43] **Heinkenschloss, M.** Formulation and Analysis of a Sequential Quadratic Programming method for the Optimal Dirichlet Boundary Control of Navier-Stokes Flow. Report TR97–14, Department of Computational and Applied Mathematics - MS134, Rice University Houston, Texas USA, 1997. see also Optimal Control: Theory, Algorithms, and Applications. Kluwer Academic Publishers B.V., 1998, pp. 178-203.

[44] **Heywood, J.G.; Rannacher, R.** & **Turek, S.** Artificial Boundaries and Flux and Pressure Conditions for the Incompressible Navier-Stokes Equations. *Int. J. Numer. Methods Fluids*, 22:325–352, 1996.

[45] **Hill, D.C.** Drag reduction strategies. CTR Annual Research Briefs, 1993. Center for Turbulence Research, Stanford University/NASA Ames Research Center, 3-14.

[46] **Hinze, M.** & **Kauffmann, A.** The instantaneous control method - convergence analysis for finite dimensional systems. Preprint No. 602/1998, Fachbereich Mathematik, Technische Universität Berlin, extended version, 1998. Former title: A new class of feedback control laws for dynamical sytems.

[47] **Hinze, M.** & **Kauffmann, A.** Control concepts for parabolic equations with an application to the control of fluid flow. Scientific Computing in Chemical Engineering II, Hamburg Harburg, Springer, 1999. see also Preprint No. 603/1998, Technische Universität Berlin, Deutschland.

[48] **Hinze, M.** & **Kauffmann, A.** Reduced order modeling and suboptimal control of a solid fuel ignition model. Preprint No. 636/1999, 1999. Technische Universität Berlin.

[49] **Hinze, M.** & **Kunisch, K.** On suboptimal Control Strategies for the Navier-Stokes Equations. ESAIM: Proceedings, Vol. 4, 1998, 181-198, http://www.emath.fr/proc/Vol.4 (1998), France.

[50] **Hinze, M.** & **Kunisch, K.** Control strategies for fluid flows - optimal versus suboptimal control. In H.G.Bock et al., editor, *ENUMATH 97*, pages 351–358. World Scientific, 1997. Singapore.

[51] **Hinze, M.** & **Kunisch, K.** Suboptimal Control Strategies for backward facing step flows. In A. Sydow, editor, *Proceedings of the 15th IMACS World Congress on Scientific Computation, Modelling and Applied Mathematics*, volume 3, p. 53-58, Berlin, 1997.

[52] **Hinze, M.** & **Kunisch, K.** Newton's method for tracking-type control of the instationary Navier-Stokes equations, 1999. ENUMATH 99, Eds. P. Neittaanmäki et al., Jyväskylä, Finland.

[53] **Hinze, M.** & **Kunisch, K.** Second order methods for optimal control of time-dependent fluid flow. *SIAM J. Control Optim.*, 40:925–946, 2001.

[54] **Hinze, M.** & **Volkwein, S.** Instantaneous control for the Burgers equation: Convergence anlysis and numerical implementation. *Nonlinear Analysis T.M.A.*, 50:1–26, 2002.

[55] **Hood, P.** & **Taylor, C.** A numerical soution of the Navier-Stokes equations using the finite element technique. *Comp. and Fluids*, 1:73–100, 1973.

[56] **Hou, L.S.; Gunzburger, M.D.; Manservisi, S.; Turner, J.** & **Yan, Y.** Computations of optimal controls for incompressible flows. In *1997 ASME Fluids Engineering Division Summer Meeting*, 1997.

[57] **Hou, L.S.** & **Svobodny, T.P.** Optimization Problems for the Navier-Stokes Equations with Regular Boundary Controls. *J. Math. Anal. Appl.*, 177:342–367, 1993.

[58] **Hou, L.S.** & **Yan, Y.** Dynamics and approximations of a velocity tracking problem for the Navier-Stokes flows with piecewise distributed controls. *SIAM J. Control Optim.*, 35:1847–1185, 1997.

[59] **Hou, L.S.** & **Yan, Y.** Dynamics for controlled Navier-Stokes Systems with distributed controls. *SIAM J. Control Optim.*, 35:654–677, 1997.

[60] **Ito, K.** & **Kang,S.** A Dissipatative Feedback Control Synthesis for Systems Arising in Fluid Dynamics. *Siam J. Control and Optimization*, 32:831–854, 1994.

[61] **Ito, K.** & **Kunisch, K.** Augmented Lagrangian-SQP-methods for nonlinear optimal control problems of tracking type. *SIAM J. Control and Optimization*, 34:874–891, 1996.

[62] **Ito, K.** & **Ravindran, S.S.** Reduced basis method for optimal control of unsteady viscous flow. *Journal of Guidance, Control and Dynamics*, 1997. submitted.

[63] **Ito, K.** & **Ravindran, S.S.** A reduced basis method for control problems governed by pdes. Control and estimation of distributed parameter systems. International conference in Vorau, Austria, July 14–20, 1996., 1998. Desch, W. (ed.) et al. Basel: Birkhaeuser. ISNM, Int. Ser. Numer. Math. 126, 153-168.

[64] **Jadabaie, A.; Yu, J.** & **Hauser, J.** Unconstrained receding horizon control of nonlinear systems. Preprint, 1999. Caltech, Electrical and Computer Engineering.

[65] **Joshi, S.S.; Speyer, J.L.** & **Kim, J.** A system theory approach to the feedback stabilization of infinitesimal and finite-amplitude disturbances in plane Poiseuille flow. *J. Fluid Mech.*, 332:157–184, 1997.

[66] **Joslin, R.D.; Gunzburger, M.D.; Nicolaides, R.A.; Erlebacher, G.** & **Hussaini, M.Y.** A methodology for the automated optimal control of flows including transitional flows. Proc. ASME Forum on Control of Transitional and Turbulent Flows FED-237, ASME, 287-294, 1996.

[67] **Justen, P.** Optimal control of thermaly coupled Navier-Stokes equations in food industry. Preprint, 1998. Fachbereich IV - Mathematik, Universität Trier.

[68] **Kauffmann, A.** *Optimal control of the solid fuel ignition model*. PhD thesis, Technische Universität Berlin, 1998.

[69] **Kelley, C.T.** & **Sachs, E.W.** Quasi-Newton methods and unconstrained optimal control problems. *SIAM J. Control and Optimization*, 25:1503–1516, 1987.

[70] **Kloucek, P.** & **Rys, F.** Stability of the fractional step θ-scheme for the nonstationary Navier-Stokes equations. *SIAM J. Numer. Anal.*, 31:1312–1335, 1994.

[71] **Koumoutsakos, P.** Active control of turbulent channel flow. Report, 1997. Center of Turbulence Research, Stanford.

[72] **Koumoutsakos. P.** Active control of vortex-wall interactions. *Phys. Fluids*, 9:3808–3816, 1997.

[73] **Koumoutsakos, P.** Vorticity flux control for a turbulent channel flow. Report, 1998. Institut für Fluidmechanik, ETH Zürich, Switzerland.

[74] **Kunisch, K.** & **Marduel, X.** Optimal control of non-isothermal viscoelastic fluid flow. Bericht 1998. Spezialforschungsbereich Optimierung und Kontrolle, Institut für Mathematik, Karl-Franzens Universität Graz, to appear in Journal of Non-Newt. Fluid Mech.

[75] **Kunisch, K.** & **Marduel, X.** Suboptimal Control of Transient Non-Isothermal Viscoelastic Fluid Flow. Bericht 1999. Spezialforschungsbereich Optimierung und Kontrolle, Institut für Mathematik, Karl-Franzens Universität Graz.

[76] **Kunisch, K.** & **Sachs, E.W.** Reduced SQP methods for parameter identification problems. *SIAM J. Numer. Anal.*, 29:1793–1820, 1992.

[77] **Kunisch, K.** & **Volkwein, S.** Control of Burgers equation by a reduced order approach using Proper Orthogonal Decomposition. *J. Optimization Theory Appl.*, 102:345–371, 1999.

[78] **Kunisch, K.** & **Volkwein, S.** Galerkin proper orthogonal decomposition methods for parabolic problems. *Numerische Mathematik*, 90:117–148, 2001.

[79] **Kupfer, F.S.** An infinite-dimensional convergence theory for reduced sqp-methods in Hilbert space. *SIAM J. Optimization*, 6:126–163, 1996.

[80] **Lee, C.; Kim, J.; Babcock, D.** & **Goodman, R.** Application of neural networks to turbulence control for drag reduction. *Phys. Fluids*, 9(6):1740–1747, 1997.

[81] **Lee, C.; Kim, J.** & **Choi, H.** Suboptimal control of turbulent channel flow for drag reduction. *J. Fluid Mech.*, 358:245–258, 1998.

[82] **Lions, P.L.** *Mathematical Topics in Fluid Mechanics*. Clarendon Press, Oxford, 1996.

[83] **Loncaric, J.** Optimal control of unsteady Stokes flow around a cylinder and the sensor/actuator placement problem. ICASE Report No. 98-18, 1998. NASA, Langley Research Center.

[84] **Ly, H.V.** & **Tran, H.T.** Modelling and control of physical processes using proper orthogonal decomposition. Report, 1998. CRSC-TR98-37, Center for Research in Scientific Computation, North Carolina State University.

[85] **Ly, H.V.** & **Tran, H.T.** Proper orthogonal decomposition for flow calculations and optimal control in a horizontal CVD reactor. Preprint, 1998. Center for Research in Scientific Computation, North Carolina State University.

[86] **Málek, J.** & **Roubiček, T.** Optimization of steady flows for incompressible viscous fluids. *Nonlinear Applied Analysis*, pages 355–372, 1999. Eds. A. Sequiera et al.

[87] **Manservisi, S.** *Optimal boundary and distributed controls for the velocity tracking problem for Navier-Stokes flows*. PhD thesis, Faculty of the Virginia Polytechnic Institute and State University, Blacksburg, 1997.

[88] **Min, C.** & **Choi, H.** Suboptimal feedback control of vortex shedding at low Reynolds nunmbers. *J. Fluid Mech.*, 401:123 – 156, 1999.

[89] **Müller-Urbaniak, S.** Eine Analyse des Zwischenschritt-θ-Verfahrens zur Lösung der instationären Navier-Stokes Gleichungen. Preprint 94-01, Sfb 359, University of Heidelberg, Germany.

[90] **Neittaanmäki, P.** & **Tiba, D.** *Optimal control of nonlinear parabolic systems - Theory, Algorithms and Applications*. Marcel Dekker, 1994.

[91] **Nevistić, V.** Optimal control: A review. Preprint, 1997. Automatic Control Laboratory, ETH Zürich, Switzerland.

[92] **Nevistić, V.** & **Primbs, J.A.** Constrained finite receding horizon control: Stability and performance analysis. Preprint, 1997. Automatic Control Laboratory, ETH Zürich, Switzerland.

[93] **Nevistić, V.** & **Primbs, J.A.** Finite receding horizon control: A general framework for stability and performance analysis. Preprint, 1997. Automatic Control Laboratory, ETH Zürich, Switzerland.

[94] **Protas, B.** & **Styczek, A.** Theoretical and computational study of the wake control problem. Preprint, 1998. Department of Aerodynamics, Warsaw University of Technology, Warsaw, Poland.

[95] **Prudhomme, S.** & **Le Letty, L.** A low order model-following strategy for active flow control. FLOWCON 1998, Book of Abstracts. Göttingen, 1998.

[96] **Rannacher, R.** On the numerical solution of the incompressible Navier-Stokes equations. *ZAMM*, 73:303–216, 1993.

[97] **Rautert, T.** & **Sachs, E.W.** Computational Design of Optimal Output Feedback Controllers. Forschungsbericht Nr. 95-12, 1995. Institut für Mathematik, Universität Trier.

[98] **Ravindran, S.S.** Proper orthogonal decomposition in optimal control of fluids. Technical report, NASA Langley Research Center, Hampton, Virginia, 1999.

[99] **Rawlings, J.B.** & **Muske, K.R.** The stability of constrained receding horizon control. *IEEE Transactions on Automatic Control*, 38(10):1512–1516, 1993.

[100] **Sirovich, L.** Turbulence and the dynamics of coherent structures, Part I-III. *Quarterly of Applied Mathematics*, 45:561–590, 1987.

[101] **Sritharan, S.**, editor. *Optimal Control of Viscous Flow*. SIAM, 1998.

[102] **Tang, K.Y.; Graham, W.R.** & **Peraire, J.** Optimal control of vortex shedding using low-order models. I: Open loop model development. II: Model based control. *Int. J. Numer. Methods Eng.*, 44:945–990, 1999. see also AIAA paper Nr. 1996-19-46.

[103] **Temam, R.** *Navier-Stokes Equations*. North-Holland, 1979.

[104] **Temam, R.** *Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Second Edition*. Springer, 1997.

[105] **Temam, R.; Bewley, T.** & **Moin, P.** Control of turbulent flows. In *Proceedings of the 18th IFIP TC7 Conference on System Modelling and Optimization*, Detroit, Michigan, 1997.

[106] **Tröltzsch, F.** & **Unger, A.** Fast solution of optimal control problems in selective cooling of steel. *ZAMM*, 81:447–456, 2001.

[107] **Volkwein, S.** Optimal Control of a Phase-Field Model Using Proper Orthogonal Decomposition. to appear in ZAMM. 1999.

[108] **Volkwein, S.** *Mesh-Indipendence of an Augmented Lagrangian-SQP Method in Hilbert Spaces and Control Problems for the Burgers Equation*. PhD thesis, Fachbereich Mathematik, Technische Universtiät Berlin, 1997.

[109] **Werner, A.** & **Arndt, H.** *Numerik gewöhnlicher Differentialgleichungen*. Springer, 1985.

[110] **Wunder, G.** Zur Konstruktion von Reglern für große dynamische Systeme. Diplomarbeit, 1998. Fachbereich Elektrotechnik, Technische Universität Berlin.

[111] **Zeidler, E.** *Nonlinear Functional Analysis and its Applications I*. Springer Verlag, 1986.