The Annals of Applied Probability
2011, Vol. 21, No. 6, 2191–2225
DOI: 10.1214/10-AAP755

OPTIMAL ARBITRAGE UNDER MODEL UNCERTAINTY

BY DANIEL FERNHOLZ AND IOANNIS KARATZAS[1,2]

Daniel Fernholz LLC and INTECH Investment Management

Dedicated to Professor Mark H. A. Davis on the occasion of his 65th birthday

In an equity market model with "Knightian" uncertainty regarding the relative risk and covariance structure of its assets, we characterize in several ways the highest return relative to the market that can be achieved using nonanticipative investment rules over a given time horizon, and under any admissible configuration of model parameters that might materialize. One characterization is in terms of the smallest positive supersolution to a fully nonlinear parabolic partial differential equation of the Hamilton–Jacobi–Bellman type. Under appropriate conditions, this smallest supersolution is the value function of an associated stochastic control problem, namely, the maximal probability with which an auxiliary multidimensional diffusion process, controlled in a manner which affects both its drift and covariance structures, stays in the interior of the positive orthant through the end of the time-horizon. This value function is also characterized in terms of a stochastic game, and can be used to generate an investment rule that realizes such best possible outperformance of the market.

1. Introduction. Consider an equity market with asset capitalizations $\mathfrak{X}(t) = (X_1(t), \dots, X_n(t))' \in (0,\infty)^n$ at time $t \in [0,\infty)$, and with covariance and relative risk rates $\alpha(t,\mathfrak{X}) = \{\alpha_{ij}(t,\mathfrak{X})\}_{1\le i,j\le n}$ and $\vartheta(t,\mathfrak{X}) = (\vartheta_1(t,\mathfrak{X}),\dots,\vartheta_n(t,\mathfrak{X}))'$, respectively. At any given time t, these rates are *nonanticipative functionals* of past-and-present capitalizations $\mathfrak{X}(s)$, $0 \le s \le t$; they are not specified with precision but are, rather, subject to "Knightian uncertainty." To wit, for a given collection

$$\mathbb{K} = \{\mathcal{K}(\mathbf{y})\}_{\mathbf{y}\in\mathfrak{S}_n}, \qquad \mathfrak{S}_n := [0,\infty)^n \setminus \{\mathbf{0}\} \tag{1.1}$$

of nonempty compact and convex subsets on $\mathbb{R}^n \times \mathbb{S}^n$, where $\mathbb{S}^n$ is the space of real, symmetric, positive definite $(n\times n)$ matrices, and $\mathbf{0}$ is the origin in $\mathbb{R}^n$, they are subject to the constraint

$$(\vartheta(t,\mathfrak{X}), \alpha(t,\mathfrak{X})) \in \mathcal{K}(\mathfrak{X}(t)) \qquad \text{for all } t \in [0,\infty). \tag{1.2}$$

Received May 2010; revised December 2010.

[1]This author is on partial leave from the Department of Mathematics, Columbia University, New York, New York 10027, USA (E-mail: ik@math.columbia.edu).

[2]Supported by NSF Grant DMS-09-05754.

MSC2010 subject classifications. Primary 60H10, 91B28; secondary 60G44, 35B50, 60J70.

Key words and phrases. Robust portfolio choice, model uncertainty, arbitrage, fully nonlinear parabolic equations, minimal solutions, maximal containment probability, stochastic control, stochastic game.

In other words, the pair (ϑ, α) must take values at time t inside the compact, convex set $\mathcal{K}(\mathfrak{X}(t))$ which is determined by the current location of the asset capitalization process; but within this range, the actual value $(\vartheta(t, \mathfrak{X}), \alpha(t, \mathfrak{X}))$ is allowed to depend on past capitalizations as well. [To put it a little differently: the constraint (1.2) is not necessarily "Markovian," as long as the sets in (1.1) are not singletons.]

Under these circumstances, what is the highest return on investment relative to the market that can be achieved using nonanticipative investment rules, and with probability one under all possible market model configurations that satisfy the constraints of (1.2)? What are the weights in the various assets of an investment rule that accomplishes this?

Answers: Subject to appropriate conditions, $1/U(T, \mathfrak{X}(0))$ and

$$X_i(t) D_i \log U\big(T - t, \mathfrak{X}(t)\big) + \frac{X_i(t)}{X_1(t) + \cdots + X_n(t)}, \qquad i = 1, \ldots, n, 0 \le t \le T, \tag{1.3}$$

respectively. Here the function $U : [0, \infty) \times (0, \infty)^n \to (0, 1]$ is the smallest nonnegative solution, in the class $\mathcal{C}^{1,2}$, of the fully nonlinear parabolic partial differential inequality

$$\frac{\partial U}{\partial \tau}(\tau, \mathbf{z}) \ge \widehat{\mathcal{L}} U(\tau, \mathbf{z}), \qquad (\tau, \mathbf{z}) \in (0, \infty) \times (0, \infty)^n \tag{1.4}$$

subject to the initial condition $U(0, \cdot) \equiv 1$, with

$$\widehat{\mathcal{L}} f(\mathbf{z}) = \sup_{a \in \mathcal{A}(\mathbf{z})} \sum_{i=1}^{n} \sum_{j=1}^{n} z_i z_j a_{ij} \left(\frac{1}{2} D_{ij}^2 f(\mathbf{z}) + \frac{D_i f(\mathbf{z})}{z_1 + \cdots + z_n} \right). \tag{1.5}$$

We use in (1.3), (1.5) and throughout this paper, the notation $D_i f = \partial f / \partial x_i$, $D_{ij}^2 f = \partial^2 f / \partial x_i \, \partial x_j$, $Df = (D_1 f, \ldots, D_n f)'$, $D^2 f = \{D_{ij}^2 f\}_{1 \le i,j \le n}$ and define

$$\mathcal{A}(\mathbf{y}) := \{a \in \mathbb{S}^n : (\theta, a) \in \mathcal{K}(\mathbf{y}), \text{ for some } \theta \in \mathbb{R}^n\}, \qquad \mathbf{y} \in \mathfrak{S}_n. \tag{1.6}$$

We call the function $U(\cdot, \cdot)$ the *arbitrage function,* as $U(T, \mathbf{x})(x_1 + \cdots + x_n)$ gives the smallest initial capital starting with which an investor, who uses nonanticipative investment rules, can match or outperform the market portfolio by time $t = T$, if the initial configuration of asset capitalizations is $\mathfrak{X}(0) = \mathbf{x} = (x_1, \ldots, x_n)' \in (0, \infty)^n$ at $t = 0$, and does so with probability one under any "admissible" market configuration that might materialize. It is perhaps worth noting that this function $U(\cdot, \cdot)$ is characterized almost entirely in terms of the prevalent covariance structure α. The relative risk ϑ enters only indirectly, namely, in determining the family of sets (1.6) which are admissible for the covariance structure. Put a bit differently, the only role ϑ plays is to ensure the asset capitalization process $\mathfrak{X}(\cdot) = (X_1(\cdot), \ldots, X_n(\cdot))'$ takes values in $(0, \infty)^n$.

Under additional regularity conditions, $U(T, \mathbf{x})$ is the value of a stochastic control problem: the maximal probability that the diffusion process $\mathfrak{Y}(\cdot) =$

$(Y_1(\cdot), \dots, Y_n(\cdot))'$ with initial configuration $\mathfrak{Y}(0) = \mathfrak{X}(0) = \mathbf{x} \in (0, \infty)^n$, values in the punctured nonnegative orthant $\mathfrak{S}_n$ of (1.1), infinitesimal generator

$$\sum_{i=1}^{n} \sum_{j=1}^{n} y_i y_j \mathrm{a}_{ij}(t, \mathbf{y}) \left(\frac{1}{2} D_{ij}^2 f(\mathbf{y}) + \frac{D_i f(\mathbf{y})}{y_1 + \cdots + y_n} \right)$$

and controlled through the choice of covariance function $\mathrm{a} : [0, \infty) \times \mathfrak{S}_n \to \mathbb{S}^n$ which satisfies $\mathrm{a}(t, \mathbf{y}) \in \mathcal{A}(\mathbf{y})$ for all $(t, \mathbf{y}) \in [0, \infty) \times \mathfrak{S}_n$, does not hit the boundary of the orthant $[0, \infty)^n$ by time $t = T$. Under appropriate conditions the function $U(\cdot, \cdot)$ satisfies then, in the notation of (1.5), the *Hamilton–Jacobi–Bellman* (*HJB*) *equation*

$$(\partial/\partial\tau) U(\tau, \mathbf{z}) = \widehat{\mathcal{L}} U(\tau, \mathbf{z}) \qquad \text{on } (0, \infty) \times (0, \infty)^n. \tag{1.7}$$

Relation to extant work: Stochastic control problems of the "maximal probability of containment" type were apparently pioneered by Van Mellaert and Dorato (1972) (see also Fleming and Rishel (1975), pages 157–158). The "Knightian uncertainty" constraint imposed in (1.2) is very similar to the formulation of stochastic control and stochastic game problems for one-dimensional diffusions pioneered by William Sudderth, that Sudderth and his collaborators developed in a series of articles that includes Pestien and Sudderth (1985), Heath et al. (1987), Orey, Pestien and Sudderth (1987), Sudderth and Weerasinghe (1989); indeed, the developments in Sections 6–8 of our paper can be construed as a multidimensional extension of the Sudderth approach.

We rely strongly on Krylov's (1989, 2002) work, which studies solutions of stochastic differential equations with constraints on the drift and diffusion coefficients in terms of "supermartingale problems" and characterizes sets of stochastic integrals via appropriate supermartingales.

The approach we adopt has a lot in common with the effort, started in the mid-1990s, to understand option pricing and hedging in the presence of uncertainty about the underlying volatility structure of assets. We have been influenced by this strand of work, particularly by the papers of Lyons (1995), Romagnoli and Vargiolu (2000), Gozzi and Vargiolu (2002), Vargiolu (2001), Talay and Zheng (2002); other important papers include Avellaneda, Lévy and Parás (1995), El Karoui, Jeanblanc-Picqué and Shreve (1998), Cvitanić, Pham and Touzi (1999), Frey (2000), Ekström and Tysk (2004), Meyer (2006), Denis and Martini (2006), whereas the recent preprints by Soner, Touzi and Zhang (2010a, 2010b) contain very relevant results. Similar in this spirit is the strand of work by Shige Peng and his collaborators, surveyed in Peng (2010), regarding the so-called "G-Brownian motion" which exhibits volatility uncertainty [see also Vorbrink (2010), as well as Nutz (2010) for extensions to settings where the range of uncertainty is stochastic]. Whereas in both these strands the relevant fully nonlinear parabolic-type partial differential (so-called "Black–Scholes–Barenblatt") equation has a typically unique solution, here the main interest arises from *lack of uniqueness* on the part of the rather similar, fully nonlinear equation (1.7).

Let us mention that optimization problems in stochastic control, mathematical economics and finance that involve model uncertainty have also been treated by other authors, among them Gilboa and Schmeidler (1989), Gundel (2005), Shied and Wu (2005), Karatzas and Zamfirescu (2005), Föllmer and Gundel (2006), Schied (2007), Riedel (2009), Bayraktar and Yao (2011), Bayraktar, Karatzas and Yao (2011) and Kardaras and Robertson (2011) [see also the survey by Föllmer, Schied and Weber (2009)].

Preview: Sections 2 and 3 set up the model for an equity market with Knightian model uncertainty regarding its volatility and market-price-of-risk characteristics, and for investment rules in its context. Section 4 introduces the notion of optimal arbitrage in this context, whereas Section 5 discusses the relevance of the fully nonlinear parabolic partial differential inequality of HJB type (1.4), (1.5) in characterizing the arbitrage function and in finding an investment rule that realizes the best outperformance of the market portfolio. Section 6 presents a verification-type result for this equation. Sections 7 and 8 make the connection with the stochastic control problem of maximizing the probability of containment for an auxiliary Itô process, controlled in a nonanticipative way and in a manner that affects both its drift and dispersion characteristics. Finally, Section 9 develops yet another characterization of the arbitrage function, this time as the min-max value of a zero-sum stochastic game; the investment rule that realizes the best outperformance of (optimal arbitrage with respect to) the market, is now seen as the investor's best response to a "least favorable" market model configuration.

2. Equity market with Knightian model uncertainty.

We shall fix throughout a canonical, filtered measurable space $(\Omega, \mathcal{F})$, $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \le t < \infty}$ and assume that Ω contains the space $\mathfrak{W} \equiv C([0,\infty); (0,\infty)^n)$ of all continuous functions $\mathfrak{w} : [0,\infty) \to (0,\infty)^n$. We shall specify this canonical space in more detail in Section 7 below, when such detail becomes necessary.

On this space, we shall consider a vector of continuous, adapted processes $\mathfrak{X}(\cdot) = (X_1(\cdot), \dots, X_n(\cdot))'$ with values in $(0,\infty)^n$; its components will represent stock capitalizations in an equity market with n assets, and thus the total market capitalization will be the sum

$$X(t) := X_1(t) + \cdots + X_n(t), \qquad 0 \le t < \infty. \tag{2.1}$$

We shall also fix throughout a collection $\mathbb{K} = \{\mathcal{K}(\mathbf{y})\}_{\mathbf{y} \in \mathfrak{S}_n}$ of nonempty, compact and convex subsets of $\mathbb{R}^n \times \mathbb{S}^n$ as in (1.1).

We shall consider $\mathbb{R}^n$-valued functionals $\vartheta(\cdot,\cdot) = (\vartheta_1(\cdot,\cdot), \dots, \vartheta_n(\cdot,\cdot))'$ and $\mathbb{S}^n$-valued functionals $\alpha(\cdot,\cdot) = (\alpha_{ij}(\cdot,\cdot))_{1 \le i,j \le n}$, all of them defined on $[0,\infty) \times \Omega$ and progressively measurable [see Karatzas and Shreve (1991), Definition 3.5.15]. We shall assume that, for every continuous function $\mathfrak{w} : [0,\infty) \to (0,\infty)^n$ and

$T \in (0, \infty)$, these functionals satisfy the constraint and integrability conditions, respectively,

$$(2.2)\qquad \begin{gathered} (\vartheta(T, \mathfrak{w}), \alpha(T, \mathfrak{w})) \in \mathcal{K}(\mathfrak{w}(T)), \\ \int_0^T \left(\|\vartheta(t, \mathfrak{w})\|^2 + \mathrm{Tr}(\alpha(t, \mathfrak{w}))\right) \mathrm{d}t < \infty. \end{gathered}$$

We shall also consider $(n \times n)$-matrix-valued functionals

$$\sigma(\cdot, \cdot) = (\sigma_{i\nu}(\cdot, \cdot))_{1 \le i, \nu \le n},$$

where

$$(2.3)\qquad \sigma(t, \mathfrak{w}) = \sqrt{\alpha(t, \mathfrak{w})} \text{ is a square root of } \alpha(t, \mathfrak{w}) : \alpha(t, \mathfrak{w}) = \sigma(t, \mathfrak{w})\sigma'(t, \mathfrak{w}).$$

2.1. *Admissible systems.* For a given collection of sets $\mathbb{K}$ as in (1.1) and a fixed initial configuration $\mathbf{x} = (x_1, \ldots, x_n)' \in (0, \infty)^n$ of asset capitalizations, we shall call *admissible system* a collection $\mathcal{M}$ consisting of the underlying filtered space $(\Omega, \mathcal{F})$, $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \le t < \infty}$, of a probability measure $\mathbb{P}$ on it, and of a pair of processes $(\mathfrak{X}(\cdot), W(\cdot))$, with $W(\cdot) = (W_1(\cdot), \ldots, W_n(\cdot))'$ an n-dimensional $\mathbb{F}$-Brownian motion and $\mathfrak{X}(\cdot) = (X_1(\cdot), \ldots, X_n(\cdot))'$ taking values in $(0, \infty)^n$. These processes have the dynamics

$$(2.4)\qquad \mathrm{d}X_i(t) = X_i(t) \sum_{\nu=1}^n \sigma_{i\nu}(t, \mathfrak{X})[\mathrm{d}W_\nu(t) + \vartheta_\nu(t, \mathfrak{X})\,\mathrm{d}t], \qquad X_i(0) = x_i > 0$$

for some progressively measurable functionals $\vartheta(\cdot, \cdot)$ and $\sigma(\cdot, \cdot)$ which satisfy (2.2) and (2.3) above.

We shall think of this admissible system $\mathcal{M}$ as a *model* subject to "Knightian" uncertainty; this is expressed by the requirement $(\vartheta(t, \mathfrak{X}), \alpha(t, \mathfrak{X})) \in \mathcal{K}(\mathfrak{X}(t))$ in (2.2), (1.1) about the market price of risk and the covariance structure of the asset capitalization vector process $\mathfrak{X}(\cdot)$. In order not to lose sight of the underlying probability space, we shall denote by $\mathbb{P}^{\mathcal{M}}$ (resp., $\mathbb{E}^{\mathbb{P}^{\mathcal{M}}}$) the probability measure (resp., the corresponding expectation operator) on this space. Finally, $\mathfrak{M}(\mathbf{x})$ will denote the collection of all such admissible systems or "models" with initial configuration $\mathbf{x} = (x_1, \ldots, x_n)' \in (0, \infty)^n$. We shall think of $\mathfrak{M}(\mathbf{x})$ as a *meta-model*, a collection of admissible models, and of the collection $\mathbf{M} = \{\mathfrak{M}(\mathbf{x})\}_{\mathbf{x} \in (0,\infty)^n}$ as a "family of meta-models."

The interpretation is that the components of the driving Brownian motion $W(\cdot)$ represent the independent factors of the resulting model; the entries of the matrix $\sigma(t, \mathfrak{X})$ are the local volatility rates of the asset capitalization vector process $\mathfrak{X}(\cdot)$ at time t; the entries of the matrix $\alpha(t, \mathfrak{X})$ as in (2.3) represent the local covariance rates; whereas the components of the vector $\vartheta(t, \mathfrak{X})$ are the market price of risk (also called *relative risk*) rates prevalent at time t. In particular,

$$(2.5)\qquad \beta(t, \mathfrak{w}) := \sigma(t, \mathfrak{w})\vartheta(t, \mathfrak{w}) = \sqrt{\alpha(t, \mathfrak{w})}\,\vartheta(t, \mathfrak{w})$$

is, in the notation of (2.3), the vector of mean rates of return for the various assets at time t, when the equations of (2.4) are cast in the more familiar form

$$\mathrm{d}X_i(t) = X_i(t)\left(\beta_i(t,\mathfrak{X})\,\mathrm{d}t + \sum_{\nu=1}^{n} \sigma_{i\nu}(t,\mathfrak{X})\,\mathrm{d}W_\nu(t)\right), \qquad i = 1,\dots,n. \tag{2.6}$$

The integrability condition of (2.2) guarantees that the process $\mathfrak{X}(\cdot)$ takes values in $(0,\infty)^n$, $\mathbb{P}$-a.s.; it implies also that the exponential process

$$L(t) := \exp\left\{-\int_0^t \vartheta'(s,\mathfrak{X})\,\mathrm{d}W(s) - \frac{1}{2}\int_0^t \|\vartheta(s,\mathfrak{X})\|^2\,\mathrm{d}s\right\}, \qquad 0 \le t < \infty \tag{2.7}$$

is well defined and a strictly positive local martingale, thus also a supermartingale.

This process $L(\cdot)$ plays the role of a state-price-density or "deflator" in the present context. Just as in our earlier works, Fernholz and Karatzas (2010a, 2010b) as well as Ruf (2011)—mostly in a Markovian context, and without model uncertainty—an important feature of this subject is that $L(\cdot)$ has to be allowed to be a *strict* local martingale, that is, that $\mathbb{E}^{\mathbb{P}}(L(T)) < 1$ be allowed to hold for some, if not all, $T \in (0,\infty)$.

2.2. *Supermartingale problems*. Constraint (1.2) brings us in the realm of the Krylov (1989, 2002) approach, which studies stochastic differential equations with constraints on the drift and diffusion coefficients in terms of "supermartingale problems." In particular, Theorem 2.2 of Krylov (2002) shows that, under a suitable regularity condition on the family of sets $\mathbb{K}$ in (1.1), solving stochastic equation (2.4) subject to the requirements of (2.2) can be cast as a *supermartingale problem,* as follows.

Consider the nonlinear partial differential operator associated with (2.6), (2.5), namely

$$\begin{aligned} \mathcal{L}f(\mathbf{z}) &= F(D^2 f(\mathbf{z}), Df(\mathbf{z}), \mathbf{z}) \\ \text{with } F(Q,p,\mathbf{z}) &:= \sup_{\substack{(\theta,a)\in\mathcal{K}(\mathbf{z}) \\ b=\sqrt{a}\theta}} \left(\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} z_i z_j a_{ij} Q_{ij} + \sum_{i=1}^{n} z_i b_i p_i\right), \\ &\qquad (Q,p,\mathbf{z}) \in \mathbb{S}^n \times \mathbb{R}^n \times (0,\infty)^n. \end{aligned} \tag{2.8}$$

The supermartingale problem is to find a probability measure $\mathbb{P} = \mathbb{P}^{\mathcal{M}}$ on the filtered measurable space $(\Omega,\mathcal{F})$, $\mathbb{F} = \{\mathcal{F}(t)\}_{0\le t<\infty}$, under which $\mathfrak{X}(\cdot)$ takes values in $(0,\infty)^n$ a.s., and the process

$$u(t,\mathfrak{X}(t)) - \int_0^t \left(\frac{\partial u}{\partial s}(s,\mathfrak{X}(s)) + \mathcal{L}u(\mathfrak{X}(s))\right)\mathrm{d}s, \qquad 0 < t < \infty$$

is a local supermartingale for every $u : (0,\infty)\times(0,\infty)^n \to \mathbb{R}$ of class $\mathcal{C}^{1,2}$ with compact support.

The regularity condition on the family $\mathbb{K}$ in (1.1) that we alluded to earlier, mandates that the function

$$F(Q, p, \cdot) \text{ in (2.8) is Borel measurable, for every } (Q, p) \in \mathbb{S}^n \times \mathbb{R}^n. \tag{2.9}$$

If, in addition, the family of sets $\mathbb{K}$ in (1.1) satisfies the linear growth condition

$$\sup_{\substack{(\theta,a)\in\mathcal{K}(\mathbf{y})\\ b=\sqrt{a}\theta}} \left(\sum_{i=1}^{n}\sum_{i=1}^{n} y_i y_j a_{ij} + \sum_{i=1}^{n}(y_i b_i)^2\right)^{1/2} \le C(1+\|\mathbf{y}\|) \tag{2.10}$$

and the upper-semicontinuity condition

$$\limsup_{[0,\infty)^n \ni \mathbf{z}\to\mathbf{y}} F(Q, p, \mathbf{z}) \le F(Q, p, \mathbf{y}) \qquad \forall (Q, p) \in \mathbb{S}^n \times \mathbb{R}^n \tag{2.11}$$

for every $\mathbf{y} \in \mathfrak{S}_n$ and some real constant $C > 0$, then Theorem 3.2 in Krylov (2002) shows that the family $\{\mathbb{P}^{\mathcal{M}}\}_{\mathcal{M}\in\mathfrak{M}(\mathbf{x})}$ is convex and sequentially compact in the topology of vague convergence of probability measures.

2.3. *Markovian admissible systems.* We shall also consider the subcollection $\mathfrak{M}_*(\mathbf{x}) \subset \mathfrak{M}(\mathbf{x})$ of *Markovian* admissible systems, for which the functionals $\vartheta(\cdot,\cdot)$ and $\alpha(\cdot,\cdot)$ as in (2.2)–(2.4) are given as

$$\vartheta(t, \mathfrak{X}) = \boldsymbol{\theta}(t, \mathfrak{X}(t)), \qquad \alpha(t, \mathfrak{X}) = \mathbf{a}(t, \mathfrak{X}(t)), \tag{2.12}$$

with measurable functions $\boldsymbol{\theta} : [0,\infty) \times (0,\infty)^n \to \mathbb{R}^n$, $\mathbf{a} : [0,\infty) \times (0,\infty)^n \to \mathbb{S}^n$ that satisfy

$$(\boldsymbol{\theta}(t, \mathbf{z}), \mathbf{a}(t, \mathbf{z})) \in \mathcal{K}(\mathbf{z}) \qquad \forall (t, \mathbf{z}) \in [0,\infty) \times (0,\infty)^n. \tag{2.13}$$

Under condition (2.10) it follows then from so-called Markovian selection results [Krylov (1973); Stroock and Varadhan (1979), Chapter 12; Ethier and Kurtz (1986), Section 4.5] that the state process $\mathfrak{X}(\cdot)$ of (2.4) can be assumed to be (strongly) Markovian under $\mathbb{P}^{\mathcal{M}}$, $\mathcal{M} \in \mathfrak{M}_*(\mathbf{x})$. We shall make this selection whenever admissible systems in $\mathfrak{M}_*(\mathbf{x})$ are invoked.

3. Investment rules. Consider now an investor who is "small" in the sense that his actions do not affect market prices. He starts with initial fortune $v > 0$ and uses a rule that invests a proportion $\varpi_i(t) = \Pi_i(t, \mathfrak{X})$ of current wealth in the ith asset of the equity market, for any given time $t \in [0,\infty)$ and all $i = 1, \dots, n$; the remaining proportion $\varpi_0(t) := 1 - \sum_{i=1}^n \varpi_i(t)$ is held in cash (equivalently, in a zero-interest money market). Here $\Pi : [0,\infty) \times \mathfrak{W} \to \mathbb{R}^n$ is a progressively measurable functional assumed to satisfy the requirement

$$\int_0^T \big(|\Pi'(t,\mathfrak{w})\sigma(t,\mathfrak{w})\vartheta(t,\mathfrak{w})| + \Pi'(t,\mathfrak{w})\alpha(t,\mathfrak{w})\Pi(t,\mathfrak{w})\big)\,\mathrm{d}t < \infty, \qquad \forall T \in (0,\infty) \tag{3.1}$$

for every continuous function $\mathfrak{w} : [0, \infty) \to (0, \infty)^n$. (Thus, the requirement (3.1) will be in force under all admissible systems.) We shall denote throughout by $\mathfrak{P}$ the collection of all such (nonanticipative) *investment rules*, and by $\mathfrak{P}_*$ the sub-collection of all *Markovian* investment rules, that is, those that can be expressed as $\varpi_i(t) = \pi_i(t, \mathfrak{X}(t))$, $0 \le t < \infty, i = 1, \dots, n$ for some measurable function $\pi : [0, \infty) \times (0, \infty)^n \to \mathbb{R}^n$.

An investment rule is called *bounded*, if the functional Π is bounded uniformly on $[0, \infty) \times \mathfrak{W}$; for a bounded investment rule, requirement (3.1) is satisfied automatically, thanks to (2.2). An investment rule is called *portfolio* if the functional Π satisfies $\sum_{i=1}^n \Pi_i = 1$ on $[0, \infty) \times \mathfrak{W}$, and a portfolio is called *long-only* if $\Pi_1 \ge 0, \dots, \Pi_n \ge 0$ also hold on this domain. A long-only portfolio is clearly bounded.

Given an initial wealth $v \in (0, \infty)$, an investment rule $\Pi \in \mathfrak{P}$ and an admissible model $\mathcal{M} \in \mathfrak{M}(\mathbf{x})$, the resulting wealth process $Z(\cdot) \equiv Z^{v,\Pi}(\cdot)$ satisfies the dynamics

$$\frac{\mathrm{d}Z(t)}{Z(t)} = \sum_{i=1}^n \Pi_i(t, \mathfrak{X}) \frac{\mathrm{d}X_i(t)}{X_i(t)} = \Pi'(t, \mathfrak{X})\sigma(t, \mathfrak{X})[\vartheta(t, \mathfrak{X})\, \mathrm{d}t + \mathrm{d}W(t)] \tag{3.2}$$

and the initial condition $Z(0) = v$. In conjunction with (2.7) in the differential form

$$\mathrm{d}L(t) = -L(t)(\vartheta(t, \mathfrak{X}))'\, \mathrm{d}W(t) \tag{3.3}$$

and the product rule of the stochastic calculus, this gives

$$L(t)Z^{v,\Pi}(t) = v + \int_0^t L(s)Z^{v,\Pi}(s)\big(\sigma'(s, \mathfrak{X})\Pi(s, \mathfrak{X}) - \vartheta(s, \mathfrak{X})\big)'\, \mathrm{d}W(s), \quad 0 \le t < \infty. \tag{3.4}$$

For any initial configuration $\mathbf{x} = (x_1, \dots, x_n)' \in (0, \infty)^n$, initial wealth $v \in (0, \infty)$, investment rule $\Pi \in \mathfrak{P}$ and admissible model $\mathcal{M} \in \mathfrak{M}(\mathbf{x})$, the product $L(\cdot)Z^{v,\Pi}(\cdot)$ is therefore under $\mathbb{P}^{\mathcal{M}}$ a continuous, positive local martingale, thus also a supermartingale. Once again, it is important that this process be allowed to be a strict local martingale.

3.1. *The market portfolio*. The choice of Markovian investment rule $\mathfrak{m} \in \mathfrak{P}_*$ given by

$$\mathfrak{m}_i(t, \mathbf{z}) = \frac{z_i}{z_1 + \cdots + z_n}, \qquad i = 1, \dots, n,\ t \in [0, \infty),\ \mathbf{z} \in (0, \infty)^n$$

leads to the long-only *market portfolio*

$$\mu(t) = \mathfrak{X}(t)/X(t) \quad \text{with weights } \mu_i(t) = X_i(t)/X(t),\ i = 1, \dots, n,\ 0 \le t < \infty \tag{3.5}$$

in the notation of (2.1). It follows from (3.2) that investing according to this portfolio amounts to owning the entire market, in proportion to the initial wealth, $Z^{v,\mathfrak{m}}(\cdot) = vX(\cdot)/X(0)$.

3.2. *Ramifications*. Reading (3.4) for the market portfolio of (3.5), and recalling (2.1), leads to

$$L(t)X(t) = X(0) + \int_0^t L(s)\big(\sigma'(s,\mathfrak{X})\mathfrak{X}(s) - \vartheta(s,\mathfrak{X})X(s)\big)'\,\mathrm{d}W(s), \tag{3.6}$$
$$0 \le t < \infty$$

or equivalently $\mathrm{d}(L(t)X(t)) = -L(t)X(t)(\widetilde{\vartheta}(t,\mathfrak{X}))'\,\mathrm{d}W(t)$, where

$$\widetilde{\vartheta}(t,\mathfrak{w}) := \vartheta(t,\mathfrak{w}) - \frac{\sigma'(t,\mathfrak{w})\mathfrak{w}(t)}{\mathfrak{w}_1(t) + \cdots + \mathfrak{w}_n(t)} \tag{3.7}$$
$$\text{satisfies } \int_0^T \|\widetilde{\vartheta}(t,\mathfrak{w})\|^2\,\mathrm{d}t < \infty$$

for all $(t,\mathfrak{w}) \in [0,\infty) \times \mathfrak{W}$, thanks to (2.2). With this notation, it follows from (3.6) that

$$L(\cdot)X(\cdot) = (x_1 + \cdots + x_n)\cdot \exp\left\{-\int_0^{\cdot} (\widetilde{\vartheta}(t,\mathfrak{X}))'\,\mathrm{d}W(t) - \frac{1}{2}\int_0^{\cdot} \|\widetilde{\vartheta}(t,\mathfrak{X})\|^2\,\mathrm{d}t\right\}.$$

On the strength of the integrability condition in (3.7), the Dambis–Dubins–Schwartz representation [e.g., Karatzas and Shreve (1991), page 174] of the $\mathbb{P}^{\mathcal{M}}$-local martingale

$$N(\cdot) := \int_0^{\cdot} (\widetilde{\vartheta}(t,\mathfrak{X}))'\,\mathrm{d}W(t)$$
$$\text{with quadratic variation } \langle N\rangle(\cdot) = \int_0^{\cdot} \|\widetilde{\vartheta}(t,\mathfrak{X})\|^2\,\mathrm{d}t < \infty$$

gives

$$L(T)X(T) = (x_1 + \cdots + x_n)\cdot e^{B(u)-(u/2)}|_{u=\langle N\rangle(T)}, \qquad 0 \le T < \infty, \tag{3.8}$$

where $B(\cdot)$ is one-dimensional, standard Brownian motion under $\mathbb{P}^{\mathcal{M}}$. Whereas the equations of (2.4) can be written as

$$\mathrm{d}X_i(t) = X_i(t)\left(\frac{\sum_{j=1}^n \alpha_{ij}(t,\mathfrak{X})X_j(t)}{X_1(t) + \cdots + X_n(t)}\,\mathrm{d}t + \sum_{\nu=1}^n \sigma_{i\nu}(t,\mathfrak{X})\,\mathrm{d}\widetilde{W}_\nu(t)\right) \tag{3.9}$$

for $i = 1,\ldots,n$, with

$$\widetilde{W}(\cdot) := W(\cdot) + \int_0^{\cdot} \widetilde{\vartheta}(t,\mathfrak{X})\,\mathrm{d}t. \tag{3.10}$$

We then have the representation

$$\Lambda(\cdot) := \frac{X(0)}{L(\cdot)X(\cdot)} = \exp\left\{\int_0^{\cdot} (\widetilde{\vartheta}(t,\mathfrak{X}))' \,\mathrm{d}W(t) + \frac{1}{2}\int_0^{\cdot} \|\widetilde{\vartheta}(t,\mathfrak{X})\|^2 \,\mathrm{d}t\right\}$$
$$= \exp\left\{\int_0^{\cdot} (\widetilde{\vartheta}(t,\mathfrak{X}))' \,\mathrm{d}\widetilde{W}(t) - \frac{1}{2}\int_0^{\cdot} \|\widetilde{\vartheta}(t,\mathfrak{X})\|^2 \,\mathrm{d}t\right\} \tag{3.11}$$

for the normalized reciprocal of the deflated total market capitalization, and

$$\mathrm{d}\mu_i(t) = \mu_i(t)\big(\mathfrak{e}_i - \mu(t)\big)'\sigma(t,\mathfrak{X})\,\mathrm{d}\widetilde{W}(t), \qquad i = 1,\dots,n \tag{3.12}$$

for the dynamics market weights in (3.5); here $\mathfrak{e}_i$ is the ith unit vector in $\mathbb{R}^n$.

4. Optimal arbitrage relative to the market. Let us consider now the smallest proportion

$$\mathfrak{u}(T,\mathbf{x}) = \inf\big\{r > 0 : \exists \Pi_r \in \mathfrak{P}, \text{ s.t. } \mathbb{P}^{\mathcal{M}}\big(Z^{rX(0),\Pi_r}(T) \geq X(T)\big) = 1, \quad \forall \mathcal{M} \in \mathfrak{M}(\mathbf{x})\big\} \tag{4.1}$$

of the initial total market capitalization $X(0) = x_1 + \cdots + x_n$ which allows the small investor, starting with initial capital $\mathfrak{u}(T,\mathbf{x})X(0)$ and through judicious choice of investment rule in the class $\mathfrak{P}$, to match or exceed the performance of the market portfolio over the time-horizon $[0,T]$, and to do this with $\mathbb{P}^{\mathcal{M}}$-probability one, under any model $\mathcal{M} \in \mathfrak{M}(\mathbf{x})$ that might materialize. We shall refer to $\mathfrak{u}(\cdot,\cdot)$ of (4.1) as the *arbitrage function* for the family of meta-models $\mathbf{M} = \{\mathfrak{M}(\mathbf{x})\}_{\mathbf{x}\in(0,\infty)^n}$, and think of it as a version of the arbitrage function studied in Fernholz and Karatzas (2010a) which is "robust" with respect to $\mathbf{M}$.

The quantity of (4.1) is strictly positive; see Proposition 1 below and the discussion following it. On the other hand, the set of (4.1) contains the number $r = 1$, so clearly

$$0 < \mathfrak{u}(T,\mathbf{x}) \leq 1.$$

If $\mathfrak{u}(T,\mathbf{x}) < 1$, then for every $r \in (\mathfrak{u}(T,\mathbf{x}), 1)$—and even for $r = \mathfrak{u}(T,\mathbf{x})$ when the infimum in (4.1) is attained, as indeed it is in the context of Theorem 1 below—there exists an investment rule $\Pi_r \in \mathfrak{P}$ such that

$$Z^{X(0),\Pi_r}(T) \geq \frac{1}{r}X(T) > X(T) = Z^{X(0),\mathfrak{m}}(T), \qquad \mathbb{P}^{\mathcal{M}}\text{-a.s.}$$

holds for every $\mathcal{M} \in \mathfrak{M}(\mathbf{x})$. In other words, the investment rule Π_r leads then to *strong arbitrage relative to the market* portfolio in the terminology of Fernholz and Karatzas (2009)—here with the extra feature that such arbitrage is now *robust*, that is, holds under any possible admissible system or "model" that might materialize. If, on the other hand, $\mathfrak{u}(T,\mathbf{x}) = 1$, then such outperformance of (equivalently, strong arbitrage relative to) the market is just not possible over all meta-models $\mathcal{M} \in \mathfrak{M}(\mathbf{x})$. In either case, the highest return on investment relative to the market

$$\mathfrak{b}(T,\mathbf{x}) := \sup\{b > 0 : \exists \Pi \in \mathfrak{P}, \text{ s.t. } \mathbb{P}^{\mathcal{M}}(Z^{X(0),\Pi}(T) \geq bX(T)) = 1, \\ \forall \mathcal{M} \in \mathfrak{M}(\mathbf{x})\},$$

achievable using (nonanticipative) investment rules, is given as $\mathfrak{b}(T,\mathbf{x}) = 1/\mathfrak{u}(T, \mathbf{x}) \geq 1$.

REMARK 1. Instances of $\mathfrak{u}(T,\mathbf{x}) < 1$ occur, when there exists a constant $\zeta > 0$ such that either

$$\inf_{a \in \mathcal{A}(\mathbf{z})} \left(\sum_{i=1}^{n} \frac{z_i a_{ii}}{z_1 + \cdots + z_n} - \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{z_i z_j a_{ij}}{(z_1 + \cdots + y_n)^2} \right) \geq \zeta \tag{4.2}$$

or

$$\left(\frac{(z_1 \cdots z_n)^{1/n}}{z_1 + \cdots + z_n} \right) \cdot \inf_{a \in \mathcal{A}(\mathbf{z})} \left(\sum_{i=1}^{n} a_{ii} - \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \right) \geq \zeta \tag{4.3}$$

holds for every $\mathbf{z} \in (0,\infty)^n$. See the survey paper Fernholz and Karatzas (2009), Examples 11.1 and 11.2 [as well as Fernholz and Karatzas (2005), Fernholz, Karatzas and Kardaras (2005) for additional examples].

PROPOSITION 1. *The quantity of* (4.1) *satisfies*

$$\mathfrak{u}(T,\mathbf{x}) \geq \Phi(T,\mathbf{x}) > 0 \\ \text{where } \Phi(T,\mathbf{x}) := \sup_{\mathcal{M} \in \mathfrak{M}(\mathbf{x})} \left(\frac{\mathbb{E}^{\mathbb{P}^{\mathcal{M}}}[L(T)X(T)]}{x_1 + \cdots + x_n} \right). \tag{4.4}$$

Furthermore, *under conditions* (2.9)–(2.11), *there exists an admissible system* $\mathcal{M}_o \in \mathfrak{M}(\mathbf{x})$ *such that*

$$\Phi(T,\mathbf{x}) = \frac{\mathbb{E}^{\mathbb{P}^{\mathcal{M}_o}}[L(T)X(T)]}{x_1 + \cdots + x_n}. \tag{4.5}$$

PROOF. Take an arbitrary element $r > 0$ of the set on the right-hand side of (4.1) and an arbitrary admissible system $\mathcal{M} \in \mathfrak{M}(\mathbf{x})$. There exists then an investment rule $\Pi_r \in \mathfrak{P}$ with the inequality $Z^{rX(0),\Pi_r}(T) \geq X(T)$ valid $\mathbb{P}^{\mathcal{M}}$-a.s. On the strength of (2.7) and (3.4), the process $L(\cdot)Z^{rX(0),\Pi_r}(\cdot)$ is a $\mathbb{P}^{\mathcal{M}}$-supermartingale; thus (3.8) and (3.7) lead to

$$r(x_1 + \cdots + x_n) = rX(0) \geq \mathbb{E}^{\mathbb{P}^{\mathcal{M}}}[L(T)Z^{rX(0),\Pi_r}(T)] \\ \geq \mathbb{E}^{\mathbb{P}^{\mathcal{M}}}[L(T)X(T)] > 0. \tag{4.6}$$

The inequality $\mathfrak{u}(T, \mathbf{x}) \geq \Phi(T, \mathbf{x})$ in (4.4) follows now from the arbitrariness of $r > 0$ and $\mathcal{M} \in \mathfrak{M}(\mathbf{x})$. The existence of an admissible system $\mathcal{M}_o \in \mathfrak{M}(\mathbf{x})$ that satisfies (4.5) follows from Theorem 3.4 in Krylov (2002), in conjunction with the dynamics of (2.4) and (3.3). □

Although strong arbitrage relative to the market may exist within the framework of the models $\mathcal{M} \in \mathfrak{M}(\mathbf{x})$ studied here (cf. Remark 1), the existence of a strictly positive supermartingale deflator process $L(\cdot)$ as in (2.7) proscribes scalable arbitrage opportunities, also known as *Unbounded Profits with Bounded Risk* (*UPBR*); this is reflected in the inequality $\mathfrak{u}(T, \mathbf{x}) > 0$ of (4.4). We refer the reader to Delbaen and Schachermayer (1995b) for the origin of the resulting *NUPBR* concept, and to Karatzas and Kardaras (2007) for an elaboration of this point in a different context, namely, the existence and properties of the numéraire portfolio.

Finally, let us write (4.4) as

$$(4.7)\qquad \Phi(T, \mathbf{x}) = \sup_{\mathcal{M} \in \mathfrak{M}(\mathbf{x})} \mathfrak{u}_{\mathcal{M}}(T, \mathbf{x}) \qquad \text{where } \mathfrak{u}_{\mathcal{M}}(T, \mathbf{x}) := \frac{\mathbb{E}^{\mathbb{P}^{\mathcal{M}}}[L(T)X(T)]}{x_1 + \cdots + x_n}.$$

We have for this quantity the interpretation

$$\mathfrak{u}_{\mathcal{M}}(T, \mathbf{x}) = \inf\{r > 0 : \exists \Pi_r \in \mathfrak{P}, \text{ s.t. } \mathbb{P}^{\mathcal{M}}(Z^{rX(0), \Pi_r}(T) \geq X(T)) = 1\}$$

as the "arbitrage function for the model $\mathcal{M} \in \mathfrak{M}(\mathbf{x})$," at least when the matrix $\sigma(t, \mathfrak{w})$ in invertible for every $(t, \mathfrak{w}) \in (0, \infty) \times \mathfrak{W}$ and when $(\mathbb{P}^{\mathcal{M}}, \mathbb{F})$-martingales can be represented as stochastic integrals with respect to the Brownian motion $W(\cdot)$ in (2.4).

5. A fully nonlinear PDI. Consider now a continuous function $U : [0, \infty) \times (0, \infty)^n \to (0, \infty)$ with

$$(5.1)\qquad U(0, \mathbf{z}) = 1, \qquad \mathbf{z} \in (0, \infty)^n,$$

which is of class $\mathcal{C}^{1,2}$ on $(0, \infty) \times (0, \infty)^n$ and satisfies on this domain the fully nonlinear partial differential inequality (PDI)

$$(5.2)\qquad \frac{\partial U}{\partial \tau}(\tau, \mathbf{z}) \geq \sum_{i=1}^n \sum_{j=1}^n z_i z_j a_{ij} \left(\frac{1}{2} D_{ij}^2 U(\tau, \mathbf{z}) + \frac{D_i U(\tau, \mathbf{z})}{z_1 + \cdots + z_n} \right) \qquad \forall a \in \mathcal{A}(\mathbf{z}).$$

We shall denote by $\mathcal{U}$ the collection of all such continuous functions $U : [0, \infty) \times (0, \infty)^n \to (0, \infty)$ which are of class $\mathcal{C}^{1,2}$ on $(0, \infty) \times (0, \infty)^n$ and satisfy (5.1) and (5.2). This collection $\mathcal{U}$ is nonempty, since we can take $U(\cdot, \cdot) \equiv 1$; however, $\mathcal{U}$ need not contain only one element.

Let us fix an initial configuration $\mathbf{x} \in (0, \infty)^n$ and consider any admissible system $\mathcal{M} \in \mathfrak{M}(\mathbf{x})$. Applying Itô's rule to the process

$$(5.3)\qquad \Xi(t) := X(t)L(t)U(T - t, \mathfrak{X}(t)), \qquad 0 \leq t \leq T$$

in conjunction with (3.6) and (2.4), we obtain its $\mathbb{P}^{\mathcal{M}}$-semimartingale decomposition as

$$\begin{aligned}\frac{\mathrm{d}\Xi(t)}{X(t)L(t)} &= \Delta(t,\mathfrak{X})\,\mathrm{d}t \\ &\quad + \sum_{\nu=1}^{n}\big[R_\nu(t,\mathfrak{X}) - U\big(T-t,\mathfrak{X}(t)\big)\widetilde{\vartheta}_\nu(t,\mathfrak{X})\big]\,\mathrm{d}W_\nu(t).\end{aligned}\tag{5.4}$$

Here we have used the notation of (3.7), and have set

$$R_\nu(t,\mathfrak{X}) := \sum_{i=1}^{n} X_i(t) D_i U\big(T-t,\mathfrak{X}(t)\big)\sigma_{i\nu}(t,\mathfrak{X}),\tag{5.5}$$

$$\begin{aligned}\Delta(t,\mathfrak{X}) &:= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} X_i(t)X_j(t)\alpha_{ij}(t,\mathfrak{X})D^2_{ij}U\big(T-t,\mathfrak{X}(t)\big) \\ &\quad + \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\frac{X_j(t)\alpha_{ij}(t,\mathfrak{X})}{X_1(t)+\cdots+X_n(t)}\right)X_i(t)D_iU\big(T-t,\mathfrak{X}(t)\big) \\ &\quad - \frac{\partial U}{\partial\tau}\big(T-t,\mathfrak{X}(t)\big).\end{aligned}\tag{5.6}$$

From the inequality of (5.2), coupled with the fact that $\alpha(t,\mathfrak{X})\in\mathcal{A}(\mathfrak{X}(t))$ holds for all $0\le t<\infty$, this last expression is clearly not positive. As a result, the positive process $\Xi(\cdot)$ of (5.3) is a $\mathbb{P}^{\mathcal{M}}$-supermartingale, namely,

$$\begin{aligned}L(t)X(t)U\big(T-t,\mathfrak{X}(t)\big) = \Xi(t) &\ge \mathbb{E}^{\mathbb{P}^{\mathcal{M}}}[\Xi(T)|\mathcal{F}(t)] \\ &= \mathbb{E}^{\mathbb{P}^{\mathcal{M}}}[L(T)X(T)|\mathcal{F}(t)]\end{aligned}\tag{5.7}$$

holds $\mathbb{P}^{\mathcal{M}}$-a.s., $\forall\mathcal{M}\in\mathfrak{M}(\mathbf{x})$ and $0\le t\le T$; in particular,

$$\begin{aligned}(x_1+\cdots+x_n)U(T,\mathbf{x}) = \Xi(0) &\ge \mathbb{E}^{\mathbb{P}^{\mathcal{M}}}[\Xi(T)] \\ &= \mathbb{E}^{\mathbb{P}^{\mathcal{M}}}[L(T)X(T)] \qquad \forall\mathcal{M}\in\mathfrak{M}(\mathbf{x}).\end{aligned}$$

With the notation of (4.4), we obtain in this manner the following analog of the inequality in Proposition 1:

$$U(T,\mathbf{x}) \ge \Phi(T,\mathbf{x}).\tag{5.8}$$

Digging in this same spot, just a bit deeper, leads to our next result; this is very much in the spirit of Theorem 5 in Fleming and Vermes (1989) and of Section II.2 in Lions (1984).

PROPOSITION 2. *For* every *horizon* $T\in(0,\infty)$, *initial configuration* $\mathbf{x}\in(0,\infty)^n$ *and function* $U:[0,\infty)\times(0,\infty)^n\to(0,\infty)$ *in the collection* $\mathcal{U}$, *we have* *the inequality*

$$U(T,\mathbf{x}) \geq \mathfrak{u}(T,\mathbf{x}) \geq \Phi(T,\mathbf{x}). \tag{5.9}$$

Furthermore, the Markovian investment rule $\pi^U \in \mathfrak{P}_*$ *generated by this function* U *through*

$$\begin{aligned} \pi_i^U(t,\mathbf{z}) &:= z_i D_i \log U(T-t,\mathbf{z}) \\ &\qquad + \frac{z_i}{z_1+\cdots+z_n}, \qquad (t,\mathbf{z}) \in [0,T]\times(0,\infty)^n \end{aligned} \tag{5.10}$$

for each $i = 1,\dots,n$, *satisfies for every admissible system* $\mathcal{M} \in \mathfrak{M}(\mathbf{x})$ *the inequality*

$$Z^{U(T,\mathbf{x})X(0),\pi^U}(T) \geq X(T), \qquad \mathbb{P}^{\mathcal{M}}\text{-a.s.} \tag{5.11}$$

PROOF. For a fixed initial configuration $\mathbf{x} \in (0,\infty)^n$, an arbitrary admissible model $\mathcal{M} \in \mathfrak{M}(\mathbf{x})$ and any function $U \in \mathcal{U}$, let us recall the notation of (5.3) and re-cast the dynamics of (5.4) as

$$\mathrm{d}\Xi(t) = \Xi(t)\left(\sum_{\nu=1}^n \Psi_\nu(t,\mathfrak{X})\,\mathrm{d}W_\nu(t) - \mathrm{d}C(t)\right). \tag{5.12}$$

Here by virtue of (5.4), (5.5) and (3.7) we have written

$$\begin{aligned} \Psi_\nu(t,\mathfrak{X}) &:= \sum_{i=1}^n \sigma_{i\nu}(t,\mathfrak{X})\left(X_i(t) D_i \log U\big(T-t,\mathfrak{X}(t)\big) + \frac{X_i(t)}{X(t)}\right) \\ &\qquad - \vartheta_\nu(t,\mathfrak{X}) \end{aligned} \tag{5.13}$$

for $\nu = 1,\dots,n$ and have introduced in the notation of (5.6) the continuous, increasing process

$$C(t) := \int_0^t \frac{(-\Delta(s,\mathfrak{X}))}{U(T-s,\mathfrak{X}(s))}\,\mathrm{d}s, \qquad 0 \leq t \leq T. \tag{5.14}$$

The expression of (5.13) suggests considering the Markovian investment rule $\pi^U \in \mathfrak{P}_*$ as in (5.10); then we cast the expression of (5.13) as

$$\Psi_\nu(t,\mathfrak{X}) = \sum_{i=1}^n \sigma_{i\nu}(t,\mathfrak{X})\pi^U(t,\mathfrak{X}(t)) - \vartheta_\nu(t,\mathfrak{X}).$$

On the strength of (3.4), the value process generated by this investment rule π^U starting with initial wealth $\xi := U(T,\mathbf{x})X(0) \equiv \Xi(0)$, satisfies the equation

$$\mathrm{d}\big(L(t)Z^{\xi,\pi^U}(t)\big) = \big(L(t)Z^{\xi,\pi^U}(t)\big)\sum_{\nu=1}^n \Psi_\nu(t,\mathfrak{X})\,\mathrm{d}W_\nu(t).$$

Juxtaposing this to (5.12), and using the positivity of $\Xi(\cdot)$ along with the nonnegativity and nondecrease of $C(\cdot)$, we obtain the $\mathbb{P}^{\mathcal{M}}$-a.s. comparison $L(\cdot)Z^{\xi,\pi^U}(\cdot) \geq \Xi(\cdot)$, thus

$$Z^{\xi,\pi^U}(t) \geq X(t)U\big(T-t, \mathfrak{X}(t)\big), \qquad 0 \leq t \leq T. \tag{5.15}$$

With $t = T$ this leads to (5.11), in conjunction with (5.1). We conclude from (5.11) that the number $U(T, \mathbf{x}) > 0$ belongs to the set on the right-hand side of (4.1), and the first comparison in (5.9) follows; the second is just a restatement of (4.4). □

COROLLARY. *Suppose that the function* $\Phi(\cdot,\cdot)$ *of* (4.4) *belongs to the collection* $\mathcal{U}$. *Then* $\Phi(\cdot,\cdot)$ *is the smallest element of* $\mathcal{U}$; *the infimum in* (4.1) *is attained*; *we can take* $U \equiv \Phi$ *in* (5.11), (5.10); *and the inequality in* (5.9) *holds as equality, that is*, $\Phi(\cdot,\cdot)$ *coincides with the arbitrage function*

$$\mathfrak{u}(T, \mathbf{x}) = \Phi(T, \mathbf{x}) \qquad \forall (T, \mathbf{x}) \in (0, \infty) \times (0, \infty)^n.$$

Interpretation: Imagine that the small investor is a manager who invests for a pension fund and tries to track or exceed the performance of an index (the market portfolio) over a finite time-horizon. He has to do this in the face of uncertainty about the characteristics of the market, including its covariance and price-of-risk structure, so he acts with extreme prudence and tries to protect his clients against the most adverse market configurations imaginable [the range of such configurations is captured by the constraints (1.2), (1.1)]. If such adverse circumstances do not materialize his strategy generates a surplus, captured here by the increasing process $C(\cdot)$ of (5.14) with $U \equiv \Phi \equiv \mathfrak{u}$, which can then be returned to the (participants in the) fund. We are borrowing and adapting this interpretation from Lyons (1995).

Similarly, the Markovian investment rule $\pi^U \in \mathfrak{P}_*$ generated by the function $U \equiv \Phi \equiv \mathfrak{u}$ in (5.10), (1.3) implements the best possible outperformance of the market portfolio, as in (5.11).

6. A verification result.

For the purposes of this section we shall impose the following *growth* condition on the family $\mathbb{A} = \{\mathcal{A}(\mathbf{y})\}_{\mathbf{y} \in \mathfrak{S}_n}$ of subsets of $\mathbb{S}^n$ in (1.6), (1.1): there exists a constant $C \in (0, \infty)$, such that for all $\mathbf{y} \in \mathfrak{S}_n$ we have

$$\sup_{a \in \mathcal{A}(\mathbf{y})} \left(\max_{1 \leq i,j \leq n} \frac{y_i y_j |a_{ij}|}{(y_1 + \cdots + y_n)} \right) \leq C(1 + \|\mathbf{y}\|). \tag{6.1}$$

We shall also need the following *strong ellipticity* condition, which mandates that for every nonempty, compact subset $\mathbf{K}$ of $(0, \infty)^n$, there exists a real constant $\lambda = \lambda_{\mathbf{K}} > 0$ such that

$$\inf_{\mathbf{z} \in \mathbf{K}} \left(\inf_{a \in \mathcal{A}(\mathbf{z})} \left(\sum_{i=1}^n \sum_{j=1}^n \xi_i \xi_j a_{ij} \right) \right) \geq \lambda_{\mathbf{K}} \|\xi\|^2 \qquad \forall \xi \in \mathbb{R}^n. \tag{6.2}$$

ASSUMPTION A. There exist a continuous function $\mathbf{a}:(0,\infty)^n \to \mathbb{S}^n$, a $\mathcal{C}^2$-function $H:(0,\infty)^n \to \mathbb{R}$, and a continuous square root $\mathbf{s}(\cdot)$ of $\mathbf{a}(\cdot)$, namely $\mathbf{a}(\cdot) = \mathbf{s}(\cdot)\mathbf{s}'(\cdot)$ such that, with the vector-valued function $\boldsymbol{\theta}(\cdot) = (\boldsymbol{\theta}_1(\cdot), \dots, \boldsymbol{\theta}_n(\cdot))'$ defined by

$$\boldsymbol{\theta}_\nu(\mathbf{z}) := \sum_{j=1}^n z_j \mathbf{s}_{j\nu}(\mathbf{z}) D_j H(\mathbf{z}), \qquad \nu = 1, \dots, n, \tag{6.3}$$

condition (2.13) is satisfied, whereas the system of stochastic differential equations

$$\begin{aligned} \mathrm{d}X_i(t) = X_i(t) \sum_{\nu=1}^n \mathbf{s}_{i\nu}(\mathfrak{X}(t))[\mathrm{d}W_\nu(t) + \boldsymbol{\theta}_\nu(\mathfrak{X}(t))\,\mathrm{d}t], \\ X_i(0) = x_i > 0, i = 1, \dots, n \end{aligned} \tag{6.4}$$

has a solution in which the state process $\mathfrak{X}(\cdot)$ takes values in $(0,\infty)^n$.

A bit more precisely, this assumption posits the existence of a Markovian admissible system $\mathcal{M}_o \in \mathfrak{M}_*(\mathbf{x})$ consisting of a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \le t < \infty}$ and of two continuous, adapted process $\mathfrak{X}(\cdot)$ and $W(\cdot)$ on it, such that under the probability measure $\mathbb{P} \equiv \mathbb{P}^{\mathcal{M}_o}$ the process $W(\cdot)$ is n-dimensional Brownian Motion, the process $\mathfrak{X}(\cdot)$ takes values in $(0,\infty)^n$ a.s. and (2.4) holds with $\vartheta_\nu(t, \mathfrak{X}) = \boldsymbol{\theta}_\nu(\mathfrak{X}(t))$ as in (6.3), and with $\sigma_{i\nu}(t, \mathfrak{X}) = \mathbf{s}_{i\nu}(\mathfrak{X}(t))$, $0 \le t < \infty$ $(1 \le i, \nu \le n)$. The system of equations (6.4) can be cast equivalently as

$$\begin{aligned} \mathrm{d}X_i(t) = X_i(t) \Bigg[\sum_{\nu=1}^n \mathbf{s}_{i\nu}(\mathfrak{X}(t))\,\mathrm{d}W_\nu(t) \\ + \left(\sum_{j=1}^n \mathbf{a}_{ij}(\mathfrak{X}(t)) X_j(t) D_j H(\mathfrak{X}(t)) \right) \mathrm{d}t \Bigg]. \end{aligned} \tag{6.5}$$

ASSUMPTION B. In the notation of the previous paragraph and under the condition

$$\sum_{i=1}^n \sum_{\nu=1}^n z_i |\mathrm{s}_{i\nu}(\mathbf{z})||\boldsymbol{\theta}_\nu(\mathbf{z})| \le C(1 + \|\mathbf{z}\|) \qquad \forall \mathbf{z} \in (0,\infty)^n, \tag{6.6}$$

we define on $(0,\infty)^n$ the continuous functions $g(\mathbf{z}) := e^{-H(\mathbf{z})} \sum_{i=1}^n z_i$ and $k(\mathbf{z}) := (1/2) \sum_{i=1}^n \sum_{j=1}^n \mathbf{a}_{ij}(\mathbf{z})[D_{ij}^2 H(\mathbf{z}) + D_i H(\mathbf{z}) D_j H(\mathbf{z})]$ and assume that the function

$$G(\tau, \mathbf{x}) := \mathbb{E}^{\mathbb{P}^{\mathcal{M}_o}} \left[g(\mathfrak{X}(\tau)) \exp\left\{ \int_0^\tau k(\mathfrak{X}(t))\,\mathrm{d}t \right\} \right], \qquad (\tau, \mathbf{x}) \in [0,\infty)^n$$

is continuous on $[0,\infty) \times (0,\infty)$ and of class $\mathcal{C}^{1,2}$ on $(0,\infty) \times (0,\infty)$.

Sufficient conditions for Assumptions A, B to hold are provided in Fernholz and Karatzas (2010a), Sections 8 and 9. It is also shown there, that we have the $\mathbb{P}^{\mathcal{M}_o}$-martingale property

$$\mathbb{E}^{\mathbb{P}^{\mathcal{M}_o}}[X(T)L(T)|\mathcal{F}(t)] = X(t)L(t) \cdot \Gamma\big(T-t, X(t)\big), \qquad 0 \le t \le T \tag{6.7}$$

for the function

$$\Gamma(\tau, \mathbf{z}) := G(\tau, \mathbf{z})/g(\mathbf{z}), \qquad (\tau, \mathbf{z}) \in [0, \infty) \times (0, \infty)^n. \tag{6.8}$$

This function is of class $\mathcal{C}^{1,2}$ on $(0, \infty) \times (0, \infty)$ and satisfies the initial condition $\Gamma(0, \cdot) \equiv 1$ on $(0, \infty)^n$ as well as the *linear* second-order parabolic equation

$$\begin{aligned} \frac{\partial \Gamma}{\partial \tau}(\tau, \mathbf{z}) = \sum_{i=1}^{n}\sum_{j=1}^{n} z_i z_j \mathbf{a}_{ij}(\mathbf{z}) \left(\frac{1}{2} D_{ij}^2 \Gamma(\tau, \mathbf{z}) + \frac{D_i \Gamma(\tau, \mathbf{z})}{z_1 + \cdots + z_n}\right), \\ (\tau, \mathbf{z}) \in (0, \infty) \times (0, \infty)^n. \end{aligned} \tag{6.9}$$

PROPOSITION 3 (Verification argument). *Under the Assumptions* A, B *and the conditions* (6.1), (6.2), *suppose that the functions* $\mathbf{a}(\cdot)$ *and* $\Gamma(\tau, \cdot)$ *satisfy the inequality*

$$\begin{aligned} &\sum_{i=1}^{n}\sum_{j=1}^{n} z_i z_j \mathbf{a}_{ij}(\mathbf{z}) \left(\frac{1}{2} D_{ij}^2 \Gamma(\tau, \mathbf{z}) + \frac{D_i \Gamma(\tau, \mathbf{z})}{z_1 + \cdots + z_n}\right) \\ &\quad \ge \sum_{i=1}^{n}\sum_{j=1}^{n} z_i z_j a_{ij} \left(\frac{1}{2} D_{ij}^2 \Gamma(\tau, \mathbf{z}) + \frac{D_i \Gamma(\tau, \mathbf{z})}{z_1 + \cdots + z_n}\right) \qquad \forall a \in \mathcal{A}(\mathbf{z}) \end{aligned} \tag{6.10}$$

for every $(\tau, \mathbf{z}) \in (0, \infty) \times (0, \infty)^n$. *Then, in the notation of* (4.1)–(4.7), *we have*:

$$\mathfrak{u}(T, \mathbf{x}) = \Phi(T, \mathbf{x}) = \Gamma(T, \mathbf{x}) = \mathfrak{u}_{\mathcal{M}_o}(T, \mathbf{x}), \qquad \forall (T, \mathbf{x}) \in (0, \infty) \times (0, \infty)^n$$

for the Markovian admissible system $\mathcal{M} \equiv \mathcal{M}_o \in \mathfrak{M}(\mathbf{x})$ *posited in Assumption* A; *the conclusions of Proposition* 2 *and its corollary for* $U \equiv \Phi$; *as well as the* $\mathbb{P}^{\mathcal{M}}$-*a.s. comparison*

$$L(t)X(t) \cdot \mathfrak{u}\big(T-t, \mathfrak{X}(t)\big) \ge \mathbb{E}^{\mathbb{P}^{\mathcal{M}}}[L(T)X(T)|\mathcal{F}(t)], \qquad 0 \le t \le T,$$

which holds for every $\mathcal{M} \in \mathfrak{M}(\mathbf{x})$ *and as equality for* $\mathcal{M} \equiv \mathcal{M}_o \in \mathfrak{M}(\mathbf{x})$.

PROOF. Under condition (6.10) the function $\Gamma(\cdot, \cdot)$ belongs to the collection $\mathcal{U}$, as (5.2) is satisfied with $U \equiv \Gamma$ on the strength of (6.9) and (6.10); thus, we deduce $\Gamma(T, \mathbf{x}) \ge \Phi(T, \mathbf{x})$ from (5.8). On the other hand, equality (6.7) with $t = 0$, and the definition of $\Phi(T, \mathbf{x})$ in (4.4), give

$$\Gamma(T, \mathbf{x}) = \frac{\mathbb{E}^{\mathbb{P}^{\mathcal{M}_o}}[L(T)X(T)]}{x_1 + \cdots + x_n} = \mathfrak{u}_{\mathcal{M}_o}(T, \mathbf{x}) \le \Phi(T, \mathbf{x}),$$

so the equality $\Gamma(T,\mathbf{x})=\Phi(T,\mathbf{x})$ follows. In other words, we identify $\mathcal{M}_o$ as a Markovian admissible system that satisfies (4.5) and attains the supremum in (4.4). The remaining claims come from Proposition 2 and its Corollary [in particular, from reading (5.9) with $U\equiv\Gamma$] and from (5.7), (6.7). $\square$

REMARK 2. Proposition 3 holds under conditions weaker than those imposed in Assumptions A and B above, at the "expense" of a certain localization. More precisely, one posits the existence of locally bounded and locally Lipschitz functions $\mathbf{s}_{i\nu}(\cdot)$ and $\boldsymbol{\theta}_\nu(\cdot)$ $(1\le i,\nu\le n)$ for which (2.13), (6.1), (6.2) and (6.6) are satisfied with $\mathbf{a}(\cdot)=\mathbf{s}(\cdot)\mathbf{s}'(\cdot)$, and for which there exists a Markovian admissible system $\mathcal{M}_o\in\mathfrak{M}_*(\mathbf{x})$ whose state process $\mathfrak{X}(\cdot)$ in (6.4) is, under $\mathbb{P}^{\mathcal{M}_o}$, a strong Markov process with values in $(0,\infty)^n$ a.s. Using results from the theory of stochastic flows [Kunita (1990), Protter (2004)] and from parabolic partial differential equations [Janson and Tysk (2006), Ekström and Tysk (2009)], Theorem 2 in Ruf (2011) shows that the function $\Gamma(\cdot,\cdot)$ is then of class $\mathcal{C}^{1,2}$ locally on $(0,\infty)\times(0,\infty)^n$, and solves there equation (6.9).

7. Maximizing the probability of containment. We have now gone as far as we could without having to specify the nature of our filtered measurable space $(\Omega,\mathcal{F})$, $\mathbb{F}=\{\mathcal{F}(t)\}_{0\le t<\infty}$. To proceed further, we shall need to choose this space carefully.

We shall take as our sample space the set Ω of right-continuous paths $\omega:[0,\infty)\to\mathfrak{S}_n\cup\{\Delta\}$. Here Δ is an additional "absorbing point"; paths stay at Δ once they get there, that is, after $\mathcal{T}(\omega)=\inf\{t\ge 0|\omega(t)=\Delta\}$, and are continuous on $(0,\mathcal{T}(\omega))$; we are employing here, and throughout this work, the usual convention $\inf\varnothing=\infty$. We also select $\mathcal{K}(\Delta)=\{(\mathbf{0},\mathrm{O}_{n\times n})\}$ and $\mathcal{A}(\Delta)=\{\mathrm{O}_{n\times n}\}$, where $\mathrm{O}_{n\times n}$ is the zero matrix. With $\mathcal{F}^\flat(t):=\sigma(\omega(s),0\le s\le t)$, the filtration $\mathbb{F}^\flat=\{\mathcal{F}^\flat(t)\}_{0\le t<\infty}$ is a *standard system* in the terminology of Parthasarathy (1967). This means that each $(\Omega,\mathcal{F}^\flat(t))$ is isomorphic to the Borel σ-algebra on some Polish space, and that for any decreasing sequence $\{A_j\}_{j\in\mathbb{N}}$ where each A_j is an atom of the corresponding $\mathcal{F}^\flat(t_j)$, for some increasing sequence $\{t_j\}_{j\in\mathbb{N}}\subset[0,\infty)$, we have $\bigcap_{j\in\mathbb{N}}A_j\ne\varnothing$ [see the Appendix in Föllmer (1972), as well as Meyer (1972) and Föllmer (1973)].

With all this in place we take $(\Omega,\mathcal{F})$, $\mathbb{F}=\{\mathcal{F}(t)\}_{0\le t<\infty}$ as our filtered measurable space, where

$$\mathcal{F}(t):=\bigcap_{\varepsilon>0}\mathcal{F}^\flat(t+\varepsilon)\quad\text{and}\quad\mathcal{F}:=\sigma\Bigg(\bigcup_{0\le t<\infty}\mathcal{F}(t)\Bigg).$$

An admissible system $\mathcal{M}\in\mathfrak{M}(\mathbf{x})$, $\mathbf{x}\in(0,\infty)^n$ defined as in Section 2 consists of this filtered measurable space $(\Omega,\mathcal{F})$, $\mathbb{F}=\{\mathcal{F}(t)\}_{0\le t<\infty}$, of a probability measure $\mathbb{P}^{\mathcal{M}}$ on it, of an n-dimensional Brownian motion $W(\cdot)$ on the resulting probability space and of the coordinate mapping process $\mathfrak{X}(t,\omega)=\omega(t)$, $0\le t<\infty$

which is assumed to satisfy (2.4), (2.2) and to take values in $(0,\infty)^n$, $\mathbb{P}$-a.s. We shall take

$$\mathcal{T} := \inf\{t \geq 0 | \Lambda(t) = 0\} = \inf\{t \geq 0 | L(t)X(t) = \infty\}$$

in the notation of (3.11), (2.7) and (2.1), and note $\mathbb{P}^{\mathcal{M}}(\mathcal{T} < \infty) = 0$.

7.1. *The Föllmer exit measure.* With this setup, there exists a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$, such that

$$\mathrm{d}\mathbb{P}^{\mathcal{M}} = \Lambda(T)\,\mathrm{d}\mathbb{Q} \qquad \text{holds on each } \mathcal{F}(T), \qquad T \in (0,\infty); \tag{7.1}$$

we express this property (7.1) by writing $\mathbb{P}^{\mathcal{M}} \ll \mathbb{Q}$. Under the measure $\mathbb{Q}$, the process $\widetilde{W}(\cdot)$ of (3.10) is Brownian motion; whereas the processes $\mu_1(\cdot), \dots, \mu_n(\cdot)$ and $\Lambda(\cdot)$ of (3.12), (3.11) in Section 3.1 are nonnegative $\mathbb{Q}$-martingales.

The "absorbing state" Δ acts here as a proxy for $\mathbb{P}^{\mathcal{M}}$-null sets to which the new measure $\mathbb{Q}$ may assign positive mass; the possible existence of such sets makes it important that the filtration $\mathbb{F}$ be "pure," that is, *not* completed by $\mathbb{P}^{\mathcal{M}}$-null sets. This probability measure $\mathbb{Q}$ satisfies

$$\begin{aligned}\frac{\mathbb{E}^{\mathbb{P}^{\mathcal{M}}}[L(T)X(T)]}{x_1 + \cdots + x_n} &= \mathbb{E}^{\mathbb{P}^{\mathcal{M}}}\big[(1/\Lambda(T)) \cdot \mathbf{1}_{\{\mathcal{T} > T\}}\big] \\ &= \mathbb{Q}(\mathcal{T} > T) \qquad \forall T \in [0,\infty)\end{aligned} \tag{7.2}$$

and

$$\mathcal{T} = \inf\left\{t \geq 0 \Big| \int_0^t \|\widetilde{\vartheta}(s, \mathfrak{X})\|^2 \,\mathrm{d}s = \infty\right\}, \qquad \mathbb{Q}\text{-a.s.} \tag{7.3}$$

We also have $\mathbb{Q}$-a.e. on $\{T < \mathcal{T} < \infty\}$

$$L(\mathcal{T} + u)X(\mathcal{T} + u) = \infty \qquad \forall u \geq 0$$

and

$$\int_0^T \|\widetilde{\vartheta}(t, \mathfrak{X})\|^2 \,\mathrm{d}t < \int_0^{\mathcal{T}} \|\widetilde{\vartheta}(t, \mathfrak{X})\|^2 \,\mathrm{d}t = \infty.$$

Whereas, $\mathbb{Q}$-a.e. on $\{\mathcal{T} = \infty\}$, we have

$$L(T)X(T) < \infty, \qquad \int_0^T \|\widetilde{\vartheta}(t, \mathfrak{X})\|^2 \,\mathrm{d}t < \infty; \qquad \forall T \in [0,\infty).$$

We deduce from (7.2) that the arbitrage function of (4.7) for the model $\mathcal{M} \in \mathfrak{M}(\mathbf{x})$ is given by the *probability of "containment" under the measure* $\mathbb{Q}$, namely, the $\mathbb{Q}$-probability that the process $\mathfrak{X}(\cdot)$, started at $\mathbf{x} \in (0,\infty)^n$, stays in $(0,\infty)^n$ throughout the time-horizon $[0, T]$.

At this point we shall impose the following requirements on $\mathbb{K} = \{\mathcal{K}(\mathbf{y})\}_{\mathbf{y}\in\mathfrak{S}_n}$, the family of compact, convex subsets of $\mathbb{R}^n \times \mathbb{S}^n$ in (1.1), (1.2): there exists a constant $0 < C < \infty$, such that for all $\mathbf{y} \in \mathfrak{S}_n$ we have the strengthening

$$\sup_{a\in\mathcal{A}(\mathbf{y})}\left(\sum_{i=1}^{n}\sum_{i=1}^{n} y_i y_j a_{ij}\right) \le C(y_1 + \cdots + y_n)^2 \tag{7.4}$$

of the growth condition in (6.1), as well as the "shear" condition

$$\sup_{(\theta,a)\in\mathcal{K}(\mathbf{y})}\left[\left(\frac{\|\theta\|^2}{1+\mathrm{Tr}(a)}\right) + \left(\frac{\mathrm{Tr}(a)}{1+\|\theta\|^2}\right)\right] \le C. \tag{7.5}$$

Then the following identity holds $\mathbb{Q}$-a.s.:

$$\mathcal{T} = \min_{1\le i\le n} \mathcal{T}_i \qquad \text{where } \mathcal{T}_i := \inf\{t \ge 0 | X_i(t) = 0\}. \tag{7.6}$$

For justification of the claims made in this subsection, we refer to Section 7 in Fernholz and Karatzas (2010a), as well as Delbaen and Schachermayer (1995a), Pal and Protter (2010) and Ruf (2011)—in addition, of course, to the seminal work by Föllmer (1972, 1973).

The special structure of the filtered measurable space $(\Omega, \mathcal{F})$, $\mathbb{F} = \{\mathcal{F}(t)\}_{0\le t<\infty}$ that we selected in this section is indispensable for this construction and for the representation (7.2); whereas the inequality $\|\theta\|^2 \le C(1+\mathrm{Tr}(a))$, $\forall(\theta, a) \in \mathcal{K}(\mathbf{y})$, $\mathbf{y} \in \mathfrak{S}_n$ from condition (7.5) is important for establishing the representation of (7.6).

7.2. *Auxiliary admissible systems.* Let us fix then an initial configuration $\mathbf{x} = (x_1, \dots, x_n)' \in (0,\infty)^n$ and denote by $\mathfrak{N}(\mathbf{x})$ the collection of stochastic systems $\mathcal{N}$ that consist of the filtered measurable space $(\Omega, \mathcal{F})$, $\mathbb{F} = \{\mathcal{F}(t)\}_{0\le t<\infty}$, of a probability measure $\mathbb{Q} \equiv \mathbb{Q}^{\mathcal{N}}$, of an $\mathbb{R}^n$-valued Brownian motion $\widetilde{\mathbb{W}}(\cdot)$ under $\mathbb{Q}$, and of the coördinate mapping process $\mathfrak{X}(t,\omega) = \omega(t)$, $(t,\omega) \in [0,\infty) \times \Omega$ which satisfies $\mathbb{Q}$-a.s. the system of the stochastic equations in (3.9)

$$\begin{aligned} \mathrm{d}X_i(t) &= X_i(t)\left(\frac{\sum_{j=1}^n \alpha_{ij}(t,\mathfrak{X})X_j(t)}{X_1(t)+\cdots+X_n(t)}\,\mathrm{d}t + \sum_{\nu=1}^{n}\sigma_{i\nu}(t,\mathfrak{X})\,\mathrm{d}\widetilde{W}_\nu(t)\right) \\ &= X_i(t)\sum_{\nu=1}^{n}\sigma_{i\nu}(t,\mathfrak{X})\left(\mathrm{d}\widetilde{W}_\nu(t) + \sum_{j=1}^{n}\frac{\sigma_{j\nu}(t,\mathfrak{X})X_j(t)}{X_1(t)+\cdots+X_n(t)}\,\mathrm{d}t\right), \qquad (7.7) \\ & \qquad i = 1, \dots, n. \end{aligned}$$

Here the elements $\sigma_{i\nu} : [0,\infty) \times \Omega \to \mathbb{R}$, $1 \le i, \nu \le n$ of the matrix $\sigma(\cdot,\cdot) = \{\sigma_{i\nu}(\cdot,\cdot)\}_{1\le i,\nu\le n}$ are progressively measurable functionals that satisfy, in the notation of (1.6),

$$\sigma(t,\omega)\sigma'(t,\omega) =: \alpha(t,\omega) \in \mathcal{A}(\omega(t)) \qquad \forall(t,\omega) \in [0,\infty) \times \Omega. \tag{7.8}$$

As in Section 2.3, we shall denote by $\mathfrak{N}_*(\mathbf{x})$ the subcollection of $\mathfrak{N}(\mathbf{x})$ that consists of *Markovian* auxiliary admissible systems, namely, those for which the equations of (7.7) are satisfied with $\alpha(t, \mathfrak{X}) = \mathbf{a}(t, \mathfrak{X}(t))$ and $\sigma(t, \mathfrak{X}) = \mathbf{s}(t, \mathfrak{X}(t))$, $0 \le t < \infty$ and with measurable functions $\mathbf{a} : [0, \infty) \times \mathfrak{S}_n \to \mathbb{S}^n$ and $\mathbf{s} : [0, \infty) \times \mathfrak{S}_n \to \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n)$ that satisfy the condition $\mathbf{s}(t, \mathbf{y})\mathbf{s}'(t, \mathbf{y}) = \mathbf{a}(t, \mathbf{y}) \in \mathcal{A}(y)$, $\forall (t, \mathbf{y}) \in [0, \infty) \times \mathfrak{S}_n$. We invoke the same Markovian selection results as in Section 2.3, to ensure that the process $\mathfrak{X}(\cdot)$ is strongly Markovian under any given $\mathbb{Q}^{\mathcal{N}}$, $\mathcal{N} \in \mathfrak{N}_*(x)$.

By analogy with (7.6), we consider

$$(7.9)\qquad \widehat{\mathcal{T}}(\omega) := \min_{1 \le i \le n} \mathcal{T}_i(\omega) \qquad \text{with } \mathcal{T}_i(\omega) = \inf\{t \ge 0 | \omega_i(t) = 0\}.$$

Then for every $\omega \in \{\widehat{\mathcal{T}} < \infty\}$ we have

$$(7.10)\qquad \int_0^T \operatorname{Tr}(\alpha(t, \omega))\, \mathrm{d}t < \int_0^{\widehat{\mathcal{T}}(\omega)} \operatorname{Tr}(\alpha(t, \omega))\, \mathrm{d}t = \infty \qquad \forall 0 \le T < \widehat{\mathcal{T}}(\omega);$$

whereas $\int_0^T \operatorname{Tr}(\alpha(t, \omega))\, \mathrm{d}t < \infty$, $0 \le T < \infty$ holds for every $\omega \in \{\widehat{\mathcal{T}} = \infty\}$.

REMARK 3. As in Section 2.2, solving the stochastic equation (7.7) subject to condition (7.8) amounts to requiring that the process

$$u(t, \mathfrak{X}(t)) - \int_0^t \left(\frac{\partial u}{\partial s}(s, \mathfrak{X}(s)) + \widehat{\mathcal{L}} u(\mathfrak{X}(s)) \right) \mathrm{d}s, \qquad 0 \le t < \infty$$

be a local supermartingale, for every continuous $u : (0, \infty) \times \mathfrak{S}_n \to \mathbb{R}$ which is of class $\mathcal{C}^{1,2}$ on $(0, \infty) \times (0, \infty)^n$ and has compact support; here $\widehat{\mathcal{L}}$ is the nonlinear second-order partial differential operator in (1.5).

REMARK 4. The total capitalization process $X(\cdot) = X_1(\cdot) + \cdots + X_n(\cdot)$ satisfies, by virtue of (7.7), the equation

$$dX(t) = X(t)[\mathrm{d}\widetilde{N}(t) + \mathrm{d}\langle \widetilde{N} \rangle(t)],$$
$$\widetilde{N}(\cdot) := \sum_{\nu=1}^n \int_0^\cdot \left(\sum_{i=1}^n (X_i(t)/X(t)) \sigma_{i\nu}(t, \mathfrak{X}) \right) \mathrm{d}\widetilde{W}_\nu(t).$$

Under the measure $\mathbb{Q}$, the process $\widetilde{N}(\cdot)$ is a continuous local martingale with quadratic variation

$$\langle \widetilde{N} \rangle(t) = \sum_{i=1}^n \sum_{j=1}^n \int_0^t X_i(s) \alpha_{ij}(s, \mathfrak{X}) X_j(s) \big(X_1(s) + \cdots + X_n(s)\big)^{-2} \mathrm{d}s \le Ct$$

from (7.4), so the total capitalization process

$$X(t) = X(0) \cdot e^{\widetilde{N}(t) + (1/2)\langle \widetilde{N} \rangle(\cdot)} = X(0) \cdot e^{\widetilde{B}(u) + (u/2)}\big|_{u = \langle \widetilde{N} \rangle(t)}, \qquad 0 \le t < \infty$$

takes values in $(0, \infty)$, $\mathbb{Q}$-a.e.; here $\widetilde{B}(\cdot)$ is a one-dimensional $\mathbb{Q}$-Brownian motion. This is in accordance with our selection of the punctured nonnegative orthant $\mathfrak{S}_n$ in (1.1) as the state-space for the process $\mathfrak{X}(\cdot)$ under $\mathbb{Q}$.

Under this measure, the relative weights $\mu_i(\cdot) = X_i(\cdot)/X(\cdot)$, $i = 1, \dots, n$ are nonnegative local martingales and supermartingales, in accordance with (3.12), and since $\sum_{i=1}^n \mu_i(\cdot) \equiv 1$ these processes are bounded, so they are actually martingales. Once any one of the processes $X_1(\cdot), \dots, X_n(\cdot)$ [i.e., any one of the processes $\mu_1(\cdot), \dots, \mu_n(\cdot)$] becomes zero, it stays at zero forever; of course, not all of them can vanish at the same time.

In Section 7.1 we started with an arbitrary admissible system $\mathcal{M} \in \mathfrak{M}(\mathbf{x})$ and produced an "auxiliary" admissible system $\mathcal{N} \in \mathfrak{N}(\mathbf{x})$, for which the property (7.2) holds. Thus, for every $(T, \mathbf{x}) \in (0, \infty) \times (0, \infty)^n$ we deduce

$$Q(T, \mathbf{x}) := \sup_{\mathcal{N} \in \mathfrak{N}(\mathbf{x})} \mathbb{Q}^{\mathcal{N}}(\mathcal{T} > T) \tag{7.11}$$
$$\geq \sup_{\mathcal{M} \in \mathfrak{M}(\mathbf{x})} \left(\frac{\mathbb{E}^{\mathbb{P}^{\mathcal{M}}}[L(T)X(T)]}{x_1 + \cdots + x_n} \right) = \Phi(T, \mathbf{x}).$$

7.3. *Preparatory steps*. We suppose from now onwards that, for every progressively measurable functional $\alpha : [0, \infty) \times \Omega \to \mathbb{S}^n$ which satisfies

$$\alpha(t, \omega) \in \mathcal{A}(\omega(t)) \qquad \text{for all } (t, \omega) \in [0, \infty) \times \Omega, \tag{7.12}$$

we can select a progressively measurable functional $\vartheta : [0, \infty) \times \Omega \to \mathbb{R}^n$ with

$$(\vartheta(t, \omega), \alpha(t, \omega)) \in \mathcal{K}(\omega(t)) \qquad \forall (t, \omega) \in [0, \infty) \times \Omega \tag{7.13}$$

[see the "measurable selection" results in Chapter 7 of Bertsekas and Shreve (1978)]. We introduce now the functional

$$\widetilde{\vartheta}(t, \omega) := \vartheta(t, \omega) - \sigma'(t, \omega)\omega(t)/\big(\omega_1(t) + \cdots + \omega_n(t)\big) \tag{7.14}$$

as in (3.7) and also, by analogy with (7.3), the stopping rule

$$\mathcal{T}(\omega) := \inf\left\{ t \geq 0 \,\Big|\, \int_0^t \|\widetilde{\vartheta}(s, \omega)\|^2 \, \mathrm{d}s = \infty \right\} \tag{7.15}$$

[cf. Levental and Skorohod (1995), where stopping rules of this type also play very important roles in the study of arbitrage].

We recall now (7.10); on the strength of the requirement (7.5), this gives

$$\int_0^T \|\vartheta(t, \omega)\|^2 \, \mathrm{d}t < \int_0^{\widehat{\mathcal{T}}(\omega)} \|\vartheta(t, \omega)\|^2 \, \mathrm{d}t = \infty, \qquad 0 \leq T < \widehat{\mathcal{T}}(\omega) \tag{7.16}$$

for every $\omega \in \{\widehat{\mathcal{T}} < \infty\}$, and $\int_0^T \|\vartheta(t, \omega)\|^2 \, dt < \infty$, $\forall T \in [0, \infty)$ for every $\omega \in \{\widehat{\mathcal{T}} = \infty\}$. In conjunction with (7.4), we obtain from (7.16) that

$$\int_0^T \|\widetilde{\vartheta}(t, \omega)\|^2 \, \mathrm{d}t < \int_0^{\widehat{\mathcal{T}}(\omega)} \|\widetilde{\vartheta}(t, \omega)\|^2 \, \mathrm{d}t = \infty, \qquad 0 \leq T < \widehat{\mathcal{T}}(\omega)$$

holds for every $\omega \in \{\widehat{\mathcal{T}} < \infty\}$, and that $\int_0^T \|\widetilde{\vartheta}(t,\omega)\|^2 \, \mathrm{d}t < \infty$, $\forall T \in [0,\infty)$ holds for every $\omega \in \{\widehat{\mathcal{T}} = \infty\}$.

We deduce for the stopping rules of (7.15) and (7.9) the identification $\widehat{\mathcal{T}}(\omega) = \mathcal{T}(\omega)$.

7.4. *The same thread, in reverse.* Let us fix now a stochastic system $\mathcal{N} \in \mathfrak{N}(\mathbf{x})$ as in Section 7.2, pick a progressively measurable functional $\alpha : [0,\infty) \times \Omega \to \mathbb{S}^n$ with $\alpha(t,\omega) \in \mathcal{A}(\omega(t))$ for all $(t,\omega) \in [0,\infty) \times \Omega$ and select a progressively measurable functional $\vartheta : [0,\infty) \times \Omega \to \mathbb{R}^n$ as in (7.13). For *this* $\vartheta(\cdot,\cdot)$ and *this* $\mathcal{N} \in \mathfrak{N}(\mathbf{x})$, we define $\widetilde{\vartheta}(\cdot,\cdot)$ by (7.14) as well as

$$\Lambda(t) = \exp\left\{\int_0^t (\widetilde{\vartheta}(s,\mathfrak{X}))' \, \mathrm{d}\widetilde{W}(s) - \frac{1}{2}\int_0^t \|\widetilde{\vartheta}(s,\mathfrak{X})\|^2 \, \mathrm{d}s\right\} \quad \text{for } 0 \le t < \mathcal{T} \tag{7.17}$$

as in (3.11), and set

$$\Lambda(\mathcal{T} + u) = 0 \qquad \text{for } u \ge 0 \text{ on } \{\mathcal{T} < \infty\} \tag{7.18}$$

in the notation of (7.15). The resulting process $\Lambda(\cdot)$ is a local martingale and a supermartingale under $\mathbb{Q}$, and we have $\mathcal{T}(\mathfrak{X}) = \inf\{t \ge 0 | \Lambda(t) = 0\}$, $\mathbb{Q}$-a.e.

We introduce also the sequence of $\mathbb{F}$-stopping rules

$$S_n(\omega) := \inf\left\{t \ge 0 \Big| \int_0^t \|\widetilde{\vartheta}(s,\omega)\|^2 \, \mathrm{d}s \ge 2\log n\right\}, \qquad n \in \mathbb{N},$$

which satisfy $\lim_{n\to\infty} \uparrow S_n(\omega) = \mathcal{T}(\omega)$ and $\exp\{\frac{1}{2}\int_0^{S_n(\omega)} \|\widetilde{\vartheta}(t,\omega)\|^2 \, \mathrm{d}t\} \le n$ ($\forall n \in \mathbb{N}$), for every $\omega \in \Omega$. From Novikov's theorem [e.g., Karatzas and Shreve (1991), page 198], $\Lambda(\cdot \wedge S_n)$ is a uniformly integrable $\mathbb{Q}$-martingale; in particular, $\mathbb{E}^{\mathbb{Q}}(\Lambda(S_n)) = 1$ holds for every $n \in \mathbb{N}$. Thus, the recipe

$$\mathbb{P}_n(A) := \mathbb{E}^{\mathbb{Q}}[\Lambda(S_n) \cdot \mathbf{1}_A], \qquad A \in \mathcal{F}(S_n)$$

defines a consistent sequence, or "tower," of probability measures $\{\mathbb{P}_n\}_{n\in\mathbb{N}}$ on $(\Omega,\mathcal{F})$. Appealing to the results in Parthasarathy (1967), pages 140–143 [see also the Appendix of Föllmer (1972)], we deduce the existence of a probability measure $\mathbb{P}$ on $(\Omega,\mathcal{F})$ such that

$$\mathbb{P}(A) = \mathbb{P}_n(A) = \mathbb{E}^{\mathbb{Q}}[\Lambda(S_n) \cdot \mathbf{1}_A] \qquad \text{holds for every } A \in \mathcal{F}(S_n), n \in \mathbb{N}. \tag{7.19}$$

(Here again, the special structure imposed in this section on the filtered measurable space $(\Omega,\mathcal{F}), \mathbb{F} = \{\mathcal{F}(t)\}_{0\le t<\infty}$ is indispensable.) Therefore, for every $T \in (0,\infty)$ we have

$$\mathbb{P}(S_n > T) = \mathbb{E}^{\mathbb{Q}}[\Lambda(S_n) \cdot \mathbf{1}_{\{S_n > T\}}] = \mathbb{E}^{\mathbb{Q}}[\Lambda(T) \cdot \mathbf{1}_{\{S_n > T\}}]$$

by optional sampling, whereas monotone convergence leads to

$$\mathbb{P}(\mathcal{T} > T) = \mathbb{E}^{\mathbb{Q}}[\Lambda(T)\mathbf{1}_{\{\mathcal{T} > T\}}]. \tag{7.20}$$

The following result echoes similar themes in Cheridito, Filipović and Yor (2005).

LEMMA 1. *The process* $\Lambda(\cdot)$ *of* (7.17), (7.18) *is a* $\mathbb{Q}$-*martingale, if and only if we have*

$$\mathbb{P}(\mathcal{T} < \infty) = 0$$

[*i.e., if and only if the process* $\mathfrak{X}(\cdot)$ *never hits the boundary of the orthant* $(0, \infty)^n$, $\mathbb{P}$-*a.s.*].

PROOF. If $\mathbb{P}(\mathcal{T} < \infty) = 0$ holds, the nonnegativity of $\Lambda(\cdot)$ and (7.20) give

$$1 = \mathbb{P}(\mathcal{T} > T) = \mathbb{E}^{\mathbb{Q}}[\Lambda(T)\mathbf{1}_{\{\mathcal{T}>T\}}] \le \mathbb{E}^{\mathbb{Q}}[\Lambda(T)] \qquad \forall T \in (0, \infty).$$

But $\Lambda(\cdot)$ is a $\mathbb{Q}$-supermartingale, so the reverse inequality $\mathbb{E}^{\mathbb{Q}}[\Lambda(T)] \le \Lambda(0) = 1$ also holds. We conclude that $\mathbb{E}^{\mathbb{Q}}[\Lambda(T)] = 1$ holds for all $T \in (0, \infty)$, so $\Lambda(\cdot)$ is a $\mathbb{Q}$-martingale.

If, on the other hand, $\Lambda(\cdot)$ is a $\mathbb{Q}$-martingale, then $\mathbb{E}^{\mathbb{Q}}[\Lambda(T)] = 1$ and (7.20) give

$$\begin{aligned}\mathbb{P}(\mathcal{T} \le T) &= \mathbb{E}^{\mathbb{Q}}(\Lambda(T)) - \mathbb{E}^{\mathbb{Q}}(\Lambda(T)\mathbf{1}_{\{\mathcal{T}>T\}}) = \mathbb{E}^{\mathbb{Q}}(\Lambda(T)\mathbf{1}_{\{\mathcal{T}\le T\}}) \\ &= \mathbb{E}^{\mathbb{Q}}(\Lambda(\mathcal{T})\mathbf{1}_{\{\mathcal{T}\le T\}}) = 0\end{aligned}$$

for every $T \in [0, \infty)$, from Optional Sampling and the fact that $\Lambda(\mathcal{T}) = 0$ holds $\mathbb{Q}$-a.e. on $\{\mathcal{T} < \infty\}$. We conclude $\mathbb{P}(\mathcal{T} < \infty) = 0$; in conjunction with the identification $\mathcal{T} \equiv \widehat{\mathcal{T}}$ and (7.9), this means that the coördinate mapping process $\mathfrak{X}(\cdot)$ never reaches the boundary of (i.e., takes values in) the strictly positive orthant $(0, \infty)^n$, $\mathbb{P}$-a.e. □

When the conditions of Lemma 1 prevail, the process

$$W(\cdot) = \widetilde{W}(\cdot) - \int_0^{\cdot} \widetilde{\vartheta}(t, \mathfrak{X})\,\mathrm{d}t$$

is Brownian motion under the probability measure $\mathbb{P} \equiv \mathbb{P}^{\mathcal{M}}$ introduced in (7.19). This measure satisfies the equations of (7.2), whereas the process $\mathfrak{X}(\cdot)$ solves $\mathbb{P}^{\mathcal{M}}$-a.s. the system

$$dX_i(t) = X_i(t)\sum_{\nu=1}^{n} \sigma_{i\nu}(t, \mathfrak{X})[\mathrm{d}W_\nu(t) + \vartheta_\nu(t, \mathfrak{X})\,\mathrm{d}t], \qquad X_i(0) = x_i > 0$$

for $i = 1, \dots, n$, as in (2.4). It is then not hard to check that $L(\cdot)$ defined by (2.7) satisfies $\mathbb{P}^{\mathcal{M}}$-a.s. the identity $L(\cdot)X(\cdot) = (x_1 + \cdots + x_n)/\Lambda(\cdot)$ in accordance with (3.11).

We formalize these considerations as follows.

ASSUMPTION C. Suppose that the collection of sets $\mathbb{K}$ in (1.1) satisfies (7.4), (7.5) and that for any given progressively measurable functional $\alpha : [0,\infty) \times \Omega \to \mathbb{S}^n$ which satisfies (7.12) and

$$\int_0^T \mathrm{Tr}(\alpha(t,\mathfrak{w}))\,\mathrm{d}t < \infty \qquad \text{for all } (T,\mathfrak{w}) \in [0,\infty) \times \mathfrak{W}, \tag{7.21}$$

there exists a progressively measurable functional $\vartheta : [0,\infty) \times \Omega \to \mathbb{R}^n$ that satisfies the condition (7.13), thus also by virtue of (7.5)

$$\int_0^T \|\vartheta(t,\mathfrak{w})\|^2\,\mathrm{d}t < \infty \qquad \text{for all } (T,\mathfrak{w}) \in [0,\infty) \times \mathfrak{W}. \tag{7.22}$$

The analysis of this subsection shows that, under Assumption C and starting with any initial configuration $\mathbf{x} = (x_1,\dots,x_n)' \in (0,\infty)^n$ and with an arbitrary "auxiliary" admissible stochastic system $\mathcal{N} = ((\Omega,\mathcal{F}),\mathbb{F},\mathbb{Q},\mathfrak{X}(\cdot),\widetilde{W}(\cdot))$ in $\mathfrak{N}(\mathbf{x})$ as in Section 7.2, the process $\Lambda(\cdot)$ of (7.18) is a $\mathbb{Q}$-martingale, and we can construct a "primal" admissible system $\mathcal{M} \in \mathfrak{M}(\mathbf{x})$ as in Section 2.1 [i.e., with the canonical process $\mathfrak{X}(\cdot)$ taking values in $(0,\infty)^n$, $\mathbb{P}^{\mathcal{M}}$-a.s.], for which (7.2) holds, and we have $\mathbb{P}^{\mathcal{M}} \ll \mathbb{Q}$ as in (7.1). We deduce

$$Q(T,\mathbf{x}) = \sup_{\mathcal{N}\in\mathfrak{N}(\mathbf{x})} \mathbb{Q}^{\mathcal{N}}(\mathcal{T} > T) \le \sup_{\mathcal{M}\in\mathfrak{M}(\mathbf{x})} \left(\frac{\mathbb{E}^{\mathbb{P}^{\mathcal{M}}}[L(T)X(T)]}{x_1+\cdots+x_n}\right) = \Phi(T,\mathbf{x}).$$

The reverse inequality $Q(T,\mathbf{x}) \ge \Phi(T,\mathbf{x})$ was established in (7.11). This way, for every function $U : [0,\infty) \times (0,\infty)^n \to (0,\infty)$ in the collection $\mathcal{U}$, we can strengthen (5.9) to

$$U(T,\mathbf{x}) \ge \mathfrak{u}(T,\mathbf{x}) \ge \Phi(T,\mathbf{x}) = Q(T,\mathbf{x}) \qquad \forall (T,\mathbf{x}) \in (0,\infty) \times (0,\infty)^n. \tag{7.23}$$

We have established the following result.

PROPOSITION 4. *Recall the functions* $\mathfrak{u}(\cdot,\cdot)$, $\Phi(\cdot,\cdot)$ *and* $Q(\cdot,\cdot)$ *defined on* $(0,\infty) \times (0,\infty)^n$ *by* (4.1), (4.4) *and* (7.11), *respectively, and impose Assumption* C. *Then* (7.23) *holds for every function* $U(\cdot,\cdot) \in \mathcal{U}$.

REMARK 5. Here is a situation where Assumption C prevails: Suppose that (7.4) holds and that, for every $\mathbf{z} \in (0,\infty)^n$ and $a \in \mathcal{A}(\mathbf{z})$, we have $(\theta, a) \in \mathcal{K}(\mathbf{z})$ for θ given by $\theta_\nu = \sum_{j=1}^n s_{j\nu}$, $\nu = 1,\dots,n$ and $ss' = a$. Then for any progressively measurable $\alpha : [0,\infty) \times \Omega \to \mathbb{S}^n$ that satisfies (7.21) we select the progressively measurable functional $\vartheta : [0,\infty) \times \Omega \to \mathbb{R}^n$ via $\vartheta_\nu(t,\omega) = \sum_{j=1}^n \sigma_{j\nu}(t,\omega)$, $\nu = 1,\dots,n$. This choice induces

$$\widetilde{\vartheta}_\nu(t,\omega) = \sum_{i=1}^n \left(1 - \frac{\omega_i(t)}{\omega_1(t)+\cdots+\omega_n(t)}\right)\sigma_{i\nu}(t,\omega), \qquad \nu = 1,\dots,n$$

which obeys $\int_0^T \|\widetilde{\vartheta}(t,\mathfrak{w})\|^2\,dt < \infty$ as in (7.22) for all $(T,\mathfrak{w}) \in [0,\infty) \times \mathfrak{W}$; the process $\Lambda(\cdot)$ of (7.17) and (7.18) is a $\mathbb{Q}$-martingale, whereas (2.4) becomes

$$\mathrm{d}X_i(t) = X_i(t)\left[\sum_{\nu=1}^n \sigma_{i\nu}(t,\mathfrak{X})\,\mathrm{d}W_\nu(t) + \left(\sum_{j=1}^n \alpha_{ij}(t,\mathfrak{X})\right)\mathrm{d}t\right], \qquad X_i(0) = x_i > 0$$

for $i = 1,\dots,n$. The condition (7.21) guarantees now that $\mathfrak{X}(\cdot)$ takes values in $(0,\infty)^n$, $\mathbb{P}^{\mathcal{M}}$-a.s. in the resulting primal admissible system $\mathcal{M} \in \mathfrak{M}(\mathbf{x})$.

8. Dynamic programming. The quantity $Q(T,\mathbf{x})$ defined in (7.11) is the value of a stochastic control problem: namely, the *maximal "containment" probability*, over all measures $\mathbb{Q}^{\mathcal{N}}$ with $\mathcal{N} \in \mathfrak{N}(\mathbf{x})$, that the process $\mathfrak{X}(\cdot)$ with dynamics (7.7), initial configuration $\mathfrak{X}(0) = \mathbf{x} \in (0,\infty)^n$, and controlled through the choice of progressively measurable functional $\alpha(\cdot,\cdot)$ as in (7.8), (7.10), does *not* hit the boundary of the positive orthant by time T.

Let us suppose that the resulting function $Q(\cdot,\cdot)$ is continuous on $(0,\infty) \times (0,\infty)^n$. Then it can be checked [as in Lions (1984), Lemma II.1 and Lions (1983a), Theorem II.4] that it satisfies as well the following dynamic programming principle: *for every initial configuration* $\mathbf{x} \in (0,\infty)^n$, *the process*

$$Q\big(T-t,\mathfrak{X}(t)\big)\mathbf{1}_{\{\mathcal{T}>t\}}, \qquad (8.1)$$
$$0 \le t \le T \textit{ is a } \mathbb{Q}^{\mathcal{N}}\textit{-supermartingale}, \forall \mathcal{N} \in \mathfrak{N}(\mathbf{x}).$$

[See El Karoui, Hŭù Nguyen and Jeanblanc-Picqué (1987), Haussmann and Lepeltier (1990), Fleming and Soner (1993) and Krylov (1980) for results in a similar vein.] Equivalently, the process $L(t)X(t)Q(T-t,\mathfrak{X}(t))$, $0 \le t \le T$ is a $\mathbb{P}^{\mathcal{M}}$-supermartingale, for every $\mathcal{M} \in \mathfrak{M}(\mathbf{x})$.

Consider now an arbitrary continuous function $\breve{U} : [0,\infty) \times (0,\infty)^n \to [0,\infty)$ which satisfies $\breve{U}(0,\cdot) \equiv 1$ on $(0,\infty)^n$, and is such that, for every $\mathcal{N} \in \mathfrak{N}(\mathbf{x})$ and $\mathbf{x} \in (0,\infty)^n$, the process $\breve{U}(T-t,\mathfrak{X}(t))\mathbf{1}_{\{\mathcal{T}>t\}}, 0 \le t \le T$ is a $\mathbb{Q}^{\mathcal{N}}$-supermartingale. We shall denote by $\breve{\mathcal{U}}$ the collection of all such functions and note that $\mathcal{U} \subseteq \breve{\mathcal{U}}$ and $Q \in \breve{\mathcal{U}}$. From optional sampling we have then for every $\mathcal{N} \in \mathfrak{N}(\mathbf{x})$ the comparisons

$$\breve{U}(T,\mathbf{x}) \ge \mathbb{E}^{\mathbb{Q}^{\mathcal{N}}}\big[\breve{U}(0,\mathfrak{X}(T))\mathbf{1}_{\{\mathcal{T}>T\}}\big] = \mathbb{Q}^{\mathcal{N}}(\mathcal{T} > T),$$

thus also $\breve{U}(T,\mathbf{x}) \ge Q(T,\mathbf{x})$, $\forall (T,\mathbf{x}) \in (0,\infty) \times (0,\infty)^n$.

In other words, *the function* $Q(\cdot,\cdot)$ *defined in* (7.11) *is the smallest element of the collection* $\breve{\mathcal{U}}$. It is also clear from this line of reasoning that $\mathcal{N}_o \in \mathfrak{N}(\mathbf{x})$ attains the supremum $Q(T,\mathbf{x}) = \sup_{\mathcal{N}\in\mathfrak{N}(\mathbf{x})} \mathbb{Q}^{\mathcal{N}}(\mathcal{T} > T)$ in (7.11), if and only if the process $Q(T-t,\mathfrak{X}(t))\mathbf{1}_{\{\mathcal{T}>t\}}, 0 \le t \le T$ is a $\mathbb{Q}^{\mathcal{N}_o}$-martingale.

THEOREM 1. *Suppose that Assumption C and conditions (2.9)–(2.11) hold and that the function $Q(\cdot,\cdot)$ of (7.11) is continuous on $(0,\infty)\times(0,\infty)^n$. Then the infimum in (4.1) is attained, and*

$$\mathfrak{u}(T,\mathbf{x}) = \Phi(T,\mathbf{x}) = Q(T,\mathbf{x}) \qquad \forall (T,\mathbf{x}) \in (0,\infty)\times(0,\infty)^n. \tag{8.2}$$

PROOF. Consider an arbitrary function $\breve{U}(\cdot,\cdot)$ in the collection $\breve{\mathcal{U}}$ just defined and fix an arbitrary pair $(T,\mathbf{x})\in(0,\infty)\times(0,\infty)^n$; then for every $\varepsilon>0$, consider a mollification $U_\varepsilon(\cdot,\cdot)\in\mathcal{U}$ of the function $\breve{U}(\cdot,\cdot)$ with $U_\varepsilon(T,\mathbf{x})\le\breve{U}(T,\mathbf{x})+\varepsilon$.

Proposition 2 gives then $\mathfrak{u}(T,\mathbf{x})\le\breve{U}(T,\mathbf{x})+\varepsilon$. Since $\varepsilon>0$ is arbitrary, this shows that $\mathfrak{u}(T,\mathbf{x})$ is dominated by $Q(T,\mathbf{x})$, the infimum of $\breve{U}(T,\mathbf{x})$ over all functions $\breve{U}(\cdot,\cdot)\in\breve{\mathcal{U}}$. But the reverse inequality $\mathfrak{u}(T,\mathbf{x})\ge Q(T,\mathbf{x})$ holds on the strength of (7.23), so (8.2) follows. □

8.1. *The HJB equation.* Under the conditions of Theorem 1, the arbitrage function $\mathfrak{u}(\cdot,\cdot)$ is equal to the function $Q(\cdot,\cdot)$ of (7.11) and is continuous on $(0,\infty)\times(0,\infty)^n$. Thanks to the dynamic programming principle of (8.1), it is also a viscosity solution of the *Hamilton–Jacobi–Bellman* (*HJB*) *equation*

$$\frac{\partial U}{\partial \tau}(\tau,\mathbf{z}) = \sup_{a\in\mathcal{A}(\mathbf{z})}\sum_{i=1}^n\sum_{j=1}^n z_i z_j a_{ij}\left(\frac{1}{2}D^2_{ij}U(\tau,\mathbf{z}) + \frac{D_i U(\tau,\mathbf{z})}{z_1+\cdots+z_n}\right) \tag{8.3}$$

on $(0,\infty)\times(0,\infty)^n$ [cf. Lions (1984), Theorem III.1 or Lions (1983b), Theorem I.1].

If in addition to being continuous, as we assumed in Theorem 1, the function $Q(\cdot,\cdot)$ of (7.11) is of class $\mathcal{C}^{1,2}$ locally on $(0,\infty)\times(0,\infty)^n$, then the arbitrage function $\mathfrak{u}(\cdot,\cdot)$ is not only a viscosity solution but actually a classical solution of the HJB equation (8.3). This is the case, for instance, under the combined conditions of Theorem 1 and Proposition 3; then the arbitrage function $\mathfrak{u}(\cdot,\cdot)$ also satisfies on the domain $(0,\infty)\times(0,\infty)^n$ the *linear* parabolic equation

$$\frac{\partial U}{\partial \tau}(\tau,\mathbf{z}) = \sum_{i=1}^n\sum_{j=1}^n z_i z_j \mathbf{a}_{ij}(\mathbf{z})\left(\frac{1}{2}D^2_{ij}U(\tau,\mathbf{z}) + \frac{D_i U(\tau,\mathbf{z})}{z_1+\cdots+z_n}\right) \tag{8.4}$$

with $\mathbf{a}:(0,\infty)^n\to\mathbb{S}^n$ as in Assumption A or Remark 2, in addition to the initial condition

$$U(0,\cdot)\equiv 1 \qquad \text{on } (0,\infty)^n. \tag{8.5}$$

In particular, the arbitrage function $\mathfrak{u}(\cdot,\cdot)$ satisfies, in this case, the requirement (6.10) and belongs to the class $\mathcal{U}$ of Section 5.

Recalling Propositions 1–3 and Theorem 1, we summarize the above discussion as follows.

THEOREM 2. *Suppose that conditions* (2.9)–(2.11), (6.1), (6.2) *and Assumptions* A, B *and* C *are in force*.

Then the arbitrage function $\mathfrak{u}(\cdot,\cdot)$ *is the smallest element of the class* $\mathcal{U}$, *as well as a classical solution of both the HJB equation* (8.3) *and of the linear parabolic equation* (8.4), *subject to* (8.5). *Furthermore* (8.2) *holds*, *the infimum in* (4.1) *is attained*, *and the Markovian investment rule* $\pi^U(\cdot,\cdot)$ *in* (5.10) *with* $U \equiv \mathfrak{u}$ *satisfies* (5.11) *for every admissible system* $\mathcal{M} \in \mathfrak{M}(\mathbf{x})$.

REMARK 6. We note that Theorem 2 is in agreement with general regularity theory for fully nonlinear parabolic equations, as in Lions (1983c), Theorem II.4 (see also Krylov (1987), Section 6.5; Krylov (1990); Wang (1992a, 1992b, 1992c), Theorems II.3.2 and III.2; or Lieberman (1996), Chapter XIV).

As we mentioned already, Assumptions A and B can be replaced in Theorem 2 by the conditions of Remark 2. We conjecture that the conclusions of Theorem 2 should hold under even weaker assumptions but leave this issue for future research.

We also remark that the function $V(t,\mathbf{z}) := (z_1 + \cdots + z_n)\mathfrak{u}(t,\mathbf{z})$, $(\tau, z) \in (0,\infty) \times (0,\infty)^n$ satisfies an HJB-type equation simpler than (8.3), namely, the *Pucci maximal equation*,

$$\frac{\partial V}{\partial \tau}(\tau,\mathbf{z}) = \frac{1}{2} \sup_{a \in \mathcal{A}(\mathbf{z})} \sum_{i=1}^{n} \sum_{j=1}^{n} z_i z_j a_{ij} D^2_{ij} V(\tau,\mathbf{z}), \tag{8.6}$$

along with the initial condition $V(0,\mathbf{z}) = z_1 + \cdots + z_n$. In the setting of Theorem 2, equation (8.6) reduces to

$$\frac{\partial V}{\partial \tau}(\tau,\mathbf{z}) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} z_i z_j \mathbf{a}_{ij}(\mathbf{z}) D^2_{ij} V(\tau,\mathbf{z}).$$

8.2. *An example*. Let us go back to the volatility-stabilized model introduced in Fernholz and Karatzas (2005), but now with some “Knightian” uncertainty regarding its volatility structure

$$1 \le \alpha_{ii}(t)\mu_i(t) \le 1 + \delta, \qquad 0 \le t < \infty$$

for some given $\delta \ge 0$. The case $\delta = 0$ corresponds to the variance structure of the model studied in Fernholz and Karatzas (2005, 2009).

More specifically let us assume that, for any given $\mathbf{y} \in \mathfrak{S}_n$, the compact, convex subset $\mathcal{A}(\mathbf{y})$ of $\mathbb{S}^n$ in (1.6) consists of all matrices $a = \{a_{ij}\}_{1\le i,j\le n}$ with $a_{ij} = 0$ for $j \ne i$ and

$$y_i a_{ii} = \eta^2 (y_1 + \cdots + y_n); \qquad i = 1, \dots, n, 1 \le \eta \le 1 + \delta.$$

The sets of (1.1) are given as

$$\mathcal{K}(\mathbf{y}) = \{(\theta, a) | a \in \mathcal{A}(\mathbf{y}), \theta = (\zeta\sqrt{a_{11}}, \dots, \zeta\sqrt{a_{nn}})' \text{ with } \zeta \in [\sqrt{C_1}, \sqrt{C_2}]\}$$

for some given constants $C_1 \in (0, 1]$, $C_2 \in (1, \infty)$; these choices satisfy (7.4), (7.5).

Condition (6.2) is satisfied in this case automatically (in fact, with $\lambda \equiv 1$), as are (6.1) and (6.6): it suffices to take

$$\mathbf{a}_{ii}(\mathbf{z}) = (z_1 + \cdots + z_n)/z_i, \qquad i = 1, \ldots, n \tag{8.7}$$

and $H(\mathbf{z}) = \sum_{i=1}^n \log z_i$, which induces $\boldsymbol{\theta}_i(z) = \sqrt{\mathbf{a}_{ii}(\mathbf{z})}$ in (6.3). These functions are all locally bounded and locally Lipschitz continuous on $(0, \infty)^n$.

The HJB equation (8.3) satisfied by the arbitrage function $\mathfrak{u}(\cdot, \cdot)$ becomes

$$\frac{\partial U}{\partial \tau}(\tau, \mathbf{z}) = \sup_{1 \le \eta \le 1+\delta} \left[\eta^2 \left\{ \frac{1}{2} \sum_{i=1}^n (z_1 + \cdots + z_n) z_i D_{ii}^2 U(\tau, \mathbf{z}) + \sum_{i=1}^n z_i D_i U(\tau, \mathbf{z}) \right\} \right],$$

and reduces to the *linear* parabolic equation

$$\frac{\partial U}{\partial \tau}(\tau, \mathbf{z}) = \frac{1}{2} \sum_{i=1}^n (z_1 + \cdots + z_n) z_i D_{ii}^2 U(\tau, \mathbf{z}) + \sum_{i=1}^n z_i D_i U(\tau, \mathbf{z}) \tag{8.8}$$

of (8.4) for the choice of variances in (8.7). The reason for this reduction is that the expression on the left-hand side of (8.8) is negative, so we have considerable simplification in this case.

REMARK 7. In this example, the arbitrage function $\mathfrak{u}(\cdot, \cdot)$ can be represented as

$$\mathfrak{u}(T, \mathbf{z}) = \frac{z_1 \cdots z_n}{z_1 + \cdots + z_n} \mathbb{E}\left[\frac{X_1(T) + \cdots + X_n(T)}{X_1(T) \cdots X_n(T)} \right]$$

in terms of the components of the $(0, \infty)^n$-valued capitalization process $\mathfrak{X}(\cdot) = (X_1(\cdot), \ldots, X_n(\cdot))'$. These are now time-changed versions $X_i(\cdot) = \Psi_i(A(\cdot))$, $i = 1, \ldots, n$ of the independent squared-Bessel processes

$$\mathrm{d}\Psi_i(u) = 4u\, \mathrm{d}u + 2\sqrt{\Psi_i(u)}\, \mathrm{d}\beta_i(u), \qquad \Psi_i(0) = z_i,$$

run with a time change $A(t) = (1/4) \int_0^t (X_1(s) + \cdots + X_n(s))\, \mathrm{d}s$ common for all components, and with $\beta_1(\cdot), \ldots, \beta_n(\cdot)$ independent standard Brownian motions [see Fernholz and Karatzas (2005, 2009), Goia (2009) and Pall (2011) for more details].

9. A stochastic game. For any given investment rule $\Pi \in \mathfrak{P}$ and admissible system $\mathcal{M} \in \mathfrak{M}(\mathbf{x})$, let us consider the quantity

$$\xi_{\Pi, \mathcal{M}}(T, \mathbf{x}) := \inf\{ r > 0 : \mathbb{P}^{\mathcal{M}}(Z^{rX(0), \Pi}(T) \ge X(T)) = 1 \}. \tag{9.1}$$

This measures, as a proportion of the initial total market capitalization, the smallest initial capital that an investor who uses the rule Π and operates within the market model $\mathcal{M}$, needs to set aside at time $t = 0$ in order for his wealth to be able to "catch up with the market portfolio" by time $t = T$, with $\mathbb{P}^{\mathcal{M}}$-probability one.

Our next result exhibits the arbitrage function $\mathfrak{u}(\cdot, \cdot)$ of (4.1) as the min–max value of a zero-sum stochastic game between two players: the investor, who tries to select the rule $\Pi \in \mathfrak{P}$ so as to make the quantity of (9.1) as small as possible and "nature," or the goddess Tyche herself, who tries to thwart him by choosing the admissible system or "model" $\mathcal{M} \in \mathfrak{M}(\mathbf{x})$ to his detriment.

THEOREM 3. *Under the conditions of Theorem* 1, *we have*

$$\mathfrak{u}(T, \mathbf{x}) = \inf_{\Pi \in \mathfrak{P}} \Big(\sup_{\mathcal{M} \in \mathfrak{M}(\mathbf{x})} \xi_{\Pi, \mathcal{M}}(T, \mathbf{x}) \Big) = \sup_{\mathcal{M} \in \mathfrak{M}(\mathbf{x})} \Big(\inf_{\Pi \in \mathfrak{P}} \xi_{\Pi, \mathcal{M}}(T, \mathbf{x}) \Big). \tag{9.2}$$

PROOF. For the quantities of (9.1) and (4.7) we claim

$$\xi_{\Pi, \mathcal{M}}(T, \mathbf{x}) \geq \mathfrak{u}_{\mathcal{M}}(T, \mathbf{x}) \qquad \forall (\Pi, \mathcal{M}) \in \mathfrak{P} \times \mathfrak{M}(\mathbf{x}). \tag{9.3}$$

Indeed, if the set on the right-hand side of (9.1) is empty, we have $\xi_{\Pi, \mathcal{M}}(T, \mathbf{x}) = \infty$ and nothing to prove; if, on the other hand, this set is not empty, then for any of its elements $r > 0$ the process $L(\cdot) V^{rX(0), \Pi}(\cdot)$ is a $\mathbb{P}^{\mathcal{M}}$-supermartingale, and therefore (4.6), that is, $r \geq \mathfrak{u}_{\mathcal{M}}(T, \mathbf{x})$, still holds and (9.3) follows again.

Taking the infimum with respect to $\Pi \in \mathfrak{P}$ on the left-hand side of (9.3), then the supremum of both sides with respect to $\mathcal{M} \in \mathfrak{M}(\mathbf{x})$, we obtain

$$\underline{G}(T, \mathbf{x}) := \sup_{\mathcal{M} \in \mathfrak{M}(\mathbf{x})} \Big(\inf_{\Pi \in \mathfrak{P}} \xi_{\Pi, \mathcal{M}}(T, \mathbf{x}) \Big) \geq \sup_{\mathcal{M} \in \mathfrak{M}(\mathbf{x})} \mathfrak{u}_{\mathcal{M}}(T, \mathbf{x}) = \Phi(T, \mathbf{x})$$

from (4.7). The quantity $\underline{G}(T, \mathbf{x})$ is the lower value of the stochastic game under consideration.

In order to complete the proof of (9.2) it suffices, on the strength of Theorem 1, to show that the upper value

$$\overline{G}(T, \mathbf{x}) := \inf_{\Pi \in \mathfrak{P}} \Big(\sup_{\mathcal{M} \in \mathfrak{M}(\mathbf{x})} \xi_{\Pi, \mathcal{M}}(T, \mathbf{x}) \Big) \geq \underline{G}(T, \mathbf{x})$$

of this game satisfies

$$\overline{G}(T, \mathbf{x}) \leq \mathfrak{u}(T, \mathbf{x}). \tag{9.4}$$

To see this, we introduce for each given investment rule $\Pi \in \mathfrak{P}$ the quantity

$$\mathfrak{h}_{\Pi}(T, \mathbf{x}) := \inf \{ r > 0 : \mathbb{P}^{\mathcal{M}} \big(Z^{rX(0), \Pi}(T) \geq X(T) \big) = 1, \forall \mathcal{M} \in \mathfrak{M}(\mathbf{x}) \};$$

that is, the smallest proportion $r > 0$ of the initial market capitalization that allows an investor using the rule Π to be able to "catch up with the market portfolio" by time $t = T$ with $\mathbb{P}^{\mathcal{M}}$-probability one, no matter which admissible system (model) $\mathcal{M}$ might materialize. We have clearly

$$\mathfrak{h}_{\Pi}(T, \mathbf{x}) \geq \mathfrak{u}(T, \mathbf{x}) \vee \xi_{\Pi, \mathcal{M}}(T, \mathbf{x}) \qquad \forall (\Pi, \mathcal{M}) \in \mathfrak{P} \times \mathfrak{M}(\mathbf{x}), \tag{9.5}$$

which leads to

$$\mathfrak{u}(T,\mathbf{x}) = \inf_{\Pi\in\mathfrak{P}} \mathfrak{h}_{\Pi}(T,\mathbf{x}) \geq \inf_{\Pi\in\mathfrak{P}} \Big(\sup_{\mathcal{M}\in\mathfrak{M}(\mathbf{x})} \xi_{\Pi,\mathcal{M}}(T,\mathbf{x}) \Big) = \overline{G}(T,\mathbf{x}) \tag{9.6}$$

and proves (9.4). □

9.1. *A least favorable model and the investor's best response.* Let us place ourselves now in the context of Theorem 2 and observe that Proposition 1, along with Proposition 2 and its Corollary, yields

$$\xi_{\Pi,\mathcal{M}_o}(T,\mathbf{x}) \geq \mathfrak{u}_{\mathcal{M}_o}(T,\mathbf{x}) = \Phi(T,\mathbf{x}) = \xi_{\Pi_o,\mathcal{M}_0}(T,\mathbf{x}) \qquad \forall \Pi \in \mathfrak{P}, \tag{9.7}$$

by virtue of (9.3) for $\mathcal{M} \equiv \mathcal{M}_o$ and of (5.11) for $U \equiv \Phi$. Here $\mathcal{M}_o$ is the "least favorable admissible system" that attains the supremum over $\mathfrak{M}(\mathbf{x})$ in (4.4), and $\Pi_o \equiv \pi^{\Phi}$ denotes the investment rule of (5.10) with $U \equiv \Phi$.

In this setting, the investment rule $\Pi_o \in \mathfrak{P}$ attains the infimum $\inf_{\Pi\in\mathfrak{P}} \mathfrak{h}_{\Pi}(T, \mathbf{x}) = \mathfrak{u}(T,\mathbf{x})$ in (9.6), and we obtain then

$$\mathfrak{h}_{\Pi_o}(T,\mathbf{x}) = \mathfrak{u}(T,\mathbf{x}) = \Phi(T,\mathbf{x}) = \xi_{\Pi_o,\mathcal{M}_0}(T,\mathbf{x}) \geq \xi_{\Pi_o,\mathcal{M}}(T,\mathbf{x}) \qquad \forall \mathcal{M} \in \mathfrak{M}(\mathbf{x}) \tag{9.8}$$

on the strength of (9.5). Putting (9.7) and (9.8) together we deduce

$$\xi_{\Pi,\mathcal{M}_o}(T,\mathbf{x}) \geq \mathfrak{u}(T,\mathbf{x}) = \xi_{\Pi_o,\mathcal{M}_0}(T,\mathbf{x}) \geq \xi_{\Pi_o,\mathcal{M}}(T,\mathbf{x}) \qquad \forall (\Pi,\mathcal{M}) \in \mathfrak{P}\times\mathfrak{M}(\mathbf{x}),$$

the saddle property of the pair $(\Pi_o, \mathcal{M}_o) \in \mathfrak{P}\times\mathfrak{M}(\mathbf{x})$.

In particular, the investment rule $\Pi_o \equiv \pi^{\Phi}$ of (5.10) with $U \equiv \Phi$ is seen to be the investor's best response to the least favorable admissible system $\mathcal{M}_o \in \mathfrak{M}(\mathbf{x})$ of Proposition 1, and vice-versa. In this sense the investor, once he has figured out a least favorable admissible system $\mathcal{M}_o$, can allow himself the luxury to "forget" about model uncertainty and concentrate on finding an investment rule $\Pi_o \in \mathfrak{P}$ that satisfies $\xi_{\Pi,\mathcal{M}_o}(T,\mathbf{x}) \geq \xi_{\Pi_o,\mathcal{M}_o}(T,\mathbf{x})$, $\forall \Pi \in \mathfrak{P}$ as in (9.7), that is, on outperforming the market portfolio with the least initial capital within the context of the least favorable model $\mathcal{M}_o$.

Acknowledgments. We are grateful to Professors Jakša Cvitanić, Nicolai Krylov, Mete Soner, Nizar Touzi, Hans Föllmer, Erhan Bayraktar, Johan Tysk, and to the two reviewers, for helpful advice on the subject matter of this paper and for bringing relevant literature to our attention. Numerous helpful discussions with Drs. Robert Fernholz, Adrian Banner, Vasileios Papathanakos, Phi-Long Nguyen-Thanh, and especially with Johannes Ruf, are gratefully acknowledged.

REFERENCES

AVELLANEDA, M., LÉVY, M. M. and PARÁS, A. (1995). Pricing and hedging derivative securities in markets with uncertain volatility. *Appl. Math. Finance* **2** 73–88.

BAYRAKTAR, E. and YAO, S. (2011). Optimal stopping for nonlinear expectations. *Stochastic Process. Appl.* To appear.

BAYRAKTAR, E., KARATZAS, I. and YAO, S. (2011). Optimal stopping for dynamic convex risk measures. *Illinois J. Math.* (special issue on honor of Don. Burkholder). To appear.

BERTSEKAS, D. P. and SHREVE, S. E. (1978). *Stochastic Optimal Control: The Discrete Time Case. Mathematics in Science and Engineering* **139**. Academic Press, New York. MR0511544

CHERIDITO, P., FILIPOVIĆ, D. and YOR, M. (2005). Equivalent and absolutely continuous measure changes for jump-diffusion processes. *Ann. Appl. Probab.* **15** 1713–1732. MR2152242

CVITANIĆ, J., PHAM, H. and TOUZI, N. (1999). Super-replication in stochastic volatility models under portfolio constraints. *J. Appl. Probab.* **36** 523–545. MR1724796

DELBAEN, F. and SCHACHERMAYER, W. (1995a). The existence of absolutely continuous local martingale measures. *Ann. Appl. Probab.* **5** 926–945. MR1384360

DELBAEN, F. and SCHACHERMAYER, W. (1995b). The no-arbitrage property under a change of numéraire. *Stochastics Stochastics Rep.* **53** 213–226. MR1381678

DENIS, L. and MARTINI, C. (2006). A theoretical framework for the pricing of contingent claims in the presence of model uncertainty. *Ann. Appl. Probab.* **16** 827–852. MR2244434

EKSTRÖM, E. and TYSK, J. (2004). Comparison of two methods for super-replication. Preprint, Uppsala Univ.

EKSTRÖM, E. and TYSK, J. (2009). Bubbles, convexity and the Black–Scholes equation. *Ann. Appl. Probab.* **19** 1369–1384. MR2538074

EL KAROUI, N., HŮÙ NGUYEN, D. and JEANBLANC-PICQUÉ, M. (1987). Compactification methods in the control of degenerate diffusions: Existence of an optimal control. *Stochastics* **20** 169–219. MR0878312

EL KAROUI, N., JEANBLANC-PICQUÉ, M. and SHREVE, S. E. (1998). Robustness of the Black and Scholes formula. *Math. Finance* **8** 93–126. MR1609962

ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes: Characterization and Convergence*. Wiley, New York. MR0838085

FERNHOLZ, E. R. and KARATZAS, I. (2005). Relative arbitrage in volatility-stabilized markets. *Annals of Finance* **1** 149–177.

FERNHOLZ, E. R. and KARATZAS, I. (2009). Stochastic portfolio theory: A survey. In *Handbook of Numerical Analysis* (A. Bensoussan and Q. Zhang, eds.). 88–168. Elsevier, Amsterdam.

FERNHOLZ, D. and KARATZAS, I. (2010a). On optimal arbitrage. *Ann. Appl. Probab.* **20** 1179–1204. MR2676936

FERNHOLZ, D. and KARATZAS, I. (2010b). Probabilistic aspects of arbitrage. In *Contemporary Mathematical Finance: Essays in Honor of Eckhard Platen* (C. Chiarella and A. Novikov, eds.) 1–17. Springer, New York.

FERNHOLZ, E. R., KARATZAS, I. and KARDARAS, C. (2005). Diversity and arbitrage in equity markets. *Finance Stoch.* **31** 37–53.

FLEMING, W. H. and RISHEL, R. W. (1975). *Deterministic and Stochastic Optimal Control. Applications of Mathematics* **1**. Springer, Berlin. MR0454768

FLEMING, W. H. and SONER, H. M. (1993). *Controlled Markov Processes and Viscosity Solutions. Applications of Mathematics (New York)* **25**. Springer, New York. MR1199811

FLEMING, W. H. and VERMES, D. (1989). Convex duality approach to the optimal control of diffusions. *SIAM J. Control Optim.* **27** 1136–1155. MR1009341

FÖLLMER, H. (1972). The exit measure of a supermartingale. *Z. Wahrsch. Verw. Gebiete* **21** 154–166. MR0309184

FÖLLMER, H. (1973). On the representation of semimartingales. *Ann. Probab.* **1** 580–589. MR0353446

FÖLLMER, H. and GUNDEL, A. (2006). Robust projections in the class of martingale measures. *Illinois J. Math.* **50** 439–472 (electronic). MR2247836

FÖLLMER, H., SCHIED, A. and WEBER, S. (2009). Robust preferences and robust Portfolio choice. In *Handbook of Numerical Analysis* (A. Bensoussan and Q. Zhang, eds.). 29–87. Elsevier, Amsterdam.

FREY, R. (2000). Superreplication in stochastic volatility models and optimal stopping. *Finance Stoch.* **4** 161–187. MR1780325

GILBOA, I. and SCHMEIDLER, D. (1989). Maxmin expected utility with nonunique prior. *J. Math. Econom.* **18** 141–153. MR1000102

GOIA, I. (2009). *Bessel and Volatility-stabilized Processes*. ProQuest LLC, Ann Arbor, MI. PhD Thesis, Columbia Univ. MR2713615

GOZZI, F. and VARGIOLU, T. (2002). Superreplication of European multiasset derivatives with bounded stochastic volatility. *Math. Methods Oper. Res.* **55** 69–91. MR1892718

GUNDEL, A. (2005). Super-replication of European multi-asset derivatives with bounded stochastic volatility. *Finance Stoch.* **9** 851–176.

HAUSSMANN, U. G. and LEPELTIER, J. P. (1990). On the existence of optimal controls. *SIAM J. Control Optim.* **28** 851–902. MR1051628

HEATH, D., OREY, S., PESTIEN, V. and SUDDERTH, W. (1987). Minimizing or maximizing the expected time to reach zero. *SIAM J. Control Optim.* **25** 195–205. MR0872458

JANSON, S. and TYSK, J. (2006). Feynman–Kac formulas for Black–Scholes-type operators. *Bull. London Math. Soc.* **38** 269–282. MR2214479

KARATZAS, I. and KARDARAS, C. (2007). The numéraire portfolio in semimartingale financial models. *Finance Stoch.* **11** 447–493. MR2335830

KARATZAS, I. and SHREVE, S. E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd ed. *Graduate Texts in Mathematics* **113**. Springer, New York. MR1121940

KARATZAS, I. and ZAMFIRESCU, I.-M. (2005). Game approach to the optimal stopping problem. *Stochastics* **77** 401–435. MR2178425

KARDARAS, C. and ROBERTSON, S. (2011). Robust maximization of asymptotic growth. Preprint, Boston Univ.

KRYLOV, N. V. (1973). The selection of a Markov process from a Markov system of processes, and the construction of quasidiffusion processes. *Izv. Akad. Nauk SSSR Ser. Mat.* **37** 691–708. MR0339338

KRYLOV, N. V. (1980). *Controlled Diffusion Processes*. *Applications of Mathematics* **14**. Springer, New York. MR0601776

KRYLOV, N. V. (1987). *Nonlinear Elliptic and Parabolic Equations of the Second Order*. *Mathematics and Its Applications* (*Soviet Series*) **7**. Reidel, Dordrecht. MR0901759

KRYLOV, N. V. (1989). A supermartingale characterization of a set of stochastic integrals. *Ukrain. Mat. Zh.* **41** 757–762, 861. MR1002711

KRYLOV, N. V. (1990). Smoothness of the value function for a controlled diffusion process in a domain. *Math. USSR Izvestiya* **34** 65–95.

KRYLOV, N. V. (2002). A supermartingale characterization of sets of stochastic integrals and applications. *Probab. Theory Related Fields* **123** 521–552. MR1921012

KUNITA, H. (1990). *Stochastic Flows and Stochastic Differential Equations*. *Cambridge Studies in Advanced Mathematics* **24**. Cambridge Univ. Press, Cambridge. MR1070361

LEVENTAL, S. and SKOROHOD, A. V. (1995). A necessary and sufficient condition for absence of arbitrage with tame portfolios. *Ann. Appl. Probab.* **5** 906–925. MR1384359

LIEBERMAN, G. M. (1996). *Second Order Parabolic Differential Equations*. World Scientific, River Edge, NJ. MR1465184

LIONS, P. L. (1983a). Optimal control of diffusion processes and Hamilton–Jacobi–Bellman equations. I. The dynamic programming principle and applications. *Comm. Partial Differential Equations* **8** 1101–1174. MR0709164

LIONS, P. L. (1983b). Optimal control of diffusion processes and Hamilton–Jacobi–Bellman equations. II. Viscosity solutions and uniqueness. *Comm. Partial Differential Equations* **8** 1229–1276. MR0709162

LIONS, P. L. (1983c). Optimal control of diffusion processes and Hamilton–Jacobi–Bellman equations. III. Regularity of the optimal cost function. In *Nonlinear Partial Differential Equations and Their Applications. Collège de France Seminar, Vol. V* (*Paris*, 1981/1982). *Res. Notes in Math.* **93** 95–205. Pitman, Boston, MA. MR0725360

LIONS, P.-L. (1984). Some recent results in the optimal control of diffusion processes. In *Stochastic Analysis* (*Katata/Kyoto*, 1982). *North-Holland Math. Library* **32** 333–367. North-Holland, Amsterdam. MR0780764

LYONS, T. J. (1995). Uncertain volatility and the risk-free synthesis of securities. *Appl. Math. Finance* **2** 117–133.

MEYER, P. A. (1972). La mesure de H. Föllmer en théorie des surmartingales. In *Séminaire de Probabilités, VI* (*Univ. Strasbourg, Année Universitaire* 1970–1971; *Journées Probabilistes de Strasbourg*, 1971). *Lecture Notes in Math.* **258** 118–129. Springer, Berlin. MR0368131

MEYER, G. H. (2006). The Black Scholes Barenblatt equation for options with uncertain volatility and its application to static hedging. *Int. J. Theor. Appl. Finance* **9** 673–703. MR2254127

NUTZ, M. (2010). Random G-expectations. Preprint, ETH Zürich.

OREY, S., PESTIEN, V. and SUDDERTH, W. (1987). Reaching zero rapidly. *SIAM J. Control Optim.* **25** 1253–1265. MR0905044

PAL, S. (2011). Analysis of the market weights under the volatility-stabilized market models. *Ann. Appl. Probab.* To appear.

PAL, S. and PROTTER, P. (2010). Analysis of continuous strict local martingales via h-transforms. *Stochastic Process. Appl.* **120** 1424–1443.

PARTHASARATHY, K. R. (1967). *Probability Measures on Metric Spaces. Probability and Mathematical Statistics* **3**. Academic Press, New York. MR0226684

PENG, S. (2010). Nonlinear expectations and stochastic calculus under uncertainty. Preprint, Shandong Univ.

PESTIEN, V. C. and SUDDERTH, W. D. (1985). Continuous-time red and black: How to control a diffusion to a goal. *Math. Oper. Res.* **10** 599–611. MR0812818

PROTTER, P. E. (2004). *Stochastic Integration and Differential Equations*, 2nd ed. *Applications of Mathematics* (*New York*) **21**. Springer, Berlin. MR2020294

RIEDEL, F. (2009). Optimal stopping with multiple priors. *Econometrica* **77** 857–908. MR2531363

ROMAGNOLI, S. and VARGIOLU, T. (2000). Robustness of the Black–Scholes approach in the case of options on several assets. *Finance Stoch.* **4** 325–341. MR1779582

RUF, J. (2011). Hedging under arbitrage. *Math. Finance*. To appear.

SCHIED, A. (2007). Optimal investments for risk- and ambiguity-averse preferences: A duality approach. *Finance Stoch.* **11** 107–129. MR2284014

SCHIED, A. and WU, C.-T. (2005). Duality theory for optimal investments under model uncertainty. *Statist. Decisions* **23** 199–217. MR2236457

SONER, H. M., TOUZI, N. and ZHANG, J. (2010a). Quasi-sure stochastic analysis through aggregation. Preprint, ETH Zürich.

SONER, H. M., TOUZI, N. and ZHANG, J. (2010b). Dual formulation of second-order target problems. Preprint, ETH Zürich.

STROOCK, D. W. and VARADHAN, S. R. S. (1979). *Multidimensional Diffusion Processes. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **233**. Springer, Berlin. MR0532498

SUDDERTH, W. D. and WEERASINGHE, A. (1989). Controlling a process to a goal in finite time. *Math. Oper. Res.* **14** 400–409. MR1008420

TALAY, D. and ZHENG, Z. (2002). Worst case model risk management. *Finance Stoch.* **6** 517–537. MR1932383

VAN MELLAERT, L. J. and DORATO, P. (1972). Numerical solution of an optimal control problem with a probability criterion. *IEEE Transactions on Automatic Control* **AC-17** 543–546.

VARGIOLU, T. (2001). Existence, uniqueness and smoothness for the Black–Scholes–Barenblatt equation. Technical report, Univ. Padova.

VORBRINK, J. (2010). Financial markets with volatility uncertainty. Technical report, Univ. Bielefeld.

WANG, L. (1992a). On the regularity theory of fully nonlinear parabolic equations. I. *Comm. Pure Appl. Math.* **45** 27–76. MR1135923

WANG, L. (1992b). On the regularity theory of fully nonlinear parabolic equations. II. *Comm. Pure Appl. Math.* **45** 141–178. MR1139064

WANG, L. (1992c). On the regularity theory of fully nonlinear parabolic equations. III. *Comm. Pure Appl. Math.* **45** 255–262. MR1151267

DANIEL FERNHOLZ LLC
1108 LAVACA, SUITE 110-440
AUSTIN, TEXAS 78701
USA
E-MAIL: df@danielfernholz.com

INTECH INVESTMENT MANAGEMENT
ONE PALMER SQUARE, SUITE 441
PRINCETON, NEW JERSEY 08542
USA
E-MAIL: ik@enhanced.com