MANAGEMENT SCIENCE
Vol. 21, No. 11, July 197
Printed in U.S.A.

OPTIMAL AND SYSTEM MYOPIC POLICIES FOR MULTI-ECHELON PRODUCTION/INVENTORY ASSEMBLY SYSTEMS*

LEROY B. SCHWARZ† AND LINUS SCHRAGE§ ¶

In this paper optimal and near optimal policies are proposed for multi-echelon production/inventory assembly systems under continuous review with constant demand over an infinite planning horizon. Costs at each stage consist of a fixed charge per order or production setup plus a linear holding cost on "echelon" inventory. The objective is minimization of average cost per unit time.

The major results of this paper are: a mathematically simple, often optimal, "system myopic" solution, a lower bound on the closeness to optimality of this solution, and a branch and bound algorithm which usually finds the optimal solution quickly.

1. Introduction

In this paper we describe optimal and near optimal policies for operating single product multi-echelon assembly production/inventory systems. In a multi-echelon assembly system each stage (which may be a production site, an assembly site, or merely a stocking site) obtains input from one or more immediate predecessors, perhaps with some delivery lag, and supplies output to a *single* successor, again with a possible delivery lag. The final stage, stage 1, satisfies the customer demand.

Figure 1 shows three possible configurations of assembly systems, including (a) the serial system, in which each stage has only one predecessor stage; and (b) the pure assembly system, in which stages $2, 3, \ldots, N$ are immediate predecessors of stage 1. Examples of assembly systems abound in the real world; e.g., the manufacture and assembly of automobiles, electrical appliances, etc. Serial systems are frequently found in processing industries; e.g., the steel or aluminum industries where the stages represent different physical and/or chemical transformations of the same basic material (ore, pig iron, sheet steel, etc.). Clark (1972) has an extensive survey of multi-echelon models.

Our objective is to select ordering policies for assembly systems which minimize (or nearly minimize) average system cost per unit time over an infinite planning horizon when the customer demand rate is constant. Costs are of two types: a setup or order cost incurred at each stage whenever a batch is ordered or produced at that stage, and a holding cost for each stage charged continuously over time which is linear in the so-called "echelon" inventory at that stage.

Clark and Scarf (1960) define the echelon stock of stage j as the number of units in the system which are in or have passed through stage j but have as yet not been sold.

* Processed by Professor Edward J. Ignall, Departmental Editor for Dynamic Programming and Inventory Theory; received July 1973, revised April 1974 and October 1974. This paper has been with the authors 6 months for revision.

† University of Rochester.

§ University of California, Los Angeles.

¶ The authors wish to acknowledge the valuable comments of Jack Williams on an earlier draft of this paper.

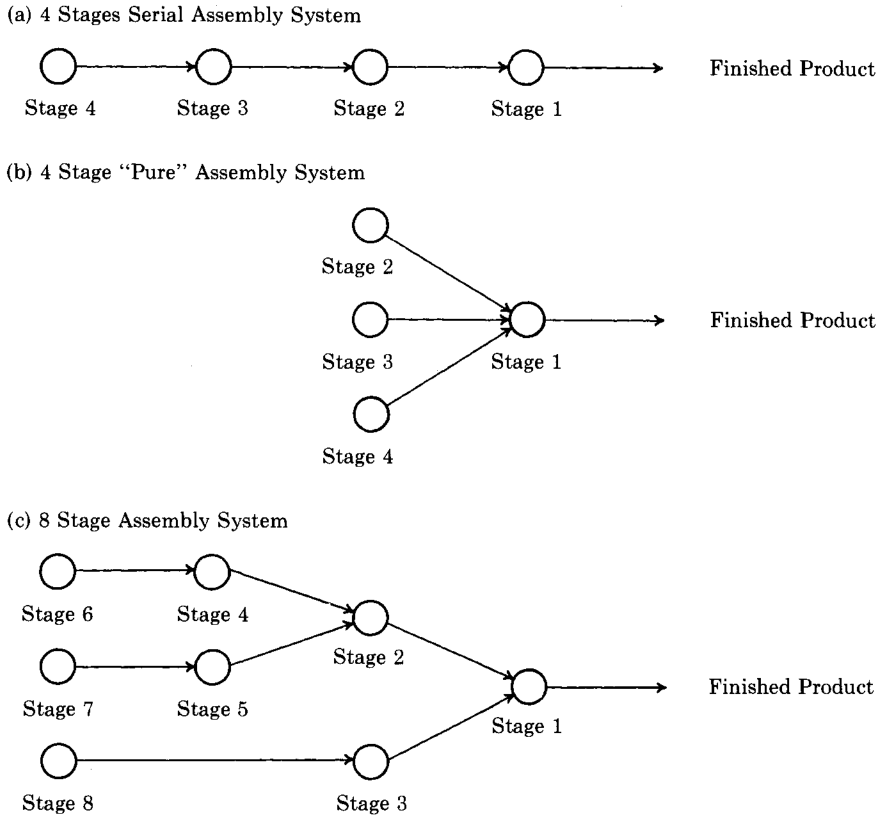

FIGURE 1

The use of echelon stock holding cost rates permits some very convenient mathematical simplifications. However, this would not be sufficient reason to use this concept unless it fit many real world situations. In fact, echelon holding cost rates can be defined for any assembly system where the installation holding cost rates are nondecreasing as the goods get closer to the customer; e.g., where value is added (or extra costs incurred) at each successive stage of the production/inventory process.

In recent years a number of algorithms have been developed for deterministic multi-echelon production/inventory assembly problems. Zangwill (1966, 1969), Veinott (1969), and Love (1972) present discrete time dynamic programming models under the assumptions of periodic review, finite time horizon, known but possibly varying demand, and concave costs. Love shows that under some conditions optimal periodic schedules can be determined for the periodic review infinite horizon serial problem when demands and costs are stationary. Kalymon (1972), using the Zangwill-Veinott approach, presents a decomposition algorithm which has been demonstrated to be computationally feasible for many problems.

There have been two relatively recent approaches to the deterministic, continuous review, infinite horizon problem. For the serial problem, Taha and Skeith (1970) consider fixed order or setup costs at each stage, linear holding costs, noninstantaneous production, delivery lags between stages, and backorders for the product at the final stage. They *assume* that in an optimal schedule the lot size at any given stage is an integer multiple of the lot size at its immediate successor stage (the integrality assumption) and suggest that the problem be solved by examining all combinations of such integer values. Under similar assumptions, but without backlogging, and using Clark and Scarf's (1960) concept of echelon stock, Crowston, Wagner, and Williams (1973) prove the optimality of the integrality assumption and present an algorithm which views the N stage assembly problem as an N stage dynamic programming problem with some appropriate computational refinements. Both of the above approaches assume that the lot size at stage 1 is an integer multiple of some basic unit, e.g., 1. The choice of the basic unit affects the computational difficulty. This assumption is not required in the model presented here; that is, Q_1, the stage 1 lot size, need not be integer valued. If there are situations in which one wishes this restriction, it is easily incorporated.

2. Problem Description

Consider an assembly production/inventory system with N stages numbered from 1 to N. Stage 1 is defined to be the final stage; it must provide output so that the outside customer demand, which occurs at a constant rate, is satisfied over an infinite horizon without backlogging or lost sales.

Define:

D = the demand rate at stage 1;
p_j = production or assembly rate at stage j when production is in progress ($p_j = \infty$ corresponds to instantaneous ordering);
$\rho_j = D/p_j$ for $j = 1, 2, \ldots, N$;
h'_j = holding cost per unit time charged against the echelon stock of stage j;
K_j = fixed cost of producing or ordering a batch at stage j;
$s(j)$ = the single immediate successor stage of stage j, $j = 2, \ldots, N$, $s(1) = \emptyset$; without loss we require $i > s(i)$;
$P(j)$ = the set of immediate predecessors of stage j; i.e., $\{k \mid s(k) = j\}$;
$A(j)$ = the set of all predecessor stages of stage j. For the serial system $A(j) = \{j+1, \ldots, N\}$; for the pure assembly system $A(1) = \{2, \ldots, N\}$, $A(j \neq 1) = \emptyset$. In Figure 1(c) $A(2) = \{4, 5, 6, 7\}$, $A(5) = \{7\}$, $A(8) = \emptyset$.
Q_j = lot size produced or ordered at stage j (to be determined);
$n_j = Q_j/Q_{s(j)}$ for $j = 2, \ldots, N$;
$m_j = n_j m_{s(j)}$ for $j = 2, \ldots, N$; $m_1 = 1$.

Crowston, Wagner, and Williams (1973) established that there exists an optimal policy in which $Q_j = n_j Q_{s(j)}$, $j = 2, \ldots, N$, where n_j is an integer. Thus, the optimal policy for the N stage problem is specified by a real positive number, Q_1, and $N - 1$ integers, n_j, $j = 2, \ldots, N$.

If one assumes that no lot splitting occurs in the shipping between stages, that is, no production is shipped from stage j to stage $s(j)$ until the entire batch is completed at j, and that shipping lead times are independent of batch sizes, then the optimization problem can be written as:

$$\text{(1)} \qquad \text{Minimize} \quad \textstyle\sum_{j=1}^{N} (K_j D/Q_j + h_j Q_j/2)$$

subject to

$$\text{(2)} \qquad Q_j = n_j Q_{s(j)} \quad \text{for} \quad j = 2, \ldots, N,$$

$$\text{(3)} \qquad n_j \geqq 1 \quad \text{and integer,}$$

where

$$h_j = (1 + \rho_j)h'_j + 2\rho_j \textstyle\sum_{k \in A(j)} h'_k . \tag{4}$$

See Appendix I for details.

If one requires Q_1 to be an integer multiple of some basic unit Q_0 (given), one simply appends the additional constraint $Q_1 = n_1 Q_0$, where n_1 is to be determined just as n_j for $j = 2, \ldots, N$.

One can make a number of useful observations about the problem when stated in the above form. First of all one can obtain an easily computed lower bound on the cost of the optimal solution by dropping constraints (2) and (3) and solving. This relaxed problem is solved by applying the standard EOQ formula at each stage. The optimal "continuous" Q_j's are then given by:

$$Q_j^c = (2K_j D/h_j)^{1/2} \quad \text{for} \quad j = 1, 2, \ldots, N. \tag{5}$$

The cost associated with this (possibly infeasible) solution is a lower bound on the cost of an optimal solution to problem (1)–(3). If the n_j's implied by the Q_j^c's are all positive integers, then of course the Q_j^c's are optimal for the original problem.

With slightly more effort a better bound may be obtained. We know that in any feasible solution Q_j must be at least as large as $Q_{s(j)}$. Therefore consider these constraints substituted for (2) and (3):

$$\text{Minimize} \quad \textstyle\sum_{j=1}^{N} (K_j D/Q_j + h_j Q_j/2) \tag{6}$$

$$\text{subject to} \quad Q_j \geqq Q_{s(j)} \quad \text{for} \quad j = 2, \ldots, N. \tag{7}$$

This is a problem with a convex objective function to be minimized over a convex set so any local minimum is also a global minimum. Denote the optimal batch sizes for this "constrained continuous" problem by Q_j^{cc} for $j = 1, 2, \ldots, N$. A simple procedure for solving (6)–(7) is:

(a) Solve (6) by itself; i.e., compute Q_j^c for $j = 1, 2, \ldots, N$.

(b) If all constraints (7) are satisfied, i.e., $Q_k^c \geqq Q_{s(k)}^c$ for all k, then stop; else find the largest i such that there is a k in $P(i)$ for which $Q_k^c < Q_i^c$. Find that j in $P(i)$ which minimizes Q_j^c and then modify (6)–(7) in the following fashion:

$$\begin{aligned} K_{s(j)} &\leftarrow K_{s(j)} + K_j ; \\ h_{s(j)} &\leftarrow h_{s(j)} + h_j ; \\ K_j &\leftarrow 0; \\ h_j &\leftarrow 0, \\ s(t) &\leftarrow i \text{ for any } t \text{ in } P(j) \text{ and} \\ P(i) &\leftarrow P(i) \cup P(j). \end{aligned}$$

All that step (b) does is select a constraint in (7) which is violated and force the violated constraint to hold at equality. This is done by "collapsing" stage j into its successor stage, $s(j)$. Steps (a) and (b) are similar in spirit to a procedure suggested by Geoffrion (1967).

To show that this procedure finds an optimal solution to (6)–(7), we must show that if a constraint in (7) is selected at any step in the procedure then it must hold as an equality in the optimal solution to (6)–(7).

PROOF. First we make three observations. They can be easily proven by analyzing the Kuhn-Tucker conditions for (6)–(7) (see Appendix II).

(i) If $Q_k^{cc} > Q_{s(k)}^{cc}$, then $Q_k^{cc} \leqq Q_k^c$;

(ii) If $Q_k^{cc} > Q_i^{cc}$ for all k in $P(i)$, then $Q_i^{cc} \geqq Q_i^c$;

(iii) For any k in the $P(i)$ selected in step (b), $Q_k^{cc} \geqq Q_k^c$.

Now assume the contrary of what is to be shown; that is, assume: (α) $Q_j^{cc} > Q_i^{cc}$ where j and i are as chosen in step (b). Now (α) and observation (i) imply: (β) $Q_j^{cc} \leqq Q_j^c$. By (iii), plus the manner of choosing j in step (b), and (β) we have: (γ) $Q_j^{cc} \leqq Q_k^{cc}$ for all k in $P(i)$. Now (α) and (γ) imply: (δ) $Q_k^{cc} > Q_i^{cc}$ for all k in $P(i)$. Combining (δ) and (ii) gives us: (θ) $Q_i^{cc} \geqq Q_i^c$. Now (β), (θ) and the fact that $Q_j^c < Q_i^c$ lead to the conclusion that $Q_j^{cc} < Q_i^{cc}$, which directly contradicts our initial assumption, (α). Therefore, we must have what was to be proven: $Q_j^{cc} = Q_i^{cc}$.

3. Optimal Integer Solutions

Optimal solutions to problem (1)–(3) can be found by a branch-and-bound procedure. There are $N - 1$ levels in the enumeration tree, corresponding to the number of n_j's which must be specified. If the $N - 1$ optimal values for the n_j's are specified then one need only determine the optimal value for Q_1 by applying the standard EOQ formula to the problem:

$$\text{Minimize}_{\text{w.r.t. } Q_1} \sum_{j=1}^{n} (K_j/m_j)/Q_1 + (Q_1/2) \sum_{j=1}^{n} m_j h_j . \tag{8}$$

This yields

$$Q_1 = [2D \sum_{j=1}^{n} (K_j/m_j) / \sum_{j=1}^{n} m_j h_j]^{1/2}. \tag{9}$$

Note that this corresponds to the standard EOQ where $\sum_{j=1}^{n} K_j/m_j$ is the average system setup cost per batch at stage 1 and $\sum_{j=1}^{n} m_j h_j$ is a composite system holding cost.

We will assume that the reader is familiar with the rudiments of branch-and-bound as described in, say, Lawler and Wood (1966). The search begins by solving problem (6)–(7) using the procedure described in §2. If the solution to (6)–(7) satisfies (3); that is, if $n_j = Q_j^{cc}/Q_{s(j)}^{cc}$ is integer for $j = 2, \ldots, N$, the optimal policy is at hand. Otherwise a noninteger n_j is chosen for branching. The branches at a particular level in the tree correspond to the possible integer values $(1, 2, \ldots)$ that can be realized by the particular n_j assigned to the level. An apparent theoretical difficulty with this tree structure is that it has an infinite number of branches. However, for practical purposes this poses no difficulty for reasons to be given below.

The Branching

When taking a branch at some level in the tree we set some variable n_j equal to some integer, say I_j, and require $Q_j = I_j Q_{s(j)}$. This will give us a condensed problem identical in form to problem (1)–(3) but without variables Q_j and n_j. The following parameter adjustments are made to (1) and (2):

$$\begin{aligned} K_{s(j)} &\leftarrow K_{s(j)} + K_j/I_j, \\ h_{s(j)} &\leftarrow h_{s(j)} + I_j h_j, \end{aligned} \tag{1'}$$

and for any k in $P(j)$ we replace the constraint

$$\begin{aligned} Q_k &= n_k Q_{s(k)} \quad \text{by} \\ Q_k &= n_k I_j Q_{s(j)} . \end{aligned} \tag{2'}$$

For any stage i such that $s(i) = j$, the constraint $Q_i \geqq Q_{s(i)}$ is modified to

$$Q_i \geqq I_j Q_{s(j)} \,. \tag{7'}$$

The bounds obtained are used in the standard fashion to limit the depth of the tree search.

The bounds can also be used to limit the breadth of the search because the bounds at a particular level in the tree *are quasiconvex* in n_j. For example, if at the level where we are branching on n_j we find that the minimum of the lower bound occurs at $n_j = 10$, then we need not examine any branches with $n_j > 15$, say, if the bound at $n_j = 15$ was sufficiently high so that the depth search could be stopped at that branch. Similar arguments would apply for $n_j < 6$, say, if the bound at $n_j = 6$ was sufficiently high to stop the depth search. We know that the bounds could only get worse for $n_j < 6$ or $n_j > 15$.

In order to show that the bounds are quasiconvex in n_j at a given level in the tree, consider the manner in which a new level is added to the enumeration tree. We solve the bounding problem at the previous level and then select a stage j for which the implied value for n_j is noninteger. Suppose that the implied noninteger value for n_j is f_j. Consider adding either of the following two constraints to the bounding problem just solved:

$$Q_j \geqq (f_j + k) Q_{s(j)} \tag{10}$$

or

$$Q_j \leqq (f_j - k) Q_{s(j)} \tag{11}$$

where k is strictly positive.

The fact that the objective function in the bounding problem is strictly convex in Q_j and $Q_{s(j)}$ implies that there must be unique minimizing values for Q_j and $Q_{s(j)}$. Therefore, if either (10) or (11) is added to the problem they will be binding. It follows that the minimal cost for the bounding problem with either constraint added is a nondecreasing function of k. This is true because the larger the value of k, the smaller is the set of feasible solutions. Therefore the bound computed at the level in the tree in which we branch on n_j is quasiconvex in n_j.

4. System Myopic Policies

Ignall and Veinott (1969) and others have suggested myopic policies for multiperiod optimization problems. Such policies optimize a given objective function with respect to the current period and ignore multiperiod interaction effects. Multistage planning systems permit a different type of nearsightedness, one which we call "system myopia." System myopic policies optimize a given objective function with respect to any two stages and ignore multistage interaction effects. The system myopic policy we chose to investigate determines the n_j values for problem (1)–(3) by considering each stage j and its $s(j)$, $j = 2, \ldots, N$, as a two-stage system. Schwarz (1973) has shown that the optimal integer $n_j = Q_j/Q_{s(j)}$ for such systems is the smallest integer n_j satisfying

$$n_j(n_j + 1) \geqq M_j \tag{12}$$

where M_j is the *myopia ratio* defined as

$$M_j = (K_j h_{s(j)}/K_{s(j)} h_j). \tag{13}$$

We shall denote these n values n_j^M, $j = 2, \ldots, N$. After the n_j^M are computed, Q_1 is computed using (9).

There is much to recommend the application of system myopic policies. First, the system myopic policy is trivially easy to determine when compared to the algorithm for determining the optimal n_j's. Second, the cost of the system myopic policy *may* be quite close to the cost of the optimal policy. In order to see this, note that the n_j^M represent one of the lattice points immediately surrounding the optimal solution to problem (1)–(2). That is, if we denote the set of optimal continuous n_j's as $n_j^c = Q_j^c/Q_{s(j)}^c = M_j^{1/2}$, where the Q_j^c are defined as in (5), it is easily shown that $| n_j^M - n_j^c | < 1$. Such proximity suggests that if we define $C(n_j)$ as the value of (1) for given values of n_j, $j = 2, \ldots, N$, and the correspondingly optimal Q_1 from (9), then $C(n_j^M)$ may be close to the value of $C(n_j^c)$, which is a lower bound on the value of $C(n_j^*)$ where n_j^* is the optimal n_j, $j = 2, \ldots, N$. Hence it follows that $C(n_j^M)$ *may* be close to $C(n_j^*)$, $j = 2, \ldots, N$. Moreover, since it can be shown that $(C(n_j^M) - C(n_j^c))/C(n_j^c)$ is in general a decreasing function of $(n_j^M - n_j^c)/n_j^c$, one can argue that the larger the M_j, the closer $C(n_j^M)$ is to $C(n_j^*)$. In other words, the larger the M_j's, the better the system myopic solution, all other things being equal.

Do we expect the M_j's to be large or small for problems with realistic parameters? If, as in many instances, the K_j's increase in j and the h_j's decrease in j, we may expect the M_j's to be larger than one, how much larger depending on the increase (decrease) in the setup (holding) costs at higher stages of the system.

In order to empirically test the goodness of system myopic policies we compared the cost of optimal and system myopic policies by computing

$$E = \frac{(C(n_j^M) - C(n_j^*))}{C(n_j^*)} \cdot 100 \tag{14}$$

for a number of test problems. In particular, 500 serial and pure assembly problems with $N = 3$, 4, and 5 stages each were generated. The closeness measure, E, which is the difference in cost between the optimal and the system myopic policy as a percentage of the cost of the optimal policy can be shown to be independent of K_1, h_1, and D. Therefore, in order to generate a problem it was only necessary to generate $K_2, \ldots, K_N$ and $h_2, \ldots, h_N$. We did so by randomly selecting values of $K_{s(j)}/K_j$ and $h_{s(j)}/h_j$ from the quantities 0.1, 0.5, 1, 2, 10. For each set of 500 problems the average

TABLE 1

Closeness of System Myopic Policies to Optimal

Stages	Number of Problems	$\bar{E}$	$E_{\max}$	σ	Optimal Frequency
			Serial Systems		
3	500	0.77	15.44	2.28	362 = 72.4%
4	500	2.61	35.05	5.80	262 = 52.4%
5	500	4.24	72.89	8.10	198 = 39.6%
			Assembly Systems		
3	500	1.05	20.76	2.63	333 = 66.6%
4	500	2.12	52.47	4.81	248 = 49.6%
5	500	4.06	62.10	8.21	153 = 36.0%

TABLE 2

Closeness to Optimality of System Myopic Policies
(all $M_j \geqq 1$)

Stages	Number of Problems	$\bar{E}$	E_{max}	σ	Optimal Frequency
			Serial Systems		
3	164	0.15	3.28	0.44	118 = 72.0%
4	106	0.21	4.88	0.55	56 = 52.8%
5	54	0.53	6.90	1.12	15 = 27.8%
			Assembly Systems		
3	215	0.02	0.62	0.07	172 = 80.0%
4	177	0.03	0.59	0.08	131 = 74.0%
5	170	0.09	1.65	0.25	84 = 49.4%

E, $\bar{E}$, the maximum E, E_{max}, the standard deviation σ, and the number of times the system myopic solution was optimal was determined. The results appear in Table 1. Computation time to determine both the optimal and system myopic policies averaged less than 0.2 seconds per problem on the IBM 360/65 computer.

As Table 1 shows, system myopic policies in general were close to optimal: the average error never exceeded 5% of the cost of the optimal policy and system myopic policies were optimal for about half of the problems generated.

If we consider the subset of problems for which all $M_j \geqq 1$, system myopic policies did even better, as is shown in Table 2. Note that the average error never exceeded 1%.

Although the above tests are by no means conclusive, they do provide evidence to support the hypothesis of the "near optimality" of system myopic policies, at least for systems with relatively few stages.

5. Summary and Conclusion

In this paper we have described optimal and near-optimal solutions for the deterministic N-stage assembly production/inventory problem. Although the largest amount of space in this paper was concerned with optimal policies, the potentially most important aspect of this paper is its concern in presenting theoretical and empirical evidence for the near optimality of system myopic policies which: (1) are easy to understand; (2) require less information; and (3) are fast and easy to compute. The implementation advantages of such policies are obvious.

Appendix I

In this appendix we will establish that the optimal policy for the N-stage assembly system corresponds to the solution to problem (1)–(3).

By arguments quite similar to those used for the one-stage EOQ problem, it can be proven that there exists an optimal policy for the N-stage assembly problem which is a stationary "cycling" policy; that is, a policy under which each stage in the system orders (produces) the same quantity each time that it orders (produces). See Schwarz (1973) for a more general discussion of cycling policies. Hence the average incremental cost per unit time (henceforth called average cost) for any stage j is the total cost accrued between orders (set-ups) divided by the time between orders (set-ups),

Q_j/D. The average cost of the system is then simply the sum of the average costs for all stages.

For the $N = 1$ stage system, the development of the average cost is straightforward. A cycle begins when $(Q_1/p_1)D = \rho_1 Q_1$ units are on hand. This inventory is used to satisfy the outside demand during the time it takes to complete the production of Q_1 units, Q_1/p_1. When production begins, stage 1's inventory increases at the rate $(p_1 - D)$, reaches a maximum of $(1 - \rho_1)Q_1 + \rho_1 Q_1$, at which time production ceases, and declines at the rate D until the cycle ends with $\rho_1 Q_1$ units on hand Thus the average cost for a one-stage system is $K_1 D/Q_1 + h_1 Q_1/2$ where $h_1 = (1 + \rho_1)h'_1$.

Similarly, when there are no delivery lags a cycle for any stage in an N-stage system begins with an installation stock of $\rho_j Q_j$ and a corresponding echelon stock of $\rho_j Q_j + \sum_{k\in S(j)} \rho_k Q_k$, where $S(j)$ is the set of *all* successors to stage j; that is, $S(j) = \{s(j), s(s(j)), \ldots, 1\}$. This stock grows at the rate $(p_j - D)$ until production ceases, at which time it declines at the rate D. Hence the average cost for stage j is $K_j D/Q_j + h'_j(\rho_j Q_j + \sum_{k\in S(j)} \rho_k Q_k + (1 - \rho_j)Q_j/2)$.

Therefore the average cost for the N-stage system is

$$\sum_{j=1}^{n} \{K_j D/Q_j + h'_j(1 - \rho_j)Q_j/2 + h'_j \rho_j Q_j + \sum_{k\in S(j)} h'_j \rho_k Q_k\}. \tag{15}$$

After rearranging (15) and substituting (4) we obtain (1). Constraints (2) and (3) follow directly for the integrality assumption established by Crowston, *et al.*

The above development was based on the assumption that the delivery lag between stage j and $s(j)$, say l_j, equals zero for all $j = 2, \ldots, N$. However, fixed delivery lags have no incremental effect on costs because a given $l_j > 0$ merely requires an additional average pipeline inventory between j and $s(j)$ of $l_j D$ units, which increases average costs by the constant $h'_j l_j D$.

Appendix II

Let us write the Kuhn-Tucker conditions associated with problem (6)–(7). Let λ_k be the Lagrange multiplier associated with the constraint $Q_k \geqq Q_{s(k)}$ for $k = 2, 3, \ldots, N$. The Q_k^{cc}'s, for $k = 1, 2, \ldots, N$, must satisfy the Kuhn-Tucker conditions:

$$-K_k D/Q_k^2 + h_k/2 - \lambda_k + \sum_{i\in P(k)} \lambda_i = 0, \tag{iv}$$

$$Q_k - Q_{s(k)} \geqq 0, \tag{v}$$

$$\lambda_k \geqq 0, \tag{vi}$$

$$\lambda_k(Q_k - Q_{s(k)}) = 0. \tag{vii}$$

The values Q_k^c are simply the values obtained for the Q_k's in (iv) when we set all the λ's $= 0$ and disregard (v).

First consider observation (i), namely, if $Q_k^{cc} > Q_{s(k)}^{cc}$ then $Q_k^{cc} \leqq Q_k^c$. Now $Q_k^{cc} > Q_{s(k)}^{cc}$ implies by (vii) that $\lambda_k = 0$. But $\sum_{i\in P(k)} \lambda_i \geqq 0$ in (iv) which implies that $Q_k^{cc} \leqq Q_k^c$.

Observation (ii) is: if $Q_i^{cc} > Q_k^{cc}$ for all i in $P(k)$, then $Q_k^{cc} \geqq Q_k^c$. Now $Q_i^{cc} > Q_k^{cc}$ for all i in $P(k)$ implies by (vii) that $\lambda_i = 0$ for all i in $P(k)$. But $-\lambda_k \leqq 0$ in (iv), which implies that $Q_k^{cc} \geqq Q_k^c$.

Observation (iii) is that for any k in the $P(i)$ selected in step (b) of the algorithm we must have $Q_k^{cc} \geqq Q_k^c$. Suppose $Q_k^{cc} < Q_k^c$ for some k in $P(i)$ as selected in step (b) of the algorithm. Consider an alternative solution wherein $Q_k^{cc} = Q_k^c$ and $Q_j^{cc} = Q_j^c$ for all $j \in P(k)$. By construction and the selection of i in step (b), the alternative solution satisfies (7). Moreover, the alternative solution has a lower cost, so $Q_k^{cc} < Q_k^c$ cannot be true.

References

1. Clark, A. J. and Scarf, H., "Optimal Policies for a Multi-Echelon Inventory Problem," *Management Science*, **6** (July 1960), 475–490.
2. ——, "An Informal Survey of Multi-Echelon Inventory Theory," *Nav. Res. Log. Quart.*, **19** (Dec. 1972), 621–650.
3. Crowston, W. B., Wagner, M. and Williams, J. F., "Economic Lot Size Determination in Multi-Stage Assembly Systems," *Management Science*, **19** (January, 1973), 517–527.
4. Geoffrion, A. M., "Reducing Concave Programs with Some Linear Constraints," *SIAM J. Appl. Math.*, **15** (May 1967), 653–664.
5. Ignall, E. J. and Veinott, A. F. Jr., "Optimality of Myopic Inventory Policies for Several Substitute Products," *Management Science*, **15** (January 1969), 284–304.
6. Kalymon, B. A., "A Decomposition Algorithm for Arborescence Inventory Systems," *Operations Research*, **20** (July–August 1972), 860–874.
7. Lawler, E. L. and Wood, D. E., "Branch and Bound Methods: A Survey," *Operations Research*, **14** (July–August 1966), 699–719.
8. Love, S., "A Facilities in Series Inventory Model with Nested Schedules," *Management Science*, **18** (January 1972), 327–338.
9. Schwarz, L. B., "A Simple Continuous Review Deterministic One-Warehouse N-Retailer Inventory Problem," *Management Science*, **19** (January 1973), 555–566.
10. Taha, H. A. and Skeith, R. W., "The Economic Lot Sizes in Multi-Stage Production Systems," *AIIE Transactions*, (June 1970), 157–162.
11. Veinott, A. F., Jr., "Minimum Concave Cost Solution of Leontief Substitution Models of Multi-Facility Inventory Systems," *Operations Research*, **17** (March–April 1969), 262–291.
12. Zangwill, W. I., "A Deterministic Multi-Period Multi-Facility Production and Inventory Model," *Operations Research*, **14** (May–June 1966), 486–507.
13. ——, "A Backlogging Model and a Multi-Echelon Model of a Dynamic Economic Lot Size Production System—A Network Approach," *Management Science*, **15** (May 1969), 506–527.