Optimal Approximation of Uniformly Rotated Images: Relationship Between Karhunen–Loeve Expansion and Discrete Cosine Transform

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***Abstract*—We will present in this correspondence that for uniformly rotated images, the optimal approximation of the images can be obtained by computing the basis vectors for the discrete cosine transform (DCT) of the original image in polar coordinates, and representing the images as linear combinations of the basis vectors.**

***Index Terms*—Discrete cosine transform, Karhunen–Loeve expansion.**

I. Introduction

There are many problems in the computer vision area where we apply convolution of an image with kernels at multiple orientations. A typical case is that of filtering in early vision stage. Template matching is another example of a problem where convolution of an image with rotated kernels is applied. In template matching, we use a collection of desired image patterns or "templates" to search for objects of interest within a given image. In this search, we use correlation as our measure of similarity. Unfortunately, correlation is sensitive to target orientation. Therefore, when the orientation of a desired target is unknown, we must prepare templates at multiple orientations and compute the correlation with each of these rotated templates at each point of the image. Since the computational expense associated with such a computation increases linearly with the number of orientations considered, true robustness with regard to target rotation is achieved only at enormous computational expense.

A technique that greatly alleviates this difficulty is to compute the best approximation of a given family consisting of the rotated images of the original image using linear combinations of a small number of basis functions [1]–[5]. The computational cost can be reduced by convolving an image with a small number of basis functions and taking linear combinations of the results. The basis functions are orthonormal sequences of functions and can be computed by singular value decomposition (SVD).

We show an interesting relationship between the eigenvectors by Karhunen–Loeve (K–L) expansion and basis vectors for the DCT. In the case of in-plane rotation, the basis vectors for the DCT of the original image in polar coordinates become the eigenvectors. The eigenvectors can be generated much more efficiently, which makes it possible to detect a target in the image, generate the approximation of its rotated images, and keep tracking the target moving and changing orientation on the fly. One possible application is aerial photo reconnaissance. The relationship between the DCT and the K–L expansion derived in this work is different from the similarity between them, which has been pointed out [5], [6], [8]. They are dependent on the fact that the autocovariance matrix is a symmetric Toeplitz matrix for a stationary Markov-1 signal. The set of data we deal with are uniformly rotated images of a pattern. They are highly correlated and the data covariance matrix does not become a Toeplitz matrix.

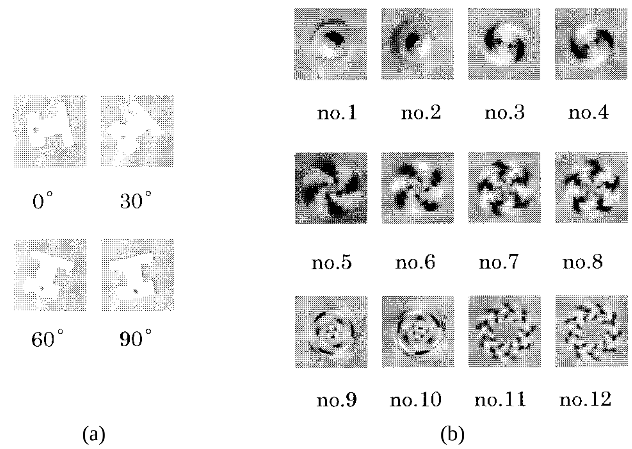

Fig. 1. Examples of eigenvectors. (a) Original image and some rotated images (30, 60, and 90°). (b) Major eigenvectors (no. 1–12).

We will show that the vector inner product matrix of uniformly rotated images becomes a symmetric periodic Toeplitz matrix. Because periodic Toeplitz matrices are circulant matrices, their eigenvectors are always the same regardless of the specific form of the matrices. We prove that cosine basis are their eigenvectors. This leads to the relationship between the eigenvectors derived from K–L expansion and the basis vectors for DCT.

The result presented in this work is strongly related to the steerable approximation derived by Perona [1], [2]. Perona [1] and Adelson [3] have shown that steerable filters allow one to use a small set of filters and still extract information corresponding to all orientations in a uniform way. Perona proved that the optimal kernels are the Fourier transform of the filter impulse response in polar coordinates. The proof is based on the SVD of the filter impulse response and the fact that the deformation involved (the rotations) is a group.

Perona's result and our result are very similar, except that his result is for the continuous case while our case is discrete. We start from some number of uniformly rotated images and derive the result based on the fact that their vector inner product matrix becomes the periodic and symmetric Toeplitz matrix. In the case of template matching, when object orientation is unknown but always limited to one out of a certain number of degrees, our result shows that it is sufficient to compute basis vectors from the rotated target object images once every certain number of degrees. This situation obtains industrial applications where parts are positioned by tools or parts feeders.

This correspondence is organized as follows. A finite-sum approximation of rotated images is explained in Section II. In Section III, the vector inner product matrix of uniformly rotated images is shown to be the periodic and symmetric Toeplitz matrix, and in Section IV, it is proved that the basis vectors for the DCT of the original image become the eigenvectors with an experimental result.

II. Uniformly Rotated Images by a Finite-Sum Linear Combination of Basis Functions

In this section, we will explain the formalism of the approximation of rotated images by a finite-sum linear combination of a smaller number of basic functions.

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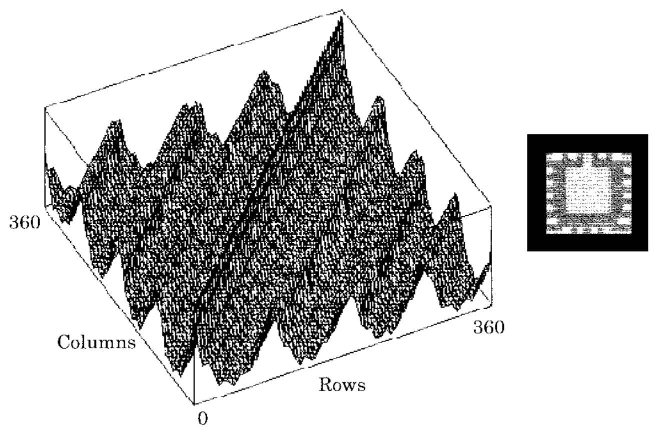


Fig. 2. (a) Example of 3-D representation of the vector inner product matrix. The z coordinate gives the element value and the x and y coordinates are the rows and the columns. The number of rotated images P is 360 (0–360°). (b) Original images used for computing the vector inner product matrix in (a).

Given a template image x_o, uniformly rotated images x_i, $i = 1, \cdots, (P-1)$ are generated by rotating the original image x_o by $(2\pi)/P$, $(4\pi)/P, \cdots, 2\pi(P-1)/P$ rad, respectively. We calculate the average $c = (1/P)\sum_{i=0}^{P-1} x_i$ of the rotated images, subtract it from all the rotated images, and normalize them to unit energy, i.e., $x_i^T x_i = 1$. The next step is to compute the covariance matrix as follows:

$$\mathbf{A} = \frac{1}{P}\sum_{i=0}^{P-1} x_i x_i^T. \tag{1}$$

SVD gives us the eigenvectors ϕ_j and eigenvalues $\lambda_j(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{P-1})$ of the matrix $\mathbf{A}$, and we obtain the optimal approximation of the rotated images by selecting the eigenvectors in decreasing order of magnitude of the eigenvalues and representing each image by a linear combination of major K eigenvectors as

$$x_i \approx \sum_{j=1}^{K} p_{ij}\phi_j \tag{2}$$

where $p_i = [\phi_1\phi_2 \cdots \phi_K]^T x_i$. Fig. 1 shows an original image and some rotated images. Major eigenvectors computed from them are displayed in Fig. 1(b). Correlation between the above rotated images and the portion of the image y is now formulated as

$$x_i \cdot y = \sum_{j=1}^{K} p_{ij}(\phi_j \cdot y). \tag{3}$$

The correlation computation is carried out by calculating the correlation between K eigenvectors and the portion of the image y, and taking the linear combination of the results. Because the computational cost for the linear combination is negligible compared to that for correlation, total cost is reduced to K/P.

III. Vector Inner Product Matrix of Rotated Images

As explained in the last section, we have the set of P training images x_i, $i = 0, 1, \cdots, (P-1)$ obtained by rotating the original image by 0, $(2\pi/P)$, $4\pi/P, \cdots, 2\pi(P-1)/P$ rad, respectively. For this case, the m, nth element of the $P \times P$ vector inner product matrix $\mathbf{R} = X^T X$ depends only on the magnitude of the difference $|(m-n)|$ [9].

$$\mathbf{R} = X^T X = \begin{bmatrix} x_0^T \\ x_1^T \\ \cdots \\ x_{P-1}^T \end{bmatrix} [x_0\, x_1 \cdots x_{P-1}] = \begin{bmatrix} x_0^T x_0 & x_0^T x_1 & \cdots & x_0^T x_{P-1} \\ x_1^T x_0 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ x_{P-1}^T x_0 & x_{P-1}^T x_1 & \cdots & x_{P-1}^T x_{P-1} \end{bmatrix}. \tag{4}$$

Proof: Let us denote the rotation matrix T_0^1 that rotates images by $(2\pi)/N$ rad.

$$T_0^1 x_i = x_{i+1}. \tag{5}$$

This is the matrix that translates each pixel from one location in the image to another. This matrix represents rotation and should not depend on each image, which gives

$$x_i = T_0^1 x_{i-1} = T_0^1 T_0^1 x_{i-2} = \cdots = T_0^{1i} x_0. \tag{6}$$

This relationship obviously leads to the following relationship between images:

$$x_0^T x_1 = x_1^T x_2 = x_2^T x_3 = x_i x_{i+1} \tag{7}$$
$$x_i^T x_j = x_{i+k}^T x_{j+k}. \tag{8}$$

It is assumed that the locations of pixels change with rotation, but no pixel disappears or appears. This is true in the continuous case and with a black background (or perfect segmentation). The real image data is discrete, so the relationship above is not accurately satisfied. However, the experimental results show that it is approximately satisfied.

An example of the vector inner product matrix is displayed in Fig. 2. In Fig. 2, the vector inner product matrix for the chip image [see Fig. 2(b) is represented in a three-dimensional (3-D) form. The x and y axes represent the rows and columns of the matrix and the z axis represents the element values. The main diagonal has values of unity because the training vectors are normalized. The elements at the locations with the same $|(m-n)|$ have the same value. This gives the matrix $\mathbf{R}$ the properties of being Toeplitz and symmetric. The property of being periodic results from the correlation repeating its values in reverse order as the angular difference goes beyond 180°. Thus, the matrix $\mathbf{R}$ is in periodic Toeplitz form.

IV. Eigenvectors of Uniformly Rotated Images

Because periodic Toeplitz matrices are circulant matrices, their eigenvectors are always the same regardless of the specific form of the matrix. Therefore, in their eigenvector/eigenvalue equation

$$\mathbf{R}\overline{\phi}_k = \overline{\lambda}_k\overline{\phi}_k \tag{9}$$

the only features distinguishing one periodic Toeplitz matrix from another are its eigenvalues.

When $c_k = x_i x_{i-k}$ is autocorrelation of the original image with respect to in-plane rotation, we can rewrite the matrix $\mathbf{R}$ as

$$\mathbf{R} = \begin{bmatrix} c_0 & c_1 & \cdots & c_{P-1} \\ c_{P-1} & c_0 & \cdots & c_{P-2} \\ \cdots & \cdots & c_0 & \cdots \\ c_1 & c_2 & \cdots & c_0 \end{bmatrix}. \tag{10}$$

Since $c_k = c_{P-k}$, the matrix $\mathbf{R}$ is real symmetric as well as a circulant matrix as mentioned above. Equation (9) can be equivalently represented as the P difference equations

$$\sum_{l=0}^{m-1} c_{P-m+l}\phi_{kl} + \sum_{l=m}^{P-1} c_{l-m}\phi_{kl} = \lambda_k \phi_{km}. \tag{11}$$

By direct substitution, it is easily verified that for any $m = 0, 1, \cdots, P-1$, that $\phi_{km} = \cos[2\pi m(k/p)]$ are solutions to (11), resulting in the eigenvalues

$$\lambda_k = \sum_{l=0}^{P-1} c_l \cos\left(2\pi k \frac{l}{P}\right). \tag{12}$$

The covariance matrix of images is represented as $\mathbf{A} = XX^T$. With the relationship between the eigenvalues/eigenvectors of XX^T and $X^T X$ [7], the eigenvectors of the covariance matrix are represented as

$$\phi_k = \lambda_k^{-(1/2)}[x_0 \cdots x_{P-1}] \begin{bmatrix} \overline{\phi}_{k0} \\ \overline{\phi}_{k1} \\ \cdots \\ \overline{\phi}_{kP-1} \end{bmatrix} = \frac{1}{\sqrt{\lambda_k}} \sum_{i=0}^{P-1} \overline{\phi}_{ki} x_i$$
$$= \frac{1}{\sqrt{\lambda_k}} \sum_{i=0}^{P-1} \cos\left(2\pi k \frac{i}{P}\right) x_i. \tag{13}$$

It can also be shown that $\lambda_k = \lambda_{P-k}$ as follows:

$$\lambda_{P-k} = \sum_{l=0}^{P-1} c_l \cos\left(2\pi l \frac{P-k}{P}\right)$$
$$= \sum_{l=0}^{P-1} c_l \cos\left(2\pi l \frac{k}{P}\right) = \lambda_k. \tag{14}$$

The above results show that the basis vectors for the DCT of the original image in polar coordinates are the eigenvectors. The basis vectors are chosen in decreasing order of magnitude of the eigenvalues, which are the discrete cosine series coefficients of the autocorrelation c_l.

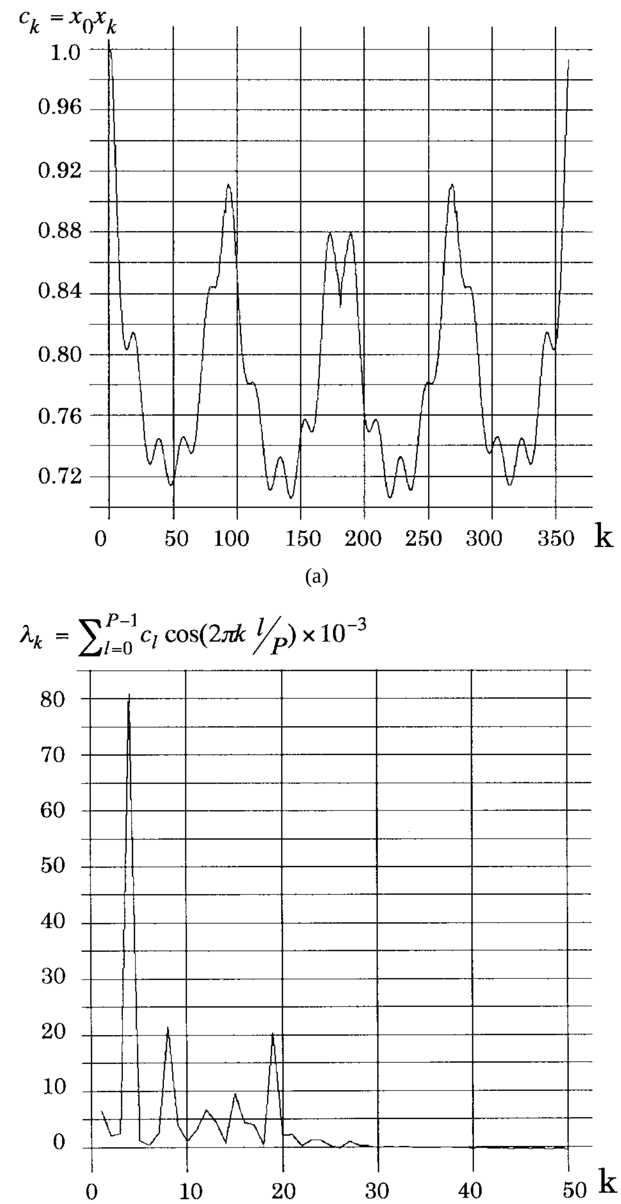
The K basis vector calculations for approximation of P rotated images are summarized as follows.

1) Compute the autocorrelation c_l of the rotated images.
2) Compute the DCT of c_l by (12). Order them by decreasing magnitude and call them λ_k.
3) Compute the basis vectors for the DCT of the rotated images by (13) whose corresponding λ_k are the highest K.

We have conducted an experiment to show that the basis vectors for the DCT become the eigenvectors. Autocorrelation $c_0, c_1, \cdots, c_{P-1}$ and its discrete cosine transform are displayed in Fig. 4. The number of data P is 360. The first 50 coefficients except for the DC component are shown in Fig. 3(b). Fig. 3(b) indicates that there are three major coefficients, $k = 4$, $k = 8$, and $k = 19$, which means there are three pairs of eigenvectors. In Fig. 4, the three major eigenvectors ϕ_j of the vector inner product matrix $\mathbf{R}$ are shown. They are cosines. There are pairs of eigenvectors of the same frequency, e.g., numbers 1, 2, 3, and 4, etc., which corresponds to (14). The eigenvectors in Fig. 4 have some offset phase. We can easily show that the basis vectors for the DCT with some offset phase Δ_k

$$\phi_{km} = \cos\left(2\pi m \frac{k}{N} + \Delta_k\right) \tag{15}$$

are also the eigenvectors with the same eigenvalues λ_k in (12).



(b)

Fig. 3. (a) Autocorrelation of the image shown in Fig. 2(b) with respect to rotation. (b) DCT of the autocorrelation above. The number of rotated images P is 360 (0–360°). Only the low-frequency part is displayed.

V. Conclusions

We have shown that the optimal approximation of uniformly rotated images is given by the basis vectors for the DCT of the original image in polar coordinates. The derived result comes from the fact that their vector inner product matrix becomes the periodic and symmetric Toeplitz matrix. Although the basis vectors can also be obtained by K–L expansion of the uniformly rotated images, DCT makes it possible to compute basis vectors much more efficiently.

Perona's result and our result are very similar, except that his result is for the continuous case while our case is discrete. We start from a certain number of uniformly rotated images and derive the result based on the fact that their vector inner product matrix becomes the periodic and symmetric Toeplitz matrix. In the case of template matching, when object orientation is unknown but always limited to one out of a certain number of degrees, our result shows that it

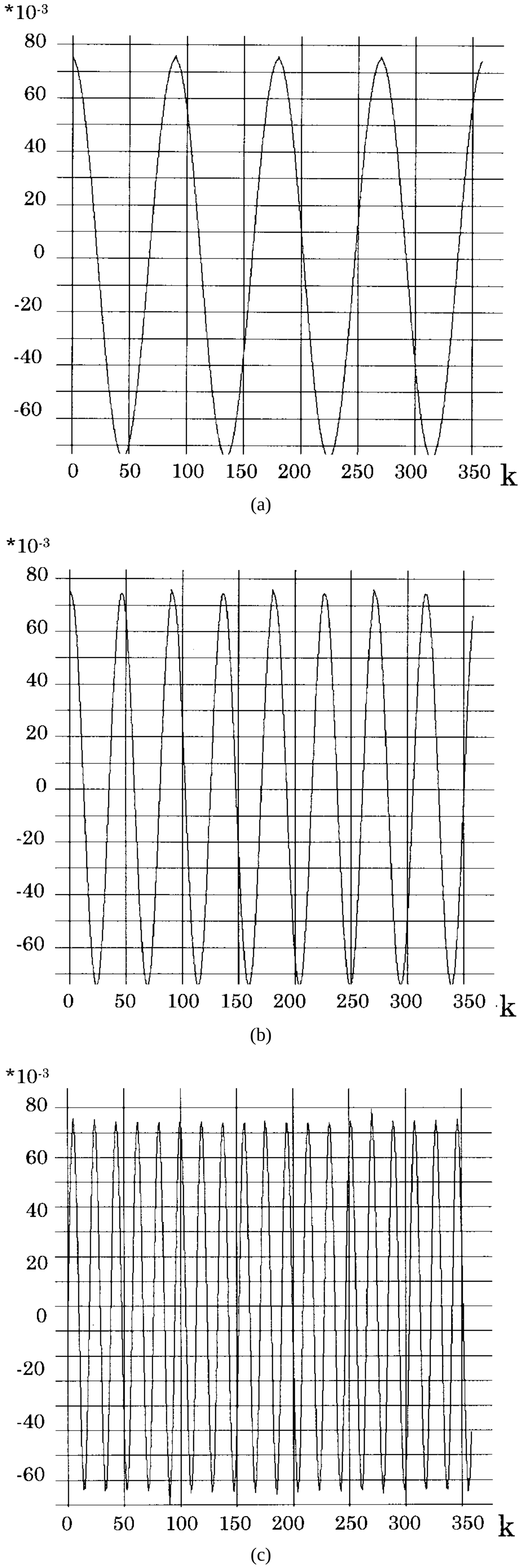


Fig. 4. Major eigenvectors of the rotated images. The original image is Fig. 2(b). The number of the rotated images P is 360.

is sufficient to compute basis vectors from the rotated target object images once every such certain number of degrees. This situation obtains industrial applications where parts are positioned by tools or parts feeders.

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A Vector Quantizer for Image Restoration

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***Abstract*— This paper presents a novel technique for image restoration based on *nonlinear interpolative vector quantization* (NLIVQ). The algorithm performs nonlinear restoration of diffraction-limited images concurrently with quantization. It is trained on image pairs consisting of an original image and its diffraction-limited counterpart. The discrete cosine transform is used in the codebook design process to control complexity. Simulation results are presented that demonstrate improvements in visual quality and peak signal-to-noise ratio of the restored images.**

***Index Terms*—Image restoration, nonlinear image processing, nonlinear interpolation, vector quantization.**

I. Introduction

Vector quantization (VQ) is another name for what Shannon called block source coding subject to a fidelity criterion [1]. Coding of this type maps consecutive, usually nonoverlapping, segments of input data to their best matching entry in a codebook of reproduction vectors. In the context of image coding, VQ is generally considered a data compression technique. However, VQ algorithms have been presented that perform other signal processing tasks concurrently with

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