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Article

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In a financial market with one riskless asset and n risky assets whose prices are lognormal, we solve in a closed form the problem of a pension fund maximizing the expected CRRA utility of its surplus till the (stochastic) death time of a representative agent. We consider a unique asset allocation problem for both accumulation and decumulation phases. The optimal investment in the risky assets must decrease during the first phase and increase during the second one. We accordingly suggest it is not optimal to manage the two phases separately, and outsourcing of allocation decisions should be avoided in both phases.

Reference

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Optimal asset allocation for pension funds under mortality risk during the accumulation and decumulation phases*

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Abstract

In a financial market with one riskless asset and n risky assets whose prices follow geometric Brownian motions, we solve the problem of a pension fund maximizing the expected CRRA utility of its surplus. We consider a unique optimization problem for both the accumulation phase and the decumulation phase, and find a closed form solution to the allocation

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problem when the stochastic death time of the fund member is distributed as a Gompertz-Makeham random variable. We show that the optimal asset allocation during these two phases must be different. In particular, the optimal portfolio starts from the allocation prescribed by Merton's theory. Then during the first phase, the investment in the risky assets must decrease through time, while during the second phase, it must increase. Our findings also suggest that it is not optimal to manage the two phases separately, and that outsourcing of allocation decisions either during the accumulation or the decumulation phase should be avoided.

JEL: G23, G11. *MSC 2000*: 62P05, 91B28, 91B30, 91B70, 93E20.

Key words: pension fund, mortality risk, asset allocation.

1 Introduction

In this paper we analyze optimal asset allocation by a pension fund which maximizes the expected utility of its surplus at the death time of a member. This surplus is defined as the difference between the total managed wealth and the retrospective mathematical reserve.

Unlike analyses dedicated to non-actuarial institutional investors (a general framework can be found in Lioui and Poncet, 2001, Menoncin, 2002, and Rudolf and Ziemba, 2004), the case of a pension fund requires the introduction of two new characteristics: (i) the different behavior of the fund wealth during the accumulation phase (hereafter APh), when contributions are paid by the member, and the decumulation phase (hereafter DPh), when the pension is paid to the member, and (ii) the mortality risk.

The link between contributions and pensions can be established inside one of the two following frameworks: the so-called defined-benefit pension plan (hereafter DB) or the so-called defined-contribution pension plan (hereafter DC). In a DB plan benefits are fixed in advance by the sponsor, and contributions are set in order to maintain the fund in balance. In a DC plan contributions are fixed, and benefits depend on the returns of the fund portfolio. The model studied here deals with the case of a pension fund which supplies its members with deterministic pension plans, namely a constant contribution rate and a constant pension rate. This type of hybrid plans are known in the US under the name "Cash Balance Plans". Note that some actuarial and consultancy service firms have recently advocated their underwriting (Da Silva, 2002).

Furthermore, a profit sharing rule is implemented for allowing redistribution of profits induced by the exposure to risky assets. In particular, the fund is assumed to pay a deterministic (but not necessarily constant) part of either its surplus or its surplus change to the members.

In our model the contribution rate and the pension rate are linked by a so-called “feasibility” condition guaranteeing it is convenient for both the pension fund and the member to contract. Such a condition is also present, for instance, in Josa-Fombellida and Rincón-Zapatero (2001) and Sundaresan and Zapatero (1997). The latter work, in particular, examines the problem of a firm which must pay both wages (before its workers retire) and pensions (after they retire). Thus, a "feasibility" condition implies the equality between the total expected value of wages and pensions paid with the total expected value of worker productivity (according to the usual economic rule equating the optimal wage with the marginal product of labor).

In our setting the demographic dimension is introduced via a survival probability for the member of the pension fund whose death time τ is stochastic. In particular, we find a closed form solution to the asset allocation problem when τ follows a Gompertz-Makeham distribution. Let us remark that the mortality risk supported by a single subscriber is much more important than the one supported by the fund, and this mutualisation effect explains why people join pension schemes. Along this work we will call "member" the individual representative agent of a set of members who have the same risk characteristics. We thus differ from an aggregated framework where both the APh and the DPh happen at the same time (see Haberman and Sung, 1994, in a discrete time setting).

The existing literature dealing with the asset allocation problem for a pension fund completely neglects the mortality risk, and partially takes into account the problem of distinguishing between the APh and the DPh. For example, Deelstra *et al.* (2000), Boulier *et al.* (2001), and Battocchio and Menoncin (2004) just deal with the investment problem during the APh while Blake *et al.* (2000) just take into account the distribution phase. Rudolf and Ziemba (2004) develop a larger setting where they do not specify any functional form for the fund liabilities. They confirm the "four-fund" theorem shown in a less general framework by Menoncin (2002).

Instead, only the actuarial literature seems to explicitly take into account the mortality risk problem (see e.g. Young and Zariphopoulou, 2002a,b, for optimal asset allocation under an exponentially distributed death time, as well as Richard, 1975, for similar results).

The single piece of work, at least to our best knowledge, which considers both the mortality risk and the difference between the APh and the DPh is the paper by Charupat and Milevsky (2002). They analyze the interaction between financial risk, mortality risk, and consumption towards the end of the life cycle. Their main result is that with Constant Relative Risk Aversion (CRRA) preferences and geometric Brownian motion dynamics, the optimal asset allocation during the DPh is identical to the APh, and coincides with the classical Merton's (1971) solution. Actually, they solve two different problems: (i) in a first step they maximize the expected utility of fund terminal wealth during the APh, and (ii) in a second step they compute the optimal consumption-portfolio for the consumer-investor during the DPh.

In the setting after Charupat and Milevsky (2002) it is up to the consumer to choose how to allocate his wealth after he retires. Accordingly, once the retirement date is reached, the fund problem of managing the remaining wealth while the annuity is being paid is not dealt with. Here, since the management of the remaining wealth is a relevant problem for a pension fund, we equate the final date of our dynamic optimization problem with the death time of the fund member (instead of the retirement date). We show that the result in Charupat and Milevsky (2002) obtained with a single geometric Brownian motion and a CRRA utility function is not robust. In fact, after solving a unique problem for the optimal asset allocation during the whole life of the fund member, we find two different portfolio compositions during the APh and the DPh, albeit remaining in the same simple framework. More precisely, we find that the optimal portfolio starts from the Merton's (1971) solution. Then during the APh, the amount of wealth invested in the risky assets decreases through time, while after the retirement date (during the DPh), the optimal portfolio becomes riskier and riskier (and, eventually, even riskier than Merton's one). Except for a long time horizon, this result indicates that ignoring the mortality risk will bias upward the risk profile.

Eventually, note that our model is not related to optimal consumption-portfolio problems, but borrows its framework from the literature about optimal portfolio allocation with stochastic expenses and labor income. In fact, neither the contribution rate nor the pension rate are considered as control variables by the fund manager. A quasi-explicit solution for this kind of problem when mortality risk is absent can be found in Menoncin (2002) while Cuoco (1997) offers an existence result for a constrained investor who is endowed with a stochastic labor income flow and optimizes his consumption.

Through this work we consider agents trading continuously in a frictionless arbitrage-free market, but not necessarily complete.

The paper is structured as follows. The framework is outlined in Section 2. First we describe the financial market. Then we compute the feasibility condition on the contribution and pension rates when the remaining lifetime follows a Gompertz-Makeham distribution. Eventually, we present the objective function for the pension fund and the dynamic budget constraint on its wealth incorporating profit sharing rules. In Section 3 we compute the optimal portfolio and give a clear allocation rule. We assess the ruin probability associated with this rule, and further discuss the main practical implications of our results for the management of a pension fund. Section 4 concludes.

2 The model

2.1 The financial market

We consider a financial market where there exist n risky assets and one riskless asset paying a constant interest rate r, whose price dynamics are described by:

$$\begin{cases} \underset{n\times 1}{dS(t)} = \underset{n\times n}{I_S}\left(\underset{n\times 1}{\mu dt} + \underset{n\times k}{\Sigma'}\underset{k\times 1}{dW(t)}\right), \quad S(t_0) = S_0, \\ dG(t) = G(t)rdt, \quad G(t_0) = G_0, \end{cases} \tag{1}$$

where I_S is a square diagonal matrix containing the elements of the vector $S(t)$ and $W(t)$ is a $k-$dimensional Wiener process. Both μ and Σ are assumed to be constant. Finally, S_0 and G_0 are deterministic positive variables.

The financial market structure (1) is very simple indeed, and we acknowledge that the prediction we obtain may be model dependent. But this simple framework allows us to obtain a closed form solution for the fund optimal asset allocation. Actually, finding the solution of the Hamilton-Jacobi-Bellman equation deriving from the dynamic stochastic optimization technique represents one of the most challenging tasks in this kind of problem. Thus, we have chosen a simple market structure allowing us to handle this explicitly. Note that our explicit results can also be taken as a benchmark for pension fund managers who want to check results given by numerical methods (such as stochastic programming methods) in more complex environments.

2.2 Contributions and pensions

In our model the retirement date T for a member is assumed to be imposed by the law. Thus, it is not a control variable as in Sundaresan and Zapatero (1997) where an employee of a firm decides when to retire by solving an optimal stopping problem.

We take into account a deterministic pension scheme where the total amount $U(t)$ of contributions to the fund follows the differential equation

$$dU(t) = udt,$$

where u is a positive constant, and the total amount $V(t)$ of pensions paid by the fund follows the differential equation

$$dV(t) = vdt,$$

where v is a positive constant.

This pension scheme is both of a DB type and of a DC type. In order for such a scheme to be viable we just need the existence of a savings account (see $G(t)$ in (1)).

As recalled by James and Vittas (1999), there are three common forms of old age retirement benefits.

1. Lump sum payments.[1] They do not require any of the complex calculations involved in scheduled withdrawals and annuities (see the two following points). Nevertheless, it happens that some workers use part of their lump sums to purchase annuities. In most OECD countries, company pension schemes allow partial commutation of future benefits into a lump sum. This varies between 25% and 33% of the discounted present value of benefits. Available evidence suggests that most workers opt for this facility.

2. Scheduled (or programmed) withdrawals.[2] The survival probability does not enter the computations, since in the event of early death, remaining account balances are inherited by dependents, accommodating a bequest motive. Unfortunately, these withdrawals are exposed to fluctuating payments as a result of the volatility of pension fund returns. In Latin American countries, scheduled withdrawals are recalculated each year on the basis of the remaining life expectancy of the family of covered workers and a stipulated rate of return. By regulation, the rate of return is equal to the average real return achieved by the pension fund concerned over the past 10 years. The life table to be used is also set by the regulators.

3. Life annuities. They are paid until subscriber death time, and they do not depend on fund performances. Among countries with mandatory second pillars, only Switzerland and Bolivia impose the use of annuities. Eastern European countries are also leaning towards compulsory annuitization. Compulsory annuitization is often advocated in order to avoid the problems caused by adverse selection. If it were not compulsory, only subscribers who know to have a long life expectancy would choose it. Accordingly, the annuity market would be greater and better developed if all workers were forced to purchase an annuity.

Clearly, our framework belongs to the third case. Furthermore, the constant contribution rate makes our case very close to the so-called "Cash Balance

Plans". These plans combine characteristics of both DC and DB schemes, and represent a third way with respect to these two schemes (Da Silva, 2002).

The constant level of the contribution and the pension rates (u and v, respectively) cannot be set separately. In particular, we know that, at time $t = t_0$ (when the member enters the fund), from the point of view of the member, resp. the pension fund, the expected present value of all pensions cannot be lower, resp. higher, than the expected present value of all contributions.

Thus, after defining

$$k(t) = u\mathbb{I}_{t<T} - v\left(1 - \mathbb{I}_{t<T}\right), \tag{2}$$

where $\mathbb{I}_E$ is the indicator function for the event E, we can argue that a pair (u, v) can be accepted by both the fund and the member if it satisfies the following condition:

$$\mathbb{E}_{t_0}^{\tau}\left[\int_{t_0}^{\tau} k(t)\, e^{-r(t-t_0)} dt\right] = 0,$$

where τ is the stochastic death time. For the sake of simplicity, we assume that the member who enters the fund is born in 0, so that the current date t_0 also coincides with his age. This equation asks for the present value of the future contributions being equal to the present value of the future benefits. This is a standard "equation-of-value" often used by actuaries.

If we call ${}_{t-t_0}p_{t_0}$ the probability that a member of the fund aged of t_0 is still alive at date t, we can write the previous condition as

$$\int_{t_0}^{\infty} \left({}_{t-t_0}p_{t_0}\right) k(t)\, e^{-r(t-t_0)} dt = 0.$$

Since both u and v are assumed to be constant, we get a simple characterization of a "feasible" pair (u, v).

Definition 1 *A pair of (positive) contribution and pension rates (u, v) is said to be feasible if*

$$\frac{v}{u} = \frac{\int_{t_0}^{T} \left({}_{t-t_0}p_{t_0}\right) e^{-rt} dt}{\int_{T}^{\infty} \left({}_{t-t_0}p_{t_0}\right) e^{-rt} dt}. \tag{3}$$

Given the age t_0 of a member and the interest rate r, a pension fund is thus able to offer a complete set of feasible values for u and v. The member will choose his preferred pair according to both his ability to sustain the contribution rate u and his wish to secure the pension rate v. Equation (3) shows that v is strictly proportional to u.

Let us finally remark that the event "death", happening at date τ, can sometimes be affected by a series of explanatory variables. To model this aspect we can rely on the so-called "proportional hazard rate model" used in statistical analysis of transition data. Fortunately, the form of the feasible ratio v/u remains unchanged, and we only need to compute the probability conditionally to the realization of the explanatory variables in (3) to accommodate this situation.

Here, we assume that the remaining lifetime of the member follows a Gompertz-Makeham distribution. The probability to be alive in t for an individual aged of t_0 is then given by

$$_{t-t_0}p_{t_0} = \exp\left\{-\lambda\left(t-t_0\right) + e^{\frac{t_0-m}{b}}\left(1-e^{\frac{t-t_0}{b}}\right)\right\}, \tag{4}$$

where λ is a positive constant measuring accidental deaths linked to non-age factors, while m and b are modal and scaling parameters of the distribution, respectively.

Accordingly, the feasibility condition (3) can be written as

$$\frac{v}{u} = \frac{\Gamma\left(-\left(\lambda+r\right)b, e^{\frac{t_0-m}{b}}\right)}{\Gamma\left(-\left(\lambda+r\right)b, e^{\frac{T-m}{b}}\right)} - 1,$$

where Γ is the incomplete Gamma function (see Appendix A for details).

Its behavior with respect to t_0, T, and r is shown in Figures 1 and 2 where we have used the values $m = 88.18$ and $b = 10.5$,[3] presented in Milevsky (2001) where the author priced all annuities using the Individual Annuity Mortality (IAM) 2000 table, dynamically adjusted using scale G, published by the Society of Actuaries. Finally, we use the value 0.01 for λ.

[Figure 1 here]

[Figure 2 here]

From Figure 1 it is clear that, given the age of the member, when the retirement date T increases, the fund can afford to pay a higher pension rate v to the member. On the contrary, given a retirement date, the older the member the lower the pension rate the fund can afford to pay. In fact, an older member pays contributions during a shorter period of time until T.

From Figure 2 the behavior of v/u with respect to the retirement date T is confirmed (the higher T the higher v/u), while we see that the higher the riskless interest rate r the higher the pension rate v the fund can afford to pay. In fact, when r increases, the discount factor $\exp(-rt)$ decreases. This means that future pensions in the feasibility condition (3) receive a lower weight, and the fund can thus increase them.

Furthermore, Figure 2 shows that a change in T (by governmental decision for example) can dramatically affect the feasible ratio v/u when the riskless interest rate r is high. This seems to suggest that an economic period when interest rates are low (and are foreseen to remain low as it is the case under the current economic situation), is the most suitable period for undertaking a reform of a pension system.

2.3 The fund objective function

The pension fund is assumed to maximize its surplus at the death time of the subscriber. The surplus is given by the difference between the total managed wealth and the retrospective mathematical reserve. We recall that a retrospective reserve is a (life or health) insurance reserve computed as the past value of assumed claims and premiums, both accumulated at an assumed interest rate. Accordingly, at each time t, the retrospective reserve $K(t)$ can be written as

$$K(t) = \int_{t_0}^{t} k(s)\, e^{-r(s-t)} ds, \tag{5}$$

where the interest rate for the discount factor is the riskless interest rate r.

In our simplified model, the use of a retrospective reserve to measure the surplus is justified by the assumptions that neither the valuation interest rate nor the mortality experience change over time. In the actuarial sciences the use of the prospective reserve is often preferred, and this could be the basis for an extension of our model with stochastic interest rates. It will be highlighted in the next section that the introduction of the retrospective reserve allows us to find a closed form solution for the Hamilton-Jacobi-Bellman equation deriving from the dynamic stochastic optimization technique we rely on.

It is worth noting that the use of either a retrospective or a prospective mathematical reserve is indifferent in our framework only if the final date (death time) is deterministic (τ_d). In this case, the feasibility condition is

$$\int_{t_0}^{\tau_d} k(s) e^{-r(s-t_0)} ds = 0,$$

and it can be written as

$$\int_{t_0}^{t} k(s) e^{-r(s-t)} ds = -\int_{t}^{\tau_d} k(s) e^{-r(s-t)} ds.$$

Under deterministic mortality we can equivalently compute the fund surplus as: (i) the difference between the managed wealth and the retrospective reserve, or (ii) the sum between the managed wealth and the prospective reserve. On the contrary, when the final date τ_d is stochastic, the prospective reserve contains the death probability, and the above equation does not hold any more. In order to obtain a closed form solution for the optimal portfolio, we need to measure the fund surplus as the difference between the managed wealth $R(t)$ and the retrospective reserve $K(t)$. Note that this definition of the pension fund surplus is different from the usual one based on an accrued liability basis.

Now, one of the most common used utility function in the literature about optimal asset allocation is the CRRA (Constant Relative Risk Aversion) utility function: $U(R) = \frac{1}{\delta} R^{\delta}$.[4] Hence, the pension fund is assumed to maximize

$$\mathbb{E}_{t_0}^{\tau}\left[\frac{1}{\delta} e^{-\rho(\tau-t_0)} \left(R(\tau) - K(\tau)\right)^{\delta}\right], \tag{6}$$

where ρ is the (positive) intertemporal discount rate. By using the independence between τ and all the other risk sources, we can rewrite this expected value as

$$\mathbb{E}_{t_0}\left[\int_{t_0}^{\infty} \frac{1}{\delta} m(t) \left({}_{t-t_0}p_{t_0}\right) e^{-\rho(t-t_0)} \left(R(t) - K(t)\right)^{\delta} dt\right],$$

where $m(t)$ is the so-called mortality force.[5] Thus, the initial problem can be restated as an intertemporal problem.

In such a framework, the retrospective reserve $K(t)$ can be viewed as liabilities the fund holds *vis-à-vis* the member while $R(t)$ represents the assets of the fund in the same spirit as Rudolf and Ziemba (2004) in their maximisation of the intertemporal expected utility of the fund surplus defined as assets net of liabilities.

One of the main advantage of using a utility function as in (6) is that we do not need to explicitly take into account the constraint $R(t) > K(t), \forall t \geq t_0$. Actually, as Merton (1990) underlines, this constraint can be omitted since there exists a positive level of wealth giving an infinite marginal utility. The marginal utility corresponding to (6) is

$$\frac{\partial U}{\partial R} = (R(t) - K(t))^{\delta-1},$$

and $R(t)$ will never fall below the value $K(t)$ since $\delta - 1 < 0$. If this were the case, then it would be optimal to invest all the wealth in the riskless asset in order to have a positive increment of wealth giving an infinite increase in utility. Thus, if the fund initial surplus (coinciding with the initial wealth) is strictly positive, then the optimal asset allocation, at each time, cannot imply a negative value for $R(t) - K(t)$. We will come back to this property in the next section. The argument is exactly the same as the one guaranteeing that for a pure CRRA utility function (i.e. $K(t) = 0, \forall t$) the optimal wealth will never be negative (i.e. will never fall below $K(t) = 0$). Besides, let us remark that the formulation (6) will also permit to obtain a closed form solution for the optimal portfolio thanks to the separability of the value function (see Section 3 and Footnote 11 for further technical details).

In the financial literature (see for instance Merton, 1990, Section 6.4) a utility function of the form (6) is known as a "state-dependent" utility. It depends on wealth as well as on other state variables (in this case contributions and pensions). We may draw an analogy between this approach and the so-called "habit formation" approach for the maximization of the intertemporal consumption when there exists a subsistence consumption level given by the weighted mean of the past consumption rates (Constantidines, 1990). In this case, before the retirement date T, $K(t)$ represents the sum of all the past contributions, and so the CRRA utility function guarantees that $R(t)$ never falls below the sum of all the contributions received before t. When the pensions start being paid ($t > T$) the fund wealth can go beyond the sum of all the contributions received by an amount equal to the sum of all the pensions paid.

2.4 The managed wealth

The fund total wealth $R(t)$ is equal to

$$R(t) = \theta(t)'S(t) + \theta_0(t)G(t),$$

where $\theta(t)$ and $\theta_0(t)$ are the number of risky assets and the number of riskless asset held, respectively. Here, we do not explicitly prevent $\theta(t)$ from being negative (short sales of risky assets), and we will provide later a discussion on this point.

The SDE associated with $R(t)$ is simply

$$dR(t) = \theta'(t)dS(t) + \theta_0(t)dG(t) + d\theta'(t)(S(t) + dS(t)) + G(t)d\theta_0(t). \qquad (7)$$

The self-financing condition implies that the two last terms in (7) must be equated to zero or, when consumption is considered, must finance the consumption rate.

To enrich our framework, we now introduce a deterministic profit sharing rule $\phi(t)$ where $0 \leq \phi(t) < 1$. This means that a proportion $\phi(t)$ of the change in the fund surplus is redistributed to the members, and allows to share profits induced by the exposure to the risky assets. The proportion may for example be: $\phi(t) = 0$ (no profit sharing) or $\phi(t) = (1 - \mathbb{I}_{t<T})\bar{\phi}$ with $0 \leq \bar{\phi} < 1$ (constant profit sharing only during the DPh).[6]

Hence, the self-financing condition in our case must ensure that the changes in portfolio composition (the two last terms in (7)) must: (i) be financed by the member contribution rate u during the APh, and (ii) finance both the pension rate v and the percentage $\phi(t)$ of the fund surplus paid to the members.

Thus, we can write the self-financing condition in the following way:

$$d\theta(t)'\left(S(t)+dS(t)\right)+G(t)d\theta_0(t)=k\left(t\right)dt-\phi\left(t\right)d\left(R(t)-K\left(t\right)\right),$$

where $d\left(R(t)-K\left(t\right)\right)$ is the differential of the fund surplus. Since we obtain from (5):

$$dK\left(t\right)=k\left(t\right)dt+rK\left(t\right)dt,$$

we can finally write:

$$d\theta(t)'\left(S(t)+dS(t)\right)+G(t)d\theta_0(t)=\left(1+\phi\left(t\right)\right)k\left(t\right)dt+\phi\left(t\right)rK\left(t\right)dt-\phi\left(t\right)dR(t).$$

Therefore, the dynamic budget constraint can be written as

$$\begin{aligned} dR(t) = & \left(\frac{1}{1+\phi\left(t\right)}\left(R(t)+\phi\left(t\right)K\left(t\right)\right)r+\frac{1}{1+\phi\left(t\right)}w(t)'M+k\left(t\right)\right)dt \\ & +\frac{1}{1+\phi\left(t\right)}w(t)'\Sigma' dW(t), \end{aligned} \tag{8}$$

where[7]

$$M=\left(\mu-r\mathbf{1}\right),\quad w(t)=I_S\theta(t),$$

and **1** is a vector of 1s.

In Charupat and Milevsky (2002) each dollar of new income flowing into the fund (u) is allocated separately and treated as a new problem. Thus, they neglect the role of u during the APh and they solve for $u=0$. In our approach, instead, we treat u as a planned flow which the fund manager can rely on. Furthermore, as both Merton (1990, Section 5.7) and Bodie *et al.* (1992) underline, it is not necessary that the new financial flows (u) can be borrowed against, since the investor behaves "as if" this were true.

We underline that another sharing rule is possible. The fund could distribute a proportion of its surplus (ψ) instead of a proportion of the change in its surplus. The self-financing condition should be accordingly written as

$$d\theta(t)'\left(S(t)+dS(t)\right)+G(t)d\theta_0(t)=k\left(t\right)dt-\psi\left(t\right)\left(R(t)-K\left(t\right)\right)dt,$$

and the evolution of fund wealth would be

$$dR(t) = (R(t)\,(r - \psi\,(t)) + w(t)'M + k\,(t) + \psi\,(t)\,K\,(t))\,dt + w(t)'\Sigma' dW(t). \tag{9}$$

Since the sharing rule ψ neither affects the total risky portfolio return $w(t)'M$ nor the wealth instantaneous variance $w(t)'\Sigma'\Sigma w(t)$, the riskiness of the optimal portfolio does not change in this case. Therefore, we prefer to keep the first formulation (ϕ) in the main text. We summarize the major results when ψ is used instead of ϕ in Appendix D. Note that as ϕ has to be comprised between 0 and 1, ψ has to be comprised between 0 and r $(0 \leq \psi\,(t) < r)$.

3 The optimal portfolio

After the presentation of the previous section, the asset allocation problem for a pension fund can be written as

$$\begin{cases} \max_w \mathbb{E}_{t_0}\left[\int_{t_0}^{\infty} f\,(t)\,\frac{1}{\delta}\,(R\,(t) - K\,(t))^{\delta}\,dt\right], \\ \text{with} \qquad dR(t) = \left(\frac{1}{1+\phi(t)}\,(R(t) + \phi\,(t)\,K\,(t))\,r + \frac{1}{1+\phi(t)} w(t)'M + k\,(t)\right) dt \\ \qquad\qquad\qquad + \frac{1}{1+\phi(t)} w(t)'\Sigma' dW(t), \\ \text{and} \qquad R\,(t_0) = R_0, \end{cases} \tag{10}$$

where

$$f\,(t) = m\,(t)\,({}_{t-t_0}p_{t_0})\,e^{-\rho(t-t_0)},$$

is the actuarial discount factor, and ρ is the (positive) intertemporal discount rate.

After defining the function

$$V\,(t, R, w) = \mathbb{E}_t\left[\int_t^{\infty} f\,(s)\,\frac{1}{\delta}\,(R\,(s) - K\,(s))^{\delta}\,ds\right],$$

the value function can be written as

$$J\,(t, R) = \max_w V\,(t, R, w)\,,$$

and it must verify the partial differential (Hamilton-Jacobi-Bellman) equation

$$J_t + \max_w \begin{bmatrix} f(t) \frac{1}{\delta} (R(t) - K(t))^{\delta} \\ +J_R \left(\frac{1}{1+\phi(t)} (R(t) + \phi(t) K(t)) r + \frac{1}{1+\phi(t)} w(t)' M + k(t) \right) \\ +\frac{1}{2} \left(\frac{1}{1+\phi(t)} \right)^2 J_{RR} w(t)' \Sigma' \Sigma w(t) \end{bmatrix} = 0.$$

The first order condition for the maximization gives

$$\frac{1}{1+\phi(t)} J_R M + \left(\frac{1}{1+\phi(t)} \right)^2 J_{RR} \Sigma' \Sigma w(t)^* = 0$$
$$\Rightarrow w(t)^* = -(1+\phi(t)) \frac{J_R}{J_{RR}} (\Sigma'\Sigma)^{-1} M.$$

A standard result in (stochastic) optimization theory guarantees that, under suitable conditions that must hold on Problem (10),[8] the value function is increasing and concave in R and so the second order conditions are satisfied.

We stress that the optimal portfolio w^* is always positive in the univariate case (single geometric Brownian motion). Instead, in the multivariate case, asset prices should not be too positively correlated if one wants to avoid the possibility of having negative weights for the optimal portfolio.[9]

We note that the optimal portfolio weights are increasing functions of the percentage $\phi(t)$ of profits that the fund shares with its members. This suggests that pension funds transferring a high percentage of their profits to their members are riskier and this is in accordance with the principle "the higher the return, the higher the risk". As shown in Appendix D, this is not true when the pension fund decides to share the level of its surplus (and not the change in it). In this case, the optimal portfolio (see Equation (15)) does not depend on the sharing rule ψ.

The Hamilton-Jacobi-Bellman equation for Problem (10) is

$$0 = J_t + f(t) \frac{1}{\delta} (R(t) - K(t))^{\delta}$$
$$+ J_R \left(\frac{1}{1+\phi(t)} (R(t) + \phi(t) K(t)) r + k(t) \right) - \frac{1}{2} \frac{J_R^2}{J_{RR}} \xi' \xi$$

where $\xi = \Sigma (\Sigma'\Sigma)^{-1} M$.[10] For the value function, we try the form $J(R,t) = g(t) f(t) U(R,t)$, where $g(t)$ needs to be determined. So, after substituting this form into the HJB equation and carrying out some simplifications, we obtain that $g(t)$ must satisfy [11]

$$0=\frac{\partial g(t)}{\partial t}+\left(\frac{\partial f(t)}{\partial t}\frac{1}{f(t)}+\frac{r\delta}{1+\phi(t)}-\frac{1}{2}\frac{\delta}{\delta-1}\xi'\xi\right)g(t)+1, \tag{11}$$

whose boundary condition must guarantee the convergence of $J(R,t)$ when t tends to infinity (the so-called transversality condition). This ODE has an infinite number of solutions, but the only one satisfying the suitable transversality condition is derived in Appendix B. Actually, the precise form of function $g(t)$ is not important for computing the optimal portfolio composition since the inverse of the Arrow-Pratt risk aversion index computed on $J(R,t)$ does not depend on $g(t)$. So, we can finally write what follows.

Proposition 1 *For u and v satisfying (3), the optimal portfolio composition solving Problem (10) is given by*

$$w(t)^{*}=\frac{1+\phi(t)}{1-\delta}\left(R(t)-K(t)\right)\left(\Sigma'\Sigma\right)^{-1}M. \tag{12}$$

We underline that the amount of wealth invested in the risky assets is a constant proportion of the global wealth, once the net flow of contributions and benefits already paid or received have been deducted. In fact, the optimization criterion is based on the surplus $R(t)-K(t)$. Accordingly, the optimal portfolio allocation in (12) is not proportional to the global wealth level $R(t)$, but it is an affine function of it, and the optimal portfolio relative composition (w^{*}/R) depends on the wealth level.

After substituting the optimal value $w(t)^{*}$ into the dynamic equation (8), we obtain

$$\begin{aligned}dR(t)=&\left(\frac{1}{1+\phi(t)}\left(R(t)+\phi(t)K(t)\right)r+\frac{1}{1-\delta}\left(R(t)-K(t)\right)\xi'\xi+k(t)\right)dt\\&+\frac{1}{1-\delta}\left(R(t)-K(t)\right)\xi' dW(t),\end{aligned}$$

and the fund surplus $R(t)-K(t)$ follows

$$\begin{aligned} d\left(R(t)-K(t)\right) &= dR(t)-k(t)dt-rK(t)dt \\ &= \left(R(t)-K(t)\right)\left(\frac{r}{1+\phi\left(t\right)}+\frac{1}{1-\delta}\xi'\xi\right)dt \\ &\quad+\frac{1}{1-\delta}\left(R(t)-K(t)\right)\xi' dW(t). \end{aligned} \tag{13}$$

When the sharing rule ψ is implemented, the term $r/\left(1+\phi\left(t\right)\right)$ is replaced with the term $r-\psi\left(t\right)$ as shown in Appendix D. This result is very intuitive since the relevant interest rate in the stochastic differential equation of the optimal wealth is given by the riskless interest rate "diminished" by the proportion of surplus paid to the subscriber. All the results that follow will be stated only for ϕ (rephrasing them in terms of ψ is easy).

Since $R(t)-K(t)$ is log-normally distributed, we can rule out the probability of getting a negative surplus when the initial wealth is strictly positive (we recall that $K\left(t_0\right)=0$ by construction).

Proposition 2 *The optimal surplus $R(t)-K(t)$ is such that*

$$\mathbb{P}\left(\left.R\left(t\right)-K\left(t\right)>0\right|R\left(t_0\right)>0\right)=1.$$

The result stated in Proposition 2 does not guarantee that the optimal wealth always remains positive. In fact, the retrospective reserve $K(t)$ may become negative if the member lives very old. This implies that there is a chance that the fund wealth $R(t)$ becomes negative at large horizons (this is known as the "longevity risk" in the actuarial literature). The next proposition summarizes the different possibilities.

Proposition 3 *The fund wealth $R(t)$ is equal to*

$$R\left(t\right)=K\left(t\right)+R\left(t_0\right)e^{\left(\frac{r}{H-t_0}\int_{t_0}^{H}\frac{1}{1+\phi(s)}ds+\frac{1}{1-\delta}\xi'\xi-\frac{1}{2}\frac{1}{(1-\delta)^2}\xi'\xi\right)(t-t_0)+\frac{1}{1-\delta}\xi'\left(W_t-W_{t_0}\right)},$$

while the retrospective reserve $K(t)$ is equal to:

$$K\left(t\right)=\frac{u}{r}\left(e^{r(t-t_0)}-e^{r\max(t-T,0)}\right)-\frac{v}{r}\left(e^{r\max(t-T,0)}-1\right).$$

(i) For any time horizon H and any feasible ratio $(v/u)^*$ such that

$$H \leq T + \frac{1}{r} \ln \left(\frac{(v/u)^*}{1 - e^{r(T-t_0)} + (v/u)^*} \right),$$

the retrospective reserve $K(H)$ is non-negative, and the ruin probability $\mathbb{P}\left(R\left(H\right) < 0 \middle| R\left(t_0\right) > 0\right)$ is equal to zero.

(ii) For any time horizon H and any feasible ratio $(v/u)^*$ such that

$$H > T + \frac{1}{r} \ln \left(\frac{(v/u)^*}{1 - e^{r(T-t_0)} + (v/u)^*} \right),$$

the retrospective reserve $K(H)$ is negative, and the ruin probability $\mathbb{P}\left(R\left(H\right) < 0 \middle| R\left(t_0\right) > 0\right)$ is given by

$$\mathcal{N}\left(\frac{\ln\left(-\frac{K(H)}{R(t_0)}\right) - \left(\frac{r}{H-t_0}\int_{t_0}^{H} \frac{1}{1+\phi(s)} ds + \frac{1}{1-\delta}\xi'\xi - \frac{1}{2}\frac{1}{(1-\delta)^2}\xi'\xi\right)(H-t_0)}{\frac{1}{1-\delta}\sqrt{\xi'\xi\,(H-t_0)}} \right),$$

where $\mathcal{N}(\bullet)$ is the cumulative distribution function of a standard normal random variable.

Proof. See Appendix C. ■

The above proposition allows to quantify the ruin probability for given values of the parameters. A direct application of this result yields the following corollary.

Corollary 1 *The initial wealth $R(t_0)$ leading to a ruin probability for a given time horizon (H) less or equal to a chosen level (α) must be*

$$R\left(t_0\right) \geq -K\left(H\right) e^{-\frac{\mathcal{N}^{-1}(\alpha)}{1-\delta}\sqrt{\xi'\xi(H-t_0)} - \left(\frac{r}{H-t_0}\int_{t_0}^{H}\frac{1}{1+\phi(s)}ds + \frac{1}{1-\delta}\xi'\xi - \frac{1}{2}\frac{1}{(1-\delta)^2}\xi'\xi\right)(H-t_0)},$$

where $\mathcal{N}^{-1}(\bullet)$ is the inverse of the cumulative distribution function of a standard normal random variable.

Let us fix the ruin probability, say to 0.01% (recall that the default probability for a AAA rating is 0.01%). We may then determine numerically what should be the amount of initial wealth compatible with this low probability level. This approach is akin to the determination of a regulatory capital matching a given loss probability level in risk management (Value-at-Risk approach).

Let us take into account the case of a subscriber who enters the fund when he is 25 (t_0) and he retires when he is 65 (T). On the financial market we take the riskless interest rate $r = 0.02$ and the square of the market price of risk $\xi'\xi = 0.24$. The risk aversion for a pension fund is often put equal to 3 (i.e. $\delta = -2$). Furthermore, we take into account the following values for the parameters of the Gompertz-Makeham distribution: $m = 88.18$, $b = 10.5$, $\lambda = 0$ (as already explained in the previous section). Finally, let us assume that the fund shares 10% ($\phi(t) = 0.1$) with its subscribers. Without any loss of generality we can put $u = 1$, and we obtain the feasible value of v equal to 11.77.

Given the values of all these parameters, we show the initial wealth compatible with different time horizons ($H > 82.5$) and with a ruin probability of 0.01% in Figure 3.

[Fig. 3 here]

Since the institutional investors are generally subject to strong minimum capital requirement, this simulation shows that, for a reasonable level of the initial wealth, the ruin probability can be considered as very small in practice. In other words, the level of the possible deficit is quite small, and can be considered as a kind of "subsistence" deficit.

Let us further remark that the optimal portfolio of Proposition 1 can be broken up into three parts:

$$w(t)^* = w_R(t)^* + w_u(t)^* + w_v(t)^*,$$

where

$$w_R(t)^* = \frac{1+\phi(t)}{1-\delta} R(t) \left(\Sigma'\Sigma\right)^{-1} M,$$
$$w_u(t)^* = -\frac{1+\phi(t)}{1-\delta}\frac{u}{r}\left(e^{r(t-t_0)} - e^{r\max(t-T,0)}\right)\left(\Sigma'\Sigma\right)^{-1} M,$$
$$w_v(t)^* = \frac{1+\phi(t)}{1-\delta}\frac{v}{r}\left(e^{r\max(t-T,0)} - 1\right)\left(\Sigma'\Sigma\right)^{-1} M.$$

The first component $w_R(t)^*$ depends on the wealth, the second component $w_u(t)^*$ depends on the contribution rate, and the third component $w_v(t)^*$ depends on the pension rate. We underline that the first component $w_R(t)^*$ coincides with Merton's portfolio.

It is interesting to stress that the actuarial risk enters the optimal portfolio via the feasible condition (3) creating a link between u and v. When this link is not considered, as in Charupat and Milevsky (2002), the portfolio composition is independent of the mortality risk.

In our framework, the optimal portfolio differs from the Merton's one by two additional components. So we are most interested in studying the behavior of the remaining portfolio $w_u(t)^* + w_v(t)^*$, namely the part directly proportional to the retrospective mathematical reserve. In particular, with a constant (or non-decreasing) sharing rule and with non-negative portfolio weights we can conclude what follows.

1. During the APh (when $w_v(t)^* = 0$ because $t < T$) the value of $w_u(t)^*$ decreases through time. Accordingly, $w_u(t)^*$ can be thought of as offsetting Merton's portfolio ($w_R(t)^*$) in order to meet payments of future pensions v.

2. During the DPh the value of $w_u(t)^* + w_v(t)^*$ increases through time. This property is very easy to check. The derivative of $w_u(t)^* + w_v(t)^*$ with respect to t is positive if
$$\frac{v}{u} > e^{r(T-t_0)} - 1,$$
but this inequality is always verified since the left hand side coincides with the lowest feasible value of v/u. For checking this, it is sufficient to

substitute ${}_{t-t_0}p_{t_0} = 1$, $\forall t > t_0$, in the feasibility condition (3) (the member never dies). Thus, we obtain the lowest feasible ratio v/u since any death probability strictly positive allows the pension fund to increase v (or to decrease u). Accordingly, $w_v(t)^*$ can be thought of as supplementing the portfolio riskyness because of closeness of death time.

So, the behavior of the optimal portfolio in absolute terms can be summarized as in the following corollary.

Corollary 2 *For an optimal portfolio with non-negative weights and a non-decreasing sharing rule, during the accumulation phase ($t < T$) the optimal portfolio is less risky than the Merton's portfolio and it becomes less and less risky as time goes on, while during the decumulation phase ($t > T$) the optimal portfolio becomes more and more risky and after time*

$$H^* = T + \frac{1}{r} \ln \left(\frac{(v/u)^*}{1 - e^{r(T-t_0)} + (v/u)^*} \right),$$

it becomes riskier than the Merton's portfolio.

Proof. For non-negative portfolio weights and a non-decreasing sharing rule, the sum $w_u(t)^* + w_v(t)^*$ is strictly and positively proportional to $-K(t)$, and $-K(t) > 0$ for $t > H^*$ (Proposition 3), which results in a riskier portfolio than Merton's one after H^*. ■

Let us provide further comments on this result. When the pension date T is still far away, the pension fund can afford to invest in a riskier portfolio (but still less risky than the Merton's one) in order to have a higher return on the managed wealth (and on the received contributions). When the pension date approaches, the payment of the pensions becomes closer, and the fund must switch to a less and less risky portfolio in order to increase the likelihood of being able to face all the payments. When the fund starts paying the pensions, the higher the number of pension installments paid the lower the probability to pay another pension installment since the death probability increases through time. Thus, after T when t increases, the riskiness of the fund portfolio can become higher and higher and, eventually, even higher than the one prescribed by the Merton's model.

The behavior described in this corollary can be seen in Figure 4 where we have plotted the function $-K(t)$ for $t_0 = 30$, $T = 60$, $r = 0.02$, and $u = 1$. As long as t is lower than the pension time T, the term in $K(t)$ makes the total portfolio less and less risky. The riskiness of the portfolio starts increasing again when t is higher than T. Furthermore, the higher the pension rate v the sharper the increase in the risky profile of the optimal portfolio. The behavior during the APh agrees with the results after Boulier *et al.* (2001) and Battocchio and Menoncin (2004).

[Fig. 4 here]

When we consider a perpetual annuity (i.e. ${}_{t-t_0}p_{t_0} = 1, \forall t \in [t_0, \infty[$) the feasibility condition (3) becomes

$$\frac{v}{u} = e^{r(T-t_0)} - 1,$$

and so the function $K(t)$ writes

$$K(t) = \frac{u}{r}\left(e^{r(t-t_0)} - e^{r\max(t-T,0)}\right) - \frac{u}{r}\left(e^{r(T-t_0)} - 1\right)\left(e^{r\max(t-T,0)} - 1\right),$$

which becomes a constant when $t > T$ as it can be seen in Figure 5. Instead, when there exists a strictly positive probability of death, the weight of the risky assets can become higher and higher after the retirement date T. Nevertheless, the ruin probability remains very little in practice as we have already shown.

[Fig. 5 here]

4 Conclusion

In this paper we have solved the asset allocation problem for a pension fund. The structure of the financial market is as follows: (i) there are n risky assets, whose prices follow geometric Brownian motions, (ii) there exists a riskless asset paying a constant interest rate, and (iii) the market is not necessarily complete. Furthermore, the fund is assumed to maximize the expected value of its surplus at the death time of its member. This surplus is given by the difference between the total managed wealth and the retrospective mathematical reserve.

We analyze the portfolio problem during both the APh and the DPh when the death time of the member is a stochastic variable (an exact solution is presented for the Gompertz-Makeham case). The contribution and pension rates are assumed to be constant, while the members also receive a deterministic percentage of either the surplus level or the surplus change.

We show that the optimal asset allocation during the APh is different from the one during the DPh. This is in agreement with conventional industry practice. In particular, the investment in the risky assets should decrease through time (and be always lower than in the typical Merton's portfolio) during the APh, for allowing the fund to meet the payment of the (constant) basic pension rate during the second phase. Instead the risky investment should increase (and eventually be higher than the one of the Merton's portfolio) during the DPh when the pensions are paid. In fact, the remaining wealth can be invested in riskier and riskier positions since the death of the member becomes more and more likely.

Finally, our model suggests that it is probably not optimal to outsource the asset allocation either during the APh or the DPh. First, optimal decisions during the APh and the Dph are intertwined, and the allocation problem cannot be disentangled into a problem corresponding to the APh and another one corresponding to the DPh. Hence we should avoid the two phases to be managed by two different entities. Second, the optimal asset allocation depends on the knowledge of the level of the fund wealth, and thus should not rely on external decisions ignoring the current situation of the fund.

A The Gompertz-Makeham distribution

Let us write the probability of being alive in t given that one is alive in t_0 as

$$_{t-t_0}p_{t_0} = \exp\left\{-\lambda\left(t-t_0\right) + y\left(1-e^{\frac{t-t_0}{b}}\right)\right\},$$

where

$$y = e^{\frac{t_0-m}{b}}.$$

The feasible ratio (3) involves the integral

$$\int_x^\infty \left(_{t-t_0}p_{t_0}\right) e^{-rt} dt,$$

which can be computed thanks to the following change of variable suggested in Charupat and Milevsky (2002):

$$z = ye^{\frac{t-t_0}{b}} \Leftrightarrow b\ln\frac{z}{y} + t_0 = t,$$
$$dz = \frac{y}{b}e^{\frac{t-t_0}{b}} dt \Leftrightarrow \frac{b}{z}dz = dt.$$

This yields

$$\int_x^\infty \left(_{t-t_0}p_{t_0}\right) e^{-rt} dt = e^{-rt_0} b e^y y^{(\lambda+r)b} \int_{ye^{\frac{x-t_0}{b}}}^\infty z^{-(\lambda+r)b-1} e^{-z} dz,$$

and, by using the incomplete Gamma function,[12]

$$\int_x^\infty \left(_{t-t_0}p_{t_0}\right) e^{-rt} dt = e^{-rt_0} b e^y y^{(\lambda+r)b} \Gamma\left(-\left(\lambda+r\right)b, ye^{\frac{x-t_0}{b}}\right).$$

Since the numerator of the feasible ratio (3) can be decomposed as follows:

$$\int_{t_0}^T \left(_{t-t_0}p_{t_0}\right) e^{-rt} dt = \int_{t_0}^\infty \left(_{t-t_0}p_{t_0}\right) e^{-rt} dt - \int_T^\infty \left(_{t-t_0}p_{t_0}\right) e^{-rt} dt,$$

we deduce that the feasible ratio is equal to

$$\frac{v}{u} = \frac{\int_{t_0}^\infty \left(_{t-t_0}p_{t_0}\right) e^{-rt} dt}{\int_T^\infty \left(_{t-t_0}p_{t_0}\right) e^{-rt} dt} - 1 = \frac{\Gamma\left(-\left(\lambda+r\right)b, y\right)}{\Gamma\left(-\left(\lambda+r\right)b, ye^{\frac{T-t_0}{b}}\right)} - 1.$$

B Transversality condition

The transversality condition is

$$\lim_{T\to\infty}\left(g\left(T\right)f\left(T\right)\frac{1}{\delta}\mathbb{E}_t\left[\left(R^*\left(T\right)-K\left(T\right)\right)^\delta\right]\right)=0.$$

We have already shown in the paper that the surplus $R-K$ follows a geometric Brownian motion. In particular, the solution of the Equation (13) is

$$\left(R\left(T\right)-K\left(T\right)\right)^\delta=\left(R\left(t\right)-K\left(t\right)\right)^\delta e^{\int_t^T\left(\frac{\delta r}{1+\phi(s)}+\frac{1}{2}\frac{\delta}{1-\delta}\xi'\xi-\frac{1}{2}\frac{\delta^2}{(1-\delta)^2}\xi'\xi\right)ds+\int_t^T\frac{\delta}{1-\delta}\xi' dW_s},$$

whose expected value is

$$\mathbb{E}_t\left[\left(R\left(T\right)-K\left(T\right)\right)^\delta\right]=\left(R\left(t\right)-K\left(t\right)\right)^\delta e^{\int_t^T a(s)ds},$$

with

$$a\left(s\right)=\frac{\delta r}{1+\phi\left(s\right)}+\frac{1}{2}\frac{\delta}{1-\delta}\xi'\xi.$$

The transversality condition thus becomes

$$\lim_{T\to\infty}\left(g\left(T\right)f\left(T\right)e^{-\int_t^T(r-a(s))ds}\right)=0.$$

Now, since the differential equation for $g\left(s\right)$ can be written as

$$0=\frac{\partial g\left(s\right)}{\partial s}+\left(\frac{\partial\ln f\left(s\right)}{\partial s}-r+a\left(s\right)\right)g\left(s\right)+1,$$

we know that, given the value of g in t, its general solution is

$$g\left(T\right)=g\left(t\right)\frac{f\left(t\right)}{f\left(T\right)}e^{-\int_t^T(-r+a(s))ds}-\int_t^T\frac{f\left(u\right)}{f\left(T\right)}e^{-\int_u^T(-r+a(s))ds}du$$

and the transversality condition becomes

$$\lim_{T\to\infty}\left(g\left(t\right)-\int_t^T\frac{f\left(u\right)}{f\left(t\right)}e^{-\int_t^u(r-a(s))\,ds}du\right)=0.$$

A suitable form of $g\left(t\right)$ is then

$$g\left(t\right)=\int_t^\infty\frac{f\left(u\right)}{f\left(t\right)}e^{-\int_t^u(r-a(s))\,ds}du.$$

This is the only solution of Equation (11) that satisfies the transversality condition.

C Proof of Proposition 3

We first need to compute the explicit expression for the fund wealth $R(t)$. Since $K(t_0) = 0$, we get from the geometric Brownian motion of Equation (13) that

$$R(t) - K(t) = R(t_0)\, e^{\left(\frac{r}{t-t_0}\int_{t_0}^{t}\frac{1}{1+\phi(s)}ds+\frac{1}{1-\delta}\xi'\xi-\frac{1}{2}\frac{1}{(1-\delta)^2}\xi'\xi\right)(t-t_0)+\frac{1}{1-\delta}\xi'\left(W_t-W_{t_0}\right)}.$$

Let us now compute the ruin probability:

$$\mathbb{P}\left(R(t) < 0 | R(t_0) > 0\right).$$

When $K(t)$ is positive, it is easy to deduce that the ruin probability is zero from Proposition 2. From Equation (5), we can deduce that

$$K(t) = \frac{u}{r}\left(e^{r(t-t_0)} - e^{r\max(t-T,0)}\right) - \frac{v}{r}\left(e^{r\max(t-T,0)} - 1\right),$$

which implies that $K(t)$ stays positive for any $t = H$ with

$$H \leq T + \frac{1}{r}\ln\left(\frac{(v/u)^*}{1 - e^{r(T-t_0)} + (v/u)^*}\right).$$

This leads to statement i) of the proposition. To get ii), let us take any $t = H$, but with

$$H > T + \frac{1}{r}\ln\left(\frac{(v/u)^*}{1 - e^{r(T-t_0)} + (v/u)^*}\right),$$

and substitute the value of the fund wealth by its expression

$$\begin{aligned}
&\mathbb{P}\left(K(H) + R(t_0)\,F(H,t_0) < 0 |\, R(t_0) > 0\right) \\
&= \mathbb{P}\left(F(H,t_0) < -\frac{K(H)}{R(t_0)}\middle|\, R(t_0) > 0\right) \\
&= \mathbb{P}\left(\ln F(H,t_0) < \ln\left(-\frac{K(H)}{R(t_0)}\right)\middle|\, R(t_0) > 0\right)
\end{aligned}$$

where

$$F(H,t_0) = e^{\left(\frac{r}{H-t_0}\int_{t_0}^{H}\frac{1}{1+\phi(s)}ds+\frac{1}{1-\delta}\xi'\xi-\frac{1}{2}\frac{1}{(1-\delta)^2}\xi'\xi\right)(H-t_0)+\frac{1}{1-\delta}\xi'\left(W_H-W_{t_0}\right)}.$$

Since $F(H,t_0)$ is lognormally distributed with mean

$$\mathbb{E}_{t_0}\left[\ln F(H,t_0)\right] = \left(\frac{r}{H-t_0}\int_{t_0}^{H}\frac{1}{1+\phi(s)}ds + \frac{1}{1-\delta}\xi'\xi - \frac{1}{2}\frac{1}{(1-\delta)^2}\xi'\xi\right)(H-t_0),$$

and variance

$$\mathbb{V}_{t_0}\left[\ln F\left(H,t_0\right)\right]=\frac{1}{\left(1-\delta\right)^2}\xi'\xi\left(H-t_0\right)$$

the ruin probability can be written as

$$\mathbb{P}\left(\eta<\frac{\ln\left(-\frac{K(H)}{R(t_0)}\right)-\left(\frac{r}{H-t_0}\int_{t_0}^{H}\frac{1}{1+\phi(s)}ds+\frac{1}{1-\delta}\xi'\xi-\frac{1}{2}\frac{1}{(1-\delta)^2}\xi'\xi\right)\left(H-t_0\right)}{\frac{1}{1-\delta}\sqrt{\xi'\xi\left(H-t_0\right)}}\right),$$

where η is a standard normal random variable.

D Sharing the surplus level

When ψ is used instead of ϕ the fund wealth follows Equation (9). The first order condition for the maximization problem leads to

$$w(t)^*=-\frac{J_R}{J_{RR}}\left(\Sigma'\Sigma\right)^{-1}M, \tag{14}$$

and the HJB equation becomes

$$\begin{aligned}0&=J_t+f\left(t\right)\frac{1}{\delta}\left(R(t)-K\left(t\right)\right)^{\delta}\\&\quad+J_R\left(R(t)\left(r-\psi\left(t\right)\right)+k\left(t\right)+\psi\left(t\right)K\left(t\right)\right)-\frac{1}{2}\frac{J_R^2}{J_{RR}}\xi'\xi.\end{aligned}$$

In this case, the same form for the value function as those presented in the text yields the following ODE for the function $g\left(t\right)$:

$$0=\frac{\partial g\left(t\right)}{\partial t}+\left(\frac{\partial f\left(t\right)}{\partial t}\frac{1}{f\left(t\right)}+\delta\left(r-\psi\left(t\right)\right)-\frac{1}{2}\frac{\delta}{\delta-1}\xi'\xi\right)g\left(t\right)+1,$$

whose solution can be derived as before. Thus, the optimal portfolio is not affected at all by the sharing rule ψ. Indeed, for u and v satisfying (3), the optimal portfolio composition solving Problem (10) with R following Equation (9) is given by

$$w(t)^*=\frac{1}{1-\delta}\left(R(t)-K\left(t\right)\right)\left(\Sigma'\Sigma\right)^{-1}M, \tag{15}$$

where the sharing rule $\left(\psi\left(t\right)\right)$ does not play any role.

Finally, the optimal wealth follows

$$dR(t) = \left(R(t)\left(r - \psi\left(t\right)\right) + \frac{1}{1-\delta}\left(R(t) - K\left(t\right)\right)\xi'\xi + k\left(t\right) + \psi\left(t\right)K\left(t\right) \right) dt + \frac{1}{1-\delta}\left(R(t) - K\left(t\right)\right)\xi' dW(t),$$

and so we can write

$$d\left(R(t) - K(t)\right) = \left(R(t) - K\left(t\right)\right)\left(r - \psi\left(t\right) + \frac{1}{1-\delta}\xi'\xi\right) dt + \frac{1}{1-\delta}\left(R(t) - K\left(t\right)\right)\xi' dW(t).$$

Notes

[1] These are extensively used in many countries: Australia, New Zealand, South Africa, Malaysia, Singapore, and Sri Lanka.

[2] These are available in many Latin American countries: Chile, Argentina, El Salvador, Mexico, and Peru, though not in Bolivia where mandatory annuitization is imposed.

[3] These values are for males. The corresponding values for females are: $m = 92.63$ and $b = 8.78$.

[4] In order to have an increasing and concave utility function, the parameter δ must be less than one.

[5] We recall that the force of mortality and the survival probability are linked by the following equation:

$$_{t-t_0}p_{t_0} = e^{-\int_{t_0}^{t} m(s)ds}.$$

[6] Even if the actual management of a pension fund is more likely to be based on a rule with a threshold on the fund surplus below which there is no sharing, we cannot afford to introduce this more "complicated" sharing rule without loosing the closed form solution for the optimal portfolio.

[7] We underline that $w \in \mathbb{R}^{n\times 1}$ contains the amount of money invested in each risky asset.

[8] Fleming and Soner (1993, Section IV.10) show that with w in a convex and compact set, with a continuous and concave objective function, with drift term of the state variable linear in both the control and the state, and the diffusion term linear in the control, then the value function is concave in the state.

[9] We recall that Farkas' lemma states that given $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^{n\times 1}$, then $\exists y \in \mathbb{R}^{n\times 1} : Ay \geq \mathbf{0}$ and $b'y < 0 \Leftrightarrow \nexists x \geq \mathbf{0} : A'x = b$. We can use this result for stating that, in order to have $w^* \geq 0$, we must assume that the variance-covariance matrix $(\Sigma'\Sigma)$ and the

vector of the risk premium (M) are such that there does not exist any $y \in \mathbb{R}^{n\times 1}$ so that $\Sigma'\Sigma y \geq \mathbf{0}$ and $M'y < 0$.

[10] When the market is complete (i.e. $\exists\Sigma^{-1}$), then ξ coincides with the Sharpe ratio.

[11] Let us suppose that the fund maximizes the expected utility of its final wealth (and not of its surplus). In this case the utility functions takes the form

$$U(R,t) = \frac{1}{\delta}R^{\delta}.$$

After applying the same form for the value function we have used in the main text: $J(R,t) = g(t)\, f(t)\, U(R,t)$ we obtain that $g(t)$ must satisfy the following ODE:

$$0 = \frac{\partial g(t)}{\partial t} + \left(\frac{\partial f(t)}{\partial t}\frac{1}{f(t)} + \frac{r\delta}{1+\phi} - \frac{1}{2}\frac{\delta}{\delta-1}\xi'\xi + \frac{\delta}{R}\left(\frac{\phi}{1+\phi}Kr + k\right)\right)g + 1,$$

which is clearly incompatible with the hypothesis that $g(t)$ does not depend on R (separability of the value function). In fact, $\frac{\phi}{1+\phi}Kr + k \neq 0$ by construction (even when $\phi = 0$).

[12] The incomplete Gamma function has the following form:

$$\Gamma(x_1, x_2) = \int_{x_2}^{\infty} e^{-s}s^{x_1-1}ds.$$

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Figure 1: Feasible ratio v/u as a function of age t_0 and retirement date T.

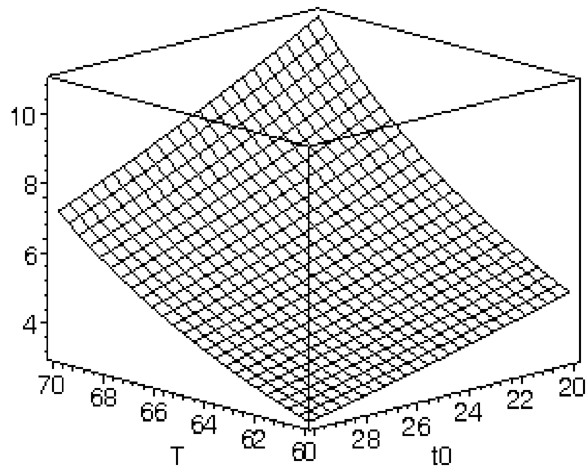

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Figure 2: Feasible ratio v/u as a function of interest rate r and retirement date T.

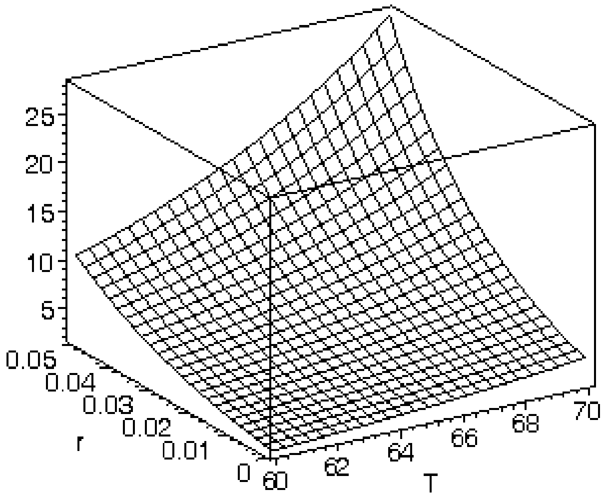

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Figure 3: Level of initial wealth $R(t_0)$ implying a default probability of 0.01% (AAA rating) as a function of time horizon H.

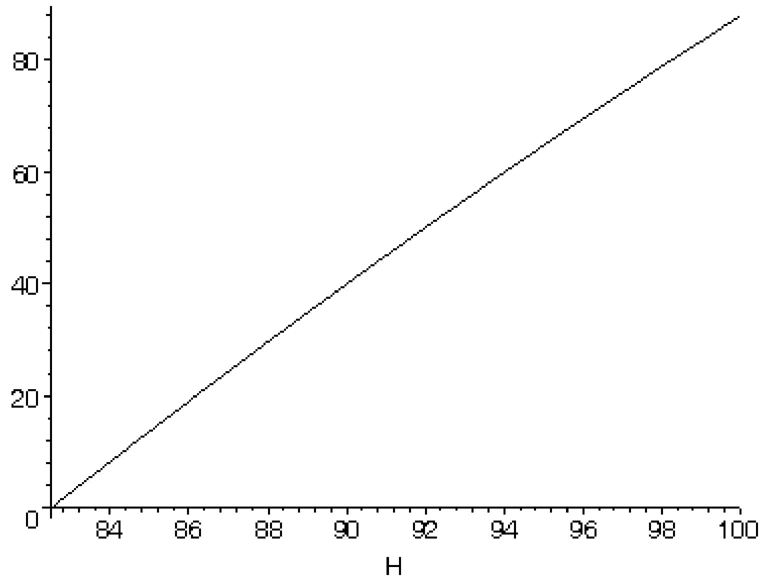

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Figure 4: Behavior of the function $-K(t)$.

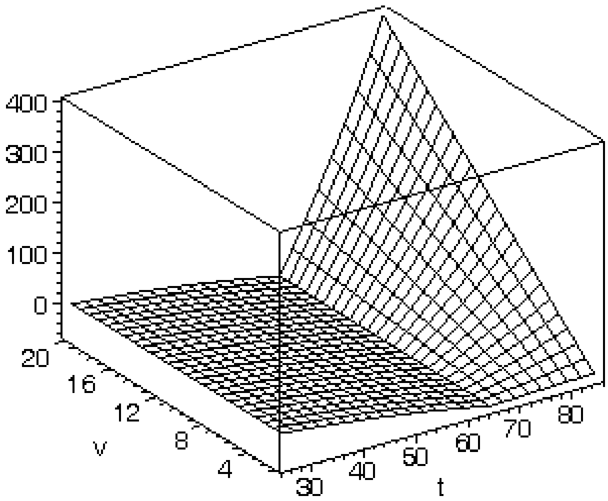

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Figure 5: Behavior of the function $-K(t)$ for a member with an infinite lifetime.