Optimal and suboptimal performance of stochastic linear tracking systems

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Synopsis

The derivation of performance indices for both optimal and suboptimal linear tracking systems is organised so that the expressions for the indices consist of terms that may be calculated from the system input and its parameters and which represent separately various identifiable components of the overall cost. The derivation of the performance index for optimal linear tracking systems is necessary to give a complete solution to the optimal linear tracking problem. A knowledge of the various cost terms in the performance index of suboptimal linear tracking systems means that the performance of existing or approximate designs, or of optimal designs either modified or having different noise inputs or performance indices, may be calculated. Further, the improvement of existing or approximate designs may be carried out by a systematic trial-and-error approach, which aims at reducing perhaps only the largest cost terms for each iteration.

List of symbols

x, x_e, x_d = state, state estimate and desired state trajectory
F, G, H = matrices describing a finite-dimensional dynamic system
v, w = additive noise
y, u, g = output, input to plant and input to the optimal system
K = feedback law
V = performance index
Q, R = weighting matrices
M, N = noise-covariance matrices
x_0, m = initial state and mean initial state
S = state-estimate covariance
S_0 = initial S
L = feedback law for state estimator
t_0, t_1 = initial and final time
$z, \bar{m}$ = state of suboptimal system and mean of its initial state
E = expectation operator
P, b = matrix and vector appearing in control problem

1 Introduction

A number of results in linear control theory derived for application to the linear regulator problem have been generalised for application to the tracking (or servomechanism) problem.[1,2] Meier and Anderson[3] have given a derivation of the performance index of both optimal and suboptimal linear regulator systems having additive noise at the input and only noisy measurements available at the outputs. The problem considered in this paper is the generalisation of these results for application to linear tracking systems.

The performances of both optimal and suboptimal stochastic linear tracking systems have terms for cost which may arise owing to uncertainty about the initial state of the plant, error in the estimate of the state, error in the control owing to additive noise at the input, or to a desired output differing from the mean initial state. A suboptimal system, in addition to having higher costs than the optimal system for these terms, has cost terms which vanish as optimal control, optimal estimation or optimal external input are used.

The derivation of the performance index of optimal linear tracking systems so that the various costs are given in separate terms is necessary to give a complete solution to the optimal linear tracking problem. The derivation of the performance index of suboptimal linear tracking systems so that the various costs are given in separate terms means that the performance of existing or approximate designs, or optimal designs either modified or having different performance indices or noise inputs, may be calculated. Further, the improvement of existing or approximate designs may be carried out by a systematic trial-and-error approach, which aims at reducing perhaps only the largest cost terms for each iteration. It may be argued that the design of an optimal system is a direct procedure and is the best design, but it must be remembered that perhaps only small changes in performance index would result for maybe gross simplifications of the controller. A further consideration is that the solution of a Riccati differential equation is avoided using the trial-and-error approach.

The systems under consideration are linear, finite-dimensional systems, with additive Gaussian noise at the input and output, and thus can be described by the state–space equations

$$\dot{x} = Fx + Gu + w \quad \quad (1a)$$

$$y = Hx + v \quad \quad (1b)$$

where x is an $n$vector (the state), u is a $p$vector (the input), y is an $m$vector (the output) and the matrices F, G and H are of the appropriate dimensions and may be time-varying. The input noise w and output noise v are independent and Gaussian with mean zero, and

$$\operatorname{cov}\{w(t), w(\tau)\} = M(t)\delta(t - \tau) \quad \quad (2a)$$

$$\operatorname{cov}\{v(t), v(\tau)\} = N(t)\delta(t - \tau) \quad \quad (2b)$$

Both M and N are nonnegative-definite symmetric matrices. A further restriction necessary for the construction of an optimal estimator for the plant described by eqns 1a and b is

(a) N is nonsingular

At the initial time t_0, it is assumed that the initial state x_0 of the plant described by eqns. 1a and b is known only to the extent that its probability distribution is Gaussian with mean m and

$$\operatorname{cov}(x_0, x_0) = S_0 \quad \quad (3)$$

where S_0 is a nonnegative-definite symmetric matrix.

The optimal performance index applicable for the stochastic tracking problem is a natural generalisation of that for the deterministic tracking problem:[1]

$$V(mt_0, t_1) = E\int_{t_0}^{t_1}\{(x' - x_d')Q(x - x_d) + u'Ru\}dt \quad \quad (4)$$

where x_d is the desired state trajectory over the interval (t_0, t_1). It will be assumed throughout that $t_1 > t_0$.

In the design of optimal linear tracking systems, a control u is selected which minimises the performance index (eqn. 4). In order for the minimisation problem to be well defined, it is required that

(b) The matrices $R(\,)$ and $Q(\,)$ are nonnegative-definite symmetric, with $R(\,)$ nonsingular

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Applying the separation theorem for linear tracking systems,* the design of optimal linear tracking systems may be conveniently considered as two separate design problems. The first is the design of a dynamic system, the state estimator,[4] with inputs $u(\,)$ and $y(\,)$, whose output $x_e(\,)$ is the minimum-variance estimate of the state $x(\,)$. The second problem is the selection of a linear controller having a feedback law K and an external input g, as for the noise-free problem,[1] where K is of dimensions $p \times n$, and g is a $p$vector. The optimum control law is then given by

$$u = -Kx_e + g \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

A review of the theorems related to the estimation problem, the deterministic tracking problem and the stochastic regulator problem is given in Section 2; this is the background for the derivation of the performance index (eqn. 4) for the optimal stochastic linear tracking problem in Section 3. Section 3 in turn serves as an introduction to the derivation of the performance index for suboptimal stochastic linear systems given in Section 4.

2 Review of estimation and control theorems

Theorem 1.[4] For the plant described by eqns. 1*a* and *b*, with noise as defined in eqn. 2, the state estimator is a finite-dimensional linear system as shown in Fig. 1. The initial state of the estimator is taken as

$$x_e(t_0) = m \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

and the gain matrix of the estimator, whose output $x_e(t)$ is the minimum-variance estimate of the state $x(t)|u_{(t_0,\,t)}, y_{(t_0,\,t)}$, is

$$L = SH'N^{-1} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

provided that (*a*) holds. Here S is the solution of the matrix Riccati differential equation

$$\dot{S} = FS + SF' - SH'N^{-1}HS + M \quad . \; . \; . \quad (8a)$$

subject to the initial condition

$$S(t_0) = S_0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (8b)$$

The state–space equation for the state estimator is

$$\dot{x}_e = Fx_e + Gu + LH(x - x_e) + Lv \quad . \; . \; . \quad (9)$$

Theorem 2.[1,2] For the tracking problem in the absence of noise, the system equation is

$$\dot{x} = Fx + Gu \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (10)$$

The optimal performance index

$$V(x_0, t_0, t_1) = \min \int_{t_0}^{t_1} \{(x' - x_d')\, Q(x - x_d) + u\, Ru\} dt \quad . \; . \; . \; . \quad (11)$$

is achieved, provided that (*b*) holds, by taking

$$u = -Kx + g \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (12)$$

where

$$K = R^{-1}G'P \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (13)$$

$$g = R^{-1}G'b, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (14)$$

P being the solution of the matrix Riccati differential equation

$$-\dot{P} = F'P + PF - PGR^{-1}G'P + Q \quad . \; . \; . \quad (15)$$

with the boundary condition

$$P(t_1) = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (16)$$

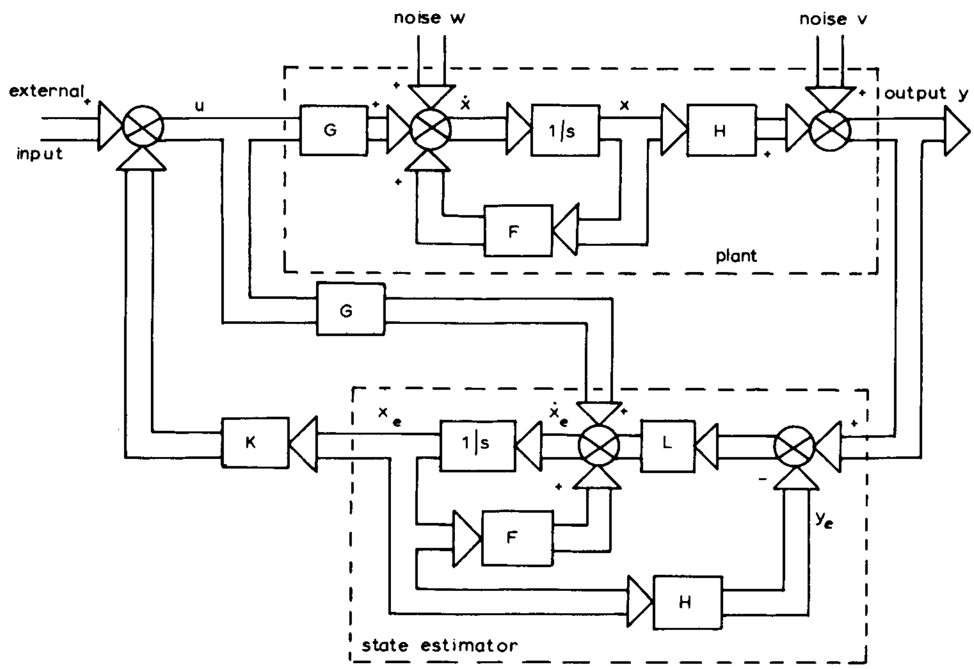

Fig. 1
Optimal tracking system

and b being the solution of the linear vector differential equation

$$-\dot{b} = (F - GK)'b + Qx_d \quad . \; . \; . \; . \; . \; . \quad (17)$$

with the boundary condition

$$b(t_1) = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (18)$$

The optimal performance index reduces to

$$V^*(x_0, t_0, t_1) = \int_{t_0}^{t_1} \left\{ -\frac{d}{dt}(x'Px) + 2\frac{d}{dt}(b'x) + x_d'Qx_d - g'Rg \right\} dt \quad . \quad (19)$$

or

$$V^*(x_0, t_0, t_1) = x_0'P(t_0)x_0 - 2b'(t_0)x_0 + \int_{t_0}^{t_1} (x_d'Qx_d - g'Rg)dt \quad . \quad (20)$$

which is well defined, since, for the conditions previously stated, P and b exist and are unique.

* RHODES, I. B., and ANDERSON, B. D. O.: 'Separation theorem for linear servomechanisms', in preparation

Theorem 3.[3] For the stochastic-regulator problem, consider the plant described by eqns. 1*a* and *b* to have noise defined by 2 and 3. The optimal control

$$u = -Kx_e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where K is given from eqns. 12, 13 and 15 and x_e is given from eqns. 6 and 9, minimises the performance index

$$V(m, t_0|x_d = 0, t_1) = E\int_{t_0}^{t_1}(x'Qx + u'Ru)dt \quad . \quad (22)$$

provided that (*b*) holds.

The optimal performance index is

$$V^*(m, t_0|x_d = 0, t_1) = E\int_{t_0}^{t_1}\left\{-\frac{d}{dt}(x'Px) + 2x'Pw + (x - x_e)'K'RK(x - x_e)\right\}dt \quad . \quad (23)$$

or

$$V^*(m, t_0|x_d = 0, t_1) = \mathrm{tr}\{S_0P(t_0)\} + m'P(t_0)m + \int_{t_0}^{t_1}\mathrm{tr}\,(SK'RK + MP)dt \quad . \quad (24)$$

All four terms of eqn. 24 are nonnegative and are due, respectively, to

(i) uncertainty about the initial state
(ii) a nonzero mean of the initial state
(iii) error in the estimate of state
(iv) error in the control due to the additive noise at the input.

Theorem 4.[3] For the plant described by eqns. 1*a* and *b*, with noise defined as in eqns. 2 and 3, suppose an estimator has feedback law L_s, while the plant has a feedback law K_s. The performance index of eqn. 22 is then

$$V_s(m, t_0|x_d = 0, t_1) = \mathrm{tr}\{S_0P_1(t_0)\} + m'P_1(t_0)m + \int_{t_0}^{t_1}[\mathrm{tr}\{S_3(GK_sP_1 + P_1K_s'G' - K_sRK_s') + MP_1 + 2(L_sNL_s' - S_3H'L_sP_2)\}]dt \quad . \quad (25)$$

where the matrix

$$P_s = \begin{bmatrix} P_1P_2 \\ P_2'P_3 \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

is the solution of the linear equation

$$-\dot{P}_s = F_s'P_s + P_sF_s + Q_s \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

with the initial condition

$$P_s(t_1) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

where

$$F_s = \begin{bmatrix} F - GK_s & -GK_s \\ 0 & F - L_sH \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad (29)$$

and

$$Q_s = \begin{bmatrix} Q + K_s'RK_s & K_s'RK_s \\ K_s'RK_s & K_s'RK_s \end{bmatrix}$$

The matrix S_s, partitioned in the same way as P, is the solution of the linear equation

$$\dot{S}_s = F_sS_s + S_sF_s' + M_s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

with the initial condition

$$S_{s0} = \begin{bmatrix} S_0 & -S_0 \\ -S_0 & S_0 \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

S_0 being given by eqn. 3. The matrix M_s is given by

$$M_s = \begin{bmatrix} M & -M \\ -M & M + L_sNL_s' \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad (32)$$

The importance of the particular expression (eqn. 25) for the performance index (eqn. 22) is that the various cost terms in the expression can be allocated to different sources. It is derived from the less complex but less instructive expression

$$V_s(m, t_0|x_d = 0, t_1) = \mathrm{tr}\{S_{s0}P_s(t_0)\} + \bar{m}'P_s(t_0)\bar{m} + \int_{t_0}^{t_1}\mathrm{tr}\,(M_sP_s)dt \quad . \quad (33)$$

where

$$\bar{m} = \begin{bmatrix} m \\ 0 \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

3 Performance of optimal linear stochastic tracking systems

For the plant described by eqns. 1*a* and *b*, state estimator (eqn. 9) and the optimal control law (eqn. 5), the equations of the combined system as shown in Fig. 1 are

$$\dot{x} = Fx + G(-Kx_e + g) + w \quad . \quad . \quad . \quad . \quad . \quad (35a)$$

$$\dot{x}_e = Fx_e + G(-Kx_e + g) + LH(x - x_e) + Lv \quad . \quad . \quad . \quad . \quad (35b)$$

The optimal performance index may be calculated from eqns. 4, 5, 12–18 and 35, using steps similar to those used in calculating eqns. 19 and 23. However, since eqn. 4 reduces to eqn. 19 for the noise-free case and to eqn. 23 when $x_d = 0$, it is not unreasonable to assume that the integral of eqn. 4 may be rearranged to give a convenient expression consisting of terms given in the integrand of eqn. 19 or 23, together with maybe other terms which vanish appropriately as $v = w = 0$ or $x_d = 0$. With this in mind, terms suggested by the integrands of eqns. 19 and 23 are expanded using eqns. 15, 17 and 35*a*:

$$-\frac{d}{dt}(x'Px) = -\dot{x}'Px - x'\dot{P}x - x'P\dot{x} = x'Qx - 2x'K'R(-Kx_e + g) - x'K'RKx - 2x'Pw \quad . \quad (36)$$

$$2\frac{d}{dt}(b'x) = 2(\dot{b}'x + b'\dot{x}) = 2g'Rg - 2x_d'Qx + 2g'RK(x - x_e) + 2b'w \quad . \quad . \quad . \quad . \quad (37)$$

and the sum of such terms suggested from the integrands of eqn. 19 or 23 is

$$-\frac{d}{dt}(x'Px) + 2\frac{d}{dt}(b'x) + (x_e - x)'K'RK(x_e - x) + x_d'Qx_d - g'Rg + 2x'Pw = (x - x_d)'Q(x - x_d) + (-Kx_e + g)'R(-Kx_e + g) + 2b'w \quad . \quad . \quad . \quad (38)$$

Thus, from eqns. 4 and 38,

$$V^*(m, t_0, t_1) = E\int_{t_0}^{t_1}\left\{-\frac{d}{dt}(x'Px) + 2\frac{d}{dt}(b'x) + (x_e - x)'K'RK(x_e - x) + x_d'Qx_d - g'Rg + 2x'Pw - 2b'w\right\}dt \quad . \quad (39)$$

since

$$E\int_{t_0}^{t_1}2b'wdt = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

and

$$E\int_{t_0}^{t_1}2\frac{d}{dt}(b'x)dt = E\left(2b'x\Big|_{t_1}^{t_0}\right) = -2b'(t_0)m \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

using eqn. 18. The evaluation of the expectation of the integral of the first, third and sixth terms of the integrand proceeds just as in Reference 3, and there results

$$V^*(m, t_0, t_1) = \mathrm{tr}\{S_0P(t_0)\} + m'P(t_0)m - 2b'(t_0)m + \int_{t_0}^{t_1}\{\mathrm{tr}\,(SK'RK + MP) + x_d'Qx_d - g'Rg\}dt \quad . \quad (42)$$

We observe that all the terms are, in fact, terms in the performance indices for the deterministic-tracking problem (eqn. 20) or the stochastic-regulator problem (eqn. 24).

4 Performance of suboptimal linear stochastic tracking systems

There are three distinct ways in which the control law of eqn. 5 may be varied to result in a suboptimal control. A suboptimal estimate of x may be used, the feedback law K may be changed or the external input g may be modified.

Referring to Fig. 1, the systems to be considered are those where L is replaced by L_s, K is replaced by K_s and g is replaced by g_s. The state-space equations of the plant, estimator and feedback-law realisation as a combined system may be written using eqn. 35 with the appropriate substitutions as

$$\dot{x} = (F - GK_s)x - GK_s(x_e - x) + w + Gg_s \quad . \quad (43a)$$

$$\dot{x}_e - \dot{x} = (F - L_sH)(x_e - x) + L_sv - w \quad . \quad . \quad (43b)$$

This may be written

$$z = F_sz + w_s + G_s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

where the state vector of the combined system is

$$z = \begin{bmatrix} x \\ x_e - x \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

The remaining terms are

$$F_s = \begin{bmatrix} F - GK_s & -GK_s \\ 0 & F - L_sH \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad (46)$$

$$w_s = \begin{bmatrix} w \\ L_sv - w \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

and

$$G_s = \begin{bmatrix} Gg_s \\ 0 \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

The cost of using laws K_s, L_s and g_s for control of the plant described by eqns. $1a$ and b is

$$V_s(m, t_0, t_1) = E\int_{t_0}^{t_1}\{(x - x_d)'Q(x - x_d) + (-K_sx_e + g_s)'R(-K_sx_e + g_s)\}dt \quad . \quad (49)$$

This may be expressed in terms of the state vector (eqn. 45) of the combined system (eqn. 43), after some elementary calculations, as

$$V_s(m, t_0, t_1) = E\int_{t_0}^{t_1}(z'Q_sz + x_d'Qx_d + g_s'Rg_s - 2x_d'[Q\,0]z - 2g_s'RK_s[I\,I]z)dt. \quad (50)$$

In order to evaluate this integral, the results of eqns. 23 and 39 suggest attempting to use P_s and b_s (the solutions of differential equations), chosen such that expansion of $-\frac{d}{dt}(z'P_sz)$ and $2\frac{d}{dt}(b_s'z)$ result in terms of the integrand of eqn. 50, quadratic and linear in z, respectively. The linear differential equation (eqns. 27 and 28) is indicated for P_s and the following linear differential equation gives b_s:

$$-\dot{b}_s = F_s'b_s + [Q\,0]'x_d - P_sG_s + [I\,I]'K_s'Rg_s \quad . \quad (51)$$

with the initial condition

$$b_s(t_1) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

Thus, using equations 17, 28, 51, 52 and 44, the following expansions are calculated:

$$-\frac{d}{dt}(z'P_sz) = z'Q_sz - 2w_s'P_sz - 2G_s'P_sz \quad . \quad . \quad (53)$$

$$2\frac{d}{dt}(b_s'z) = -2x_d'[Q\,0]z + 2b_sw_s + 2G_s'P_sz + 2b_s'G_s - 2g_s'RK_s[I\,I]z \quad . \quad (54)$$

Eqn. 50 may now be expressed, using eqns. 53 and 54, as

$$V_s(m, t_0, t_1) = E\int_{t_0}^{t_1}\left\{-\frac{d}{dt}(z'P_sz) + 2\frac{d}{dt}(b_s'z) + x_d'Qx_d + g_s'Rg_s - 2b_s'w_s - 2b_s'G_s + 2w_s'P_sz\right\}dt \quad . \quad . \quad . \quad . \quad (55)$$

Using calculations like those of Reference 3 for deriving eqn. 24 from eqn. 23, and as used in Section 3 for deriving eqn. 42 from eqn. 39, eqn. 55 may be rewritten as

$$V_s(m, t_0, t_1) = \mathrm{tr}\{S_{s0}P_s(t_0)\} + \bar{m}'P_s(t_0)\bar{m} - 2b_s'(t)\bar{m} + \int_{t_0}^{t_1}\{\mathrm{tr}(M_sP_s) + x_d'Qx_d - 2b_s'G_s + g_s'Rg_s\}dt \quad . \quad (56)$$

Applying the results of the derivation of eqn. 25 from eqn. 33, and considering an even partitioning of b_s as

$$b_s = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

eqn. 56 may be written as

$$V_s(m, t_0, t_1) = \mathrm{tr}\{S_0P_1(t_0)\} + m'P_1(t_0)m - 2b_1'(t_0)m + \int_{t_0}^{t_1}[\mathrm{tr}\{S_3(GK_sP_1 + P_1K_s'G' - K_sRK_s') + MP_1 + 2(L_sNL_s' - S_3H'L_sP_2)\} + x_d'Qx_d - g_s'Rg_s + 2(g_s'R - b_1'G)g_s]dt \quad . \quad . \quad (58)$$

Eqn. 58 is the most general result of the paper, since, by setting K_s, L_s and g_s to the optimal values given in eqns. 13, 7 and 14, the suboptimal performance index (eqn. 58) reduces to the optimal performance index (eqn. 42). This is readily shown using the following results from eqns. 27, 28, 15, 16, 26 and 13:

$$P_1 = P \mid K_s = K \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59a)$$

$$P_2 = 0 \mid K_s = K \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59b)$$

and from eqns. 51, 52, 17, 18, 13, 57 and 14:

$$b_1 = b \mid K_s = K, g_s = g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

and also using eqns. 30, 31, 7 and 8:

$$S_3 = S \mid L_s = L \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

The suboptimal performance index (eqn. 58) is also the most general performance index discussed in the paper, since, with the appropriate substitutions, all the previously mentioned performance indices are readily calculated. For example, setting g_s and x_d to zero means that from eqns. 51 and 52, b_1 equals 0, and the tracking-problem suboptimal performance index (eqn. 58) reduces to the regulator-problem suboptimal index (eqn. 25). This means that the following terms may be identified as in Reference 1. The term $m'P_1(t_0)m$ is the performance index of a deterministic linear regulator when the feedback law K_s is used instead of the optimal law K. Note, for this case, that $m = x_0$. The term $\mathrm{tr}\{S_0P_1(t_0)\}$ is the additional contribution if the initial state x_0 is only known to the extent of its mean m and covariance S_0. The first term in the integrand of eqn. 58 corresponds to error in the estimate of the state for the optimal case, and the second term corresponds to error in the control from additive noise at the input, also for the optimal case. The third term arises from suboptimality for the regulator problem and is dependent on S_3, a modified estimation error covariance, and a matrix P_2 which depends on the feedback law chosen. The remaining terms are identified using eqn. 20. The terms $m'P_1(t_0)m - 2b_1'm + \int_{t_0}^{t_1}(x_d'Qx_d - g_s'Rg_s)dt$ give the performance index for the deterministic tracking problem with optimal g_s and K_s, while the term $\int_{t_0}^{t_1}(g_s'R - b_1'G)g_sdt$ results from suboptimality of the external input g_s and the feedback law K_s.

Thus we see that all the various terms are identifiable costs, and, since they are in a form suitable for digital computation, the derivation of the tracking-problem performance index as given in eqn. 58 is justified. Then too, if not all g_s, K_s and L_s are suboptimal, simplifications to the expression of eqn. 58 may be made using the appropriate eqns. 59, 60 or 61.

A particular case which may correspond to a practical situation is when K_s and L_s are suboptimal and g_s may be selected to minimise the performance index (eqn. 58). From eqn. 58, the optimal g_s minimises

$$-b_1'(t_0)m + \int_{t_0}^{t_1}(g_s'Rg_s - 2b_1'Gg_s)dt \quad . \quad . \quad . \quad . \quad (62)$$

subject to the constraints given by eqns. 51, 52 and 57:

$$-\dot{b}_1 = (F - GK_s)'b_1 + Qx_d + (K_s'R - P_1G)g_s \quad . \quad (63a)$$

and

$$b_1(t_1) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63b)$$

Evidently this is a nonstandard quadratic loss-minimisation problem. If it is not convenient to solve the minimisation problem, a reasonable g_s may be calculated from an equation suggested by the completely optimal case:

$$g_s = R^{-1}G'b_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

When this is inserted in eqn. 63 the following equation results for b_1:

$$-\dot{b}_1 = F'b_1 - P_1GR^{-1}G'b_1 + Qx_d \quad . \quad . \quad . \quad (65a)$$

$$b_1(t_1) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65b)$$

Expr. 62 then reduces to

$$-b_1(t_0)m - \int_{t_0}^{t_1}(g_s'Rg_s)dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

This corresponds to the elimination of the last term in the performance index (eqn. 58), previously identified as a term introduced due to suboptimality of the external input g_s.

The main result of this section is now summarised as theorem 5.

Theorem 5. For the plant of eqn. 1 with noise defined as in eqns. 2 and 3, suppose an estimator has feedback law L_s, while the plant has feedback law K_s and the external input is g_s. The performance index (eqn. 4) is then eqn. 58, where b_1 is given from eqns. 63 (being derived from eqns. 51 and 52), P_1 and P_2 are linear differential equations, being derived from eqns. 27 and 28 as

$$-\dot{P}_1 = (F - GK_s)'P_1 + P_1(F - GK_s) + Q + K_s'RK_s \quad . \quad (67)$$

$$-\dot{P}_2 = (F - GK_s)'P_2 + P_2(F - L_sH) - P_1GK_s + K_s'RK_s \quad . \quad . \quad . \quad . \quad (68)$$

$$P_1(t_1) = P_2(t_1) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

and S_3 is given from a linear differential equation derived from eqns. 30 and 31 as

$$\dot{S}_3 = (F - L_sH)S_3 + S_3(F - L_sH)' + M + L_sNL_s' \quad . \quad . \quad . \quad . \quad (70a)$$

$$S_3(t_0) = S_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70b)$$

5 Conclusions

A performance index having many terms has been derived for suboptimal stochastic linear tracking systems. The number of terms may be reduced only as the complexity of the problem is reduced or as the feedback law and input signals become optimal. For example, if the desired state trajectory x_d is zero, the index reduces to that for the stochastic-regulator problem, or, if the deterministic optimal tracking problem is considered, the index reduces to the well known optimal tracking-problem index.[2]

The significance of the particular form of the result is that, since each term is an identifiable cost and is suitable for digital computation, the systematic design of linear stochastic tracking system controllers with controller-complexity constraints is now possible.

6 References

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