

# Optimal Asset Allocation and Risk Shifting in Money Management\*

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July 2006

\*We are grateful to Robert McDonald (the editor) and two anonymous referees for valuable suggestions. We would also like to thank colleagues at MIT Sloan, NYU Stern, and LBS, and especially Denis Gromb, Roberto Rigobon, and Antoinette Schoar, as well as Joe Chen, Glenn Ellison, Mike Gallmeyer, David Musto, Brad Paye, Ludovic Phalippou, Paola Sapienza, Kari Sigurdsson, Lucie Tepla, Peter Tufano, Dimitri Vayanos, and the seminar participants at the Bank of England, BI, Bilkent, Copenhagen, Harvard, HEC, LBS, LSE, MIT, NHH, NYU, Northwestern, Rutgers, UCD, UCLA, UNC, USI Lugano, Washington St. Louis, Yale, ASAP conference, American Finance Association, Blaise Pascal International Conference, European Finance Association, and Econometric Society winter meetings for their comments. Special thanks to Martijn Cremers. Pavitra Kumar, Dmitry Makarov, Juha Valkama, and Jialan Wang provided excellent research assistance. Parts of this work are drawn from the paper that was previously circulated under the title "Offsetting the Incentives: Risk Shifting and Benefits of Benchmarking in Money Management" (2005, CEPR DP 5006). Research support from the Q Group is gratefully acknowledged. We are indebted to Will Goetzmann, Geert Rouwenhorst, and the International Center for Finance at Yale SOM for kindly providing the data. All errors are solely our responsibility.

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## Abstract

This paper investigates a fund manager's risk-taking incentives induced by an increasing and convex fund-flows to relative-performance relationship. In a dynamic portfolio choice framework, we show that the ensuing convexities in the manager's objective give rise to a finite risk-shifting range over which she gambles to finish ahead of her benchmark. Such gambling entails either an increase or a decrease in the volatility of the manager's portfolio, depending on her risk tolerance. In the latter case, the manager reduces her holdings of the risky asset despite its positive risk premium. Our empirical analysis lends support to the novel predictions of the model. Under multiple sources of risk, with both systematic and idiosyncratic risks present, we show that optimal managerial risk shifting may not necessarily involve taking on any idiosyncratic risk. Costs of misaligned incentives to investors resulting from the manager's policy are demonstrated to be economically significant.

**JEL Classifications:** G11, G20, D60, D81.

**Keywords:** Fund Flows, Implicit Incentives, Risk Taking, Relative Performance, Risk Management, Portfolio Choice.

# 1. Introduction

“The real business of money management is not managing money, it is getting money to manage.”<sup>1</sup> Indeed, with the number of mutual funds in the US exceeding the number of stocks, fund managers are increasingly concerned with attracting investors’ money. Recent empirical evidence (e.g., Gruber (1996), Chevalier and Ellison (1997), Sirri and Tufano (1998)) offers simple insight to a manager: money tends to flow into the fund if it performs well relative to a benchmark. With her compensation typically linked to the value of assets under management, this positive fund-flows to relative-performance relationship creates an *implicit incentive* for the manager to distort her asset allocation choice so as to increase the likelihood of future fund inflows. Our objective is to analyze the effects of these incentives within a familiar dynamic portfolio choice framework.

We consider a risk-averse fund manager, whose compensation depends on the fund’s value at some terminal date (e.g., end of the year). This fund value is determined by the portfolio choice of the manager during the year and by capital inflows/outflows at year-end. The fund flows depend on the fund’s performance over the year relative to a benchmark – a reference portfolio of stock and money markets. We consider several types of the flow-performance relationship, which all give rise to local convexities in the manager’s objective.<sup>2</sup> While the importance of such local convexities has been noted by numerous studies, our analysis uncovers several novel implications of these convexities that are at odds with conventional views.

Absent implicit incentives, the manager in our setting maintains a constant risk exposure, independent of her performance relative to the benchmark (Merton (1971)). But when she accounts for the flow-performance relationship, she takes on additional risk over a certain range of relative performance, the “risk-shifting” range. We show that implicit incentives can lead the manager to either increase or decrease volatility over this range. The volatility increase corresponds to the common wisdom (Jensen and Meckling (1976)). The possibility of a decrease is somewhat unexpected: how can “gambling” to finish ahead of the benchmark be consistent with a decrease in portfolio volatility? Simply, in the context of relative performance evaluation, any strategy entailing a deviation from the benchmark is inherently risky. By taking on more systematic risk than that of the benchmark (boosting portfolio volatility), the manager gambles to improve her relative standing when the benchmark goes up. Similarly, by taking on less systematic risk than that of the benchmark (reducing volatility), the manager bets on improving her relative performance when the benchmark falls. The direction of the manager’s deviation from the benchmark depends on her risk aversion: a more risk tolerant manager boosts volatility, while a relatively risk averse

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<sup>1</sup>As eloquently put by Mark Hurley in the famous Goldman, Sachs and Co. report on the evolution of the investment management industry (see WSJ 11/16/95 and e.g., <http://assetmag.com/story/20010601/10438.asp>).

<sup>2</sup>We attempt to reinterpret some of these specifications as forms of managerial compensation contracts (Section 2.3). Our examples include the so-called 80/120 plans offered to US executives and the asymmetric fee structures, common amongst the European mutual funds.

manager does the opposite, despite the positive risk premium offered by the risky asset.

Since the manager is risk averse, the range over which she takes excessive risk, as well as her ensuing risk exposure, are finite. No risk shifting takes place when the manager is far behind or ahead of the benchmark. Overall, the strength of managerial risk-taking incentives is highly time- and state-dependent, with a maximum positioned somewhere deep in the underperformance region and a minimum differing across the specifications of the flow-performance relationship we consider. These implications are typically in disagreement with the predictions of extant work arguing that risk taking is most pronounced next to convex kinks or discontinuities in a manager's payoff (see, e.g., Chevalier and Ellison (1997), Murphy (1999)). In fact, in one of our specifications, risk-taking incentives are minimized around a discontinuity. These sharp differences in implications are due to the differences in adopted measures of risk taking. The traditional definition of a risk-taking incentive (e.g., standard in corporate finance) is the sensitivity of the manager's payoff to volatility. This measure captures the strength of the manager's desire to increase her risk exposure relative to some fixed status quo asset allocation. Our measure of risk taking is the optimal risk exposure as defined in the portfolio choice literature: the fraction of the fund optimally invested in the risky asset. Intuitively, instead of just taking a partial derivative of the manager's value function with respect to volatility, we take this derivative and equate it to zero to derive the optimal volatility for each level of relative performance.

For most of our analysis we adopt the simplest possible setting, the Black and Scholes (1973) economy with a single source of risk. To investigate the manager's portfolio allocation across different stocks and exposure to systematic versus idiosyncratic risk, we extend our baseline model to multiple sources of uncertainty. The overall behavior of the manager is akin to that in the baseline model. However, the manager's incentive to deviate from the benchmark portfolio when underperforming now manifests itself not in increasing or decreasing the fund's volatility but in tilting the weight in each risky security away from the benchmark. We also show that risk shifting does not necessarily involve taking on idiosyncratic risk. In fact, when faced with both systematic and idiosyncratic risks, the manager may very well optimally expose herself to no idiosyncratic risk, while engaging in her optimal risk shifting via systematic risk only.

The costs of misaligned incentives resulting from the manager's policy are shown to be potentially economically significant.<sup>3</sup> We compare the manager's policy with that maximizing investors' utility alone. For example, we find that if the investor's relative risk aversion is 2 and the manager's is 1, the cost to the investor is nearly 10% of his initial wealth. The costs are particularly severe when

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<sup>3</sup>While we focus on the costs arising from misalignment of objectives, there could also be benefits to hiring a manager. After all, the investors have chosen active management. For example, the manager might have lower transactions, market participation or informational costs (Merton (1987)), lower opportunity cost of time (time constraints and other forms of investors' bounded rationality (Rubinstein (1998))), better investments-specific education, or better information or ability. Modeling the benefits together with the costs requires a full-fledged model of interaction between managers and investors, which is beyond the scope of this paper.

the manager’s and investor’s attitudes towards risk differ substantially, when the flow-performance relationship increases steeply, or when the benchmark is very risky.

Finally, we collect daily returns of US mutual funds to examine empirically the testable implications of our model. Since we are not the first to study risk taking of US fund managers, we focus on testing the implications that are novel to our model.<sup>4</sup> Our first hypothesis is that underperforming managers boost the deviation of their portfolio from the benchmark, i.e., increase the tracking error variance. We find support for this hypothesis, and no evidence that underperforming managers increase the volatility of their portfolios. The latter result is consistent with Busse (2001). The second hypothesis is that managers who are sufficiently risk averse decrease their portfolio beta when underperforming the market. We find that managers who chose lower risk portfolios in a previous year, decrease their portfolio betas in the subsequent year when underperforming. We also find that funds whose current year’s betas are below one decrease their portfolio beta when underperforming the market. However, while the signs of these responses are consistent with our theory, their magnitudes are considerably smaller, most likely reflecting the risk-management constraints imposed in practice (see Almazan, Brown, Carlson, and Chapman (2004)). Finally, we investigate how the extent to which funds have fallen behind the benchmark influences their risk-taking choices. We find that funds increase risk when moderately behind the benchmark, and cease to do so when they have fallen far behind.

Our work is related to the literature on money managers’ incentives. Chevalier and Ellison (1997) study flows-induced risk taking by mutual fund managers. Unlike us, they define risk-taking incentive as the sensitivity of a fund’s value to its volatility. Our analysis in a dynamic setting revisits Chevalier and Ellison and offers considerably different implications for managerial risk taking.<sup>5</sup> Within a dynamic asset allocation framework like ours, Carpenter (2000) examines the risk-taking behavior of a fund manager with a convex, option-based compensation. Such a payoff structure is not a subclass of the flow functions we consider since our manager always incurs a penalty (fund outflows) when her performance deteriorates, while a poor-performing manager in Carpenter enjoys a fixed “safety net” independent of her fund value. This safety net suppresses the poor-performing manager’s risk aversion considerations, leading Carpenter to conclude that the fund volatility becomes unbounded. This contrasts our findings that, in the underperformance

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<sup>4</sup>Empirical work on risk taking by US fund managers includes Brown, Harlow, and Starks (1996), Chevalier and Ellison (1997), Busse (2001), and more recently, Reed and Wu (2005). Reed and Wu present a broad set of tests consistent with the main predictions of this paper, and argue that risk-shifting behavior of mutual fund managers is due to the benchmark- and not tournaments-induced incentives.

<sup>5</sup>A related area in corporate finance is the work on risk averse managers’ risk-taking incentives induced by executive stock options (Lambert, Larcker, and Verrecchia (1991), Hall and Murphy (2000), Hall and Murphy (2002), Lewellen (2006)). Here, a risk-taking incentive is given by the sensitivity of the manager’s certainty equivalent wealth to volatility. There are some similarities between the results obtained in this context and ours (see, especially Lewellen (2006)), however unlike in our model the manager is assumed to hold a pre-specified portfolio, and may affect risk exposure only through manipulating the company’s stock price. The notion of implicit incentives was introduced by Fama (1980) and Holmstrom (1999), and applied to other related problems in corporate finance by, for example, Zwiebel (1995) and Huddart (1999).

region, the fund volatility (i) exhibits an interior extremum, and (ii) may decrease.

Building on the analysis of Carpenter and our paper, Hodder and Jackwerth (2005) analyze a hedge fund manager, incorporating further realistic features in the compensation structure along the lines of Goetzmann, Ingersoll, and Ross (2003). They also provide a thorough comparison of managerial risk taking arising in Carpenter, Goetzmann, Ingersoll, and Ross, and our analysis. Brennan (1993), Cuoco and Kaniel (2003), and Gomez and Zapatero (2003) study equilibrium asset prices in an economy with agents compensated based on their performance relative to a benchmark, Arora and Ou-Yang (2006) study a fund manager's career concerns problem. Hugonnier and Kaniel (2004) endogenize (marketable) fund flows in a dynamic economy with a small investor and a noncompetitive fund manager. Under the derived flow function, however, the manager's optimal policy does not depend on her risk aversion, and does not involve any risk shifting. Berk and Green (2004) develop a model with a competitive capital market and decreasing returns to scale in active portfolio management. In the context of fund-value-maximizing managers, they deduce an empirically plausible fund flows-performance relationship even with no persistence in performance, but do not study risk-shifting behavior. Lynch and Musto (2003) suggest that the convexity in the flows-performance relationship is due to an abandonment option (replacement of personnel or technique after bad performance). Ross (2004) investigates the relation between the manager's fee structure and her attitude toward risk within a general class of preferences and compensation structures. However, he focuses on fee schedules under which the objective function remains globally concave. Chen and Pennacchi (2005), like us, argue that relative-performance based compensation may sometimes give the manager an incentive to decrease portfolio volatility; however, in their model the objective function is globally concave, which rules the type of risk-shifting behavior studied in this paper. Becker, Ferson, Myers, and Schill (1999) also model a manager concerned with her relative performance, but their primary focus is on testing for market timing in mutual funds. More in the spirit of our analysis, in a dynamic portfolio choice framework, Cadenillas, Cvitanic, and Zapatero (2004) consider a principal-agent problem in which a risk-averse manager compensated with options chooses the riskiness of the projects she invests in. There is also a recent literature examining asset allocation under benchmarking constraints. In a dynamic setting like ours, Tepla (2001), and Basak, Shapiro, and Tepla (2006) study the optimal policies of an agent subject to a benchmarking restriction.

The rest of the paper proceeds as follows. Section 2 describes the model, solves for the optimal risk exposure of the manager under various fund-flow to relative-performance specifications, and computes the potential costs of active management to the investor due to managerial incentives. Section 3 discusses the extension of our analysis to multiple sources of uncertainty and multiple stocks. Section 4 provides some empirical support for our theory. Section 5 concludes. Proofs and further details are in the Appendix.

## 2. Fund Manager’s Implicit Incentives

### 2.1 The Economic Setting

We adopt the familiar Black and Scholes (1973) economy for the financial investment opportunities. We consider a continuous-time, finite horizon,  $[0, T]$  economy, in which uncertainty is driven by a Brownian motion  $w$ . Available for investing are a riskless money market account and a risky stock. The money market provides a constant interest rate  $r$ . The stock price,  $S$ , follows a geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dw_t,$$

with constant mean return,  $\mu$ , and volatility,  $\sigma$ . Throughout,  $\sigma^Z$  denotes the volatility (instantaneous standard deviation) of an Itô process  $Z$  satisfying  $dZ_t/Z_t = \mu_t^Z dt + \sigma_t^Z dw_t$ .

We consider a fund manager who dynamically allocates the fund’s assets, initially valued at  $W_0$ , between the risky stock and the money market. Her portfolio value process,  $W$ , follows

$$dW_t = [(1 - \theta_t)r + \theta_t\mu] W_t dt + \theta_t\sigma W_t dw_t, \quad (1)$$

where  $\theta$  denotes the fraction of the portfolio invested in the risky stock, or the *risk exposure*.<sup>6</sup> Consistent with the leading practice, the manager’s compensation, due at the horizon  $T$ , is proportional to the terminal value of assets under management. Such compensation provides the manager with *implicit incentives* arising from the well-documented fund-flows to relative-performance relationship (see e.g., Chevalier and Ellison (1997)). If the manager does well relative to some benchmark (e.g., the stock market), her assets under management multiply due to the inflow of new investors’ money; if she does poorly, a part of assets under her management gets withdrawn. The *benchmark*  $Y$  relative to which her performance is evaluated is a value-weighted portfolio with a fraction  $\beta$  invested in the stock market and  $(1 - \beta)$  in the money market, satisfying<sup>7</sup>

$$dY_t = (1 - \beta)rY_t dt + \beta(Y_t/S_t)dS_t = [(1 - \beta)r + \beta\mu]Y_t dt + \beta\sigma Y_t dw_t.$$

We denote the continuously compounded returns on the manager’s portfolio and on the benchmark over the period  $[0, t]$  by  $R_t^W = \ln \frac{W_t}{W_0}$  and  $R_t^Y = \ln \frac{Y_t}{Y_0}$ , respectively, and normalize  $Y_0 = W_0$ , without

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<sup>6</sup>In this baseline analysis, we assume no investment restrictions. While hedge fund managers face little regulation, equity mutual fund managers are likely to face borrowing and short-sale constraints, as well as restricted trading in derivatives. However, in reality a fund manager can invest in a large number of stocks with different characteristics (beta, volatility), and so may implement a wide range of risk exposures. Indeed, in our empirical analysis we find that equity fund managers do change their risk exposures, and these effects are statistically significant, as also documented by Chevalier and Ellison (1997).

<sup>7</sup>Alternatively, the benchmark  $Y$  can be interpreted as a peer group performance. The optimal policies we derive in this paper would then constitute the best response of an individual manager. One may in principle attempt to go further and solve for a Nash equilibrium in the game amongst managers evaluated on their performance relative to the peer group. This, however, would require a manager to have full knowledge of other managers’ preferences, portfolio compositions, etc.

loss of generality. At the terminal date, the fund receives flows at a rate  $f_T$ , specified as follows:

$$f_T = \begin{cases} f_L & \text{if } R_T^W - R_T^Y < \eta_L, \\ f_L + \psi(R_T^W - R_T^Y - \eta_L) & \text{if } \eta_L \leq R_T^W - R_T^Y < \eta_H, \\ f_H \equiv f_L + \psi(\eta_H - \eta_L) & \text{if } R_T^W - R_T^Y \geq \eta_H, \end{cases} \quad (2)$$

with  $f_L, \psi > 0, \eta_L \leq \eta_H \in \mathbb{R}$ . Our model of this flow-performance relationship draws on the estimation by Chevalier and Ellison (Figure 1). It is flat for managers who are well below the market. When the relative performance reaches about  $\eta_L = -8\%$ , the flow function displays a convex “kink” followed by an upward-sloping approximately linear segment. At about  $\eta_H = 8\%$ , the relationship again becomes flat.<sup>8</sup> Hereafter, we refer to this flow specification as a *collar-type*, as it resembles a collar or a bull spread of option pricing, with a *lower threshold*  $\eta_L$  and an *upper threshold*  $\eta_H$ . The flow rate  $f_T$  is understood in the proportion-of-portfolio terms; for example if  $f_T > 1$ , the manager gets an inflow, otherwise if  $f_T < 1$ , gets an outflow. The manager is guided by constant relative risk aversion (CRRA) preferences, defined over the overall value of assets under management at time  $T$ :

$$u(W_T f_T) = \frac{(W_T f_T)^{1-\gamma}}{1-\gamma}, \quad \gamma > 0, \quad (3)$$

where  $f_T$  directly enters through the utility, and not through the budget constraint, because future (time- $T$ ) fund flows are nontradable. We note that this payoff is consistent with a linear fee structure, predominantly adopted by mutual fund companies (e.g., Das and Sundaram (2001), Elton, Gruber, and Blake (2003)).<sup>9</sup> It turns out that this simple way of modeling fund flows is able to capture most of the insights pertaining to risk-taking incentives of a risk-averse manager that we attempt to highlight. In Section 2.3, we discuss how our results extend to other fund-flows to relative-performance relationships, and also reinterpret the resulting payoff function of the manager as a compensation contract.

Absent implicit incentive considerations ( $f_T = 1$ ), the manager’s optimal risk exposure,  $\theta^N$ , henceforth the *normal risk exposure*, is given by (Merton (1971)):

$$\theta_t^N = \frac{1}{\gamma} \frac{\mu - r}{\sigma^2}.$$

By analogy, we define the risk exposure of the benchmark portfolio,  $\theta^Y$ , as the fraction of the

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<sup>8</sup>Of course, our form of  $f_T$  is an overly simplified version of the specification revealed by Chevalier and Ellison’s estimation. Here, for expositional clarity, we do not consider two other “kinks” corresponding to the regions of extremely good and extremely bad performance relative to the benchmark. This is because, as will become clear, our analysis is of a local type: focusing on one convex region at a time. Alternative empirical estimations (e.g., Sirri and Tufano (1998)) find that the flow-performance relationship is flat in the underperformance region, but becomes convex for overperforming funds. We consider this and other possible fund-flow specifications in Section 2.3.

<sup>9</sup>In particular, the fund company may have a linear fee structure,  $\alpha W_T f_T$ ,  $\alpha > 0$ . Such a linear structure would be optimal in our model absent implicit and explicit incentives, i.e., if  $f_T = 1$  and a hypothetical investor had CRRA preferences with  $\gamma$ . However, the presence of incentives leads to a considerable cost to investors, as demonstrated in Section 2.4. Our specification does not capture the case of fulcrum fees, which are less common, but the model can be extended to incorporate them.



benchmark invested in the risky asset:

$$\theta_t^Y = \beta.$$

## 2.2 Manager’s Risk Taking Incentives

The optimization problem of the manager is given by:

$$\max_{\theta} E[u(W_T f_T)] \quad (4)$$

subject to the budget constraint (1) and  $f_T$  given by (2). This problem is nonstandard in that it is nonconcave over a range of  $W_T$ , where the range is dependent on the performance of the stochastic benchmark  $Y$ . The empirical literature on fund-flows to relative-performance relationship clearly indicates that convexities are inherent in the mutual fund managers’ problems. A convexity in the manager’s objective gives rise to a range of terminal portfolio values  $W_T$  over which the manager is risk-loving (and the objective lies below its concavified version). In our model such a range, which we refer to it as the *risk-shifting range*, is finite. In contrast to standard concave maximization problems, the shape of the manager’s objective function in this range plays no role for the risk-taking choices of the manager. For the collar-type fund flow specification, the occurrence of the risk-shifting range is due to a convex kink at the lower threshold  $\eta_L$ . The shape of the upward-sloping part of the flow function turns out to not matter for the manager’s behavior under the following condition:

**Condition 1.**  $\frac{\gamma}{1-\gamma} \left( \frac{f_H + \psi}{f_L} \right)^{1-1/\gamma} + \frac{f_H + \psi}{f_H} - \frac{1}{1-\gamma} \geq 0,$

where  $\psi = (f_H - f_L)/(\eta_H - \eta_L)$  as in (2). This is satisfied for empirically plausible parameter values ( $f_L = 0.8, f_H = 1.5, \beta = 1.0, \eta_L = -0.08, \eta_H = 0.08$ ), calibrated from Chevalier and Ellison’s Figure 1, and a range of managerial risk aversion  $\gamma \in (0, 1.56)$ . More generally, Condition 1 is satisfied for low enough risk aversion  $\gamma$  or threshold differential  $|\eta_H - \eta_L|$ . For example, with low enough risk aversion, the manager’s incentives to “gamble” are so pronounced in the risk-shifting range that they dominate any incentives induced by the upward-sloping part of the flow function, rendering it immaterial for the manager’s optimization. We first focus on the case where Condition 1 holds and an explicit characterization for the optimal policy obtains; the case when it is violated is considered in Section 2.3.

As is well known (e.g., Karatzas and Shreve (1998)), the driving economic state variable in an agent’s dynamic investment problem is the so-called state price density. In the complete-markets Black and Scholes (1973) economy, the state price density process,  $\xi$ , is given by  $d\xi_t = -r\xi_t dt - \kappa\xi_t dw_t$ , where  $\kappa \equiv (\mu - r)/\sigma$  is the constant market price of risk in the economy. Proposition 1 characterizes the solution to (4) in terms of  $\xi$ .<sup>10</sup>

<sup>10</sup>Our proof of Proposition 1 exploits the dependence of the benchmark  $Y$  on the economic state variable  $\xi$ . There is an alternative to this direct method, which is presented in the proof of Proposition 2 dealing with multiple sources of uncertainty and multiple stocks.

**Proposition 1.** *Under Condition 1, the optimal risk exposure and terminal wealth of a fund manager facing implicit incentives are given by*

(a) *for economies with  $\theta^N > \theta^Y$ ,*

$$\begin{aligned}\hat{\theta}_t &= \theta^N + \left[ \mathcal{N}(d(\hat{\kappa}, \xi_a)) - \mathcal{N}(d(\hat{\kappa}, \hat{\xi})) \right] (\gamma/\hat{\kappa} - 1) A \theta^N Z(\hat{\kappa}) \xi_t^{-1/\hat{\kappa}} / \hat{W}_t \\ &\quad + \left\{ \left[ \phi(d(\hat{\kappa}, \xi_a)) - \phi(d(\hat{\kappa}, \hat{\xi})) \right] A Z(\hat{\kappa}) \xi_t^{-1/\hat{\kappa}} \right. \\ &\quad \left. + \left[ \phi(d(\gamma, \hat{\xi})) f_H^{(1/\gamma-1)} - \phi(d(\gamma, \xi_a)) f_L^{(1/\gamma-1)} \right] Z(\gamma) (\hat{y} \xi_t)^{-1/\gamma} \right\} \frac{\gamma \theta^N}{\kappa \sqrt{T-t} \hat{W}_t}, \\ \hat{W}_T &= \frac{1}{f_H} J \left( \frac{\hat{y}}{f_H} \xi_T \right) 1_{\{\xi_T < \hat{\xi}\}} + e^{\eta_H} Y_T 1_{\{\hat{\xi} \leq \xi_T < \xi_a\}} + \frac{1}{f_L} J \left( \frac{\hat{y}}{f_L} \xi_T \right) 1_{\{\xi_a \leq \xi_T\}}\end{aligned}$$

(b) *for economies with  $\theta^N < \theta^Y$ ,*

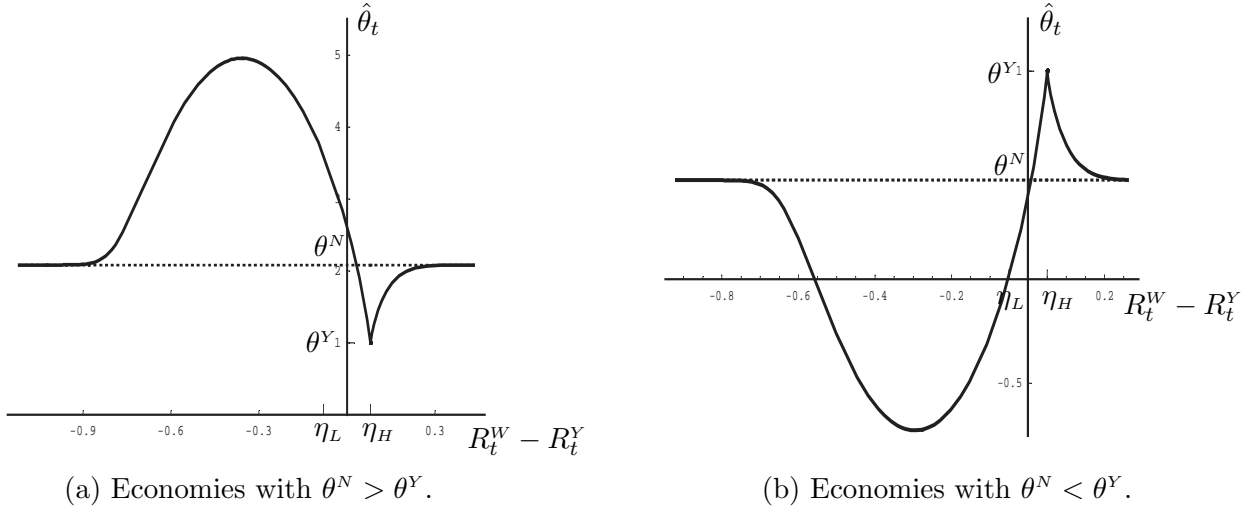
$$\begin{aligned}\hat{\theta}_t &= \theta^N + \left[ \mathcal{N}(d(\hat{\kappa}, \hat{\xi})) - \mathcal{N}(d(\hat{\kappa}, \xi_b)) \right] (\gamma/\hat{\kappa} - 1) A \theta^N Z(\hat{\kappa}) \xi_t^{-1/\hat{\kappa}} / \hat{W}_t \\ &\quad + \left\{ \left[ \phi(d(\hat{\kappa}, \hat{\xi})) - \phi(d(\hat{\kappa}, \xi_b)) \right] A Z(\hat{\kappa}) \xi_t^{-1/\hat{\kappa}} \right. \\ &\quad \left. + \left[ \phi(d(\gamma, \xi_b)) f_L^{(1/\gamma-1)} - \phi(d(\gamma, \hat{\xi})) f_H^{(1/\gamma-1)} \right] Z(\gamma) (\hat{y} \xi_t)^{-1/\gamma} \right\} \frac{\gamma \theta^N}{\kappa \sqrt{T-t} \hat{W}_t}, \\ \hat{W}_T &= \frac{1}{f_L} J \left( \frac{\hat{y}}{f_L} \xi_T \right) 1_{\{\xi_T < \xi_b\}} + e^{\eta_H} Y_T 1_{\{\xi_b \leq \xi_T < \hat{\xi}\}} + \frac{1}{f_H} J \left( \frac{\hat{y}}{f_H} \xi_T \right) 1_{\{\hat{\xi} \leq \xi_T\}}\end{aligned}$$

where in all economies  $\hat{y}$  solves  $E[\xi_T \hat{W}_T] = W_0$ ,  $J(\cdot)$  is the inverse function of  $u'(\cdot)$ ,  $\mathcal{N}(\cdot)$  and  $\phi(\cdot)$  the standard-normal cumulative distribution and density functions respectively,  $\hat{\kappa} = \kappa/(\beta\sigma)$ ,  $\hat{\xi} = (\hat{y} A^\gamma / f_H^{1-\gamma})^{1/(\gamma/\hat{\kappa}-1)}$ ,  $A = W_0 e^{[\eta_H/T + (1-\beta)r + \beta(\mu - \beta\sigma^2/2 - (r + \kappa^2/2)\sigma/\kappa)]T}$ ,  $Z(v) = e^{\frac{1-v}{v}(r + \frac{\kappa^2}{2v})(T-t)}$ ,  $g(\xi) = \left( \gamma \left( \frac{\hat{y}}{f_L} \xi \right)^{1-1/\gamma} - \left( \frac{\xi^{1/\hat{\kappa}}}{A f_H} \right)^{\gamma-1} \right) / (1-\gamma) + \hat{y} A \xi^{1-1/\hat{\kappa}}$ ,  $d(v, x) = \left( \ln \frac{x}{\xi} + \left( r + \frac{2-v}{2v} \kappa^2 \right) (T-t) \right) / (\kappa \sqrt{T-t})$ , in (a)  $\xi_a > \hat{\xi}$  satisfies  $g(\xi_a) = 0$ , in (b)  $\xi_b < \hat{\xi}$  satisfies  $g(\xi_b) = 0$ , and  $\hat{W}_t$  is as given in the proof. The knife-edge economies with  $\theta^N = \theta^Y$  are described in the proof.

Proposition 1 reveals that the manager's optimal behavior has a different pattern depending on whether the risk exposure of the benchmark is lower than the manager's normal risk exposure (economies (a)) or not (economies (b)). We note that both types of economies, (a) and (b), are empirically plausible since each economy is identified by conditions involving the manager-specific risk aversion  $\gamma$ , which need not equal that of a representative agent. The implications for optimal risk taking are best highlighted by plotting the manager's state-dependent risk exposure as a function of her performance relative to the benchmark, as presented in Figure 1.

Two considerations drive the manager's behavior: her risk aversion and the risk-shifting incentive induced by the kink in her payoff when her relative performance is at the lower threshold  $\eta_L$ . As emphasized in the vast risk-shifting literature (originating from Jensen and Meckling (1976)), to increase her portfolio value, the manager has an incentive to distort her normal policy by boosting her portfolio volatility. Indeed, the manager's risk-loving behavior over a range of terminal payoffs

gives rise to a hump in her optimal risk exposure, as clearly pronounced in both panels of Figure 1. In panel (a), this hump reflects an optimal choice to increase her portfolio volatility over the risk-shifting range. In panel (b), in contrast, the volatility is reduced. The latter direction is somewhat unexpected: how can “gambling” to finish ahead of the benchmark be consistent with a decrease in portfolio volatility? Simply, in the context of relative performance evaluation, any strategy entailing a deviation from the benchmark is inherently risky. By taking on more systematic risk than that of the benchmark (boosting portfolio volatility), the manager gambles to improve her relative standing when the benchmark goes up. Similarly, by taking on less systematic risk than that of the benchmark (reducing volatility), the manager bets on improving her relative performance when the benchmark falls. The direction of the manager’s deviation from the benchmark depends on her risk aversion: a more risk tolerant manager, whose normal policy is riskier than the benchmark, decides to boost volatility, while, in utility terms, it is cheaper for a relatively risk averse manager to do the opposite, despite the positive risk premium offered by the risky asset.



**Figure 1. The manager’s optimal risk exposure.** The solid plots are for the optimal risk exposure, and the dotted plots are for the manager’s normal risk exposure. In economies (a) parameter values are  $\gamma = 1.5$ ,  $f_L = 0.8$ ,  $f_H = 1.5$ ,  $\beta = 1.0$ ,  $\eta_L = -0.08$ ,  $\eta_H = 0.08$ ,  $\mu = 0.1$ ,  $r = 0.02$ ,  $\sigma = 0.16$ ,  $W_0 = 1$ ,  $t = 0.75$ ,  $T = 1$ ; in economies (b)  $\gamma = 2$ ,  $f_L = 0.8$ ,  $f_H = 1.5$ ,  $\beta = 1.0$ ,  $\eta_L = -0.08$ ,  $\eta_H = 0.08$ ,  $\mu = 0.1$ ,  $r = 0.02$ ,  $\sigma = 0.29$ ,  $W_0 = 1$ ,  $t = 0.75$ ,  $T = 1$ .<sup>11</sup>

We note that this implication is very general, and holds for all alternative flow-performance specifications considered in Section 2.3. This suggests that volatility is not the most appropriate measure of risk under relative performance evaluation. As follows directly from our closed-form expressions, the quantity measuring risk taking in our model is the distance between the volatility of the manager’s portfolio and that of the benchmark  $|\sigma_t^W - \sigma_t^Y| = \sigma|\theta_t - \theta_t^Y|$ .<sup>12</sup> In both panels of Figure 1,

<sup>11</sup>The figure is typical. The parameter values of the flow function are calibrated according to the estimation in Chevalier and Ellison.

<sup>12</sup>Indeed, in the proof of Proposition 2, we provide a formal justification for focusing on this distance, where we demonstrate that the relevant state variable in the optimization problem of the manager (4) is her *relative* portfolio

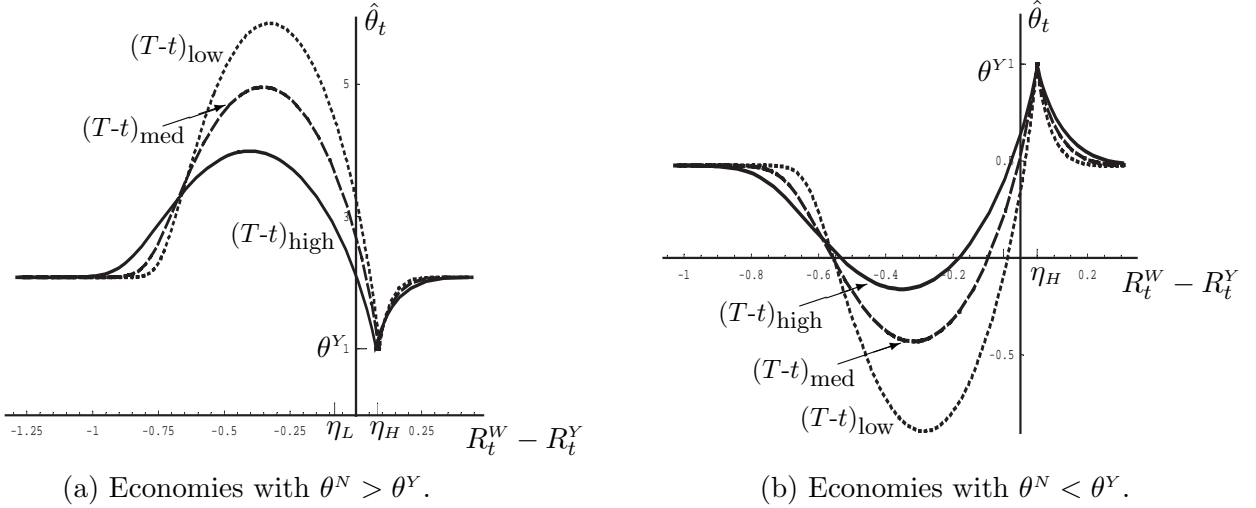
$|\sigma_t^W - \sigma_t^Y|$  increases sharply over the risk-shifting range, while the manager’s portfolio volatility  $\sigma_t^W$  may or may not increase.

Furthermore, when the manager is far behind or ahead of her benchmark, her flows-induced implicit incentives are weak, and so her optimal policy converges to her normal policy. Consequently, the optimal risk-shifting range is finite, positioned in the neighborhood of the convex kink in the flow-performance relationship. Over that range, the optimal risk exposure is not infinite, reflecting a tradeoff between the managerial risk shifting and risk aversion. The maximum risk taking obtains somewhere inside this range; in Figure 1, the maximum corresponds to a point deep in the underperformance region, well below the lower threshold  $\eta_L$ . The minimum is around the upper threshold  $\eta_H$ , where the manager “locks in” her gains by mimicking the benchmark. Here, it is useful to contrast our results on the manager’s optimal risk-taking to measures of risk-taking incentives as defined in the corporate finance literature, typically under the assumption of agents’ risk neutrality. For example, Green and Talmor (1986), in the context of the asset substitution problem, define the risk-taking incentive as the sensitivity of the value of the equityholders’ option-like payoff to “changes in investment risk” (variability of the underlying cash flow). In option pricing, this measure is referred to as vega, the partial derivative of an option’s (portfolio) value with respect to the underlying volatility. The risk-taking incentive then captures the strength of the (value-maximizing) manager’s desire to increase her portfolio volatility relative to some state-independent status quo asset allocation, and is strongest when the manager’s payoff vega achieves its maximum. Adopting this measure for the relative-performance-based payoff, Chevalier and Ellison (1997) argue that the “incentive to increase or decrease risk will always be maximized at the point at which the flow-performance relationship has a kink and will decline smoothly to zero at extreme performance levels.” As evident from our Figure 1, our measure of risk-taking, the optimal risk exposure, conflicts with this prediction – in fact, in panel (b) the risk exposure is around zero at the convex kink  $\eta_L$ . Intuitively, instead of merely taking a partial derivative of the manager’s value function with respect to volatility, we take this partial derivative and equate it to zero to derive the optimal volatility for each level of relative performance.<sup>13</sup>

The wave-shape optimal policy, depicted in Figure 1, finances the manager’s terminal portfolio value. From Proposition 1, this terminal value displays three distinct patterns depending on the state of the world, given by the state price density,  $\xi_T$ , with low  $\xi_T$  representing good states and high value  $W/Y$ , whose volatility is given by  $|\sigma_t^W - \sigma_t^Y|$ . We also note that in this Section, the only type of gambles the manager is allowed to take on are systematic gambles. A natural question is whether the optimal risk-taking behavior we derive here would persist if the manager were allowed access to a market for idiosyncratic gambles. We examine this possibility in Section 3.

<sup>13</sup>Note that our endogenous wave-shape pattern of risk exposure does not converge to the corporate-finance (bell-shaped) measure of risk-taking incentives even as the risk aversion coefficient of the manager tends to zero. This is because the proper limit of the preferences of our manager is a linear function over the range of positive values of terminal wealth, coupled with a restriction that wealth cannot fall below zero (the negativity of wealth is ruled out by the Inada conditions). This function is (weakly) concave, so we do not get a risk-neutral (linear) objective even in the limit.

$\xi_T$  bad states. In the extreme states (low  $\xi_T$  or high  $\xi_T$ ), the manager behaves as if the fund flows were constant at the low  $f_L$  or high  $f_H$  rate. In addition, there is an extended intermediate region in which the manager mimics the benchmark when her performance reaches the upper threshold  $\eta_H$ . The nonconcavity of the manager's problem gives rise to a discontinuity in the optimal wealth profile at a critical level of the state  $\xi_T = \hat{\xi}$ , triggering the risk-shifting range.



**Figure 2. Dynamics of the manager's optimal risk exposure.**  $(T-t)_{high} = 0.5$ ,  $(T-t)_{med} = 0.25$ , and  $(T-t)_{low} = 0.15$ . The remaining parameter values are the same as in Figure 1.

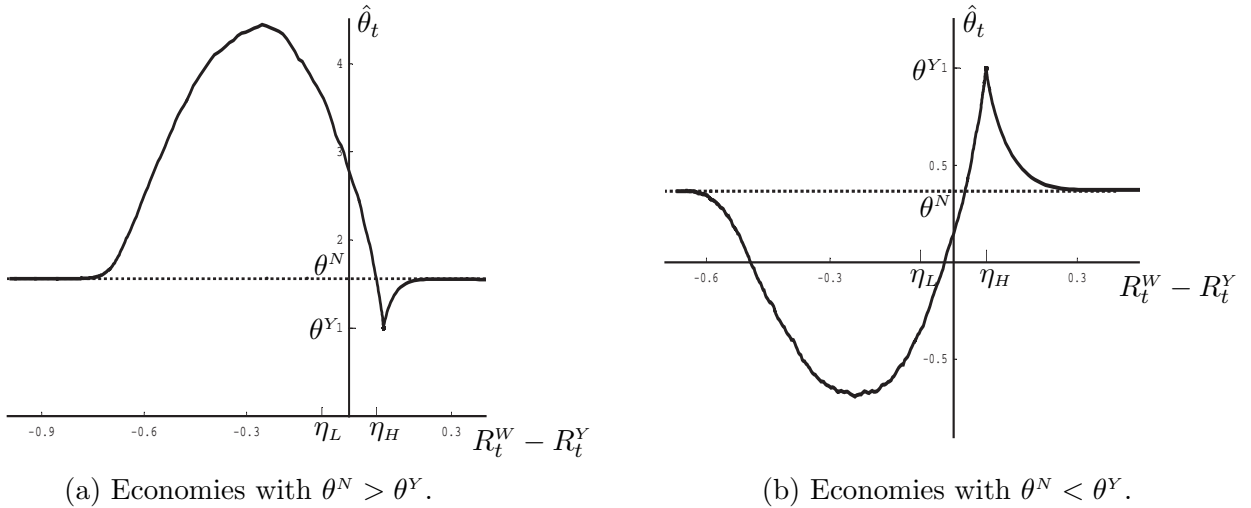
The manager's optimal trading strategy throughout the year reflects the anticipation of the year-end drive to avoid the suboptimal range of the terminal portfolio returns. As evident from Figure 2, she does not wait until the year-end to see how her returns turn out, to then take a gamble right before the terminal date if necessary. Rather, she starts tilting her risk exposure around the suboptimal range well in advance, giving rise to a hump in the risk exposure as in Figure 1. However, the more opportunities she has to adjust her portfolio in the future, the less risk exposure she is willing to bear today. Risk aversion (normal policy) considerations dominate early in the year, tempering the risk-shifting considerations and bringing the optimal policy closer to normal, but over time the risk-shifting motive grows stronger, and hence the magnitude of risk-taking around the suboptimal range of portfolio returns increases. Additionally, the range over which the risk exposure displays a risk-shifting-induced hump shrinks monotonically as the horizon approaches, with the point at which the manager's policy is closest to mimicking the benchmark converging to  $\eta_H$ .

### 2.3 Alternative Flow-Performance Specifications and Applications

This Section considers generalizations of our baseline analysis to alternative specifications of the fund-flow to relative-performance relationship and also attempts to reinterpret the ensuing manager's payoff functions as compensation contracts.

### A. Collar-Type (continued)

We first keep the collar-type fund-flow specification (2) but assume that Condition 1 is violated. In this case, the shape of the upward-sloping region of the fund flow function affects the manager’s optimal policy. An analytical solution is not available for this case because the manager’s optimal terminal portfolio value displays four regions of distinct behavior, and so we solve for the optimal policy numerically (see Appendix B for an elaboration on the form of the solution and Appendix C for the details of our numerical procedure). Figure 3 presents the solution.



**Figure 3. The manager’s optimal risk exposure when Condition 1 is not satisfied.** The solid plots are for the optimal risk exposure, and the dotted plots are for the manager’s normal risk exposure. In panel (a),  $\eta_H = 0.10$ , and in both panels  $\mu = 0.08$ . The remaining parameter values are as in Figure 1.

We again see that the optimal risk exposure displays two different patterns, depending on whether the economy is of type (a) or (b). In the latter case, the manager optimally decreases the volatility of her portfolio. Overall, despite the more complex nature of the solution, the manager’s optimal behavior is quite similar to that in Section 2.2. Although theoretically the minimum of risk exposure is achieved away from  $\eta_H$ , it is not so visible in Figure 3.

It is useful to draw a parallel between the collar-type flow function and executive compensation contracts in the US. The most prevalent form of annual bonuses offered to executives is the so-called 80/120 plan (Murphy (1999)). Under an 80/120 plan, the manager receives only a base salary if her performance does not exceed 80% of a pre-specified performance standard. The point 20% below the performance standard is known as the “performance threshold,” at which the manager’s compensation jumps up, and then increases continuously over an “incentive zone”. The compensation schedule in the incentive zone is typically, although not always, linear. Annual bonuses are capped once the performance exceeds 120% of the performance standard. Under the plausible assumption of the base salary and hence the bonus (typically a percentage of base) being

proportional to company’s size,<sup>14</sup> adopting a performance standard based on performance relative to a benchmark, we can attempt to reinterpret our manager’s payoff function as an executive’s base salary plus an annual bonus. Our collar-type specification is indeed very similar to an 80/120 plan; the only feature that is missing is the discontinuity at the performance threshold, which has been argued to affect a manager’s incentives in a significant way. Since the literature on compensation typically looks at sensitivity-based measures of risk-taking incentives, our measure would potentially yield rather different conclusions pertaining to risk taking. Moreover, Proposition 1 shows that, for the case when Condition 1 holds, the shape of the incentive zone – concave, convex or any other – is irrelevant to the manager’s optimal choice, potentially defeating the intended purpose of the incentive zone.

## B. Digital

As discussed above, a manager’s payoff is typically discontinuous at a performance threshold. We isolate the effects of such a jump by specializing the fund flow function  $f_T$  as follows:

$$f_T = \begin{cases} f_L & \text{if } R_T^W - R_T^Y < \eta, \\ f_H & \text{if } R_T^W - R_T^Y \geq \eta, \end{cases} \quad 0 < f_L \leq f_H, \eta \in \mathbb{R}.$$

If the manager’s year-end relative performance is above the threshold  $\eta$ , she gets a high bonus  $f_H$ , otherwise a low bonus  $f_L$ . For this payoff, we again obtain an analytical characterization of the optimal risk exposure, which is given by the same expression as in Proposition 1 and the plot in Figure 1, but with  $\eta_H$  now replaced by the performance threshold  $\eta$ .

Some comments are in order. As in Figures 1–2, the risk-taking incentives induced by a jump in the bonus are minimized around the performance threshold  $\eta$ . This is somewhat surprising in light of the literature arguing that a discontinuity induces a peak in risk-taking since the slope at the performance threshold is infinite (see e.g., Murphy (1999)). This contrast is again due to the differences in sensitivity- versus optimality-based measures such as ours. The manager being risk averse, strives to take on as little risk as necessary to exceed the performance threshold. A small risk increase suffices to achieve this when the performance is right below the threshold, where the payoff’s sensitivity to volatility is extremely high. As the manager falls further behind, she needs a bigger gamble to catch up with the benchmark, and so her risk-taking keeps increasing

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<sup>14</sup>This assumption rules out the case where the manager’s utility does not depend on how badly she is doing when underperforming. That case is considered by Carpenter (2000), who considers a manager paid with a call option on firm assets. To obtain Carpenter’s “safety net” at poor fund performance within our model, we may consider a fund flow function, which equals  $K/W_T$  until a threshold level, and then becomes a constant. This provides the manager with a floor  $K$  for terminal portfolio values below  $K$ , and a payoff linear in  $W_T$  for levels above  $K$ . Note that an interpretation of such a payoff as a flow-performance relationship is counterfactual because it rewards the manager for underperformance. Moreover, the fund inflows tend to infinity at zero fund value. This in turn induces an underperforming manager protected by a safety net to optimally choose an unbounded risk exposure even if she is risk averse. In contrast, our specification increasingly penalizes the manager for destroying firm value (e.g., reflecting an increase in the likelihood of being fired) and leads to a finite risk exposure when performing extremely badly.

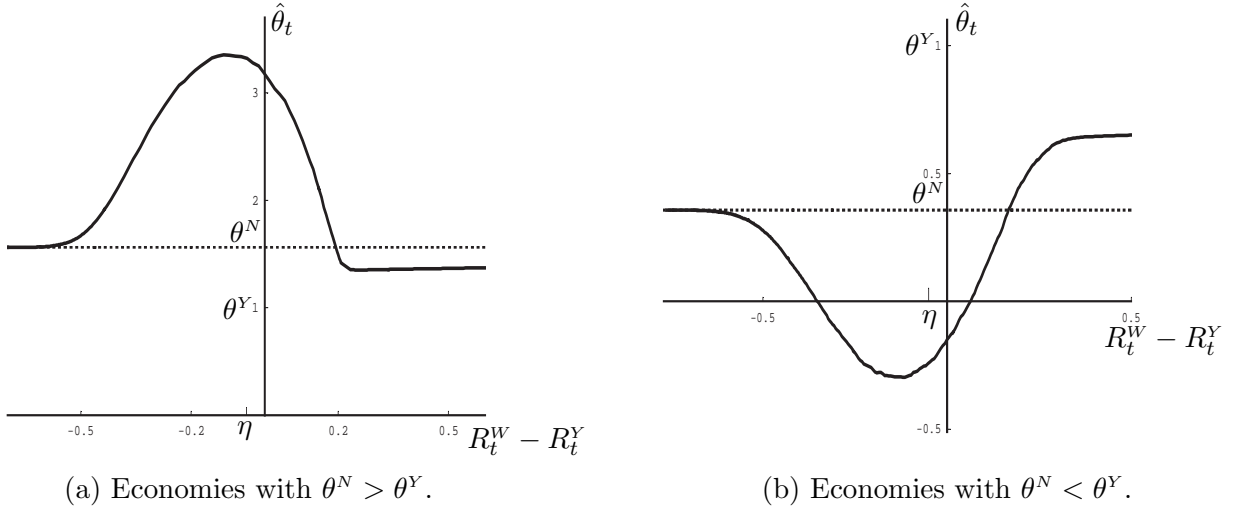
in the underperformance region until it reaches a maximum where the risk-shifting incentives are mitigated by her risk aversion.

### C. Linear-Convex and Other Specifications

Sirri and Tufano (1998) document that the flows become increasingly sensitive to performance in the region of good performance. We model this as follows:

$$f_T = \begin{cases} f_L & \text{if } R_T^W - R_T^Y < \eta, \\ f_L + \psi \left( e^{(R_T^W - R_T^Y)} - e^\eta \right) & \text{if } R_T^W - R_T^Y \geq \eta, \end{cases}$$

with  $f_L, \psi > 0$  and  $\eta \in \mathbb{R}$ .<sup>15</sup> Under this linear-convex specification, the flows are not sensitive to bad performance, however, they are very sensitive to good performance, increasing exponentially above the threshold  $\eta$ . Figure 4 reports the numerical solution for the manager's optimal behavior.



**Figure 4. The manager's optimal risk exposure under a linear-convex flow-performance specification.** The solid plots are for the optimal risk exposure, and the dotted plots are for the manager's normal risk exposure. The parameter values  $f_L = 0.97$ ,  $\psi = 1.6$ ,  $\eta = -0.05$  are calibrated to match the estimated relation in Sirri and Tufano (1998). The remaining parameter values are the same as in Figure 3.

We again note the familiar pattern: a hump over the risk-shifting range and two types of deviations from the benchmark, distinguishing panels (a) and (b). The main difference here occurs in the overperformance region, where the flows are always convex. When far ahead of the benchmark, the manager's policy no longer tends to her normal. First, a convex flow function combined with a concave objective alters her effective risk aversion, which now tends to  $2\gamma - 1$  since in the

<sup>15</sup>For brevity, we omit a technical condition, imposing a lower bound on  $\eta$ , guaranteeing that the manager's objective function is strictly concave in the overperformance region. This condition is easily satisfied under our parameterization.



limit her payoff is quadratic in portfolio value  $W$ . Second, even in the overperformance limit, the manager's payoff depends on the level of the benchmark, a feature absent in our previous specifications. The flows being insensitive to bad performance, however, the limit of the risk exposure in the underperformance region coincides with the manager's normal policy, as before.

We have also considered a linear-linear specification of the flow-performance relationship, where the second segment has a higher slope, hence resembling a call option. Such a specification is motivated by asymmetric fees becoming more prevalent in European mutual funds. The optimal risk exposure is similar to that in Figure 4, the only difference being that the manager again behaves as if her risk aversion were  $\gamma$  in the limit of good performance. We omit the figure for brevity. Finally, we conjecture that for a general payoff structure  $W_T f_T$ , the bulk of our results holds locally for every region in which  $u(W_T f_T)$  is nonconcave. If such a region includes  $W_T = 0$  or  $W_T = \infty$ , then at the global maximum (or minimum), the manager's risk exposure can be infinite (a corner solution) or not well-defined. Otherwise, the manager's risk exposure is bounded from above and below for each  $t$ , and the wave-shape pattern of risk-taking incentives is along the lines of that described in the preceding analysis.

Our analysis may be applied to other problems, where payoff structures are nonconcave and agents are (effectively) risk averse. Beyond executive bonuses, other possible applications include stock-option-based executive compensation and option-like compensation of hedge fund managers. Nonconcave payoffs also arise in: banking due to deposit insurance; corporate finance due to shareholders having a nonconcave payoff. Since in most of these applications nonconcavities in the payoff do not arise over a stochastic range as in our problem with a stochastic benchmark, it is worth pointing that a riskless benchmark is a special case of our analysis. The optimal risk exposure will resemble that in Figure 1a. An additional application that fits our digital payoff specification is an election, with the two possibilities representing the outcomes of being elected or not. Our optimal risk-shifting range would then represent the behavior of a risk averse candidate who is behind in the polls.

## 2.4 Costs of Active Management

In this Section, we assess the economic significance of the manager's adverse behavior. Towards this, we consider an investment policy associated with the manager acting in the best interest of fund investors, fully ignoring her own incentives. A hypothetical fund investor is assumed to have CRRA preferences,  $u_I(W_T) = \frac{W_T^{1-\gamma_I}}{1-\gamma_I}$ ,  $\gamma_I > 0$ , over the horizon wealth  $W_T$ . The investor is passive in that he delegates all his initial wealth,  $W_0$ , to the manager to invest. The decision to delegate, exogenous here, captures in a reduced form the choice to abstain from active investing due to various imperfections associated with money management (participation and information costs, time required to implement a dynamic trading strategy, transaction costs, behavior limitations),

or simply because he believes that the manager has better information or ability. The manager's investment policy that maximizes the investor's utility is given by  $\theta^I = \frac{1}{\gamma_I} \frac{\mu - r}{\sigma^2}$ . Note there is an additional conflict of interest arising due to the differences in attitudes towards risk of the manager and the investor. The manager has an *explicit incentive* to manage assets in line with her own appetite for risk.

We compute the utility loss to the investor of the manager's deviating from the policy  $\theta^I$ .<sup>16</sup> Following Cole and Obstfeld (1991), we define a cost-benefit measure,  $\hat{\lambda}$ , reflecting the investor's gain/loss quantified in units of his initial wealth:

$$V^I((1 + \hat{\lambda})W_0) = \hat{V}(W_0),$$

where  $V^I(\cdot)$  denotes the investor's indirect utility under the policy absent incentives  $\theta^I$ , and  $\hat{V}(\cdot)$  his indirect utility under the optimal policy accounting for incentives  $\hat{\theta}$ . To disentangle the implications of explicit and implicit incentives, we decompose the total cost-benefit measure into two components:  $\lambda^N$  and  $\lambda^Y$ . The former captures the effects of the manager's attitude towards risk driving her normal policy, while the latter the effects of implicit incentives. In particular,  $\lambda^N$  solves  $V^I((1 + \lambda^N)W_0) = \hat{V}(W_0; f_T = 1)$ , where  $\hat{V}(W_0; f_T = 1)$  denotes the investor's indirect utility absent implicit incentives only, and  $\lambda^Y$  solves  $1 + \hat{\lambda} = (1 + \lambda^N)(1 + \lambda^Y)$ .

The gain/loss due to explicit incentives is determined by the manager's risk aversion,  $\gamma$ . Absent implicit incentives, the further  $\gamma$  deviates from the investor's risk aversion,  $\gamma_I$ , the larger the discrepancy between the optimal risk exposure of the manager,  $\theta^N$ , and that optimal for the investor,  $\theta^I$ , and consequently the higher the loss to the investor. As reported in Tables 1a and 1b, the loss due to explicit incentives,  $\lambda^N$ , is zero when the manager and the investor have the same attitude towards risk,  $\gamma = \gamma_I (= 2)$ . However, for  $\gamma = 0.5$  such a loss can be quite significant: 28.86% in economies (a) and 8.13% in economies (b).<sup>17</sup>

The gain/loss due to implicit incentives is driven by the parameters of the flow-performance relationship. The more the manager gambles when underperforming, the more she deviates from the investor's desired risk exposure. Table 1 reveals that the loss to the investor due to implicit incentives alone,  $\lambda^Y$ , increases with (i) the reward for outperformance,  $f_H - f_L$ , (ii) typically the riskiness of the benchmark,  $\theta^Y$ , and (iii) the flow threshold differential,  $\eta_H - \eta_L$ . For the largest implicit reward considered in Table 1,  $f_H - f_L = 1.1$ , the loss is 10.26% in economies (a) and 6.88% in economies (b). Additionally, the effects of implicit incentives are most pronounced for relatively risky benchmarks. For example, Table 1a reports the cost due to implicit incentives to be 11.79% for the riskiest benchmark we consider ( $\theta^Y = 1.5$ ), and in Table 1b a corresponding cost of 8.45%.

<sup>16</sup>This measure is, of course, subject to a caveat that investors *have chosen* active management. This measure does not take into account the potential costs to investors associated with implementing their optimal strategy themselves.

<sup>17</sup>The values reported in Table 1 are for the model parameters, calibrated to conform with the observed market dynamics and capturing the observed flow-performance relationship for mutual funds. The reported range of managerial risk aversion coefficient  $\gamma$  is smaller in Table 1a than in Table 1b because the condition for economies (a) to occur imposes an upper bound on  $\gamma$ , equaling 2.34 for our baseline calibration.

The combined effect of the explicit and implicit incentives can be considerable. Table 1 reports the total lost to the investors,  $\hat{\lambda}$ , ranges from 2.27% to 48.16%.

### 3. Multiple Sources of Risk and Multiple Stocks

Until now, we have assumed the Black and Scholes (1973) economy in which the fund manager decides how to allocate her portfolio between a risky and a riskless asset. This setting has served as the simplest possible one to highlight the most important insights pertaining to risk-taking incentives. In real life, however, managers must often allocate their portfolios between different stocks, rather than between stocks and bonds. Moreover, unlike in our baseline model, managers may wish to adjust their portfolio riskiness through taking on idiosyncratic rather than systematic risk. Thus, it is of interest to examine a setup in which one can make a distinction between the effects of systematic and idiosyncratic risks on the manager's decisions.

To analyze these issues, we extend our model to multiple sources of uncertainty and multiple stocks. Each stock price,  $S_i$ , follows

$$dS_{it} = \mu_i S_{it} dt + \sigma_i S_{it} dw_t, \quad i = 1, \dots, n,$$

where the stock mean returns  $\mu \equiv (\mu_1, \dots, \mu_n)^\top$  and the nondegenerate volatility matrix  $\sigma \equiv \{\sigma_{ij}, i, j = 1, \dots, n\}$  are constant, and  $w = (w_1, \dots, w_n)^\top$  is an  $n$ -dimensional standard Brownian motion. The benchmark  $Y$  is now a value-weighted portfolio with fractions  $\beta \equiv (\beta_1, \dots, \beta_n)^\top$  invested in the stocks and  $1 - \beta^\top \mathbf{1}$  in the money market, where  $\mathbf{1} \equiv (1, \dots, 1)^\top$ . The manager's optimization problem, as before, is given by (4).

Allowing for multiple sources of uncertainty increases the dimensionality of the problem, introducing certain technical difficulties, and our proof of Proposition 1 does not readily extend. The proof of Proposition 2 employs a method to reduce the multi-state variable problem to a single one via a change of variable (accounting for benchmarking) and a change of measure (accounting for risk aversion). As in Proposition 1, we present Proposition 2 in terms of the state-price density process  $\xi$ , following  $d\xi_t = -\xi_t r dt - \xi_t \kappa^\top dw_t$ , where now the market price of risk  $\kappa \equiv \sigma^{-1}(\mu - r\mathbf{1})$  is  $n$ -dimensional. For brevity, we present only the case where Condition 1 is satisfied.

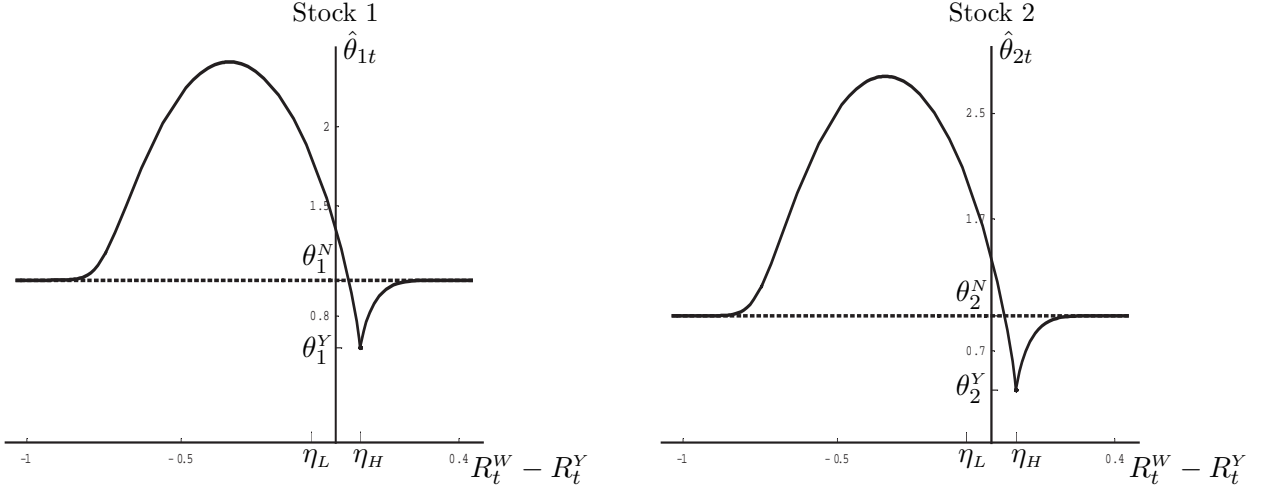
**Proposition 2.** *Under Condition 1, the optimal fractions of the manager's portfolio invested in*

risky assets and her terminal wealth are given by

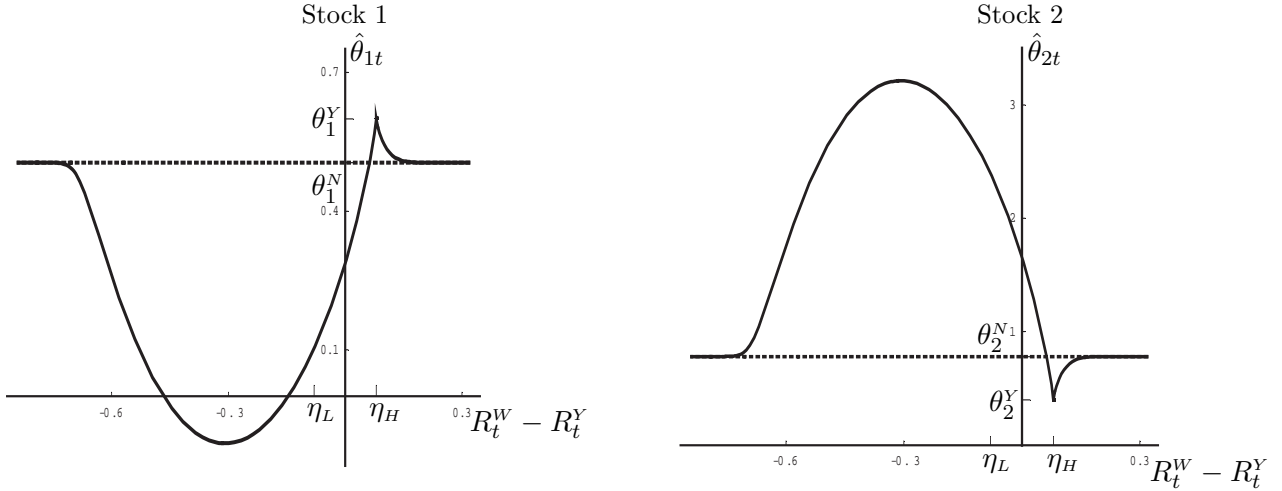
$$\begin{aligned}
\hat{\theta}_t &= \theta^N + (\theta^N - \theta^Y) \left\{ \left[ \mathcal{N}(d_2(\gamma, \pi_*)) - \mathcal{N}(d_2(\gamma, \pi^*)) \right] e^{\eta_H} \right. \\
&\quad + \left[ (\pi^{*1/\gamma} \phi(d_1(\gamma, \pi^*)) - \pi_*^{1/\gamma} \phi(d_1(\gamma, \pi_*))) e^{\eta_H} \right. \\
&\quad \left. \left. + (f_H^{(1/\gamma-1)} \phi(d_1(\gamma, \pi_*)) - f_L^{(1/\gamma-1)} \phi(d_1(\gamma, \pi^*))) y^{-1/\gamma} \right] \frac{\gamma \xi_t^{-1/\gamma} \hat{Z}(\gamma)}{\|\kappa^\top - \gamma \sigma^\top \beta\| \sqrt{T-t} \hat{W}_t} \right\}, \quad (5) \\
\hat{W}_T &= \frac{1}{f_H} J \left( \frac{y}{f_H} \xi_T \right) 1_{\{\xi_T Y_T^\gamma < \pi_*\}} + e^{\eta_H} Y_T 1_{\{\pi_* \leq \xi_T Y_T^\gamma < \pi^*\}} + \frac{1}{f_L} J \left( \frac{y}{f_L} \xi_T \right) 1_{\{\pi^* \leq \xi_T Y_T^\gamma\}},
\end{aligned}$$

where  $y$  solves  $E[\xi_T \hat{W}_T] = W_0$ ,  $J(\cdot)$ ,  $\mathcal{N}(\cdot)$ ,  $\phi(\cdot)$  are as given in Proposition 1,  $\pi_* = f_H^{1-\gamma} e^{-\gamma \eta_H} / y$ ,  $\pi^* > \pi_*$  satisfies  $\hat{g}(\pi) = 0$ ,  $\hat{g}(\pi) = \left( \gamma \left( \frac{y}{f_L} \pi \right)^{1-1/\gamma} - (f_H e^{\eta_H})^{1-\gamma} \right) / (1 - \gamma) + e^{\eta_H} y \pi$ ,  $\hat{Z}(\gamma) = e^{\frac{1-\gamma}{2\gamma^2} \|\kappa - \gamma \sigma^\top \beta\|^2 (T-t)}$ ,  $\hat{d}_1(\gamma, \hat{\pi}) = \frac{\ln \frac{\hat{\pi}}{\xi_t Y_t^\gamma} + \frac{2-\gamma}{2\gamma} \|\kappa - \gamma \sigma^\top \beta\|^2 (T-t)}{\|\kappa - \gamma \sigma^\top \beta\| \sqrt{T-t}}$ ,  $\hat{d}_2(\gamma, \hat{\pi}) = \hat{d}_1(\gamma, \hat{\pi}) - \frac{1}{\gamma} \|\kappa - \gamma \sigma^\top \beta\| \sqrt{T-t}$ ,  $\theta^N = \frac{1}{\gamma} (\sigma^\top)^{-1} \kappa$ ,  $\theta^Y = \beta$ , and  $\hat{W}_t$  is as given in the proof. The case of  $\theta^N = \theta^Y$  is described in the proof.

The solution has a similar structure to the single stock case, with our earlier insights going through component-by-component. This can be viewed as “tilting” positions in individual stocks. Thus, at the threshold  $\eta_H$ , the manager selects portfolio weights in individual stocks close to those of the benchmark portfolio. When outperforming, the manager tilts each portfolio weight away from the benchmark and in the direction of her normal policy, converging to it in the limit. When underperforming, she deviates from the benchmark by tilting the investment in each stock  $i$  in the direction dictated by the sign of  $\theta_i^N - \theta_i^Y$ ; i.e., whether the manager’s normal policy assigns a larger or smaller weight to the stock than its weight in the benchmark. For each stock’s portfolio weight, we now obtain two typical investment patterns, where the underperforming manager either increases or decreases her weight in the stock, analogous to economies (a) and (b) of Proposition 1. Figure 5 illustrates this for the case of two risky stocks with parameters chosen such that  $\theta_1^N > \theta_1^Y$ ,  $\theta_2^N > \theta_2^Y$  (panel (a)) and  $\theta_1^N < \theta_1^Y$ ,  $\theta_2^N > \theta_2^Y$  (panel (b)). The remaining cases are mirror images of the ones presented. Note that even with constant security price parameters ( $r$ ,  $\mu$ ,  $\sigma$ ), the manager’s investment policy does not display two-fund separation. We may rewrite the optimal risk exposure expression in Proposition 2 as  $\hat{\theta}_t = (\sigma^T)^{-1} \kappa (1 + x_t) / \gamma - \beta x_t$ , where  $x_t$  denotes the expression in  $\{\cdot\}$  of equation (5). Hence, we see that the optimal portfolio satisfies a three-fund separation property, with the three funds being the instantaneous mean-variance efficient portfolio of risky assets  $(\sigma^T)^{-1} \kappa / (\mathbf{1}^T (\sigma^T)^{-1} \kappa)$ , the benchmark portfolio  $\theta^Y$ , and the riskless asset.



(a) Economies with  $\theta_1^N > \theta_1^Y$  and  $\theta_2^N > \theta_2^Y$ .



(b) Economies with  $\theta_1^N < \theta_1^Y$  and  $\theta_2^N > \theta_2^Y$ .

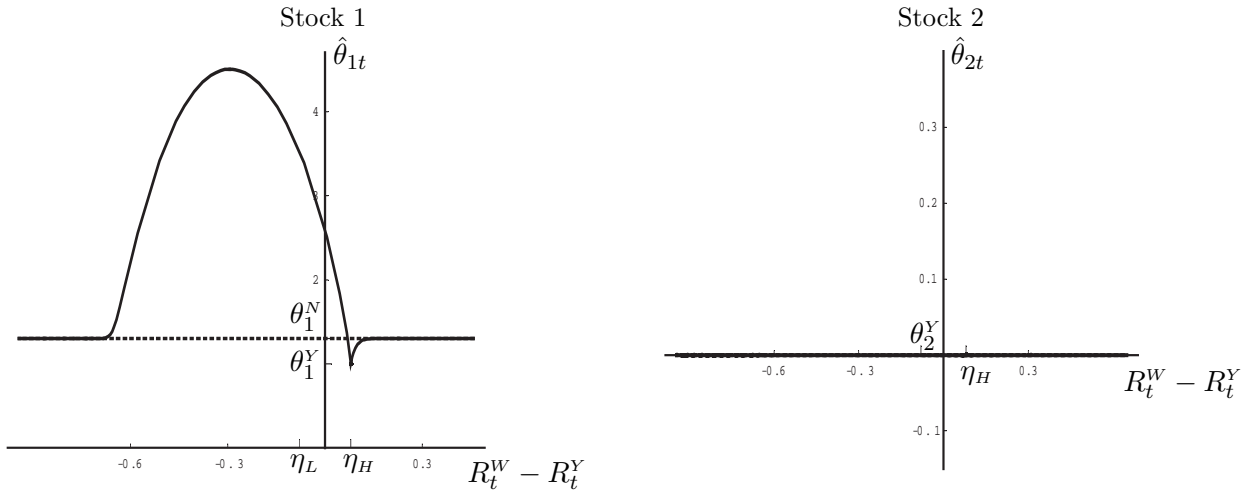
**Figure 5. The manager’s optimal portfolio weights in Stocks 1 and 2.** The solid plots are for the optimal portfolio weights, and the dotted plots are for the manager’s normal weights. We parameter values are  $\gamma = 1.5$ ,  $f_L = 0.8$ ,  $f_H = 1.5$ ,  $\beta = (0.6, 0.4)^\top$ ,  $\eta_L = -0.08$ ,  $\eta_H = 0.08$ ,  $\mu = (0.07, 0.09)$ ,  $r = 0.02$ ,  $\sigma_1 = (0.18, 0.0)$  (panel (a)),  $\sigma_1 = (0.18, 0.08)$  (panel (b)),  $\sigma_2 = (0.0, 0.22)$ ,  $t = 0.75$ ,  $T = 1$ .

We also examine whether the manager achieves her optimal risk-taking profile by taking on systematic versus idiosyncratic risk, given the current discussion in the literature (e.g., Chevalier and Ellison (1997)). When underperforming, does the manager really take on systematic risk as suggested by our baseline model, or rather, opts for idiosyncratic gambles, more in line with the perceived wisdom? To shed some light on this issue, we present a special case of our model, in which the manager has a choice between systematic and idiosyncratic risks. We consider a two-stock

economy with the following parameterization:

$$\mu = \begin{pmatrix} \mu_1 \\ r \end{pmatrix}, \quad \sigma = \begin{pmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Here, the relative performance benchmark  $Y$  is given by stock 1, which is driven solely by the Brownian motion  $w_1$ . In contrast, stock 2 is purely driven by  $w_2$  and does not command any risk premium, with mean return equal to the riskless rate. Consequently, the market price of risk is given by  $\kappa = \left(\frac{\mu_1 - r}{\sigma_{11}}, 0\right)^\top$ . In other words, by investing in stock 1 the manager is exposed to systematic risk, and by investing in stock 2 the manager takes on idiosyncratic risk. We report the ensuing optimal portfolio weights in stocks 1 and 2 in Figure 6.



**Figure 6. The manager’s optimal portfolio weights in Stocks 1 and 2.** Stock 1 is driven by systematic risk, while stock 2 is driven by idiosyncratic risk. The solid plots are for the optimal portfolio weights, and the dotted plots are for the manager’s normal weights. The parameter values are We parameter values are  $\gamma = 1.5$ ,  $f_L = 0.8$ ,  $f_H = 1.5$ ,  $\beta = (1.0, 0.0)^\top$ ,  $\eta_L = -0.08$ ,  $\eta_H = 0.08$ ,  $\mu = (0.07, 0.02)$ ,  $r = 0.02$ ,  $\sigma_1 = (0.16, 0.0)$   $\sigma_2 = (0.0, 0.22)$ ,  $t = 0.75$ ,  $T = 1$ .

As evident from Figure 6, the manager chooses to invest *nothing* in stock 2, and hence is not exposed to any idiosyncratic risk. Instead, she is only exposed to systematic risk and engages in risk-shifting via adjusting her position in stock 1, in a manner consistent with Section 2. The above example is not pathological. We have considered a sequence of economies parameterized by  $\mu_2^m$ , with  $\mu_2^m \rightarrow r$  as  $m \rightarrow \infty$ . In each economy  $m$ , the market price of risk due to  $w_2$  is nonzero, and the weights in each risky asset are similar to those in Figure 6. As we increase  $m$ , the investment in the second stock uniformly converges to zero.

## 4. Empirical Analysis

Some support for our model’s implications can be drawn from the extant empirical literature on risk shifting by mutual funds. However, existing empirical evidence is insufficient to verify the central implications of our model, which are novel. Namely, no empirical paper has drawn a distinction between our economies (a), in which the benchmark is less risky than the manager’s normal policy, and (b), in which the benchmark is riskier. In the latter, according to the model, the manager’s gambling entails reducing the volatility (systematic risk) of her portfolio when underperforming. Moreover, the measure of risk-taking incentives arising from our model is the deviation of the manager’s portfolio from the benchmark – the manager’s portfolio *tracking error variance*  $\text{Var}(R_t^W - R_t^Y)$  – and not the portfolio variance  $\text{Var}(R_t^W)$ . An underperforming manager is predicted to always increase the tracking error variance, but not necessarily the variance.

The existing literature has been sparked by Brown, Harlow, and Starks (1996) who find that managers who underperform relative to their peers increase the volatility of their portfolios towards the year-end. However, there remains some controversy about this result, stemming from Busse (2001), who finds no such increase. Nevertheless, the papers above offer tests of our model only if the world is of type (a), since the measure of portfolio riskiness adopted in these studies is the portfolio’s total variance, and not the tracking error variance. Chevalier and Ellison (1997) also find that underperforming managers take bigger risk towards the year-end. Their paper provides stronger support for our theory because they measure risk as the sample variance of a fund’s excess return over the market. However, as they use monthly data, their estimates of year-to-date standard deviations could be quite imprecise. Using daily data, Reed and Wu (2005) argue that it is not underperformance relative to the peers but underperformance relative to the S&P 500 index that induces fund managers to gamble. Mutual fund managers appear to take larger risks before beating the S&P 500; after beating they tend to increase the correlation of their funds’ assets with the market. None of these papers examines the risk-taking behavior of managers who normally desire lower risk exposure than their benchmark (our economies (b)), and thus gamble by decreasing the risk of their portfolios.

We here provide a set of simple empirical tests that focus on the implications novel to our analysis, and thus complement the existing body of work. We combine daily returns data on US mutual funds from the International Center for Finance at Yale SOM with CRSP mutual-fund database. Using CRSP objective codes, we leave only actively managed US equity mutual funds in the aggressive growth, growth and income, and long-term growth categories. The data ranges from 1970 to 1998. This data range is comparable to those used by Chevalier and Ellison and Sirri and Tufano (1998), who document convexities in the flow-performance relationship, required for our risk-taking implications.

Our first hypothesis is that the tracking error variance is higher for managers whose year-to-date

returns are below that of the benchmark than for those whose return are above the benchmark. To examine this hypothesis, for each year, we use funds' daily returns to estimate monthly sample standard deviations of each fund's *excess* return—a total of 12 entries per fund per year. For comparison, we also estimate monthly sample standard deviations of fund returns, as in Busse. On a monthly basis, for each fund, we compute an OVER indicator that equals one if the fund was overperforming the S&P 500 index, a benchmark also used by Chevalier and Ellison and others, as of the end of the preceding month. To reduce the noise in the data coming from the funds that are right at the benchmark-beating boundary, we require that a fund's year-to-date return was above the market for four out of last five trading days of the preceding month. Finally, we drop fund-year entries with fewer than 200 daily observations. We then regress the tracking error standard deviation, our risk measure, on the OVER indicator. The regression includes fund-year fixed effects. For comparison, we repeat this exercise for standard deviation of fund returns as the dependent variable.

Table 2 reveals that the OVER indicator comes out significant in the tracking error regression. Consistent with our theory, managers tend to reduce their tracking error variance when overperforming the market, and gamble by increasing it when underperforming. In contrast, the effect on the variance of fund returns has the opposite sign, however, it ceases to be statistically significant throughout many of our robustness checks. This finding is in line with Busse, who argues that underperforming managers do not seem to manipulate their portfolio standard deviation towards the year end. We have performed a number of robustness checks and the message was the same.<sup>18</sup>

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<sup>18</sup>We have repeated the exercise with an alternative, more contemporaneous, measure for the OVER indicator, OVER1, which equals one if the fund is overperforming the S&P 500 index for more than three quarters of the concurrent month, and zero otherwise. We have also tried the lagged version of it as a possible alternative. Throughout all these tests the effect of overperforming the S&P 500 index was always negative and significant at the 1% level. We have also repeated the exercise winsorizing our fund returns data to reduce the impact of outliers and data errors. Additionally, we have put lagged dependent variable in the regression, as we were concerned about autocorrelation in the dependent variable, and the results were very similar to those in Table 2. We have further repeated the regressions clustering by month and fund objective code, guarding against potential problems due to cross-sectional correlation. Significance of all of our possible OVER indicators has dropped, but has still remained at the 5% level. In the standard deviation regressions, however, all possible OVER indicators have become insignificant. Finally, we have rerun the analysis on a restricted dataset where we dropped fund-year entries for funds that were either above or below the S&P 500 for 90% of the year. The reason for this is that our identification strategy relies on the transition between the two regimes (above/ below the S&P500). Including funds that spent less than 10% of the year in one of the regimes could leave us with too few observations in that regime, making our estimates less precise. The results for this restricted dataset were qualitatively the same.



	Dependent Variable:	
	$\sigma_m(R_{i,t}^W - R_t^Y)$	$\sigma_m(R_{i,t}^W)$
$\text{OVER}_{i,m} \times 10^3$	-0.1549 (-4.78)	0.1020 (2.78)
Fund-year fixed effects	Yes	Yes
$R^2$	0.38	0.36
Number of observations	111810	111810

**Table 2. Tracking error standard deviation and fund return standard deviation tests.**  $R_{i,t}^W$  is fund  $i$ 's return on day  $t$  of month  $m$ ,  $R_t^Y$  is the return of the S&P 500 index on day  $t$  of month  $m$ ,  $\text{OVER}_{i,m}$  is an overperformance indicator for month  $m$  defined above, and  $\sigma_m(\cdot)$  is the sample standard deviation for month  $m$ . t-statistics are in parentheses. All standard errors are corrected for heteroscedasticity.

Our second set of tests aims at providing evidence for economies of type (b), in which a manager gambles through reducing the systematic risk of her portfolio, rather than increasing it. Recall that, according to our model, a manager would choose this risk exposure pattern if she is sufficiently risk averse, i.e., normally chooses a portfolio that is less risky than her benchmark. To identify such managers, we look at (i) their portfolio betas in the preceding year, (ii) their betas in the current year, and construct a subsample of funds whose betas are below one. Then, for each fund for each week, we construct an UNDER indicator that equals one when the fund's year-to-date return is below that of the S&P 500 as of the end of the preceding week. To reduce noise coming from the funds on the benchmark-beating boundary, we require that additionally the fund was above the S&P 500 for four out of five days of the preceding week. Our hypothesis is that sufficiently risk-averse managers decrease their portfolio betas when underperforming the market. Of course, there are many other influences potentially affecting funds' betas. We therefore also include fund-year fixed effects, as well as month fixed effects. The latter are to control for seasonality in betas, documented by, e.g., Lewellen and Nagel (2006). Because of the computational restriction to keep the size of the dummy-variable set manageable, we do not include any fixed effects for the intercept terms, but those are too small and insignificant to make any appreciable difference. We again drop fund-years with fewer than 200 entries. We then regress daily funds' excess returns on the excess return on the S&P 500 interacted with the UNDER indicator, as well as the excess return on the S&P 500 interacted with the month and fund-year dummies, and report the results in Table 3.

	Dependent Variable: $R_{i,t}^W - R_t^F$			
	Beta $_T$ below 1		Beta $_{T-1}$ below 1	
	(1)	(2)	(3)	(4)
UNDER $_{i,w} \times (R_t^Y - R_t^F)$	-0.013 (-5.18)	-0.015 (-5.86)	-0.015 (-5.87)	-0.019 (-7.13)
Month fixed effects	No	Yes	No	Yes
Fund-year fixed effects	Yes	Yes	Yes	Yes
$R^2$	0.34	0.34	0.34	0.34
Number of observations	1874449		1869813	

**Table 3. Beta tests.**  $R_{i,t}^W$  is fund  $i$ 's return on day  $t$   $R_t^Y$  is the return of the S&P 500 index on day  $t$ ,  $R_t^F$  is the riskfree interest rate on day  $t$ , and UNDER $_{i,w}$  is an underperformance indicator for week  $w$  defined above. The fixed effects dummies are interacted with  $(R_t^Y - R_t^F)$ . t-statistics are in parentheses.

In both tests, the underperformance indicator interacted with the market is negative and significant. Underperforming managers who fall into our type-(b) category, i.e., whose betas are below one, choose to reduce their portfolio betas. The effect is similar in magnitude and significance for the two approaches to isolating managers who normally desire lower risk exposure than the market: based on their betas this year or last year. We have performed a number of robustness checks and the message was the same.<sup>19</sup> However, while the signs of our estimates are consistent with our theory, their magnitudes are considerably smaller, most likely reflecting the risk-management constraints imposed in practice (see Almazan, Brown, Carlson, and Chapman (2004)).

Finally, we test whether the “gambling” behavior of a fund manager is less pronounced if she has fallen far behind the benchmark.<sup>20</sup> Towards that end, we expand the menu of our underperformance indicators, now distinguishing between the funds that are currently trailing the benchmark by more than 5%, 10%, 15%, or 20%. We again look only at economies of type (b), identifying sufficiently risk averse managers by the same procedure as in our previous test. In the interest of space, we only report the results for funds whose previous year’s beta was below one and always include month fixed effects; the results for funds whose current year’s beta is below one and results without month

<sup>19</sup>We have tried alternative definitions of the UNDER indicator, basing it solely on performance as of Friday of the preceding week, or just on the preceding week, in which case UNDER equals one when the fund was trailing the S&P 500 index for four days or more of the preceding week. Additionally, we have put several lags of the dependent variable in the regression, as we were concerned about autocorrelation in funds daily returns. The lagged dependent variable was highly significant, but the significance of the interaction term with UNDER was virtually unchanged. Furthermore, we have repeated the analysis on a restricted dataset where we dropped fund-year entries for funds that were either above or below the S&P 500 for 90% of the year. The results for this restricted dataset were qualitatively the same. Due to the large size of our sample, we have been able to implement White correction for heteroscedasticity only on random 20% subsamples of funds in our dataset. Additionally, in these subsamples we have clustered errors by day and fund objective code. In all these subsamples the pertinent estimates were significant at the 5% level.

<sup>20</sup>We are grateful to an anonymous referee for suggesting this test.

fixed effects are very similar. The criteria for including observations in our analysis are as before. Table 4 reports the findings.

	Dependent Variable: $R_{i,t}^W - R_t^F$				
	(1)	(2)	(3)	(4)	(5)
$\text{UNDER}(0\%)_{i,w} \times (R_t^Y - R_t^F)$	-0.015 (-5.57)	-0.020 (-7.76)	-0.023 (-8.81)	-0.022 (-8.63)	-0.018 (-6.85)
$\text{UNDER}(-5\%)_{i,w} \times (R_t^Y - R_t^F)$	-0.020 (-7.19)				-0.019 (-6.73)
$\text{UNDER}(-10\%)_{i,w} \times (R_t^Y - R_t^F)$		-0.008 (-2.32)			0.005 (-1.37)
$\text{UNDER}(-15\%)_{i,w} \times (R_t^Y - R_t^F)$			0.002 (-0.37)		-0.015 (-2.51)
$\text{UNDER}(-20\%)_{i,w} \times (R_t^Y - R_t^F)$				0.025 (4.03)	0.036 (4.79)
Month fixed effects	Yes	Yes	Yes	Yes	Yes
Fund-year fixed effects	Yes	Yes	Yes	Yes	Yes
$R^2$	0.35	0.35	0.35	0.35	0.35
Number of observations	1792678	1823597	1840330	1848641	1730435

**Table 4. Beta tests for varying degree of underperformance.**  $R_{i,t}^W$  is fund  $i$ 's return on day  $t$   $R_t^Y$  is the return of the S&P 500 index on day  $t$ ,  $R_t^F$  is the riskfree interest rate on day  $t$ , and  $\text{UNDER}_{i,w}(-p\%)$  is an underperformance indicator for week  $w$ , which equals 1 if the fund is trailing the S&P 500 by  $p\%$ . The fixed effects dummies are interacted with  $(R_t^Y - R_t^F)$ . t-statistics are in parentheses.

We have seen from our earlier results that sufficiently risk averse fund managers who fall behind the benchmark decrease their betas. Table 4 reveals that if they fall further behind, by 5%, they tend to decrease their betas even further. When the funds' underperformance reaches the 10%-15% range, the "gambling" incentives of the managers subside. The statistical significance of the estimates in rows three and four is weak, and it weakens further in several robustness checks. So, the maximum deviation from the managers' normal policies appears to be occurring at a level of underperformance somewhere between -15% and -5%. This is consistent with our theoretical finding that at some level of underperformance, the incentives to gamble are counterbalanced by a manager's risk aversion, and the deviation from the normal policy reaches its maximum. Below the maximum deviation point, at -20%, the reward to "gambling" is small relative to the utility costs due risk aversion, and the managers raise the betas of their funds. This pattern is consistent with the predictions of the theory: throughout all specifications of the flow-performance relationship considered in Section 2, the optimal risk exposure of the manager is always hump-shaped.<sup>21</sup>

<sup>21</sup>This finding is also supported by the tracking error standard deviation test, where we repeat the analysis sum-

## 5. Conclusion

We have attempted to isolate one of the most important adverse incentives of a fund manager: an implicit incentive to perform well relative to a benchmark. This incentive introduces a nonconcavity in the manager’s problem, akin to nonconcavities observed in many corporate finance applications (e.g., asset substitution problem, “gambling for resurrection,” executive compensation, hedge fund managers compensation). It has been argued that in some of these applications, agents do not behave as though they are risk-neutral, but effectively as risk averse. Our methodology of dealing with nonconcavities in the presence of risk aversion may then help shed some light on these and other issues of interest. We solve the manager’s problem within a standard complete-markets framework, which offers considerable tractability, allowing us to derive the manager’s optimal policy in closed-form. In many real world applications, nonconcavities in the payoff structure go hand-in-hand with capital markets imperfections, frictions, for example with restrictions against trading the underlying security designed to induce the “right” incentives to the manager. Our story of the optimal interaction of risk-shifting with risk aversion would then be further compounded by the effects of such frictions.

In our setup, the need to constrain the manager’s investment choice arises naturally as the potential costs of misaligned incentives resulting from the manager’s policy turn out to be economically significant. In related work, Basak, Pavlova, and Shapiro (2006) demonstrate how a simple risk management practice that accounts for benchmarking can ameliorate the adverse effects of managerial incentives. We believe that endogenizing investment restrictions in the context of active money management is a fruitful area for future research. It would also be of interest to endogenize within our model the fund-flows to relative-performance relationship that we have taken as given. Finally, while we focus on the fund manager’s changing her portfolio composition in response to incentives, this may not be the only way the manager can increase her relative performance. As Christoffersen (2001) documents, about half of money fund managers voluntarily waive fees so as to increase their net return, and variation in these fee waivers is largely driven by relative performance. It would be interesting to incorporate such a feature into our model.

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marized in Table 2 for a range of underperformance indicators. In the interest of space, we do not report these results here.

## Appendix A: Proofs

**Proof of Proposition 1.** Before proceeding with the proof, we present for completeness the results that for brevity were not included in the body of the proposition. First, note that for  $\gamma = 1$ ,  $g(\cdot)$  takes the form:  $g(\xi) = -\left(\ln \frac{\hat{y}\xi}{f_L} + 1 - \ln \frac{\xi^{1/\hat{\kappa}}}{A f_H}\right) + y A \xi^{1-1/\hat{\kappa}}$ . Second, since  $\hat{W}_t \xi_t$  is a martingale (given the dynamics of  $\hat{W}_t$  and  $\xi_t$ ), the time- $t$  wealth is obtained by evaluating the conditional expectation of  $\hat{W}_T \xi_T$ . In the economies described in (a):

$$\begin{aligned} \hat{W}_t &= E_t \left[ \hat{W}_T \xi_T / \xi_t \right] \\ &= \left[ \mathcal{N}(d(\gamma, \hat{\xi})) f_H^{(1/\gamma-1)} + \mathcal{N}(-d(\gamma, \xi_a)) f_L^{(1/\gamma-1)} \right] Z(\gamma) (\hat{y} \xi_t)^{-1/\gamma} \\ &\quad + \left[ \mathcal{N}(d(\hat{\kappa}, \xi_a) - \mathcal{N}(d(\hat{\kappa}, \hat{\xi})) \right] A Z(\hat{\kappa}) \xi_t^{-1/\hat{\kappa}}. \end{aligned} \quad (\text{A1})$$

Similarly, in the economies described in (b):

$$\begin{aligned} \hat{W}_t &= \left[ \mathcal{N}(d(\gamma, \xi_b)) f_L^{(1/\gamma-1)} + \mathcal{N}(-d(\gamma, \hat{\xi})) f_H^{(1/\gamma-1)} \right] Z(\gamma) (\hat{y} \xi_t)^{-1/\gamma} \\ &\quad + \left[ \mathcal{N}(d(\hat{\kappa}, \hat{\xi}) - \mathcal{N}(d(\hat{\kappa}, \xi_b)) \right] A Z(\hat{\kappa}) \xi_t^{-1/\hat{\kappa}}. \end{aligned} \quad (\text{A2})$$

Finally, when  $\theta^N = \theta^Y$ , if  $\eta_H \leq 0$  (so that  $e^{\eta_H} Y_T$  and hence  $R_T^Y + \eta_H$  are feasible) or  $\eta_H \geq \tilde{\eta}$  (so that  $e^{\eta_H} Y_T$  and hence  $R_T^Y + \eta_H$  are above a critical level of infeasibility), then  $\hat{\theta}_t = \theta^N$  and  $\hat{W}_T = J(\hat{y} \xi_T)$ ; otherwise  $\hat{W}_T = \left\{ \frac{1}{f_L} J \left( \frac{\bar{y}}{f_L} \xi_T \right) \text{ or } e^{\eta_H} Y_T \right\}$ , with the indifference solution alternating between the two values in any way that satisfies the budget constraint, where (using the steps outlined below) one can show that  $\tilde{\eta}$  solves  $g \left( \xi = 1, y = f_L^{1-\hat{\kappa}} e^{(1-\hat{\kappa})(r+\beta(\mu-r)/2)T} / W_0^{\hat{\kappa}}, \eta_H \right) = 0$ , and  $\bar{y} > f_H^{1-\gamma} / A^\gamma$  solves  $g(\xi = 1, y) = 0$ .

We now proceed with the steps of the proof. To obtain the risk exposure expressions in the proposition, note that from (1), the diffusion term of the manager's optimal portfolio value process is  $\hat{\theta}_t \sigma \hat{W}_t$ . Equating the latter term with the diffusion term obtained by applying Itô's Lemma to (A1) and (A2) yields the expressions for  $\hat{\theta}_t$  in economies (a) and (b), respectively. Therefore, to complete the proof, it is sufficient to establish optimality of the given terminal wealth  $\hat{W}_T$ . Methodologically, most related to this proof is the proof of Proposition 2 of the constrained model in Basak, Shapiro, and Tepla (2006), but the setting here is notably different and the optimization problem is unconstrained. The non-concavity of the problem, arising due to (4), has an extra dimension of complexity, as not only does the expression we employ for the convex conjugate has a discontinuous non-concavity at a stochastic location, but the magnitude of the discontinuity is stochastic as well. Therefore, to our knowledge, the way the proof below adapts the martingale representation and the convex-duality techniques (see, e.g., Karatzas and Shreve (1988)) to a non-concave problem, has not been previously used in the literature.

Appealing to the martingale representation approach, the dynamic budget constraint (1) of the manager's optimization problem can be restated using the terminal value of the state price density

process as  $E[\hat{W}_T \xi_T] = W_0$ . The manager thus effectively solves a static variational problem (in which, without loss of generality,  $\xi_0 = 1$ ). For a given value of  $\theta^N$  and  $\theta^Y$ , let  $W_T$  be any candidate optimal solution, satisfying the static budget constraint  $E[W_T \xi_T] \leq W_0$ . Consider the following difference in the manager's expected utility:

$$\begin{aligned} E[u(\hat{W}_T f_T)] - E[u(W_T f_T)] &= E[u(\hat{W}_T f_T)] - \hat{y}W_0 - (E[u(W_T f_T)] - \hat{y}W_0) \\ &\geq E[u(\hat{W}_T f_T)] - E[\hat{y}\hat{W}_T \xi_T] + (E[u(W_T f_T)] - E[\hat{y}W_T \xi_T]) = E[v(\hat{W}_T, \xi_T) - v(W_T, \xi_T)], \end{aligned} \quad (\text{A3})$$

where the inequality is due to  $\hat{W}_T$  satisfying the budget constraint with equality, while  $W_T$  satisfying the budget constraint with inequality, and where

$$v(W, \xi) = u(W f_L 1_{\{R^W - R^Y < \eta_H\}} + W f_H 1_{\{R^W - R^Y \geq \eta_H\}}) - \hat{y}W\xi. \quad (\text{A4})$$

To show optimality of  $\hat{W}_T$ , it is left to show that the right-hand side of (A3) is non-negative. Under the geometric Brownian motion dynamics of  $Y_T$  and  $\xi_T$ , and using the normalization of  $Y_0$  and  $\xi_0$ , it is straightforward to verify that  $Y_T = Ae^{-\eta_H} \xi_T^{-\beta\sigma/\kappa}$ . The expression in (A4) is thus simplified to

$$v(W, \xi) = u(W f_L 1_{\{W < A\xi^{-\beta\sigma/\kappa}\}} + W f_H 1_{\{W \geq A\xi^{-\beta\sigma/\kappa}\}}) - \hat{y}W\xi. \quad (\text{A5})$$

Given the manager's CRRA preferences, to establish the non-negativity of (A3), one needs to account for the relation between the parameters  $\gamma$ ,  $\beta$ ,  $\sigma$ , and  $\kappa$  in (A5). To avoid repetition of technical details, we provide the proof for optimality of  $\hat{W}_T$  for the economies in (a) with  $\gamma > 1$ . The logic of the proof applies to the remaining subdivisions of the parameter space, as identified in the Proposition. Therefore, we now show that for the case in which  $\kappa/(\beta\sigma) > \gamma > 1$ ,

$$\arg \max_W v(W, \xi) = f_H^{1/\gamma-1} (\hat{y}\xi)^{-1/\gamma} 1_{\{\xi < \hat{\xi}\}} + A\xi^{-\beta\sigma/\kappa} 1_{\{\hat{\xi} \leq \xi < \xi_a\}} + f_L^{1/\gamma-1} (\hat{y}\xi)^{-1/\gamma} 1_{\{\xi_a \leq \xi\}}.$$

Indeed, in the above convex conjugate construction, there are three local maximizers of  $v(W, \xi)$ :  $W_H \equiv \frac{1}{f_H} J\left(\frac{\hat{y}}{f_H} \xi\right) = f_H^{1/\gamma-1} (\hat{y}\xi)^{-1/\gamma}$ ,  $W_L \equiv \frac{1}{f_L} J\left(\frac{\hat{y}}{f_L} \xi\right) = f_L^{1/\gamma-1} (\hat{y}\xi)^{-1/\gamma}$ , and  $W_A \equiv A\xi^{-\beta\sigma/\kappa}$ , where each of the three can become the global maximizer of  $v(W, \xi)$  for different values of  $\xi$ . When  $\xi = \hat{\xi}$ , then  $W_H(\hat{\xi}) = W_A(\hat{\xi})$ . When  $\xi < \hat{\xi}$ , then  $W_L > W_H > W_A$  holds under the given subdivision of the parameter space, and so for  $W \in \{W_L, W_H, W_A\}$ , we get  $v(W, \xi) = u(W f_H) - \hat{y}W\xi$ , establishing  $W_H$  as the global maximizer. When  $\xi > \hat{\xi}$ , then  $W_H < \min(W_A, W_L)$ , and  $v(W_H, \xi) = u(W_H f_L) - \hat{y}W_H \xi$ , establishing that  $W_H$  cannot be the global maximizer, because for  $W_L < W_A$  by the local optimality of  $W_L$ , we have  $u(W_H f_L) - \hat{y}W_H \xi < u(W_L f_L) - \hat{y}W_L \xi = v(W_L, \xi)$ , and for  $W_L \geq W_A$  accounting for the local optimality of  $W_A$  due to the stochastic non-concavity, we have  $u(W_H f_L) - \hat{y}W_H \xi < u(W_A f_L) - \hat{y}W_A \xi = v(W_A, \xi)$ . Moreover, for  $\xi > \xi$ , where  $\xi = (\hat{y}A^\gamma/f_L^{1-\gamma})^{1/(\gamma/\hat{\kappa}-1)} > \hat{\xi}$ , we have  $W_L < W_A$ ; whereas for the range  $\hat{\xi} \leq \xi < \xi$ , we get  $W_L > W_A$ , and for this range  $W_A$  is the global maximizer. Finally, note that for  $\xi > \xi$ , we obtain  $v(W_L, \xi) = v(W_A, \xi) + g(\xi)$ . Then, using  $\kappa/(\beta\sigma) > \gamma > 1$ , and the fact that  $g(\xi) < 0$ ,  $g(\infty) = \infty$ , it is straightforward to verify that  $g(\xi) > 0$  if and only if  $\xi > \xi_a$ , where  $g(\xi_a) = 0$ , and  $\xi_a > \xi$ ,

thereby completing the proof for the case of interest in the parameter space. Note that since  $W_L(\xi) = W_A(\xi)$ , having  $\hat{W} = W_A$  for  $\hat{\xi} < \xi < \xi_a$  gives rise to a discontinuity in  $\hat{W}_T$  as a function of  $\xi_T$  at  $\xi_a$ . The discontinuity arises in the other subcases in (a) as well, and analogously, under the parameter values in (b), the optimal policy is discontinuous at  $\xi_b$ . ■

**Proof of Proposition 2.** To establish optimality of the stated terminal wealth for the manager's optimization problem in (4) under  $n$  sources of uncertainty and  $n$  risky assets, it is still sufficient to establish the non-negativity of the right-hand side of (A3), as in the proof of Proposition 1. However, because for  $n > 1$ ,  $\xi$  and  $Y$  each span a different subspace of the state space,  $v(W, \xi)$  as given in (A4) does not in general simplify to the parametric form in (A5). Therefore, to gain further tractability, note that in (A4),  $v(W, \xi) = Y^{1-\gamma} \hat{v}(\frac{W}{Y}, \xi Y^\gamma)$ , where  $Y^{1-\gamma} \geq 0$ , and

$$\hat{v}(V, \pi) = u(V f_L 1_{\{V < e^{\eta_H}\}} + V f_H 1_{\{V \geq e^{\eta_H}\}}) - y\pi V.$$

Using this change of variables ( $V \equiv \frac{W}{Y}$ ,  $\pi \equiv \xi Y^\gamma$ ), one obtains:

$$\arg \max_V \hat{v}(V, \pi) = f_H^{1/\gamma-1} (y\pi)^{-1/\gamma} 1_{\{\pi < \pi_*\}} + e^{\eta_H} 1_{\{\pi_* \leq \pi < \pi^*\}} + f_L^{1/\gamma-1} (y\pi)^{-1/\gamma} 1_{\{\pi^* \leq \pi\}} \quad (\text{A6})$$

$$= \left[ \frac{1}{f_H} J\left(\frac{y}{f_H} \xi\right) 1_{\{\pi < \pi_*\}} + e^{\eta_H} Y_T 1_{\{\pi_* \leq \pi < \pi^*\}} + \frac{1}{f_L} J\left(\frac{y}{f_L} \xi\right) 1_{\{\pi^* \leq \pi\}} \right] \frac{1}{Y}. \quad (\text{A7})$$

The equality in (A6) is readily verified by following steps analogous to those in Proposition 1, to show that each of the three local maximizers in the above convex conjugate construction is indeed a global maximizer in designated ranges of  $\pi$ . The equality in (A7) holds due to the change of variables. The expression in brackets in (A7) is the terminal wealth,  $\hat{W}$ , stated in the Proposition, thereby verifying its optimality. Note from (A6)–(A7) that  $\hat{V}_T \equiv \frac{\hat{W}_T}{Y_T}$  is a function of only  $\pi_T \equiv \xi_T Y_T^\gamma$ . Also note that the diffusion component of  $\xi_T Y_T^\gamma$  is a function of

$$-\kappa^\top w_T + \gamma \beta^\top \sigma w_T = -\gamma w_T^\top \sigma^\top \left( \frac{1}{\gamma} (\sigma^\top)^{-1} \kappa - \beta \right) = -\gamma w_T^\top \sigma^\top (\theta^N - \theta^Y).$$

The latter indicates that the optimal policy is driven by the component-wise relation between the manager's normal weights in each stock,  $\theta_i^N$ , compared with the benchmark weights  $\theta_i^Y$ ,  $i = 1, \dots, n$ . With  $n = 1$ , Proposition 1, for expositional purposes, separately examines economies (a) ( $\theta^N < \theta^Y$ ) and (b) ( $\theta^N > \theta^Y$ ), to highlight the economic intuition of each case. With  $n > 1$ , Proposition 3 does not refine the expressions to account for all possible relations between  $\gamma$ ,  $\beta$ ,  $\sigma$ , and  $\kappa$ . We present results in their general form, and discuss how the basic intuition, with  $n = 1$ , extends to the case where various components of the optimal policy behave according to their counterparts in economies (a) or (b). Similarly, when for some  $i$ ,  $\theta_i^N = \theta_i^Y$ , then  $\hat{\theta}_i = \theta_i^N$ . However, in the case where  $\theta_i^N = \theta_i^Y$ , for *all*  $i$ , then as discussed in Proposition 1,  $\hat{W}_T$  behaves either as in the normal case, or alternates between the normal and the benchmark levels.

Given the optimal terminal wealth, we proceed to derive the wealth dynamics and the trading strategy. Relying on the martingale property of  $\xi W$ ,  $\hat{W}_t = E_t \left[ \hat{W}_T \xi_T / \xi_t \right]$ , but unlike in Proposition 1, one should account here for the joint distribution of  $Y^\gamma$  and  $\xi$ , under the physical probability

measure  $P$ . To circumvent this, it is helpful to employ a change of measure. The new measure,  $G$ , is defined by a Radon-Nikodym derivative, which accounts for the manager's risk aversion and the composition of the benchmark:

$$\frac{dG}{dP} \equiv e^{-\frac{1}{2}(1-\gamma)^2\|\beta^\top\sigma\|^2T+(1-\gamma)\beta^\top\sigma w_T}.$$

Combining this change of measure with the change of variables, the martingale property of  $\xi W$  implies that

$$\hat{V}_t \equiv \frac{\hat{W}_t}{Y_t} = E_t \left[ \frac{\xi_T Y_T}{\xi_t Y_t} \hat{V}(\pi_T) \right] = E_t^G \left[ \frac{\rho_t}{\pi_t} \hat{V}(\pi_T) \right], \quad (\text{A8})$$

where  $E^G$  is the expectation under the new measure, and  $\rho_t = e^{(1-\gamma)(r+\beta^\top\sigma\kappa-\gamma\|\beta^\top\sigma\|^2/2)(T-t)}$  is a deterministic function of time. The first equality in (A8) emphasizes the aforementioned fact that  $\frac{\hat{W}_T}{Y_T}$  is only a function of  $\pi_T$  ( $\pi_T$  is log-normally distributed). The second equality in (A8) follows using  $Y_T = Y_t e^{(r+\beta^\top\sigma\kappa-\|\beta^\top\sigma\|^2/2)(T-t)+\beta^\top\sigma(w_T-w_t)}$  and  $w_t = (1-\gamma)\sigma^\top\beta t + w_t^G$ , where  $w_t^G$  is an  $n$ -dimensional standard Brownian motion under  $G$ . After restating the problem in terms of  $\pi$ , the time- $t$  wealth is straightforwardly obtained by evaluating the conditional expectation in (A8) under  $G$ :

$$\begin{aligned} \hat{W}_t &= Y_t E_t^G \left[ \pi_T \hat{V}(\pi_T) (\rho_t/\pi_t) \right] \\ &= \left[ \mathcal{N}(\hat{d}_1(\gamma, \pi_*)) f_H^{(1/\gamma-1)} + \mathcal{N}(-\hat{d}_1(\gamma, \pi_*)) f_L^{(1/\gamma-1)} \right] \hat{Z}(\gamma) (y \xi_t / \rho_t)^{-1/\gamma} \\ &\quad + \left[ \mathcal{N}(\hat{d}_2(\gamma, \pi_*)) - \mathcal{N}(\hat{d}_2(\gamma, \pi_*)) \right] e^{\eta H} Y_t. \end{aligned} \quad (\text{A9})$$

Note from (A9) that  $\frac{\hat{W}_t}{Y_t}$ , and hence  $R_t^W - R_t^Y$ , is a function of only  $\pi_t \equiv \xi_t Y_t^\gamma$ .

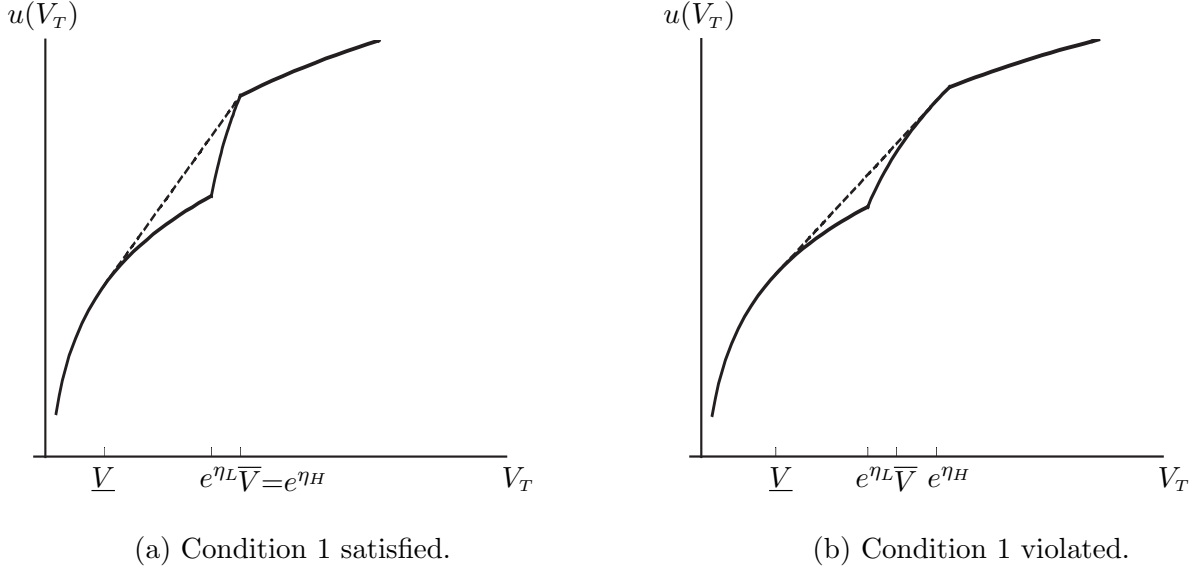
The expression for  $\hat{\theta}_t$  in the Proposition follows directly from an application of Itô's Lemma to (A9). It is helpful to note the fact that like  $\frac{\hat{W}_t}{Y_t}$ ,  $\hat{\theta}_t$  is also a function of only  $\pi_t$ , which is evident since  $\hat{\theta}_t$  depends on  $\pi_t$  via the  $\hat{d}$  terms and via  $\frac{\xi_t^{-1/\gamma}}{W_t} = (\xi_t Y_t^\gamma)^{-1/\gamma} \frac{Y_t}{W_t} = \pi_t^{-1/\gamma} / \hat{V}(\pi_t)$ . Finally, we observe that (A8) can be restated as

$$\hat{V}_t(\pi_t) = \int_{-\infty}^{\infty} \rho_t e^{h_1(t,z)} \hat{V}_T(\pi_t e^{h_1(t,z)}) h_2(t,z) dz,$$

where,  $h_1(t,z) = e^{-\frac{1}{2}\|k-\gamma\sigma^\top\beta\|(T-t)-z} \geq 0$ ,  $h_2(t,z) \geq 0$  is a probability density function of a normal random variable with zero mean and a variance that is a deterministic function of time, and  $\hat{V}_T(\cdot)$  is the non-increasing function in (A6) with a zero derivative over  $[\pi_*, \pi^*)$  and a negative derivative over  $(0, \pi_*) \cup (\pi^*, \infty)$ . Differentiating under the integral with respect to  $\pi_t$  (assuming appropriate regularity conditions), establishes that  $\forall 0 < t < T$ ,  $\hat{V}_t(\pi_t)$  is a monotonically decreasing function of  $\pi_t$ . Therefore, regardless of the dimensionality of the structure of uncertainty in the economy, we can use the isomorphism between  $\pi_t$  and  $\frac{\hat{W}_t}{Y_t}$  to plot  $\hat{\theta}_t$  as a function of  $R_t^W - R_t^Y$ . ■



## Appendix B: Nonconcavities and Risk Shifting



**Figure 7. The manager's terminal utility.** The solid plots are for the manager's utility as given by (3), with its concavification superimposed with the dashed line, where  $\underline{V}$ ,  $\bar{V}$  denote the concavification points and  $V_T = W_T/Y_T$  denotes the relative portfolio value. Condition 1 is violated when  $\bar{V} < e^{\eta H}$ .

To further illustrate the role of nonconcavities, we consider a simple static setting with two periods  $(0, T)$ . As shown in the proof of Proposition 2, via a change of variable and a change of measure one can replace the manager's objective function by that defined over the relative portfolio value  $V_T = W_T/Y_T$ , and replace the budget constraint by an equivalent representation in terms of  $V_t$ . In particular, for the collar-type fund flow specification (2), the manager's objective function becomes:

$$u(V_T) = \begin{cases} u_L(V_T) \equiv (V_T f_L)^{1-\gamma}/(1-\gamma) & \text{if } V_T < e^{\eta L}, \\ u_M(V_T) \equiv (V_T f_L + \psi V_T (\log V_T - \eta_L))^{1-\gamma}/(1-\gamma) & \text{if } e^{\eta L} \geq V_T < e^{\eta H}, \\ u_H(V_T) \equiv (V_T f_H)^{1-\gamma}/(1-\gamma) & \text{if } V_T \geq e^{\eta H}. \end{cases}$$

The concavification of this function involves solving for the concavification points  $\underline{V}$  and  $\bar{V}$  and the chord between  $\underline{V}$  and  $\bar{V}$ . Formally, this requires solving the system of equations

$$u_L(\underline{V}) = a + b\underline{V}, \quad u_M(\bar{V}) = a + b\bar{V}, \quad (\text{A10})$$

$$u'_L(\underline{V}) = b, \quad u'_M(\underline{V}) = b, \quad (\text{A11})$$

for  $a$ ,  $b$ ,  $\underline{V}$ , and  $\bar{V}$ . The tangency point  $\bar{V}$  may or may not belong to the range  $(e^{\eta L}, e^{\eta H})$ . If it does, the concavification occurs within the range where  $u(\bar{V})$  is given by  $u_M(\bar{V})$ . Otherwise, the concavification occurs at the corner, at  $e^{\eta H}$ . The condition distinguishing the latter case is

$\bar{V} \geq e^{\eta H}$ , which is equivalent to Condition 1 (after some algebra). If Condition 1 holds, equations (A10)–(A11) need to be modified accordingly to account for the corner solution.

Figure 7 depicts the transformed utility function  $u(V_T)$  with its concavification superimposed on the plot by the dashed line. In both panels, one can see a finite range of relative portfolio values over which the utility is nonconcave, and hence the manager has an incentive to gamble. That is, she would always prefer adding a zero present value gamble  $\{+\varepsilon$  with probability 50%,  $-\varepsilon$  with probability 50%\} to her status quo portfolio (defined by some fixed  $\hat{\theta}$ ) over ending up with a value in the suboptimal range  $(\underline{V}, \bar{V})$ . Indeed, the optimal terminal fund portfolio value derived in Proposition 1 features a discontinuity, responsible for the manager’s risk-shifting behavior discussed in Section 2.2. Figure 7 also highlights the role of Condition 1. In panel (a), the intermediate segment of the objective function is subsumed within the suboptimal range, and hence its shape is immaterial for the manager’s choice of her optimal terminal portfolio value and hence the risk exposure. In panel (b), part of the intermediate segment of the objective function  $u_M(\cdot)$  falls outside the suboptimal range, and hence the functional form defining this segment enters in the solution of the manager’s optimal policy.

How would the manager, whose investment opportunity set consists of assets with continuous distributions, achieve a discontinuous optimal wealth profile? Simply, by taking advantage of continuous trading and thus synthetically replicating a 50/50 gamble, or its close substitute, a binary option. One can see from the expressions for the optimal trading strategies that they indeed contain binary option-type components. What if, perhaps more realistically, the manager is unable to synthetically create a binary option, as would be the case, for example, in the popular two-period model with continuous state space but with a finite number of securities available for investment? The above argument indicates that in such an economy the manager would clearly benefit from introduction of specific securities into her investment opportunity set: binary options. We note that this discussion applies generally to any preferences exhibiting a nonconcavity, and is not driven by the fact that in our setting the manager’s payoff essentially includes a binary option.

## Appendix C: Numerical Procedure

For the cases where an analytical solution is not available, we solve the model numerically. Our numerical solution utilizes a Monte Carlo simulation (e.g., Boyle, Broadie, and Glasserman (1997)). The simulation presented here is for the economic setting of Section 2, with  $Y_t = S_t$ . We first simulate the relative portfolio value  $\hat{V}_t(\pi_t)$  using equation (A8). This simulation requires knowledge of the distribution of  $\pi_T$  under the  $G$  measure (proof of Proposition 2), which in our setup is

$$\pi_T = \pi_t e^{[-(1-\gamma)(\mu-\gamma\sigma^2/2)-(\gamma\sigma-\kappa)^2/2](T-t)+(\gamma\sigma-\kappa)\sqrt{T-t}z},$$

where  $z$  is a standard normal random variable. The second step is to compute the trading strategy financing  $\hat{V}_t(\pi_t)$ . Following standard arguments, we obtain the optimal risk exposure expression as

$$\hat{\theta}_t(\pi_t) = 1 + \frac{\partial \hat{V}_t(\pi_t)}{\partial \pi_t} \frac{\pi_t}{\hat{V}_t} \left( \gamma - \frac{\kappa}{\sigma} \right). \quad (\text{A12})$$

The derivative  $\frac{\partial \hat{V}_t(\pi_t)}{\partial \pi_t}$  is computed numerically from the simulated  $\hat{V}_t(\pi_t)$  schedule.

Briefly, the details of the procedure are as follows.

- (a) Fix  $\pi_t$ . Simulate  $M$  normal variates,  $z^1, \dots, z^M$ , and for each normal variate  $z^i$  compute

$$F_{t,T}^i \equiv e^{[-(1-\gamma)(\mu-\gamma\sigma^2/2)-1/2(\gamma\sigma-\kappa)^2](T-t)+(\gamma\sigma-\kappa)\sqrt{T-t}z^i} \quad \text{and} \quad \pi_T^i = \pi_t F_{t,T}^i.$$

- (b) Determine the relevant critical values of  $\pi_T$ . This step depends on the form of the fund flow function. Use the concavification points  $\underline{V}$  and  $\bar{V}$  (Appendix B) to determine the value of  $\pi_T$  at which the risk-shifting range occurs. For example, for the collar-type fund flow with Condition 1 violated, compute

$$\pi^* = u'_L(\underline{V}) = u'_M(\underline{V}).$$

- (c) Determine  $V_T^i$  for a given  $\pi_T^i$ . This step again depends on the flow specification. Continuing with the collar-type fund flow with Condition 1 violated, we compute the optimal  $V_T^i$  for each segment of  $u(\cdot)$  (Appendix B), as well as its value over the risk-shifting range. Compute  $V_T^i$  from the following formulas:

$$V_T^i = \begin{cases} \frac{1}{f_H} \left( \frac{\pi_T^i}{f_H} \right)^{-1/\gamma} & \text{if } \pi_T^i < \pi_{**}, \\ e^{\eta H} & \text{if } \pi_{**} \leq \pi_T^i < \pi_*, \\ \text{Solve } u'_M(V_T) = \pi_T^i \text{ numerically for } V_T & \text{if } \pi_* \leq \pi_T^i < \pi^*, \\ \frac{1}{f_L} \left( \frac{\pi_T^i}{f_L} \right)^{-1/\gamma} & \text{if } \pi_T^i \geq \pi^*, \end{cases}$$

where  $\pi_* = u'_M(e^{\eta H})$  and  $\pi_{**} = u'_H(e^{\eta H})$ . The flat segment over  $(\pi_{**}, \pi_*)$  is due to the kink in the utility function upon transitioning from  $u_M(\cdot)$  to  $u_H(\cdot)$ , and the discontinuity at  $\pi^*$  is due to the risk-shifting range. The initial portfolio value is chosen such that the Lagrange multiplier on the budget constraint  $\hat{y}$  is unity.

- (d) Compute  $\hat{V}(\pi_t) = \frac{1}{M} \sum_{i=1}^M \rho_t F_{t,T}^i V_T^i$ .
- (e) Repeat the procedure (steps (a)-(d)) for a range of  $\pi_t$ , step size  $\Delta\pi$ . Save the array  $\hat{V}(\pi_t)$ .
- (f) Compute the trading strategy using (A12), approximating the derivative  $\frac{\partial \hat{V}_t(\pi_t)}{\partial \pi_t}$  by the finite difference  $\frac{\partial \hat{V}_t(\pi_t)}{\partial \pi_t} \approx \frac{\hat{V}(\pi_t + \Delta\pi) - \hat{V}(\pi_t)}{\Delta\pi}$ .
- (g) Plot  $\theta_t(\pi_t)$  as a function of  $\log \hat{V}(\pi_t)$  parametrically. This is the desired optimal risk exposure, depicted in Figure 3.

**Table 1a**  
**Costs and benefits of active management in economies (a)**

The investor's gain/loss quantified in units of his initial wealth,  $\hat{\lambda}$ , solves  $V^I((1 + \hat{\lambda})W_0) = \hat{V}(W_0)$ , where  $V^I(\cdot)$  denotes the investor's indirect utility under his optimal policy  $\theta^I$ , and  $\hat{V}(\cdot)$  his indirect utility under delegation. The gain due to explicit incentives,  $\lambda^N$ , solves  $V^I((1 + \lambda^N)W_0) = \hat{V}(W_0; f_T = 1)$ , where  $\hat{V}(W_0; f_T=1)$  denotes the investor's indirect utility under delegation absent implicit incentives. The gain due to implicit incentives,  $\lambda^Y$ , solves  $1 + \hat{\lambda} = (1 + \lambda^N)(1 + \lambda^Y)$ . The fixed parameter values are (where applicable)  $\gamma = 1$ ,  $\gamma_I = 2$ ,  $f_L = 0.8$ ,  $f_H = 1.5$ ,  $f_L + f_H = 2.3$ ,  $\beta = 1$ ,  $\eta_L = -0.08$ ,  $\eta_H = 0.08$ ,  $\eta_L + \eta_H = 0$ ,  $\mu = 0.08$ ,  $r = 0.02$ ,  $\sigma = 0.16$ ,  $W_0 = 1$ ,  $T = 1$ .

Effects of		Cost-benefit measures				
		$\lambda^Y, \lambda^N$ $\hat{\lambda}$ (%)				
Managerial risk aversion	$\gamma$	<u>0.5</u>	<u>1.0</u>	<u>1.5</u>	<u>2.0</u>	<u>2.25</u>
		-28.86, -27.12	-6.68, -3.45	-5.05, -0.39	-3.45, 0.00	-2.80, -0.04
		-48.16	-9.90	-5.42	-3.45	-2.83
Implicit reward for outperformance	$f_H - f_L$	<u>0.3</u>	<u>0.5</u>	<u>0.7</u>	<u>0.9</u>	<u>1.1</u>
		-2.82, -3.45	-4.80, -3.45	-6.68, -3.45	-8.48, -3.45	-10.26, -3.45
		-6.17	-8.09	-9.90	-11.64	-13.36
Risk exposure of the benchmark	$\theta^Y$	<u>0.50</u>	<u>0.75</u>	<u>1.00</u>	<u>1.25</u>	<u>1.50</u>
		-3.99, -3.45	-5.00, -3.45	-6.68, -3.45	-8.97, -3.45	-11.79, -3.45
		-7.31	-8.28	-9.90	-12.12	-14.83
Flow threshold differential	$\eta_H - \eta_L$	<u>0.08</u>	<u>0.12</u>	<u>0.16</u>	<u>0.20</u>	<u>0.24</u>
		-2.06, -3.45	-4.49, -3.45	-6.68, -3.45	-8.65, -3.45	-10.42, -3.45
		-5.44	-7.79	-9.90	-11.81	-13.52

**Table 1b**  
**Costs and benefits of active management in economies (b)**

The investor's gain/loss quantified in units of his initial wealth,  $\hat{\lambda}$ , solves  $V^I((1 + \hat{\lambda})W_0) = \hat{V}(W_0)$ , where  $V^I(\cdot)$  denotes the investor's indirect utility under his optimal policy  $\theta^I$ , and  $\hat{V}(\cdot)$  his indirect utility under delegation. The gain due to explicit incentives,  $\lambda^N$ , solves  $V^I((1 + \lambda^N)W_0) = \hat{V}(W_0; f_T = 1)$ , where  $\hat{V}(W_0; f_T=1)$  denotes the investor's indirect utility under delegation absent implicit incentives. The gain due to implicit incentives,  $\lambda^Y$ , solves  $1 + \hat{\lambda} = (1 + \lambda^N)(1 + \lambda^Y)$ . The fixed parameter values are (where applicable)  $\gamma = 1$ ,  $\gamma_I = 2$ ,  $f_L = 0.8$ ,  $f_H = 1.5$ ,  $f_L + f_H = 2.3$ ,  $\beta = 1$ ,  $\eta_L = -0.08$ ,  $\eta_H = 0.08$ ,  $\eta_L + \eta_H = 0$ ,  $\mu = 0.06$ ,  $r = 0.02$ ,  $\sigma = 0.29$ ,  $W_0 = 1$ ,  $T = 1$ .

<b>Effects of</b>		Cost-benefit measures				
		$\lambda^Y, \lambda^N$ $\hat{\lambda}$ (%)				
Managerial risk aversion	$\gamma$	<u>0.5</u>	<u>1.0</u>	<u>2.0</u>	<u>3.0</u>	<u>4.0</u>
		-8.13, -4.19	-5.12, -0.47	-3.31, 0.00	-2.56, -0.05	-2.15, -0.11
		-11.98	-5.61	-3.31	-2.61	-2.27
Implicit reward for outperformance	$f_H - f_L$	<u>0.3</u>	<u>0.5</u>	<u>0.7</u>	<u>0.9</u>	<u>1.1</u>
		-3.33, -0.47	-4.29, -0.47	-5.12, -0.47	-6.01, -0.47	-6.88, -0.47
		-3.79	-4.74	-5.61	-6.46	-7.32
Risk exposure of the benchmark	$\theta^Y$	<u>0.50</u>	<u>0.75</u>	<u>1.00</u>	<u>1.25</u>	<u>1.50</u>
		-5.43, -0.47	-4.63, -0.47	-5.12, -0.47	-6.69, -0.47	-8.45, -0.47
		-5.88	-5.08	-5.61	-7.13	-8.88
Flow threshold differential	$\eta_H - \eta_L$	<u>0.08</u>	<u>0.12</u>	<u>0.16</u>	<u>0.20</u>	<u>0.24</u>
		-4.33, -0.47	-4.70, -0.47	-5.12, -0.47	-5.67, -0.47	-6.21, -0.47
		-4.78	-5.15	-5.61	-6.12	-6.65

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