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Abstract

This paper presents a general method for finding optimal blocking arrangements of pure-level and mixed-level orthogonal designs of resolution 3. The method requires enumeration of all non-isomorphic 'parent designs' that include the treatment factors as well as the future blocking factor. For all these designs, word-length frequency vectors expressing the aliasing between main effects and two-factor interactions as well as the aliasing among two-factor interactions are calculated. The same is done for all projections into 'child designs' obtained by dropping any possible blocking factor. The word-length frequency vectors of the parent and child designs then allow the selection of blocking arrangements that are optimal with respect to five criteria appropriate for blocking resolution-3 orthogonal designs. We provide optimal blocking arrangements for orthogonal pure-level and mixed-level designs of 12, 16, 18, 20 and 27 runs.

KEY WORDS: Aliasing; Confounding; Generalized Word-Length Pattern; Orthogonal Array; Strength 2

1 Introduction

Effect estimates from a series of experimental runs can be sensitive to heterogeneous operating conditions. For example, chemical experiments are generally conducted with a limited number of reactors, where each of the reactors is used for a subset of the experimental runs. The challenge then is to group the runs so that uncontrollable differences between the reactors do not affect the estimates of the effects of the treatment factors. We call the groups of experimental runs blocks and the allocation of the runs to the blocks blocking. In this article, we focus on blocking pure-level and mixed-level orthogonal designs of resolution 3 or strength 2.

Early work on blocking designs for industrial experiments was carried out in the late 1950s. This resulted in blocking arrangements of regular two-level designs produced by the National Bureau of Standards (1957), and of regular three-level designs listed by Connor and Zelen (1959). Later work concentrated on defining optimality criteria for blocking (Bisgaard, 1994; Sun et al., 1997; Sitter et al., 1997; Cheng and Wu, 2002). These criteria are based on the defining contrast subgroup, a mathematical construct that only exists for the sub-class of regular fractional factorial designs. Criteria for blocking two-level nonregular designs, which generalize those for regular two-level designs, were proposed by Cheng et al. (2004).

Finding blocking arrangements for nonregular three-level designs and mixed-level designs requires new methodology. The same is true for all two-level designs with a number of blocks other than a power of two. In this paper, we propose five different criteria for selecting blocking arrangements for general fractional factorial designs of resolution 3, as well as a method for finding blocking arrangements that are optimal with respect to these criteria. Two of the five blocking criteria we use are new, while the remaining three are adaptations of existing criteria in the literature. The blocking criteria we propose are based on the assumption that a blocking factor does not interact with the treatment factors, while treatment factors may or may not interact with other treatment factors.

Like Cheng et al. (2004), we start from an exhaustive enumeration of designs. However, unlike the method of Cheng et al. (2004), which can only be used to find arrangements of two-level designs in a number of blocks that is a power of two, our approach permits the blocking of general factorial designs of resolution 3 in any number of blocks that is compatible with orthogonal blocking. For instance, our approach can be used to arrange 20-run two-level orthogonal designs in five blocks of four runs, to arrange a 12-run two-level orthogonal design in three blocks of four runs, and to arrange a 27-run design in nine blocks of three runs.

To illustrate our blocking approach, consider the design of a 27-run chemical experiment with eight three-level treatment factors and three blocks of nine runs. Such a design is suitable when three reactors are employed for the experiment. In total, the experiment has nine three-level factors, eight of which are treatment factors and one of which is the blocking factor. We call the design for the treatment factors a 'child design' and the full design, involving the treatment factors and the blocking factor, the 'parent design'.

Three parent designs are displayed in Table 1. The levels of the treatment factors are represented by 0, 1, and 2, whereas the three levels of the blocking factor are labeled 'Reactor 1', 'Reactor 2' and 'Reactor 3'. In each of the designs, the treatment factors are orthogonal to each other and to the blocking factor. Because of the orthogonality of the treatment factors to the blocking factor, the estimates of the main effects of the treatment factors are not affected by block differences.

The three parent designs in Table 1 are nonregular orthogonal designs with nine three-level factors and 27 runs. Schoen et al. (2010) showed that there are 9793 different designs of this type. The blocking

Table 1: Three optimally blocked 3^8 designs in 3 blocks of size 9

ID	Reactor 1	Reactor 2	Reactor 3
1	0 0 0 0 0 0 0 0	0 1 0 1 2 0 2 1	0 2 0 1 1 2 0 2
	0 0 1 1 1 1 1 1	0 1 1 0 0 2 1 2	0 2 1 2 2 0 1 0
	0 0 2 2 2 2 2 2	0 1 2 2 1 1 0 0	0 2 2 0 0 1 2 1
	1 1 0 0 1 1 2 2	1 2 0 1 0 1 1 0	1 0 0 2 2 1 0 1
	1 1 1 1 2 2 0 0	1 2 1 2 1 2 2 1	1 0 1 1 0 0 2 2
	1 1 2 2 0 0 1 1	1 2 2 0 2 0 0 2	1 0 2 0 1 2 1 0
	2 2 0 0 2 2 1 1	2 0 0 2 1 0 1 2	2 1 0 2 0 2 2 0
	2 2 1 2 0 1 0 2	2 0 1 0 2 1 2 0	2 1 1 0 1 0 0 1
	2 2 2 1 1 0 2 0	2 0 2 1 0 2 0 1	2 1 2 1 2 1 1 2
2	0 0 0 0 0 0 0 0	0 0 0 2 2 2 2 2	0 1 1 1 1 2 2 0
	0 0 0 1 1 1 1 1	0 2 2 0 0 2 2 1	0 1 1 2 2 0 0 1
	0 1 1 0 0 1 1 2	0 2 2 1 2 0 1 0	0 2 2 2 1 1 0 2
	1 1 2 0 2 0 2 2	1 0 1 0 1 2 0 2	1 0 1 1 0 0 2 1
	1 1 2 2 1 2 1 1	1 0 1 2 2 1 1 0	1 2 0 0 1 1 2 0
	1 2 0 1 2 2 0 1	1 1 2 1 0 1 0 0	1 2 0 2 0 0 1 2
	2 0 2 2 1 0 2 0	2 1 0 1 1 0 0 2	2 0 2 0 2 1 0 1
	2 2 1 1 2 1 2 2	2 1 0 2 0 1 2 1	2 0 2 1 0 2 1 2
	2 2 1 2 0 2 0 0	2 2 1 0 1 0 1 1	2 1 0 0 2 2 1 0
3	0 0 0 0 0 0 0 0	0 0 2 2 2 2 2 1	0 1 2 2 0 2 0 2
	0 0 0 1 1 1 1 1	0 1 1 2 1 1 1 0	0 2 1 0 2 2 1 1
	0 1 1 0 0 1 2 2	0 2 0 1 2 0 2 2	0 2 2 1 1 0 0 0
	1 0 1 1 2 2 0 2	1 0 2 0 0 0 1 1	1 0 2 0 1 1 2 2
	1 1 2 2 2 0 1 0	1 1 0 1 0 2 2 0	1 1 1 1 1 0 2 1
	1 2 0 2 1 2 1 2	1 2 1 0 2 1 0 0	1 2 0 2 0 1 0 1
	2 1 2 1 2 1 0 1	2 0 1 2 1 0 0 2	2 0 0 2 2 1 2 0
	2 2 1 2 0 0 2 1	2 1 0 0 1 2 0 1	2 0 1 1 0 2 1 0
	2 2 2 0 1 2 2 0	2 2 2 1 0 1 1 2	2 1 0 0 2 0 1 2

arrangements presented in the table are optimal according to the different criteria that we utilize in this paper to select optimally blocked designs.

The remainder of this paper is organized as follows. In Section 2, we review the optimality criteria for completely randomized designs on which our criteria for blocking are based. We introduce our optimality criteria for blocking in Section 3. In Section 4, we describe how we can use different ways of sorting completely randomized designs for the purpose of blocking. We apply that approach in Section 5 to generate the best blocking arrangements for mixed-level and pure-level designs with 12, 16, 18, 20, and 27 runs, and describe the results. We conclude the paper with a brief discussion of the limitations of our approach, and two technical appendices in which we compare some of our results for 16 and 27 runs to those of Cheng et al. (2004) and Xu et al. (2004), respectively.

2 Criteria for completely randomized designs

The goal of this paper is to find optimal blocking arrangements of general resolution-3 designs with N runs and k treatment factors. Hedayat et al. (1999) show that these designs are orthogonal arrays of strength 2. Formally, an orthogonal array is a rectangular arrangement of symbols in N rows and k columns, with s_j different symbols in its jth column. An array of strength 2 has the property that, for all pairs of columns, all pairs of symbols occur equally often (Rao, 1947). We denote the class of strength-2 arrays with N rows, k columns and s_j levels for the jth column by $\mathrm{OA}(N; s_1 \ldots s_k)$.

Xu and Wu (2001) propose the generalized word-length pattern (GWLP) to classify orthogonal arrays of any strength, number of factors, and numbers of factor levels. The GWLP of a design with k factors and strength $t \geq 2$ is a vector $(A_3, A_4, A_5, \ldots, A_k)$. We call an element A_i of the GWLP the generalized word count of length i. It is calculated as follows. First, for each factor j in the design, $s_j - 1$ orthogonal main-effect contrast vectors with length $\sqrt{N}$ are constructed. Second, the element-wise products are calculated of all sets of i main-effect contrast vectors corresponding to i different main effects. Third, for each set of i contrasts vectors, the element-wise products are summed and the resulting sum is squared. Finally, the squared sums corresponding to all sets are summed and divided by N^2.

The A_3 value quantifies the extent to which main-effect contrast vectors are aliased with two-factor interaction contrast vectors, and each other A_i value, where $i > 3$, measures the extent to which main-effect contrast vectors are aliased with $(i-1)$-factor interaction contrast vectors and q-factor interaction contrast vectors are aliased with $(i-q)$-factor interaction contrast vectors (for q ranging from 2 to $i - 2$). Both the word-length pattern for regular prime-level designs (Box et al., 2005; Wu and Hamada, 2009) and the extended word-length pattern for nonregular two-level designs (Tang and Deng, 1999; Deng and Tang, 1999) are special cases of the GWLP.

The resolution of a design is defined as the smallest p for which $A_p > 0$. All the designs discussed in this paper have at least resolution 3. To distinguish between designs of equal resolution, Xu and Wu (2001) use a generalized aberration (GA) criterion similar to the aberration criterion for regular fractional factorial two-level designs (Fries and Hunter, 1980). A minimum GA design sequentially minimizes the entries of the GWLP from left to right.

For the designs studied in this paper, aliasing of main-effect contrast vectors with two-factor interaction contrast vectors and among two-factor interaction contrast vectors are key issues. This is because, usually, the interest in industrial experimentation is in main effects and two-factor interactions, while higher-order interaction effects are considered negligible. We therefore restrict our attention to the first two elements of the GWLP, A_3 and A_4. As an illustration, the A_3 and A_4 values of the three eight-factor child designs in

Table 2: Properties of the eight-factor child designs in Table 1. The elements of the FA_3^c vector correspond to A_3 values of 2, 2/3, 14/27, 4/9, 8/27, and 0.

ID	A_3^c	A_4^c	FA_3^c					
1	16	60	2	18	0	0	0	36
2	16	60	2	18	0	0	0	36
3	20.30	45.85	0	1	11	16	23	5

Table 1 are shown in the second and third column of Table 2. We call these values A_3^c and A_4^c values, where the superscript 'c' is short for child design.

The eight-factor child designs from parent designs 1 and 2 have minimum GA. Each one of the two child designs can be obtained from the other by switching columns and relabeling levels. Therefore, the two child designs are isomorphic. They outperform the eight-factor child design obtained from parent design 3 in terms of the GA criterion because they have a smaller A_3^c value.

Designs of resolution 3 are very often used for screening purposes, in which case only a few of the k treatment factors are expected to be active. Therefore, the statistical properties of low-order projections of the k-factor designs are of utmost importance. The lower the order of the projections, the more important their properties for screening purposes. Since all projections of any orthogonal design into one factor result in an equireplicated one-factor design and all projections of any resolution-3 design into two factors result in at least one replicate of a full factorial design, projections into one and two factors cannot be used to distinguish between different designs of resolution 3. Therefore, the most important projection is the projection into three factors.

To quantify the projection properties of a design into three factors, Xu et al. (2004) propose to determine the A_3 value of the projections. If there are k factors in total, then the number of projections into three factors equals $k(k-1)(k-2)/6$. Each of these projections yields a certain A_3 value. Usually, there are only a limited number of different A_3 values. We call the vector of frequencies with which each of these A_3 values occurs in a three-factor projection FA_3. This vector is a generalization of the confounding frequency vector introduced by Deng and Tang (1999) for nonregular two-level designs.

The FA_3 vectors of the three eight-factor child designs in Table 1 are listed in the last six columns of Table 2. We denote these vectors by FA_3^c to stress that they correspond to child designs. The six different A_3 values obtained for the 56 3-factor projections of the three designs were 2, 2/3, 14/27, 4/9, 8/27 and 0. The value of 2 is particularly unfavorable. It means that, in the corresponding three-factor projection, the main effect of any one factor is completely aliased with a component of the two-factor interaction of the two other factors.

The child designs obtained from parent designs 1 and 2 both have a frequency distribution equal to $FA_3^c = (2, 18, 0, 0, 0, 36)$. For the child design from parent design 3, the frequency distribution is given by $FA_3^c = (0, 1, 11, 16, 23, 5)$. So, the worst three-factor projections of the child designs from parent designs 1 and 2 have A_3 values of 2 and $2/3$. Two of the 56 projections into three factors have an A_3 value of 2, and 18 of the projections have an A_3 value of $2/3$. The child design from parent design 3 has better projection properties because none of its projections into three factors has an A_3 value of 2, and only one of its projections has an A_3 value of $2/3$. In general, designs that have small frequencies for large A_3 values are preferable. The design that has the best projection properties, as measured by the FA_3 distribution, is said to have minimum projection aberration or minimum PA. Clearly, the child design from parent design 3 is the better option in Table 2 in terms of the PA criterion.

What is most interesting about the GA and PA criteria is that they favor different designs. The child designs from parents 1 and 2 are the best in terms of the GA criterion, whereas the child design from parent 3 is better in terms of the PA criterion.

In Table 2, the last element of the FA_3^c vector gives the frequency of three-factor projections with an A_3 value of zero. In the remainder of this paper, we omit this frequency, because it is redundant. Indeed, it can always be calculated as the difference between $k(k-1)(k-2)/6$ and the sum of all other frequencies in the FA_3 vector.

3   Blocking optimality

A key feature of screening experiments is that not all the main effects and two-factor interaction effects can be estimated simultaneously. When the experiment is blocked, this undesirable feature of screening experiments is even worse because a proper data analysis requires the estimation of block effects on top of the main effects of the treatment factors and the two-factor interaction effects. In this paper, we consider only blocking arrangements for which the main effects of the treatment factors can be estimated independently from the block effects. This is because we wish to prioritize estimation of main effects. In a blocked experiment, besides being aliased with main-effect contrast vectors and other two-factor interaction contrast vectors, any two-factor interaction contrast vector can also be confounded with block contrast vectors. As a result, the challenge is to find blocking patterns that have as little aliasing as possible between main-effect contrast vectors and two-factor interaction contrast vectors and among the two-factor interaction contrast vectors (as in the absence of blocking), and an optimal confounding pattern between the two-factor interaction contrast vectors and the block contrast vectors.

3.1 Pure- and mixed-type words

When a design involves a blocking factor, there are contrast vectors involving the blocking factor and those that do not involve the blocking factor. Therefore, any A_i value of a parent design, which we denote by A_i^p (where the superscript 'p' is short for parent design), partly corresponds to the first type of contrast vectors and partly to the second type. Following Cheng and Wu (2002), we call the portion of A_i that corresponds to contrasts involving the blocking factor mixed-type word counts and the portion of A_i that corresponds to contrasts not involving the blocking factor pure-type word counts. This distinction is important for the evaluation of a given blocking arrangement, because it is less important to have low mixed-type word counts than it is to have low pure-type word counts. This is due to the fact that blocking factors are assumed not to interact with treatment factors. Mixed-type word counts quantify the extent to which there is confounding between the treatment factors' interaction contrast vectors on the one hand, and block contrast vectors on the other hand, but also to what extent there is confounding between treatment $\times$ block contrast vectors and treatment contrast vectors. Mixed-type word counts are therefore only partially relevant.

A crucial insight for the purpose of this paper is that the pure-type word count of length i of a parent design equals A_i^c, the word count of length i of the child design. In the remainder of this paper, we denote the mixed-type word count of length i by $A_{i-1,1}$. It is easy to see that $A_i^p = A_i^c + A_{i-1,1}$.

3.2 Five criteria

There has been extensive discussion in the literature about the importance of mixed-type and pure-type word counts in design selection criteria. An overview of this discussion is given in Cheng et al. (2004) and Ai and Zhang (2004). While all authors agree that the A_3^c value should be minimized first, there is considerable debate about the order of importance of the A_4^c, A_{21}, and A_{31} values.

Assuming that words of length $i \geq 5$ can be ignored, we identified five sensible design selection criteria. The first four criteria are based on two different orders of importance of the A_3^c, A_4^c, A_{21}, and A_{31} values, while the fifth criterion is based on the properties of the designs when projected into three factors. We list the five criteria in Table 3. Using the criteria requires minimizing the entries in the table from left to right.

W_1- and W_2-criteria

The criteria labeled W_1 and W_2 in Table 3 were used by Cheng and Wu (2002) and Cheng et al. (2004). These criteria are classical criteria because they both seek to minimize the most severe types of aliasing and confounding. Which of the two criteria is preferable depends on the experimental context. Sometimes, confounding of two-factor interaction effects with blocks is considered more serious than aliasing among

Table 3: Five criteria for selecting optimal blocking arrangements. Entries have to be minimized from left to right.

Criterion	Minimization Vector			
W_1	A_3^c	A_4^c	A_{21}	A_{31}
W_2	A_3^c	A_{21}	A_4^c	A_{31}
W_1^-	A_3^c	A_4^c	$-A_{21}$	A_{31}
W_2^-	A_3^c	$-A_{21}$	A_4^c	A_{31}
W_3	FA_3^c	FA_{21}		

two-factor interaction effects. In that case, criterion W_2 is preferred. In other experiments, criterion W_1 might be the preferred criterion.

W_1^-- and W_2^--criteria

To the best of our knowledge, the W_1^-- and W_2^--criteria are new to the literature. The novel feature of these criteria is that they seek to maximize the confounding between the two-factor interaction effects and the blocking factor, while they still aim at minimizing the aliasing of main effects of treatment factors with their two-factor interactions, aliasing of two-factor interactions with other two-factor interactions and confounding of three-factor interactions with blocks. In Table 3, we have made this clear by using the entry $-A_{21}$ instead of A_{21} for the W_1^-- and W_2^--criteria. As a matter of fact, minimizing $-A_{21}$ is the same as maximizing the mixed-type word count of length 3, A_{21}.

The W_1^-- and W_2^--criteria are useful when we use a main-effects only model to detect active factors, in cases with insufficient degrees of freedom to estimate two-factor interaction effects. This is because, when the A_{21} value is large, there are fewer two-factor interaction effects that may inflate the residual error variance in a model involving block effects and main effects. In that case, all other things being equal, the power for detecting active factors is increased.

We illustrate the advantage of maximizing A_{21} with a small simulation study that involves the first two designs of Table 1. Both designs are arrangements of an eight-factor 27-run design in three blocks of size nine. The design with ID 1 in the table is W_1-optimal, whereas the design with ID 2 is W_1^--optimal. For each of the two designs, we first normalized the interaction contrast vectors to a length of one. We then generated 5000 sets of responses assuming that the two-factor interaction coefficients are from a standard normal distribution. For each of the 5000 data sets, we estimated a model including block effects and main effects, and quantified the mean squares value for the blocks, the main effects and the residual error. Side-by-side box plots of the mean squares for the 5000 data sets are shown in Figure 1.

The figure shows that using the W_1^--optimal blocking arrangement leads to a larger value of the mean
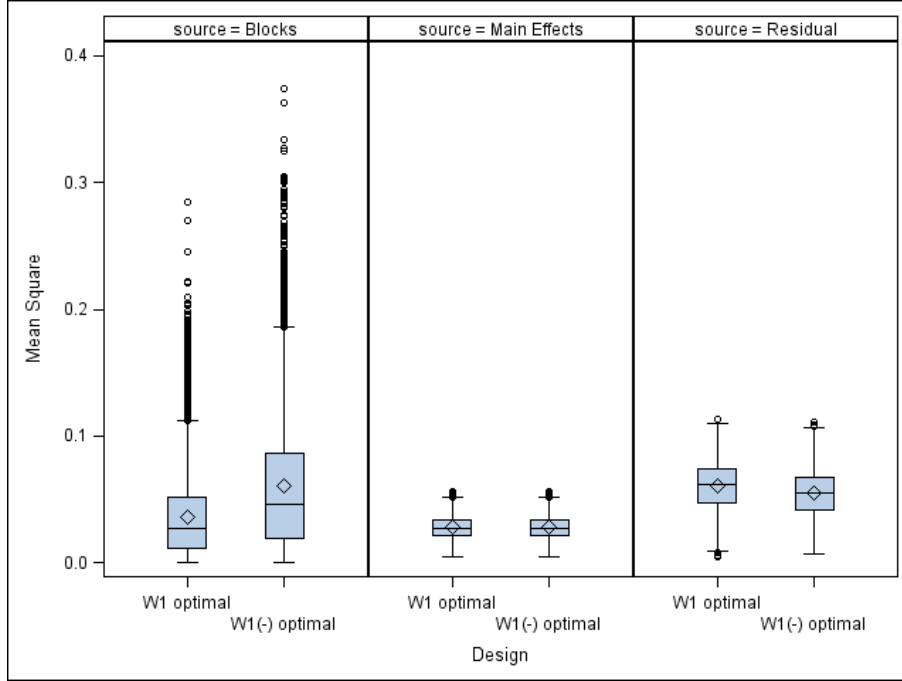
Figure 1: Comparison of mean squares for blocks, main effects and residual error for a W_1-optimal and a W_1^--optimal blocking arrangement of a 27-run design involving eight three-level factors, assuming a model involving block effects and main effects is fitted.

squares for the blocks than the W_1-optimal blocking arrangement. Conversely, the W_1^--optimal blocking arrangement leads to a smaller value of the mean squares for the residual error than the W_1-optimal blocking arrangement. Finally, the mean squares for the main effects remain unchanged. In 56% of the simulations, the error mean squares for the W_1-optimal design were higher than the error mean squares for the W_1^--optimal alternative. In 50% of the simulations, the increase in mean squares was 11% or more. A drop in the mean squares for the residual error results in a smaller estimate for the residual error variance, and therefore in a larger value for the test statistics for the main effects. As a result, the power for detecting main effects increases. For this reason, the W_1^--optimal blocking arrangement is more suitable for detecting active main effects than the W_1-optimal arrangement. Since the child designs of the W_1- and W_1^--optimal designs are isomorphic in this example, the differences in mean squares are entirely due to the blocking.

Note that, in Figure 1, the drop in the mean squares for the residual error is smaller than the increase in the mean squares for the blocks. This is due to the fact that, in the main-effects model we utilized to analyze the simulated data, eight degrees of freedom are associated with the mean squares for the residual error, while only two degrees of freedom are associated with blocks. In a case with a larger number of blocks and a smaller number of degrees of freedom for the residual error, the decrease in the mean squares value for the residual error would be larger relative to the increase in the mean squares value for the blocks.

Table 4: GWLP-based characteristics of the blocked designs in Table 1.

ID	A_3^c	A_4^c	A_{21}	A_{31}	A_3^p	A_4^p
1	16	60	8	48	24	108
2	16	60	14	24	30	84
3	20.30	45.85	10.11	36.82	30.41	82.67

W_3-**criterion**

The final criterion we use is based on the projections of designs into three factors. We label that criterion W_3. The W_3-criterion is similar in spirit to the PA criterion discussed in the Section 2, but it distinguishes between the frequency vector for the pure-type word counts of length 3, FA_3^c, and the frequency vector for the mixed-type word counts of that length, FA_{21}. More specifically, the W_3-criterion requires sequentially minimizing the elements of FA_3^c, followed by sequentially minimizing the elements of FA_{21}. The criterion is a simplified version of the W_2^G-criterion proposed by Cheng et al. (2004) for blocking two-level designs.

Application

To illustrate the application of the five blocking criteria, consider again the three designs in Table 1. The GWLP-based characteristics of these designs are shown in Table 4. Of the three designs, the one labeled 1 performs best in terms of W_1- and W_2-criteria. In terms of the W_1^-- and W_2^--criteria, the design labeled 2 is best.

The word-length frequency vectors required to determine the designs that are best with respect to the W_3-criterion are given in Table 5. Of the three designs, the design labeled 3 is best according to the pure-type word-length frequency vector FA_3^c, because it has a zero frequency for $A_3 = 2$. As a result, for each projection into three factors, the model with all main effects and all two-factor interaction effects is estimable. Design 2 has a zero frequency of $A_3 = 2$ in the mixed-type word-length frequency vector FA_{21}. Although that design has a frequency of two for $A_3 = 2$ in the pure-type word-length frequency vector FA_3^c, it is a compromise between the two other designs, because it has a better FA_{21} vector than design 1 (see Table 5) and a better A_3^c value than design 3 (see Table 4).

4   Finding optimal blocking arrangements

In this section, we explain how to determine the optimal blocking arrangements according to the W_1-, W_1^--, W_2-, W_2^-- and W_3-criteria. Two key concepts in our approach are the child design and the parent design. The child design is an orthogonal array denoted by $\text{OA}(N; s_1 \ldots s_k)$. The parent design is an orthogonal

Table 5: Pure-type and mixed-type word-length frequency vectors FA_3^c, FA_{21} and FA_3^p for the three-factor projections of the designs in Table 1.

ID		A_3 values				
		2	2/3	14/27	4/9	8/27
1	FA_3^c	2	18	0	0	0
	FA_{21}	1	9	0	0	0
	FA_3^p	3	27	0	0	0
2	FA_3^c	2	18	0	0	0
	FA_{21}	0	7	14	0	7
	FA_3^p	2	25	14	0	7
3	FA_3^c	0	1	11	16	23
	FA_{21}	0	1	7	5	13
	FA_3^p	0	2	18	21	36

array with one extra column for the blocking factor. Since we denote the number of blocks by b, that extra column should have b levels. We denote this kind of array by $\mathrm{OA}(N; s_1 \ldots s_k b)$. This is important because all possible ways of blocking k-factor child designs can be obtained by enumerating all non-isomorphic parent designs of the type $\mathrm{OA}(N; s_1 \ldots s_k b)$, and studying all possible projections of these parent designs into designs with one fewer b-level column. A key feature of this approach, which is known as single replacement in the literature, is that it ensures orthogonality of the blocking factor with respect to the main effects of the k treatment factors. This is consistent with our prioritization of the main-effects estimation.

Our use of single replacement for any number of blocks b is possible due to the work of Schoen et al. (2010), whose enumeration algorithm enabled us to obtain complete catalogs of all non-isomorphic pure-level and mixed-level orthogonal arrays, which can serve as parent and child designs.

4.1 Determining A_3^c, A_{21}, A_4^c and A_{31} values and FA_3^c and FA_{21} vectors

The first step in finding optimal blocking arrangements is to determine the A_3^c, A_{21}, A_4^c and A_{31} values and the pure-type and mixed-type word-length frequency vectors FA_3^c and FA_{21}. We start by computing the A_3^p and A_4^p values, and the word length frequency vector, FA_3^p, of the parent designs. The challenge then is to split the A_3^p values into A_3^c and A_{21} values, the A_4^p values into A_4^c and A_{31} values, and the FA_3^p vectors into FA_3^c and FA_{21} vectors. The most convenient way to achieve this is by computing the A_3 and A_4 values of any possible child design that can be obtained from the parent designs by dropping a b-level factor. The A_{21} and A_{31} values are then calculated as $A_{21} = A_3^p - A_3^c$ and $A_{31} = A_4^p - A_4^c$, respectively. In a similar fashion,

we computed the mixed-type word-length frequency vector FA_{21} as the element-wise difference between the FA_3 vectors of the parent designs and those of the child designs.

We illustrate this approach for parent design 1 in Table 1. This design involves nine three-level factors in total, eight treatment factors and one blocking factor. The nine-factor design is a parent design with $A_3^p = 24$ and $A_4^p = 108$. Its FA_3^p frequency vector is $(3, 27, 0, 0, 0)$, with elements corresponding to the A_3 values 2, 2/3, 14/27, 4/9 and 8/27 for the three-factor projections. As shown in Table 2, the child design obtained by dropping the blocking factor for design 1 in Table 1 has $A_3^c = 16$ and $A_4^c = 60$. Therefore, the blocking arrangement for design 1 in Table 1 has $A_{21} = 24 - 16 = 8$ and $A_{31} = 108 - 60 = 48$.

Table 2 also shows that the FA_3^c frequency vector is $(2, 18, 0, 0, 0)$. As a result, the frequency vector for the mixed-type counts, FA_{21}, is $(3 - 2, 27 - 18, 0, 0, 0)$ or $(1, 9, 0, 0, 0)$. This is shown in Table 5, along with the results for parent designs 2 and 3.

4.2 Determining the optimal blocking arrangements

Design 1 in Table 1 is a minimum GA design for nine three-level factors and 27 runs. In addition, its eight-factor projection obtained by dropping the blocking factor is a minimum GA design for eight three-level factors and 27 runs. Therefore, the design sequentially minimizes $(A_3^c, A_4^c, A_3^p, A_4^p)$. This is equivalent to sequentially minimizing $(A_3^c, A_4^c, A_3^p - A_3^c, A_4^p - A_4^c)$, or $(A_3^c, A_4^c, A_{21}, A_{31})$. As a result, the blocking arrangement given in Table 1 for design 1 is the best blocking arrangement in terms of the W_1-criterion.

The blocking arrangement obtained from parent design 1 is also optimal in terms of the W_2-criterion. Optimization of this criterion requires sequential minimization of $(A_3^c, A_{21}, A_4^c, A_{31})$, which is equivalent to sequential minimization of $(A_3^c, A_3^p, A_4^c, A_4^p)$. The fact that the blocking arrangement is optimal in terms of both the W_1-criterion and the W_2-criterion is due to the fact that all eight-factor designs with minimum A_3^c value have the same A_4^c value. Therefore, sequentially minimizing $(A_3^c, A_4^c, A_3^p, A_4^p)$ to find the best design in terms of the W_1-criterion gives exactly the same results as sequentially minimizing $(A_3^c, A_3^p, A_4^c, A_4^p)$ to find the best design in terms of the W_2-criterion. However, it will not generally be the case that the W_1- and W_2-criteria favor the same design.

The nine-factor parent design 2 in Table 1 has A_3^p and A_4^p values of 30 and 84, respectively. Dropping the blocking factor of the design results in a minimum GA design for eight factors. Design 2 has the worst possible A_3^p value among all parent designs that project into the minimum GA eight-factor child design. So, the blocking arrangement is such that the vector $(A_3^c, A_4^c, -A_3^p, A_4^p)$ is sequentially minimized. Therefore, it is optimal in terms of the W_1^--criterion. Due to the fact that all eight-factor designs with minimum A_3^c value have the same A_4^c value, the blocking arrangement is also optimal in terms of the W_2^--criterion.

Finally, the nine-factor parent design 3 in Table 1 has an FA_3^p vector equal to $(0, 2, 18, 21, 36)$. It therefore

sequentially minimizes the FA_3^p vector. Dropping the blocking factor of the design results in an eight-factor projection that is ranked third in the FA_3 ranking of all eight-factor child designs. The resulting eight-factor child design is, however, the best ranked one among all eight-factor designs that can be obtained from any nine-factor design by means of projection. In other words, it sequentially minimizes the FA_3^c vector among all child designs. Because parent design 3 has the best FA_3^p ranking among all possible nine-factor parent designs and projects into the best child design, its blocking arrangement in Table 1 sequentially minimizes (FA_3^c, FA_3^p). This is equivalent to minimizing $(FA_3^c, FA_3^p - FA_3^c)$ or (FA_3^c, FA_{21}). Using the blocking factor of design 3 therefore leads to the best blocking arrangement according to the W_3-criterion. The FA_{21} frequency vector is $(0, 2 - 1, 18 - 11, 21 - 16, 36 - 23)$ or $(0, 1, 7, 5, 13)$.

The different minimization vectors in Table 3 make it easy to determine optimal blocking arrangements. For given numbers of treatment factors k and numbers of levels $s_1, \ldots, s_k$, we first create all non-isomorphic parent designs of the type $OA(N; s_1 \ldots s_k b)$ and determine their A_3^p, A_4^p and FA_3^p values. We then create all possible child designs by dropping a b-level factor and determine their A_3^c and A_4^c values, the corresponding A_{21} and A_{31} values, the FA_3^c vector and the corresponding FA_{21}-vector. The optimal blocking arrangements according to the W_1-, W_1^--, W_2-, W_2^-- and W_3-criteria can then be found by sequentially minimizing the entries in Table 3.

In many of the scenarios we investigated, we found that there are several blocking arrangements which have the same values for A_3^c, A_4^c, A_3^p and A_4^p, but different word-length frequency vectors FA_3^c and FA_3^p. Generally, designs that sequentially minimize the FA_3^c and FA_3^p vectors increase the quality of the three-factor projections. Therefore, we used these vectors as tie-breakers to distinguish between designs that are optimal in terms of the W_1-, W_1^--, W_2- and W_2^--criteria. Of all W_1-, W_1^--, W_2- and W_2^--optimal designs, we only report the ones that minimize the FA_3^c and FA_3^p vectors. This explains why, in a few cases, we report fewer designs than Cheng et al. (2004).

5   Optimal blocking arrangements for designs with 12-27 runs

In this section, we apply our blocking method to catalogs of all non-isomorphic orthogonal designs with 12, 16, 18, 20, and 27 runs that include at least one factor at more than two levels. We do not address arrangements of two-level designs in two blocks because these were studied earlier by Cheng et al. (2004, CLY). However, we do address the arrangement of two-level 16-run designs in four blocks, because these can be derived from designs of the type $OA(16; 4^1 2^n)$. We use this type of design to point out differences between our methodology and CLY's methodology. Files containing all optimal blocking arrangements reported in this section are available electronically from the authors.

The design tables in this section identify all parent designs by three basic characteristics. First, a simple ID is given. Second, the R_1 ranking gives the rank of a design after sequentially sorting the designs in ascending order of the A_3 and A_4 values and the FA_3 vector. Third, the R_2 ranking gives the rank of a design after sequential sorting the designs in ascending order of the FA_3 vector and the A_4 value. The designs with $R_1 = 1$ are recommended when more than three active factors are expected. The designs with $R_2 = 1$ are recommended when at most three factors are expected to be active.

In the tables, the basic characteristics of the designs are followed by the A_3 and A_4 values and the FA_3 vector. Finally, each table involves one or two columns describing the blocking arrangements that can be obtained from the parent designs. In Section 5.1, we explain in detail how the tables with optimal blocking arrangements can be used.

5.1 Designs with 12 runs

Wang and Wu (1992) were the first to list the complete series of non-isomorphic designs of the type $\mathrm{OA}(12; 3 \times 2^n)$. Table 6 shows the designs we recommend for blocking purposes. The R_1 and R_2 rankings of the designs are identical. The A_3 values for three-factor projections corresponding to the three elements of the FA_3 vector are 1, 2/3, and 1/9.

A design of the type $\mathrm{OA}(12; 3 \times 2^n)$ can be used to arrange a $3 \times 2^{n-1}$ design in two blocks of size 6, or to arrange a 2^n design in three blocks of size 4. The columns labeled '2 blocks' and '3 blocks' give the blocking arrangements that are optimal with respect to each of the five optimality criteria we defined. We

Table 6: Optimal blocking arrangements based on parent designs of the type $\mathrm{OA}(12; 3 \times 2^n)$. Elements of the FA_3 vector correspond to A_3 values of 1, 2/3, and 1/9. An 'x' indicates W_1- and W_2-optimal arrangements; a 'y' indicates W_1^-- and W_2^--optimal arrangements; a 'z' indicates W_3-optimal arrangements.

n	ID	R_1, R_2	A_3	A_4	FA_3			2 blocks	3 blocks
4	1	1	1.778	1	0	2	4	1:2;xyz	xyz
3	1	1	0.111	0.889	0	0	1	1:1;xz	xz
		2	2	0.778	0.222	0	1	1	
2	1	1	0	-	0	0	0	1:1;xz	xz

Table 7: Properties of two-level child designs obtained from parent designs of the type $\mathrm{OA}(12; 3 \times 2^n)$. Elements of the FA_3 vector correspond to A_3 values of 1, 2/3, and 1/9.

n	R_1^p	A_3	A_4	FA_3		
4	1	0.444	0.111	0	0	4
3	1	0.111	-	0	0	1

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explain how to use the table by example.

The entries in the column labeled '2 blocks' are of the form 'p:q;{criteria}'. The first digit, p, indicates which column of the parent design has to be used for the blocking factor. The second digit, q, is the R_1 rank of the child design obtained by dropping the column corresponding to the blocking factor, and {criteria} specifies the set of criteria for which the resulting blocking arrangement is optimal. Consider, first, the single design of the type $OA(12; 3 \times 2^4)$. It is impossible to project this design into the design of the type $OA(12; 3 \times 2^3)$ with $R_1 = 1$ by dropping one of its two-level columns. However, by dropping the first two-level column, the $OA(12; 3 \times 2^4)$ design projects into the $OA(12; 3 \times 2^3)$ design with $R_1 = 2$. This explains the entry '1:2' in the column labeled '2 blocks'. The $OA(12; 3 \times 2^3)$ child design is reported in Table 6 as well, in the row for $n = 3$ and $R_1 = 2$. Dropping any other two-level column than the first one from the $OA(12; 3 \times 2^4)$ design also results in the $OA(12; 3 \times 2^3)$ child design with $R_1 = 2$. So, there is a single parent design, and it projects into a single child design. Therefore, the blocking arrangement is optimal according to all five optimality criteria considered here. This is indicated by the letters x, y, and z after the semicolon in the column labeled '2 blocks'. The A_{21} value and the FA_{21} vector for this blocking arrangement can be found by subtracting the A_3 value and the FA_3 frequencies of the child design from the A_3 value and the FA_3 frequencies of the parent design, respectively. This yields $A_{21} = 1.778 - 0.778 = 1$ and $FA_{21} = (0 - 0, 2 - 1, 4 - 1) = (0, 1, 3)$.

By dropping the three-level factor of the $OA(12; 3 \times 2^4)$ parent design, we obtain a child design of the type $OA(12; 2^4)$. We have displayed the A_3 and A_4 values and the FA_3 vectors of the child design in Table 7. The 2^4 child design has A_3 and A_4 values of 0.444 and 0.111, respectively, and an FA_3 vector equal to $(0, 0, 4)$. Again, the A_{21} value and the FA_{21} vector can be found by subtracting the entries of the child design from those of the parent design. This yields $A_{21} = 1.778 - 0.444 = 1.333$ and $FA_{21} = (0 - 0, 2 - 0, 4 - 4) = (0, 2, 0)$.

There are three different designs of the type $OA(12; 3 \times 2^3)$. Two of these are listed in Table 6. Their two-level parts consist of a full 2^3 design combined with one regular half-fraction of that design. The $OA(12; 3 \times 2^3)$ design with $R_1 = 1$ can be used to obtain the best blocking arrangement in terms of the W_1-, W_2- and W_3-criteria, for two as well as for three blocks. The properties of the $OA(12; 2^3)$ child design in the three-blocks case are given in Table 7. The $OA(12; 3 \times 2^3)$ design with $R_1 = 2$ is reported only because it is the child design corresponding to the $OA(12; 3 \times 2^4)$ parent design. In other words, the $OA(12; 3 \times 2^3)$ design with $R_1 = 2$ can be arranged in two blocks of size 6, using the $OA(12; 3 \times 2^4)$ as a parent design.

Using the three-level factor of the $OA(12; 3 \times 2^3)$ design with $R_1 = 2$ as a blocking factor results in an optimal blocking arrangement in terms of the W_1^-- and W_2^--criteria. However, we do not recommend these criteria in this case, for the following reason. The rationale of the W_1^-- and W_2^--criteria is to minimize the inflation of the residual error by two-factor interactions in a model containing main effects and block effects

only. However, any $OA(12; 2^3)$ design arranged in three blocks of size four allows us to fit a model containing all main effects and all two-factor interactions, with three degrees of freedom for estimating the residual error variance. Hence, in this case, the W_1^-- and W_2^--criteria, which assume that there are not enough degrees of freedom for estimating a model including all two-factor interaction effects, are not needed.

The $OA(12; 3 \times 2^3)$ design with $R_1 = 3$ is not shown in Table 6, even though it can be used to obtain a W_1^-- and W_2^--optimal arrangement of a $OA(12; 3 \times 2^2)$ design in two blocks. Here too, we do not recommend the W_1^-- and W_2^--criteria, because the number of runs is large enough to estimate the block effects, the main effects and the two-factor interaction effects, with one degree of freedom for the residual error variance.

Finally, there are two designs of the type $OA(12; 3 \times 2^2)$. The recommended one is the full factorial design, which can be arranged in three blocks of size four by assigning the three replicates of the 2^2 factorial design to the three blocks, and in two blocks of size six by assigning the two replicates of the 3×2 factorial design to the two blocks. The resulting blocking arrangements are optimal in terms of the W_1-, W_2- and W_3-criteria.

The second design of the type $OA(12; 3 \times 2^2)$ results in blocking arrangements that are optimal in terms of the W_1^-- and W_2^--criteria. However, these criteria again are not appropriate because the number of runs is sufficiently large to estimate a model including all two-factor interaction effects. Therefore, the second $OA(12; 3 \times 2^2)$ design is not shown in Table 6.

Table 8: Optimal blocking arrangements based on parent designs of the type $OA(16; 4^3 2^n)$. Elements of the FA_3 vector correspond to A_3 values of 3, 1, 1/2, and 1/4. An 'x' indicates W_1- and W_2-optimal arrangements; a 'y' indicates W_1^-- and W_2^--optimal arrangements; a 'z' indicates W_3-optimal arrangements.

n	ID	R_1	R_2	A_3	A_4		FA_3			2 blocks	4 blocks
6	1	1	1	32	51	1	21	12	8	1:1;xyz	1:1;xyz
5	1	1	1	25	32	1	17	8	4	4:1;xyz	1:1;xyz
4	1	1	2	19	18	1	14	4	0		3:1;xy
	2	3	1	19	18.5	1	13	5	2	1:3;z	1:2;z
	3	4	3	19	19	1	16	0	0	1:1;xy	
3	1	1	2	13	12	1	10	0	0	1:1;xz	1:2;x
	2	2	1	13.75	9.75	1	9	3	1		
	3	3	3	14	9	1	10	2	0		1:1;yz
	4	4	4	14	9	1	11	0	0	1:1;y	
2	1	1	1	9	5	1	6	0	0	1:1;xz	1:1;xz
	2	3	3	10	3	1	7	0	0	1:1;y	1:1;y
1	1	1	1	6	1	1	3	0	0	1:1;xyz	1:1;xyz
0	1	1	1	3	0	1	0	0	0		1:1;xyz

5.2 Designs with 16 runs

The blocking of pure two-level 16-run designs has been studied extensively by CLY. In many studies, however, experimenters also want to include factors with more than two levels. Tables 8–10 present parent designs as well as optimal blocking arrangements for cases with two-level factors and one or more additional four-level factors. In addition, Table 10 also shows the optimal arrangements of pure two-level designs in four blocks. While these blocking arrangements have been studied by CLY already, we decided to report a table with the best blocking arrangements found using our five criteria for this case for three reasons. First, the table facilitates a comparison of our methodology with the methodology of CLY. Second, we found an attractive blocking arrangement for the well-known resolution-5 2^{5-1} design that was not given by CLY. Third, two of our criteria were not used by CLY.

A design is listed in Tables 8–10 if it is best according to the R_1 or R_2 ranking, if it yields an optimal blocking arrangement, or if it is the child design of an optimal blocking arrangement. Table 8 contains

Table 9: Optimal blocking arrangements based on parent designs of the type $OA(16; 4^2 2^n)$. Elements of the FA_3 vector correspond to A_3 values of 1, 1/2, and 1/4. An 'x' indicates W_1- and W_2-optimal arrangements; a 'y' indicates W_1^-- and W_2^--optimal arrangements; a 'z' indicates W_3-optimal arrangements.

n	ID	R_1	R_2	A_3	A_4		FA_3		2 blocks	4 blocks
9	1	1	1	33	69	15	24	24	1;1;xyz	1:1;xyz
8	1	1	1	26	47	12	20	16	1:1;xy,3:9;z	2:1;xyz
7	1	1	3	20	30	10	16	8		2:1;xy,1:5;z
	2	9	1	20	31	10	14	12		
	3	10	4	20	31	11	12	12	3:1;xyz	
6	1	1	1	14	21	8	8	8	1:1;xz	
	2	5	8	15	17	9	12	0		2:1;xy
	3	12	14	15	18	12	4	4	1:2;y	
	4	16	6	15	18.5	9	9	6		1:3;z
5	1	1	1	10	12	6	6	4	1:2;z	
	2	2	4	10	12	7	4	4	3:1;x	
	3	12	10	11	9	7	8	0		1:1;xyz
	4	21	21	11	10	11	0	0	1:1;y	
4	1	1	8	6	8	6	0	0	1:2;x	1:2;x
	2	2	1	6.5	7	4	4	2	1:1;z	
	3	12	7	7.5	4.5	5	5	0		1:1;z
	4	19	19	8	4	8	0	0	1:1;y	1:1;y
3	1	1	1	4	3	3	2	0	1:1;xz	1:1;xz
	2	2	5	4	3	4	0	0	1:1;x	
	3	8	9	5	1	5	0	0	1:1;y	
	4	10	10	6	1	6	0	0		1:1;y
2	1	1	1	2	1	2	0	0	1:1;xz	1:1;xz
	2	3	3	3	0	3	0	0	1:1;y	
1	1	1	1	1	–	1	0	0	1:1;xyz	1:1;xz

the results for designs of the type $OA(16; 4^3 2^n)$. Such designs can be used to arrange designs of the type $OA(16; 4^2 2^n)$ in four blocks of size four, or to arrange $OA(16; 4^3 2^{n-1})$ designs in two blocks of size eight. The A_3^c and A_4^c values, and the FA_3^c vector, for the arrangements in two blocks can be found in Table 8 as well. The A_3^c and A_4^c values, and the FA_3^c vector, for the arrangements in four blocks are given in Table 9.

Even in the small series of $OA(16; 4^3 2^n)$ designs, there is a remarkable differentiation in optimality features. For example, there are four non-isomorphic $OA(16; 4^3 2^3)$ designs. The design with ranking $R_2 = 1$ does not yield an optimal blocking arrangement according to any of the criteria, while the remaining three designs are needed to obtain an optimal blocking arrangement for each of our five optimality criteria.

Table 9 presents the results for the designs of the type $OA(16; 4^2 2^n)$. An $OA(16; 4^2 2^n)$ design can be used to arrange $OA(16; 4 \times 2^n)$ designs in four blocks and to arrange $OA(16; 4^2 2^{n-1})$ designs in two blocks. The child designs for the cases involving four blocks are given in Table 10, whereas those for the cases involving two blocks are given in Table 9 itself. For the $OA(16; 4^2 2^n)$ parent design for eight two-level factors, i.e. for $n = 8$, Table 9 gives two different optimal arrangements in two blocks, depending on the column used for the blocking factor. Using the design's first column for the blocking factor results in a blocking arrangement that is W_1-, W_2-, W_1^- and W_2^--optimal, whereas using the design's third column for the blocking factor results in a blocking arrangement that is W_3-optimal. This explains the entry '1:1;xy,3:9;z' in Table 9. Similarly, the first parent design for seven two-level factors ($n = 7$) gives two different optimal arrangements in four blocks.

Table 10 presents the results for designs of the type $OA(16; 4 \times 2^n)$. The parent designs for optimal arrangements in four blocks have child designs whose characteristics are given in Table 11.

The arrangements in four blocks in Table 10 permit a direct comparison of our results with those of CLY. In Table 10, we mark the designs not found by these authors with an asterisk. Remarkably, our W_1-optimal design with five two-level factors is one of these designs. Its treatment factors form a regular resolution-5 2^{5-1} design. The W_1-optimal blocking arrangement, however, is nonregular because its FA_{21} vector is $(0, 6, 0)$, where the frequency of 6 corresponds to an A_3 value of 0.5. The best blocking arrangement of the regular resolution-5 2^{5-1} design reported by CLY has an FA_{21} vector of $(1, 4, 0)$, and therefore confounds one two-factor interaction effect with the blocks. Our W_1-optimal design does not confound any two-factor interaction effect with the blocks completely, and is therefore preferable. We provide an extensive comparison of all other arrangements of 16-run two-level designs in four blocks we obtained with these of CLY in Appendix A.

The case involving five two-level factors ($n = 5$) and four blocks in Table 10 shows, for the first time in this paper, that W_1^--optimal designs can differ from W_2^--optimal designs. The W_1-optimal design for this case is also different from the W_2-optimal design.

We do not list W_1^-- or W_2^--optimal designs for cases involving three and four two-level factors in four

blocks, because it is possible to estimate all the main effects and two-factor interactions using the W_1- or W_2-optimal designs. As a result, there is no added value in utilizing the W_1^-- or W_2^--optimal blocking arrangements in these scenarios.

There is also a set of designs of the type $OA(16; 8 \times 2^n)$ involving up to eight two-level factors, which can be used to run an $OA(16; 8 \times 2^{n-1})$ design in two blocks of size eight or an $OA(16; 2^n)$ design in eight blocks of size two. The only non-isomorphic $OA(16; 8 \times 2^n)$ designs with three, five, six, seven or eight two-level factors can be constructed by folding over an eight-run full factorial or fractional factorial design,

Table 10: Optimal blocking arrangements based on parent designs of the type $OA(16; 4 \times 2^n)$. Elements of the FA_3 vector correspond to A_3 values of 1, 1/2, and 1/4. An 'x' indicates W_1- and W_2-optimal arrangements; an 'x$_1$' indicates W_1-optimal arrangements; an 'x$_2$' indicates W_2-optimal arrangements; a 'y' indicates W_1^-- and W_2^--optimal arrangements; a 'y$_1$' indicates W_1^--optimal arrangements; a 'y$_2$' indicates W_2^--optimal arrangements; a 'z' indicates W_3-optimal arrangements. Two-level designs in four blocks marked with an asterisk, and all mixed-level designs in two-blocks, were not found previously.

n	ID	R_1	R_2	A_3	A_4		FA_3		2 blocks	4 blocks
12	1	1	1	34	87	6	24	64	1:1;xyz	xyz
11	1	1	1	27	62	3	24	48	1:1;xyz	xyz
10	1	1	1	21	42	1	24	32	1:11;z	xyz
	2	16	2	21	43	3	18	36	5:1;xy	
9	1	1	3	15	30	3	12	24	1:1;x	
	2	11	1	16	26	0	24	16	1:37;z	xyz
	3	23	2	16	28	1	18	24	4:1;y	
8	1	1	3	11	19	1	12	16		
	2	3	11	11	19	3	8	16	5:1;x	
	3	34	2	12	14	0	24	0		xyz
	4	37	1	12	16	0	18	12	2:5;z	
	5	66	28	12	18	4	8	16	1:1;y	
7	1	1	24	7	13	3	4	8	3:1;xy	
	2	3	6	7.5	12	1	8	10	1:3;z	
	3	5	1	8	10	0	12	8		
	4	64	5	9	7	0	18	0		xyz
6	1	1	46	3	11	3	0	0	1:2;x	x
	2	3	1	4.5	8	0	6	6		
	3	4	3	5	6	0	8	4	2:1;yz	
	4	46	7	6	3	0	12	0		z
	5	71	47	7	3	3	8	0		y*
5	1	1	1	2	5	0	4	0	1:1;xz	x$_2$z
	2	2	30	2	5	2	0	0		
	3	6	6	3	3	0	6	0		x$_1$y$_1$*
	4	43	48	4	2	4	0	0	1:2;y	
	5	50	38	6	1	6	0	0		y$_2$*
4	1	1	1	1	2	0	2	0	1:1;xz	xz
	2	2	11	1	2	1	0	0		
	3	15	21	2	1	2	0	0	1:1;y	
3	1	1	1	0	1	0	0	0	1:1;xz	xz

and linking each fold-over pair to a different level of the eight-level factor. There are two non-isomorphic OA($16; 8 \times 2^4$) designs, both of which have an A_3 value of 6. The two-level part of one of these designs is a replicated half fraction of the 2^4 design. Therefore, this design has an A_4 value of 1. The other design, which is the better of the two, has a full factorial 2^4 design for the two-level factors, which results in a zero A_4 value. For that design, the eight-level factor can be constructed using any three two-factor interactions of the two-level factors.

In each case, there is only one possible way to obtain an orthogonally blocked design involving two blocks of size eight from the OA($16; 8 \times 2^n$) design. That arrangement can be obtained by using any two-level factor as the blocking factor. In case $n = 4$, the best blocking pattern is obtained from the OA($16; 8 \times 2^4$) with a zero A_4 value. For constructing 16-run designs involving eight blocks of size two, there is only one possibility, that is by using the eight-level factor as the blocking factor. In case $n = 4$, it is again best to use the OA($16; 8 \times 2^4$) design with a zero A_4 value.

5.3 Designs with 18 runs

In this subsection, we provide optimal blocking arrangements of 18-run designs. These designs permit the study of up to seven three-level factors, and an additional two-level factor, and the study of one six-level factor with up to six three-level factors. The designs, denoted by OA($18; 3^n \times 2$), OA($18; 3^n$) and OA($18; 6 \times 3^n$), have been studied extensively before. Schoen (2009) provides a recent overview of the literature on 18-run designs and a comprehensive characterization of the designs. The blocking of these 18-run designs has,

Table 11: Properties of two-level child designs obtained from parent designs of the type OA($16; 4 \times 2^n$). Elements of the FA_3 vector correspond to A_3 values of 1, 1/2, and 1/4. The column labeled R_1^p gives the R_1 ranks of the parent designs in Table 10.

n	R_1^p	A_3	A_4	FA_3		
12	1	16	39	0	0	64
11	1	12	26	0	0	48
10	1	8	18	0	0	32
9	11	4	14	0	0	16
8	34	0	14	0	0	0
7	64	0	7	0	0	0
6	1	0	3	0	0	0
	46	0	3	0	0	0
	71	0	3	0	0	0
5	1	0	1	0	0	0
	6	0	0	0	0	0
	50	0	1	0	0	0
4	1	0	0	0	0	0
3	1	0	0	0	0	0

however, not received any attention in the literature. The optimal blocking arrangements we obtained are given in Tables 12–14.

Table 12 shows the best blocking arrangements obtained from designs of the type $OA(18; 3^n \times 2)$. These designs can be used to arrange $OA(18; 3^n)$ designs in two blocks and to arrange $OA(18; 3^{n-1} \times 2)$ designs in three blocks. Remarkably, the parent designs that give the best arrangements in two blocks are different from the parent designs that give the best arrangements in three blocks in all but one case. Another noteworthy result is that Taguchi's L_{18} array, which is the design with $n = 7$ and $R_1 = 3$ in Table 12, cannot be used to obtain an optimal arrangement in two blocks. The child designs of the optimal arrangements in two blocks are the three-level designs with $R_1 = 1$ in Table 13. Table 12 shows that the corresponding parent designs also have $R_1 = 1$ for $n = 7$ and 6, but, for $n = 5$ and 4, R_1 equals 5 and 4, respectively. Finally, the optimal arrangement in three blocks derived from a parent design with $n = 3$ has a full factorial $3^2 \times 2$ child design. There is no extensive confounding of the two-factor interactions with the blocks for this arrangement. Since two degrees of freedom are available for estimating the residual error variance in case a model including two-factor interactions is estimated, we do not recommend using the W_1^-- or W_2^--criterion in this scenario.

Except for the case where $n = 7$ and $R_1 = 1$ and the case where $n = 4$ and $R_1 = 4$, all the designs in Table 12 can be derived from Taguchi's L_{18} array by dropping one or more columns.

Optimal arrangements of pure three-level 18-run designs in three blocks of size six are given in Table 13. For all designs, the first column is used for the blocking factor. The child designs of all optimal blocking

Table 12: Optimal blocking arrangements based on parent designs of the type $OA(18; 3^n \times 2)$. Elements of the FA_3 vector correspond to A_3 values of 2, 1, 2/3, 1/2, and 4/9. An 'x' indicates W_1- and W_2-optimal arrangements; a 'y' indicates W_1^-- and W_2^--optimal arrangements; a 'z' indicates W_3-optimal arrangements.

n	ID	R_1	R_2	A_3	A_4			FA_3			2 blocks	3 blocks
7	1	1	1	28	52.5	1	0	21	16	9	1:1;xyz	
	2	3	3	28	52.5	1	6	9	28	0		1:1;xyz
6	1	1	1	16	28.5	0	0	9	20	0	1:1;xyz	
	2	3	3	16.67	25.83	0	0	15	8	6		1:3;z
	3	5	5	17	24.5	0	6	6	14	0		3:1;xy
5	1	1	11	8.5	12	0	3	3	7	0		5:1;x
	2	3	1	8.67	11.5	0	0	6	4	6		
	3	4	2	8.78	11.17	0	0	6	6	4		1:4;z
	4	5	3	9	10.5	0	0	6	10	0	1:1;xyz	
	5	19	13	10	7.5	0	6	3	4	0		3:1;y
4	1	1	33	3.5	4.5	0	3	0	1	0		1:1;xz
	2	4	1	3.78	3.94	0	0	0	4	4	1:1;xz	
	3	10	8	4	3.5	0	0	3	2	2		1:1;y
	4	26	12	4.67	2.17	0	0	4	4	0	1:1;y	
3	1	1	1	0.5	1.5	0	0	0	1	0	1:1;xz	1:1;xz
	2	10	8	1.83	0.17	0	0	2	1	0	1:1;y	

arrangements have $R_1 = R_2 = 1$. The seven-factor design with $R_1 = R_2 = 1$ in Table 13 does not yield any optimal blocking pattern, but it is included in the table because it is the child design of the first optimal blocking arrangement involving two blocks in Table 12.

Finally, Table 14 gives optimal blocking arrangements of $OA(18; 3^n)$ designs in six blocks of size three and of $OA(18; 6 \times 3^{n-1})$ designs in three blocks of size six. Dropping any three-level column of any design in Table 14 results in the same projection, so that every possible arrangement in three blocks is optimal with respect to each of our five criteria. As there is only one six-level factor, using that factor as a blocking factor yields designs that are optimal with respect to each of our five criteria as well.

5.4 Designs with 20 runs

In this section, we discuss optimal blocking arrangements for 20-run designs. Optimal blocking arrangements of pure two-level $OA(20; 2^n)$ designs in two blocks of size ten are given by CLY. In Table 15, we present optimal arrangements of $OA(20; 2^n)$ designs in five blocks of size four, and optimal arrangements of $OA(20; 5 \times 2^n)$ designs in two blocks of size ten. In Table 16, we give the A_3 and A_4 values and the FA_3 vector of

Table 13: Optimal blocking arrangements based on parent designs of the type $OA(18; 3^n)$. Elements of the FA_3 vector correspond to A_3 values of 2, 1, 2/3, and 1/2. An 'x' indicates W_1- and W_2-optimal arrangements; a 'y' indicates W_1^-- and W_2^--optimal arrangements; a 'z' indicates W_3-optimal arrangements.

n	ID	R_1, R_2	A_3	A_4		FA_3			3 blocks
7	1	1	22.0	34.5	1	0	18	16	
	2	3	22.0	34.5	1	6	0	28	1:1;xyz
6	1	1	10.0	22.5	0	0	0	20	1:1;xz
	2	8	13.0	13.5	1	3	0	16	1:1;y
5	1	1	5.0	7.5	0	0	0	10	1:1;xz
	2	10	8.0	1.5	0	6	0	4	1:1;y
4	1	1	2.0	1.5	0	0	0	4	1:1;xz
	2	9	3.5	0.0	1	0	0	3	1:1;y
3	1	1	0.5	0.0	0	0	0	1	1:1;xyz

Table 14: Optimal blocking arrangements based on parent designs of the type $OA(18; 6 \times 3^n)$. Elements of the FA_3 vector correspond to A_3 values of 2 and 1/2. An 'x' indicates W_1- and W_2-optimal arrangements; a 'y' indicates W_1^-- and W_2^--optimal arrangements; a 'z' indicates W_3-optimal arrangements.

n	R_1	A_3	A_4	FA_3		3 blocks	6 blocks
6	1	40.0	52.5	15	20	1:1;xyz	1:1;xyz
5	1	25.0	22.5	10	10	1:1;xyz	1:1;xyz
4	1	14.0	7.5	6	4	1:1;xyz	1:1;xyz
3	1	6.5	1.5	2	2	1:1;xyz	1:1;xyz
2	1	2.0	-	1	0	1:1;xyz	1:1;xyz

the two-level child designs obtained by dropping the five-level column from the parent designs that give the optimal arrangements in blocks of size four.

Table 15 shows one case where the W_1^--optimal blocking arrangement differs from the W_2^--optimal one. For six two-level factors ($n = 6$) and five blocks, the design with $R_1 = 3$ allows a blocking arrangement with A_3^c, A_4^c, A_{21} and A_{31} values of 0.8, 0.6, $4.8 - 0.8 = 4$ and $5.8 - 0.6 = 5.2$, respectively, while the design ranked ninth allows an arrangement with A_3^c, A_4^c, A_{21} and A_{31} values of 0.8, 0.92, $5.2 - 0.8 = 4.4$ and $4.92 - 0.92 = 4$, respectively. The former blocking arrangement is W_1^--optimal (because it has a smaller A_4^c value), whereas the latter is W_2^--optimal (because it has a larger A_{21} value).

Table 15: Optimal blocking arrangements based on parent designs of the type $OA(20; 5 \times 2^n)$. Elements of the FA_3 vector correspond to A_3 values of 1, 4/5, 2/5, 9/25, and 1/25. An 'x' indicates W_1- and W_2-optimal arrangements; a 'y' indicates W_1^-- and W_2^--optimal arrangements; a 'y_1' indicates W_1^--optimal arrangements; a 'y_2' indicates W_2^--optimal arrangements; a 'z' indicates W_3-optimal arrangements.

n	ID	R_1	R_2	A_3	A_4			FA_3			2 blocks	5 blocks
8	1	1	1	10.88	16.24	0	0	20	2	54	1:1;xyz	xyz
7	1	1	1	7.72	9.64	0	0	15	1	34	2:6;xz	xyz
	2	2	2	8.04	9.32	0	0	15	2	33	1:6;y	
6	1	1	1	4.72	6.2	0	0	9	1	19		
	2	3	5	4.8	5.8	0	0	10	0	20	1:1;xyz	xy_1z
	3	6	5	5.12	5.4	0	0	10	1	19		
	4	9	12	5.2	4.92	0	0	11	0	20		y_2
5	1	1	1	2.4	3.8	0	0	5	0	10	1:1;xz	xz
	2	9	10	3.2	2.6	0	0	7	0	10	1:1;y	
	3	16	18	3.6	1.8	0	2	4	0	10		y
4	1	1	1	1.36	1.48	0	0	3	0	4	1:3;xz	xz
	2	9	5	1.76	1.08	0	0	4	0	4	1:3;y	
3	1	1	1	0.04	0.96	0	0	0	0	1	1:1;xz	xz
	2	3	3	0.44	0.56	0	0	1	0	1		
2	1	1	1	0	-	0	0	0	0	0	1:1;xz	xz

Table 16: Properties of two-level child designs obtained from parent of designs of the type $OA(20; 5 \times 2^n)$. Elements of the FA_3 vector correspond to A_3 values of 1, 4/5, 2/5, 9/25, and 1/25. The column labeled R_1^p gives the R_1 ranks of the parent designs in Table 15

n	R_1^p	A_3	A_4			FA_3		
8	1	2.88	4.72	0	0	0	2	54
7	1	1.72	2.36	0	0	0	1	34
6	3	0.80	0.60	0	0	0	0	20
	9	0.80	0.92	0	0	0	0	20
5	1	0.40	0.20	0	0	0	0	10
	16	0.40	0.20	0	0	0	0	10
4	1	0.16	0.04	0	0	0	0	4
3	1	0.04	-	0	0	0	0	1

Finally, we neither give W_1^--optimal or W_2^--optimal five-block arrangements for up to four two-level factors, nor W_1^--optimal or W_2^--optimal two-block arrangements with up to two two-level factors and one five-level factor, as it is possible to fit a model with all main effects and two-factor interaction effects in these cases.

5.5 Designs with 27 runs

Three-level designs with 27 runs can be used to study the effects of up to 13 factors. There are 68 non-isomorphic 13-factor designs in 27 runs (Lam and Tonchev, 1996). Their projective properties in the absence of blocking have been extensively studied by Xu et al. (2004). To find the optimal blocking arrangements, we started from a complete enumeration of all non-isomorphic designs for four up to 13 factors. While evaluating these designs, we encountered several 27-run designs that have better FA_3 vectors than the designs listed in Xu et al. (2004). We discuss this in detail in Appendix B. In this section, we focus on the best blocking arrangements.

Table 17 lists the optimal blocking arrangements for 27-run three-level designs in three blocks of size nine. One remarkable result is that, in five cases, three different parent designs are needed to cover the five blocking optimality criteria. Another remarkable result is that the child designs of the W_1-, W_2-, W_1^-- and W_2^--optimal blocking arrangements for $n = 6$ have $R_1 = 4$ only. This clearly demonstrates that starting from the best child design does not guarantee finding an optimal blocking arrangement.

We obtained similar results for the W_3-optimal blocking arrangements of ten-factor, eight-factor and seven-factor child designs. The child design of the W_3-optimal arrangement obtained from the 11-factor parent design with $R_1 = R_3 = 3$ is only 275th in the R_1 ranking and only 5th in the R_2 ranking of all ten-factor designs. This is due to the fact that none of the ten-factor designs with a better R_1 or R_2 ranking can be obtained by projection from an 11-factor design. Therefore, they cannot be arranged in blocks so that the main effects can be estimated independently from the block effects.

The eight-factor child design corresponding to the W_3-optimal blocking arrangement obtained from the nine-factor parent design with $R_1 = 2645$ and $R_2 = 1$ is 5259th in the R_1 ranking and third in the R_2 ranking of all eight-factor designs. Neither the first nor the second design in the R_2 ranking of all eight-factor designs can be obtained from a nine-factor parent design by projection. Finally, the seven-factor child design corresponding to the W_3-optimal blocking arrangement obtained from the eight-factor parent design with $R_1 = 7153$ and $R_2 = 360$ is 2903rd in the R_1 ranking and 89th in the R_2 ranking of all seven-factor designs. None of the seven-factor designs with a better R_1 or R_2 ranking can be obtained by projection from an eight-factor design, and, hence, no desirable blocking arrangement exists for them.

In all other cases, the child designs corresponding to optimal blocking arrangements are listed first in

the R_1 or R_2 rankings. So, in many cases, it is possible to find desirable blocking arrangements for the best possible child designs.

Table 18 lists the orthogonal blocking arrangements obtained from designs of the type $\text{OA}(27; 9 \times 3^n)$. These designs allow us to arrange $\text{OA}(27; 3^n)$ designs in nine blocks of size three, and to arrange $\text{OA}(27; 9 \times 3^{n-1})$ designs in three blocks of size nine. For the arrangements in three blocks of size nine, the characteristics of the child designs are listed in Table 18 too. For the arrangements in nine blocks of size three, the child

Table 17: Optimal blocking arrangements based on parent designs of the type $\text{OA}(27; 3^n)$. Elements of the FA_3 vector correspond to A_3 values of 2, 20/27, 2/3, 14/27, 4/9, and 8/27. An 'x' indicates W_1- and W_2-optimal arrangements; a 'y' indicates W_1^-- and W_2^--optimal arrangements; a 'z' indicates W_3-optimal arrangements.

n	ID	R_1	R_2	A_3	A_4			FA_3				3 blocks
13	1	1	1	104	468	0	0	52	0	156	0	1:1;xyz
12	1	1	1	80	324	0	0	40	0	120	0	1:1;xyz
11	1	1	1	60	216	0	0	30	0	90	0	1:275;z
	2	78	78	60	216	2	0	40	0	60	9	7:1;xy
10	1	1	607	42	144	2	0	33	0	36	0	
	2	6	1272	42	144	3	0	36	18	0	9	4:1;xy
	3	275	5	43.33	138.67	0	0	21	0	66	0	
	4	1214	1	44	136	0	0	3	27	27	54	1:2645;z
9	1	1	4885	24	108	3	0	27	0	0	0	1:1;x
	2	680	4003	30	84	2	0	25	14	0	7	4:1;y
	3	2645	1	30.44	82.67	0	0	2	18	21	36	1:5259;z
8	1	1	13835	16	60	2	0	18	0	0	0	1:1;x
	2	2530	1	19.85	47.41	0	0	0	16	12	21	
	3	5259	3	20.3	45.85	0	0	1	11	16	23	
	4	7153	360	20.52	45.41	0	1	1	12	15	21	1:2903;z
	5	14783	14394	22	42	3	0	24	0	0	0	1:1;y
7	1	1	71407	10	30	1	0	12	0	0	0	
	2	3	77742	10	30	2	0	9	0	0	0	5:1;x
	3	2903	89	12	24.44	0	0	0	10	15	6	
	4	7745	7	12.3	23.93	0	0	0	4	7	24	1:1944;z
	5	8841	1	12.3	25.04	0	0	0	0	15	19	
	6	78098	78908	16	18	8	0	0	0	0	0	1:1;y
6	1	1	62510	4	18	2	0	0	0	0	0	1:4;x
	2	7	15037	5.33	14	0	0	8	0	0	0	
	3	599	67	10	6	0	0	0	1	4	13	4:42;z
	4	1944	1	6.15	11.78	0	0	0	0	3	17	
	5	62548	62550	6.37	11.56	5	0	0	0	0	0	1:4;y
5	1	1	849	2	6	0	0	1	0	3	0	
	2	2	2110	2	6	0	0	3	0	0	0	1:1;x
	3	4	6060	2	6	1	0	0	0	0	0	
	4	42	1	2.67	4.67	0	0	0	0	0	9	
	5	44	66	2.67	4.67	0	0	0	0	6	0	1:1;z
	6	49	2362	2.67	4.67	0	0	4	0	0	0	
	7	2591	6111	4	2	2	0	0	0	0	0	1:1;y
4	1	1	1	0	2	0	0	0	0	0	0	1:1;xz

designs' A_3 and A_4 values and their FA_3 vector are listed in Table 17.

All but one W_1-, W_2-, W_1^-- and W_2^--optimal blocking arrangements have child designs with $R_1 = 1$. The only exception is for $n = 5$ and nine blocks. The 3^5 child design for that case has an FA_3 vector of $(1, 0, 0, 0, 0, 0)$, while the best 5-factor design in the R_1 ranking has an FA_3 vector of $(0, 0, 1, 0, 3, 0)$. Note from the list of six-factor designs in Table 17 that the latter design cannot be blocked orthogonally in three blocks either.

Most W_3-optimal blocking arrangements in Table 18 have child designs with $R_2 = 1$. The two exceptions are the W_3-optimal arrangements for six and five three-level factors and nine blocks. The six-factor child design has $R_2 = 15037$, and the five-factor child design has $R_2 = 2362$. Clearly, there is a high price to be paid here for orthogonal blocking in nine blocks in terms of the R_2 ranking of the child designs.

6   Discussion

In this paper, we proposed a simple method to arrange orthogonal designs of resolution 3 or strength 2 in blocks. The starting point of our method is an exhaustive catalog of non-isomorphic parent designs. We then use single replacement of a column in the designs by the blocking factor.

The method works well even when the catalogs have tens of thousands of arrays. However, it is computationally infeasible for cases involving even larger numbers of arrays. For instance, for 24 runs, there are 52,912,678 non-isomorphic designs for 11 two-level factors and 51,154,497 non-isomorphic designs for

Table 18: Optimal blocking arrangements based on parent designs of the type $\mathrm{OA}(27; 9 \times 3^n)$. Elements of the FA_3 vector correspond to A_3 values of 2 and 2/3. An 'x' indicates W_1- and W_2-optimal arrangements; a 'y' indicates W_1^-- and W_2^--optimal arrangements; a 'z' indicates W_3-optimal arrangements.

n	ID	R_1	R_2	A_3	A_4	FA_3		3 blocks	9 blocks
9	1	1	1	96	252	39	27	1:1;xyz	1:1;xyz
8	1	1	1	72	156	30	18	1:1;xyz	1:1;xyz
7	1	1	1	52	90	22	12	1:3;z	1:1;xyz
	2	3	3	52	90	23	9	7:1;xy	
6	1	1	6	34	54	17	0	1:1;x	1:1;xy
	2	3	1	35.33	48.67	15	8	1:3;z	1:7;z
	3	7	7	36	46	18	0	6:1;y	
5	1	1	4	22	24	11	0	1:1;xz	1:4;xy
	2	3	1	22.67	22	10	4		1:49;z
	3	7	7	24	18	12	0	1:1;y	
4	1	1	1	12	10	6	0	1:1;xz	1:1;xyz
	2	5	5	14	6	7	0	1:1;y	
3	1	1	1	6	2	3	0	1:1;xz	1:1;xyz
	2	5	5	8	0	4	0	1:1;y	
2	1	1	1	2	-	1	0	1:1;xyz	1:1;xyz

12 two-level factors (Schoen et al., 2010). These numbers are too large to identify all optimal blocking arrangements with our method.

Applying our method will also be problematic for finding optimal blocking arrangements of resolution-4 or strength-3 orthogonal designs. This is because it is sufficient for the parent design to have resolution 3, but the child design must have resolution 4. For example, to arrange a resolution-4 design of the type $OA(54; 3^5)$ in three blocks of size 18, we would need a complete catalog of all resolution-3 non-isomorphic parent designs of the type $OA(54; 3^6)$. Not only is generating such a catalog computationally very intensive, but the resolution of every possible child design would have to be determined as well. This is also computationally demanding. As a result, for designs with a resolution of 4 or more, alternative blocking strategies are needed.

Appendix A. Comparison with Cheng et al. (2004)

We listed the W_1-, W_2-, W_1^--, W_2^-- and W_3-optimal arrangements of 16-run two-level designs in four blocks of size four in Table 10. In this appendix, we compare our results with those of Cheng et al. (2004, CLY).

A1. The FA_3 vector of designs of the type $OA(16; 4 \times 2^n)$

At first sight, there is a discrepancy between our work and that of CLY. This is due to the fact that our method sometimes results in an FA_{21} vector with a non-zero entry for $A_3 = 0.5$, while the possible A_3 values of three-factor designs are either 1, 0.25, or 0 for pure two-level designs in 16 runs. Therefore, none of the designs in CLY have A_3 values of 0.5.

As an example, consider the difference of the FA_3 vector of the $OA(16; 4 \times 2^n)$ parent design with ID 1 in Table 10 with the corresponding vector of the child design in Table 11. In our approach, the FA_{21}-vector is $(6 - 0, 24 - 0, 64 - 64)$ or $(6, 24, 0)$, with elements corresponding to the A_3 values 1, 0.5 and 0.25. In the approach of CLY, the same blocked design has an FA_{21}-vector of $(6, 48)$, with elements corresponding to the A_3 values 1 and 0.25. It turns out that CLY's frequency for an A_3 value of 0.25 is the sum of our frequency for an A_3 value of 0.25 and twice our frequency for an A_3 value of 0.5.

To see this, it is important to realize that we started from a catalog of non-isomorphic orthogonal 16-run designs, involving n two-level factors and one four-level factor. We applied the single replacement method to this catalog of designs. CLY, however, started from a catalog of two-level designs, replacing two two-level factors with a four-level blocking factor.

In our approach, we encountered two types of three-factor projections. The first type involved three two-level factors, whereas the second type involved two two-level factors and the four-level factor. It is the

Table 19: A_3 values and FA_3 vectors of the three possible non-isomorphic projections into two two-level factors and one four-level factor.

Projection		This Paper			CLY		
	A_3	1	0.5	0	1	0.25	0
(i)	FA_3	0	0	1	0	0	4
(ii)	FA_3	0	1	0	0	2	2
(iii)	FA_3	1	0	0	1	0	3

second type of projection that causes the difference between our approach and that of CLY.

There are three non-isomorphic three-factor projections involving the four-level factor. If we denote the two-level pseudo-factors corresponding to the four-level factor by a_1 and a_2 and the ordinary two-level factors by B and C, then the three-factor projections are either (i) a full factorial 4×2^2 design, (ii) an eight-run fractional factorial design with defining relation $I = a_1BC$ combined with an eight-run fractional factorial design with defining relation $I = a_2BC$, and (iii) a duplicated eight-run fractional factorial design with defining relation $I = a_1BC$. The A_3 values of these projections are 0, 0.5, and 1, respectively.

We illustrate the calculation of these A_3 values using projection (ii), because this is where the difference between our approach and that of CLY occurs. The A_3 value is obtained as follows. The sum of the element-wise products of the contrast columns a_1, B and C equals 8 because the three-factor interaction contrast vector a_1BC is aliased with the intercept in one half of the design and orthogonal to the intercept in the other half of the design. Following the definition of the A_3 value in Xu and Wu (2001), this contributes an amount of $(8/16)^2 = 0.25$ to the A_3 value of the projection. Similarly, the sum of the element-wise products of the contrast columns a_2, B and C equals 8 because the three-factor interaction contrast vector a_2BC is also aliased with the intercept in one half of the design and orthogonal to the intercept in the other half of the design. This also contributes an amount of $(8/16)^2 = 0.25$ to the A_3 value of the projection. Finally, the element-wise products of the contrast columns a_1, a_2 and B is zero, as is the sum of the element-wise product of the contrast columns a_1, a_2 and C. As a result, the total A_3 value for this projection is 0.25+0.25+0+0=0.5.

In contrast with our approach, CLY do not sum the four A_3 values obtained from a_1, a_2, B and C. So, for the projection (ii) into one four-level factor and two two-level factors, they use the four A_3 values of 0, 0, 0.25 and 0.25 separately. As a result, they do not have a value of 0.5.

The FA_3 vectors and A_3 values for the projection (i) into a full factorial design and the projection (iii) into a duplicated fractional factorial design can be derived in a similar fashion. For each of the projections, we compare our approach with the approach of CLY in Table 19. We conclude that the frequency of the A_3 value of 0.25 in CLY can be obtained by adding twice our frequency for an A_3 value of 0.5 to our frequency

for an A_3 value of 0.25. The frequency of the A_3 value of 1 in our approach matches that in CLY's approach.

A2. Design comparison

For each of the cases with 12, 9, 8 and 7 two-level factors, Table 10 contains one optimal arrangement in four blocks. In contrast, CLY report two blocking arrangements, because they consider rankings based on the A_i values entirely separately from rankings based on the FA_i vectors. Their first arrangement in four blocks is best according to their A_i-based ranking, and their second arrangement is optimal according to both their A_i-based ranking and their FA_i-based ranking. Our R_1 ranking, which forms the basis for selecting W_1- and W_2-optimal designs, uses the FA_3 vector to distinguish between designs that have the same A_3 and A_4 values. Therefore, the 12-, 9-, 8- and 7-factor designs we report correspond to the designs that are optimal according to CLY's FA_i-based criterion.

For the 11-factor case involving four blocks, our Table 10 again contains a single blocking arrangement, whereas CLY give two designs. Here too, our design corresponds to the second one listed by CLY. We did not identify their first design because we did not use the FA_4 vector in our work. This is because we found that the different A_4 values of four-factor projections in designs with factors acting at more than two levels can be very close to each other. Therefore, the practical importance of a classification based on an FA_4 vector is questionable for these designs. Note that two-level designs, as opposed to multi-level designs, can only have a few clearly distinct A_4 values, and classification based on FA_4 is indeed useful.

For the 10-factor case involving four blocks, CLY list three blocking arrangements. Our Table 10 reports their third design only. For the reason why we did not identify their first design, we refer to the 12-, 9-, 8- and 7-factor cases. For the reason why we did not identify their second design, we refer to the 11-factor case.

For the case involving six factors, we identified a W_1- and W_2-optimal blocking arrangement, as well as a design that gives a W_3-optimal blocking arrangement. Both designs were also given by CLY. Compared to CLY, we identify an additional blocking arrangement which is W_1^-- and W_2^--optimal. The W_1^-- and W_2^--criteria were not used by CLY, which explains why they did not report that blocking arrangement. Note that, in all cases with more than six factors, the W_1-, W_2-, W_1^--, W_2^-- and W_3-optimal blocking arrangements coincide. This is no longer the case when the number of factors is at most six.

For the case involving five factors, CLY list four blocking arrangements, while we report only three. Only one of our designs is listed by CLY. The first and third blocking arrangement in CLY are both W_1-optimal, provided only A_i values are used to rank the blocking arrangements. Our approach to identify W_1-optimal blocking arrangements is based on A_3 and A_4 values, and on the FA_3 vector to break possible ties, and resulted in a single W_1-optimal blocking arrangement that outperforms those identified by CLY;

see the discussion of Table 10 in Section 5.2. That blocking arrangement is also optimal in terms of the W_1^--criterion.

The second and fourth five-factor blocking arrangement given by CLY are W_2-optimal. The second blocking arrangement is only optimal when the ranking is based only on A_i values, while the fourth design is optimal both when using A_i values and FA_i vectors. Our mixed ranking identified a W_2-optimal design that corresponds with the fourth CLY design. Note that the FA_{21} and FA_{31} vectors for the second five-factor CLY design are incorrect. The correct FA_{21} vector is (2,0) and the correct FA_{31} vector is (4,0), where the elements correspond to the A_3 values 1 and 0.25, respectively, following the notation of CLY.

Finally, our third five-factor blocking arrangement is based on the W_2^-- criterion, which was not considered by CLY.

Appendix B. Comparison with Xu et al. (2004)

Xu et al. (2004) studied the A_3 values and the FA_3 vectors of all projections from the 68 non-isomorphic $OA(27; 3^{13})$ designs reported by Lam and Tonchev (1996) as well as of 1000 randomly generated orthogonal designs with 5 to 10 factors, and listed the best ones. In our search for optimal blocking arrangements, we used the complete enumeration of all 27-run non-isomorphic orthogonal designs instead. For this reason, we were able to identify several designs with better properties in the absence of blocking than those listed by Xu et al. (2004).

Our global search confirmed that Xu et al. (2004) found designs with minimum A_3 values. For each number of factors, n, in Table 17, the minimum A_3 value is given in the row with $R_1 = 1$. For the case of 10 factors, we were able to find several designs that have a better FA_3 vector than the best design reported by Xu et al. (2004). Their design has a frequency of three for $A_3 = 2$, a frequency of nine for $A_3 = 30/27$ and a zero frequency for $A_3 = 20/27$ among all three-factor projections. Two of the 10-factor designs we found with better FA_3 vectors are shown in Table 17. The one with $R_1 = 1$ has a frequency of two for $A_3 = 2$, a zero frequency for $A_3 = 30/27$ and a frequency of 33 for $A_3 = 20/27$. The one with $R_1 = 6$ has a frequency of three for $A_3 = 2$, a zero frequency for $A_3 = 30/27$ and a frequency of 36 for $A_3 = 20/27$. Note that none of the three-factor projections for any of the designs in Table 17 resulted in an A_3 value of 30/27. Finally, all the designs with 5-10 factors that are first in the R_2 ranking in Table 17 have a better FA_3 vector than the corresponding designs reported by Xu et al. (2004).

In conclusion, by going through the complete catalog of all non-isomorphic 27-run three-level designs, we found several attractive designs that were hitherto not known.

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