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Optimal condition for non-simultaneous blow-up in a reaction-diffusion system

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Abstract. We study the positive blowing-up solutions of the semilinear parabolic system: $u_t - \Delta u = v^p + u^r$, $v_t - \Delta v = u^q + v^s$, where $t \in (0, T)$, $x \in \boldsymbol{R}^N$ and $p, q, r, s > 1$. We prove that if $r > q+1$ or $s > p+1$ then one component of a blowing-up solution may stay bounded until the blow-up time, while if $r < q+1$ and $s < p+1$ this cannot happen. We also investigate the blow up rates of a class of positive radial solutions. We prove that in some range of the parameters p, q, r and s, solutions of the system have an uncoupled blow-up asymptotic behavior, while in another range they have a coupled blow-up behavior.

1. Introduction.

In this paper we are concerned with the study of the behavior of positive blowing-up solutions of the semilinear parabolic system:

$$\begin{cases} u_t - \Delta u = v^p + u^r, & 0 < t < T,\ x \in \boldsymbol{R}^N, \\ v_t - \Delta v = u^q + v^s, & 0 < t < T,\ x \in \boldsymbol{R}^N, \end{cases} \tag{1}$$

where $u = u(t, x)$, $v = v(t, x)$. Throughout this paper, we assume that $p, q, r, s > 1$, N is a positive integer and we consider initial data

$$u(0, x) = u_0(x), \quad v(0, x) = v_0(x), \tag{2}$$

where u_0, v_0 are assumed to be nonnegative and bounded. Also $\|\cdot\|_\infty$ will denote the norm in $L^\infty(\boldsymbol{R}^N)$.

It is known that (1)–(2) has a unique nonnegative maximal solution on $[0, T) \times \boldsymbol{R}^N$, classical for $t > 0$. This follows by standard contraction mapping argument. Moreover, if $T < \infty$, then

$$\lim_{t \nearrow T} (\|u(t)\|_\infty + \|v(t)\|_\infty) = \infty, \tag{3}$$

and we say that the solution blows up in finite time with blow-up time T. We say that the blow-up is *simultaneous* if

$$\limsup_{t \nearrow T} \|u(t)\|_\infty = \limsup_{t \nearrow T} \|v(t)\|_\infty = \infty \tag{4}$$

and that it is *non-simultaneous* if (4) does not hold, i.e. if one of the two components remains bounded on $[0, T) \times \boldsymbol{R}^N$.

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System (1) can be viewed as a combination of the following two systems

$$\begin{cases} u_t - \Delta u = v^p, & 0 < t < T,\ x \in \boldsymbol{R}^N, \\ v_t - \Delta v = u^q, & 0 < t < T,\ x \in \boldsymbol{R}^N, \end{cases} \tag{5}$$

and

$$\begin{cases} u_t - \Delta u = u^r, & 0 < t < T,\ x \in \boldsymbol{R}^N, \\ v_t - \Delta v = v^s, & 0 < t < T,\ x \in \boldsymbol{R}^N. \end{cases} \tag{6}$$

It is well known [**7**] that (5) has only simultaneous blowing-up solutions. Of course, since (6) is completely uncoupled, non-simultaneous blowing-up solutions clearly exist for (6). It is therefore natural to ask whether the blow-up is simultaneous or not for the system (1).

Intuitively, one might expect that only simultaneous blow-up should occur if r, s are small as compared with p, q in some sense, and that non-simultaneous blow-up could occur in the opposite case. An almost complete answer to this question is given in the following theorem, which is the main result of this paper.

Theorem 1 (Simultaneous or non-simultaneous blow-up). *Let* (u, v) *be a positive blowing-up solution of* (1)–(2).

(i) *If* $r < q + 1$ *and* $s < p + 1$, *then only simultaneous blow-up occurs, that is* (4) *holds.*

(ii) *If* $r > q + 1$ *or* $s > p + 1$, *then there exist* u_0 *and* v_0 *such that non-simultaneous blow-up occurs.*

In Proposition 2.1 below we indicate which component of the solution (u, v) must blow-up in the case of non-simultaneous blow-up. The phenomenon of non-simultaneous blow-up for parabolic systems seems to have first been suggested in [**23**] (see [**23**, pp. 467–472]). It was observed numerically there for a quasilinear system coupled by products of power nonlinearities and explicitly computed in the spatially homogeneous case (system of ODE's). Further mathematical study was carried out in [**20**], [**21**] for a system of two porous medium equations coupled by nonlinear boundary conditions, and for a semilinear parabolic system in $\boldsymbol{R}^N$ coupled by products of power nonlinearities.

Our second aim in this paper is to investigate the blow-up rates of solutions of (1). The blow-up rate is essentially known in the scalar case (6) [**26**], [**11**], [**13**] and for the purely coupled system (5) (see [**1**], [**5**], [**9**], and also [**4**], [**6**] for the bounded domain case and [**10**] for general unbounded domains). More precisely, for the scalar equation $u_t - \Delta u = u^r$ in $\boldsymbol{R}^N$ it always holds

$$C_1(T - t)^{-1/(r-1)} \le \|u(t)\|_\infty \le C_2(T - t)^{-1/(r-1)}$$

assuming $(N - 2)r < N + 2$, see [**13**]. On the other hand, for large classes of solutions of (5), it holds

$$\|u(t)\|_\infty \le C(T - t)^{-(p+1)/(pq-1)}$$

$$\|v(t)\|_\infty \le C(T - t)^{-(q+1)/(pq-1)}$$

and it is not too difficult to show that the corresponding lower bounds are also true.

We prove that for some values of the parameters p, q, r, s, nonglobal solutions have the same blow-up rate as for (5). For some other values of p, q, r, s we obtain an uncoupled asymptotic blow-up behavior: each component either remains bounded or blows up at the same rate as solutions of $u_t - \Delta u = u^r$ or $v_t - \Delta v = v^s$. However, we are presently unable to give the blow-up rates for all values of p, q, r and s. See Proposition 3.2 below, where lower estimates of blow-up rates are established.

We will usually consider positive nonglobal solutions of (1) which are radially symmetric and radially nonincreasing, and which are nondecreasing in time. In other words:

$$\begin{cases} u(t,x) = u(t,\rho), \qquad v(t,x) = v(t,\rho), \quad \text{where } \rho = |x|, \\ u, v > 0, u_t, v_t \geq 0, \quad u_\rho, v_\rho \leq 0 \quad \text{on } (0,T) \times \boldsymbol{R}^N. \end{cases} \tag{7}$$

Note that the existence of solutions to (1) satisfying (3) and (7) can be obtained for initial data $(\lambda u_0, \lambda v_0)$, with $\lambda > 0$ large enough, whenever

$$\begin{cases} u_0, v_0 \text{ are positive radially symmetric, radially nonincreasing,} \\ \Delta u_0 + v_0^p + u_0^r \geq 0, \\ \Delta v_0 + u_0^q + v_0^s \geq 0. \end{cases} \tag{8}$$

We have obtained the following result.

Theorem 2 (Blow-up rates). *Let (u, v) be a positive blowing-up solution of* (1)–(2). *Then we have the following*:

(i) *Let $r < p(q+1)/(p+1)$ and $s < q(p+1)/(q+1)$ and assume* (7). *Then there exist $C_1, C_2 > 0$ such that*

$$C_1 \leq (T-t)^{(p+1)/(pq-1)} \|u(t)\|_\infty \leq C_2, \quad 0 < t < T \tag{9}$$

and

$$C_1 \leq (T-t)^{(q+1)/(pq-1)} \|v(t)\|_\infty \leq C_2, \quad 0 < t < T, \tag{10}$$

provided that

$$\frac{N-2}{N} < \frac{1}{p+1} + \frac{1}{q+1}. \tag{11}$$

(ii) *Let $r > q+1$ or $s > p+1$, and assume that non-simultaneous blow-up occurs.*

(a) *if $\lim_{t \nearrow T} \|u(t)\|_\infty = \infty$, then there exist $C_1, C_2 > 0$ such that*

$$C_1 \leq (T-t)^{1/(r-1)} \|u(t)\|_\infty \leq C_2, \quad 0 < t < T, \tag{12}$$

provided that $(N-2)r < N+2$.

(b) *if $\lim_{t \nearrow T} \|v(t)\|_\infty = \infty$, then there exist $C_1, C_2 > 0$ such that*

$$C_1 \leq (T-t)^{1/(s-1)} \|v(t)\|_\infty \leq C_2, \quad 0 < t < T, \tag{13}$$

provided that $(N-2)s < N+2$.

(iii) *Let $r > q+1$, $s > p+1$, $N = 1$ and assume* (7). *Then* (12) *and* (13) *hold.*

Note that in view of Theorem 1, non-simultaneous blow-up can occur in the case (ii) of Theorem 2. The result of Theorem 2 (i) still holds for the system (5) (with same proof). Although for the system (5) the upper estimates in (9)–(10) are known from [**4**], [**1**], [**5**], our result is new even for (5). Indeed, unlike [**4**], we do not impose the condition $u_{tp}, v_{tp} \le 0$ on the solution (cf. (v) in [**4**, Theorem, p. 266]). Also, the results of [**1**], [**5**] are obtained under different assumptions on N, p and q.

Let us briefly describe the main ideas of our proofs. The proof of simultaneous blow-up in Theorem 1 relies on a scaling argument, in the spirit of [**5**] for system (5), which enables one to estimate the maximum of one component up to time t in terms of the other component (see Proposition 2.1). The examples of non-simultaneous blow-up rely on a combination of an upper bound for T, obtained by estimating one component from below, with an upper estimate for the other component. To prove the upper blow-up rates in Theorem 2, we use some arguments from [**25**] based on scaling and integration by parts in time (see also [**26**]). Unlike other methods for blow-up rate estimates, it has the advantage to apply to problems without variational structure.

The system (1) was considered in [**24**] and a similar system with reaction and absorption terms was treated in [**3**]. Global existence and large time behavior are studied in [**24**]. The problem of global or nonglobal existence is considered in [**3**].

The rest of the paper is organized as follows. Section 2 contains the proof of Theorem 1 and some additional results. Section 3 is devoted to the proof of Theorem 2. We also establish a result concerning the limiting case when $r = p(q+1)/(p+1)$ and $s = q(p+1)/(q+1)$. In this paper C_1, C_2 and C denote positive generic constants, not necessarily the same at different places.

2. Simultaneous and non-simultaneous blow-up.

In this section we are concerned with the proof of Theorem 1. The main ingredient is the following result, which makes it possible to compare the components of positive blowing-up solution of the system (1). In particular it indicates which component of the solution (u, v) must blow-up in case of non-simultaneous blow-up.

Proposition 2.1. *Let (u, v) be a positive solution of* (1) *with $T < \infty$, and define $Q_t = (0, t) \times \boldsymbol{R}^N$.*

(i) *If $r < q+1$, then there exists $C > 0$ such that*

$$\sup_{Q_t} u^\alpha \le C \sup_{Q_t} v, \quad T/2 < t < T, \tag{14}$$

where

$$\alpha = \min\left(q+1-r, \frac{q+1}{p+1}, \frac{q}{s}\right).$$

In particular, $\limsup_{t \nearrow T} \|v(t)\|_\infty = \infty$.

(ii) *If $s < p+1$, then there exists $C > 0$ such that*

$$\sup_{Q_t} v^\beta \le C \sup_{Q_t} u, \quad T/2 < t < T, \tag{15}$$

where

$$\beta = \min\left(p + 1 - s, \frac{p+1}{q+1}, \frac{p}{r}\right).$$

In particular, $\limsup_{t \nearrow T} \|u(t)\|_\infty = \infty$.

PROOF. Let (u, v) be a blowing-up solution of (1). Then (3) holds. As in [**5**], let us introduce the functions U, V defined by:

$$U(t) = \sup_{Q_t} u \quad \text{and} \quad V(t) = \sup_{Q_t} v. \tag{16}$$

Then U and V are positive continuous and nondecreasing on $(0, T)$. Also, by (3) at least U or V diverges as $t \nearrow T$.

We first prove (i). We argue by contradiction. Assume that (14) is not true. Then there exists a sequence $t_n \nearrow T$ as $n \to \infty$ such that

$$V(t_n)[U(t_n)]^{-\alpha} \to 0 \quad \text{as } n \to \infty.$$

Since $\alpha > 0$, it follows that $U(t)$ diverges as $t \nearrow T$. Let

$$\lambda_n = [U(t_n)]^{-(q-\alpha)/2}.$$

Since $\alpha < q$ (note that $\alpha \le q/s$) then $\lambda_n \to 0$ as $n \to \infty$.

Let $(t_n', x_n') \in (0, t_n] \times \boldsymbol{R}^N$ be such that $u(t_n', x_n') \ge (1/2)U(t_n)$. We have $t_n' \to T$ as $n \to \infty$. Define the rescaled functions φ_n and ψ_n by:

$$\varphi_n(s, y) = \lambda_n^{2/(q-\alpha)} u(\lambda_n^2 s + t_n', \lambda_n y + x_n'),$$

$$\psi_n(s, y) = \lambda_n^{2\alpha/(q-\alpha)} v(\lambda_n^2 s + t_n', \lambda_n y + x_n').$$

where $(s, y) \in (-\lambda_n^{-2} t_n', \lambda_n^{-2}(T - t_n')) \times \boldsymbol{R}^N \equiv D_n$.

If we restrict s to $(-\lambda_n^{-2} t_n', 0]$, then we have

$$0 \le \varphi_n \le 1, \ \varphi_n(0,0) \ge 1/2 \quad \text{and} \quad 0 \le \psi_n \le V(t_n)[U(t_n)]^{-\alpha} \to 0, \quad \text{as } n \to \infty. \tag{17}$$

On the other hand, since (u, v) is solution of (1) then (φ_n, ψ_n) is solution of the system

$$\begin{cases} \varphi_s - \Delta\varphi = \lambda_n^{2((p+1)/(q-\alpha))((q+1)/(p+1)-\alpha)} \psi^p + \lambda_n^{(2/(q-\alpha))(q+1-r-\alpha)} \varphi^r, \\ \psi_s - \Delta\psi = \varphi^q + \lambda_n^{(2s/(q-\alpha))(q/s-\alpha)} \psi^s \end{cases}$$

on D_n. By the choice of α, all the powers of λ_n in the right hand sides of the equations are nonnegative and then the coefficient converges to zero or one as $n \to \infty$. As in [**5**], by using interior parabolic estimates, there exists a subsequence, still denoted by (φ_n, ψ_n), converging uniformly on compact subsets of $(-\infty, 0] \times \boldsymbol{R}^N$ to a solution (φ, ψ) of

$$\begin{cases} \varphi_s - \Delta\varphi = \varepsilon_1 \psi^p + \varepsilon_2 \varphi^r, \\ \psi_s - \Delta\psi = \varphi^q + \varepsilon_3 \psi^s \end{cases}$$

on $(-\infty, 0] \times \boldsymbol{R}^N$, with $\varepsilon_i = 0$ or 1, $i = 1, 2, 3$.

But (17) implies that $\psi \equiv 0$ and then by the second equation $\varphi \equiv 0$. This leads to a contradiction with the fact that φ must satisfy $\varphi(0,0) \geq 1/2$. This proves (14) and then (i) of Proposition 2.1.

Statement (ii) follows by interchanging the role of u and v, α and β, r and s and p and q. This finishes the proof of the proposition. □

We now give results of non-simultaneous blow-up. Let $\varphi_1(x) = c\sin(\pi x)$ be the eigenfunction of $-d^2/dx^2$ in $H_0^1(0,1)$ corresponding to the first eigenvalue $\lambda_1 = \pi^2$ with $\int_0^1 \varphi_1(x)\,dx = 1$. Also, define the constants

$$a = \frac{1}{r-1}, \quad c_1 = \frac{1}{r-1-q} \quad \text{and} \quad c_2 = \left(\frac{r-1}{s-1}\right)^{1/(s-1)}.$$

We have obtained the following.

PROPOSITION 2.2. *Assume that* $r > q+1$.

(i) *If* u_0, v_0 *are two positive constant functions such that*

$$v_0 + c_1 u_0^{q+1-r} < c_2 u_0^{(r-1)/(s-1)}, \tag{18}$$

then the (*spatially homogeneous*) *solution* (u,v) *of* (1)–(2) *satisfies*

$$\lim_{t\to T} u(t) = \infty \quad \textit{and} \quad \sup_{t\in(0,T)} v(t) < \infty$$

for some finite T.

(ii) *Assume* $N = 1$. *There exist constants* $c_3, c_4 > 0$ *depending only on* q, r, s, *such that for all* $u_0, v_0 \in L^\infty(\boldsymbol{R})$ *satisfying* (7) *and,*

$$\|v_0\|_\infty + c_3\left(\int_0^1 u_0\varphi_1\right)^{q+1-r} \leq c_4\left(\int_0^1 u_0\varphi_1\right)^{(r-1)/(s-1)} \tag{19}$$

$$\textit{and} \quad \int_0^1 u_0\varphi_1 \geq (2\lambda_1)^{1/(r-1)},$$

the solution (u,v) *of* (1)–(2) *satisfies*

$$\lim_{t\to T} \|u(t)\|_\infty = \infty \quad \textit{and} \quad \sup_{(0,T)\times\boldsymbol{R}} v < \infty$$

for some finite T.

REMARK 2.1. For $s > p+1$, the analogue of Proposition 2.2 obviously holds by exchanging the roles of u, v.

PROOF OF PROPOSITION 2.2. (i) It is clear that the solution (u,v) of (1)–(2) is independent from the space variable x and satisfies

$$\begin{cases} u_t = v^p + u^r \\ v_t = u^q + v^s. \end{cases} \tag{20}$$

Since $u_t \geq u^r$, it follows that the maximum existence time T of (u,v) satisfies $T \leq T_1 := u_0^{-(r-1)}/(r-1)$ and that

$$u(t) \leq ((1/a)(T-t))^{-a}, \quad 0 < t < T.$$

Then integrating the second equation of (20), in view of $aq < 1$, we get

$$\begin{aligned} v(t) &\le v_0 + \int_0^t v^s(\sigma)\,d\sigma + a^{aq}\int_0^t (T-\sigma)^{-aq}\,d\sigma \\ &\le v_0 + \frac{a^{aq}}{1-aq}T^{1-aq} + \int_0^t v^s(\sigma)\,d\sigma, \quad 0 < t < T. \end{aligned}$$

We deduce that v is bounded on $[0, T')$ for any $T' \le T$ such that

$$T' < T_2 := \frac{1}{(s-1)[v_0 + (a^{aq}/(1-aq))T^{1-aq}]^{s-1}}.$$

Then, if (18) holds (which for fixed v_0 is satisfied for all u_0 large enough), it follows that $T_2 > T_1$, so that v is bounded on $[0, T)$.

(ii) Since v is nonnegative, by the first equation of (1), u satisfies

$$u_t \ge u_{xx} + u^r, \quad \text{on } (0,T)\times \boldsymbol{R}. \tag{21}$$

Multiplying (21) by φ_1, integrating by parts over $(0,1)$, using Jensen's inequality and the fact that $\varphi_1'(0) > 0$, $\varphi_1'(1) < 0$ yields

$$\begin{aligned} \frac{d}{dt}\left(\int_0^1 u(t,x)\varphi_1(x)\,dx\right) &\ge \int_0^1 u_{xx}\varphi_1 + \int_0^1 u^r\varphi_1 \\ &\ge \int_0^1 u\varphi_1'' - [u\varphi_1']_0^1 + \left(\int_0^1 u\varphi_1\right)^r \\ &\ge -\lambda_1\int_0^1 u\varphi_1 + \left(\int_0^1 u\varphi_1\right)^r. \end{aligned}$$

Since $\int_0^1 u_0\varphi_1 \ge (2\lambda_1)^{1/(r-1)}$, it follows that $\int_0^1 u(t,x)\varphi_1(x)\,dx \ge (2\lambda_1)^{1/(r-1)}$ for all $t \in [0, T)$. Hence by the last inequality we have that

$$\frac{d}{dt}\left(\int_0^1 u(t,x)\varphi_1(x)\,dx\right) \ge \frac{1}{2}\left(\int_0^1 u\varphi_1\right)^r, \quad 0 < t < T. \tag{22}$$

Integrating (22) gives that T satisfies:

$$T \le T_1 := \frac{2}{r-1}\left(\int_0^1 u_0(x)\varphi_1(x)\,dx\right)^{-(r-1)}.$$

On the other hand, by the second equation of (1), v satisfies:

$$v(t) = e^{t\Delta}v_0 + \int_0^t e^{(t-\sigma)\Delta}v^s(\sigma)\,d\sigma + \int_0^t e^{(t-\sigma)\Delta}u^q(\sigma)\,d\sigma, \quad 0 < t < T,$$

where $e^{t\Delta}$ is the heat semigroup. Then, estimating this integral equation in L^∞-norm yields

$$\|v(t)\|_\infty \le \|v_0\|_\infty + \int_0^t \|v(\sigma)\|_\infty^s\,d\sigma + \int_0^t \|u(\sigma)\|_\infty^q\,d\sigma, \quad 0 < t < T. \tag{23}$$

Now, since u satisfies (7) and (21), it follows from [**25**, Theorem 4 (i)] that there exists a constant $C > 0$ such that $\|u(t)\|_\infty \le C(T-t)^{-1/(r-1)}$, $0 < t < T$. Actually, the proof of [**25**, Theorem 4 (i)] in the case of $\boldsymbol{R}^N$ shows that this constant C depends only on r. By combining this estimate and (23) and arguing as in case (i), we prove that v is bounded on $[0, T') \times \boldsymbol{R}^N$ for any $T' \le T$ satisfying

$$T' < T_2 := \frac{1}{(s-1)[\|v_0\|_\infty + CT^{1-aq}]^{s-1}}.$$

In particular, if

$$\|v_0\|_\infty + c_3\left(\int_0^1 u_0\varphi_1\right)^{q+1-r} \le c_4\left(\int_0^1 u_0\varphi_1\right)^{(r-1)/(s-1)},$$

where c_3, c_4 are positive constants depending only on q, r, s, then $T_1 < T_2$ so that v is bounded on $[0, T) \times \boldsymbol{R}^N$ proving a non-simultaneous blow-up. $\square$

REMARK 2.2. The result of Proposition 2.2 part (ii) would still hold in higher dimensions (with similar proof) provided that one could prove that $\|u(t)\|_\infty \le C(T-t)^{-1/(r-1)}$ for all $t \in [0, T)$, with C a constant independent on u_0 and v_0.(*)

PROOF OF THEOREM 1. Let (u, v) be a blowing-up solution of (1)–(2).

We first prove (i). Assume that $r < q+1$ and $s < p+1$. Then by Proposition 2.1 part (ii), $\limsup_{t\nearrow T}\|u(t)\|_\infty = \infty$ and by Proposition 2.1 part (i), $\limsup_{t\nearrow T}\|v(t)\|_\infty = \infty$. Then u and v blow up and (4) holds. Hence, non-simultaneous blow-up does not occur.

We now prove (ii). Assume that $r > q+1$. Then by Proposition 2.2 there exist u_0, v_0 such that $\lim_{t\to T}\|u(t)\|_\infty = \infty$ and $\sup_{(0,T)\times\boldsymbol{R}^N} v < \infty$ for some finite T. Hence, non-simultaneous blow-up occurs. For $s > p+1$, the analogue of Proposition 2.2 holds by exchanging the roles of u and v. This finishes the proof of Theorem 1. $\square$

3. Blowup rates.

In this section we establish the blow-up rates and some additional results. In view of the proof of the upper estimates in Theorem 2 (i), we prepare the following lemma.

LEMMA 3.1. *Put* $a = (pq-1)/2(p+1)$, $b = (pq-1)/2(q+1)$. *Under the hypothesis* (11), *there exist* $R > 0$, $\eta > 0$ *such that the problem*

$$\begin{cases} z_1'' + \dfrac{N-1}{\rho}z_1' + |z_2|^{p-1}z_2 = \eta_1(\rho), \quad 0 < \rho < R, \\ z_2'' + \dfrac{N-1}{\rho}z_2' + |z_1|^{q-1}z_1 = \eta_2(\rho), \quad 0 < \rho < R, \\ z_1'(0) = z_2'(0) = 0, \\ z_1(0) = \alpha > 0, \quad z_2(0) = \beta > 0, \quad \alpha^a + \beta^b = 1 \end{cases} \tag{24}$$

admits no solution (z_1, z_2) *of class* C^2 *such that* $z_1, z_2 \ge 0$ *on* $[0, R)$, *whenever*

$$\int_0^R |\eta_1(\rho)|\,d\rho + \int_0^R |\eta_2(\rho)|\,d\rho < \eta. \tag{25}$$

(*) *Note added in proof:* results in that direction have been obtained recently by J. Rossi and the first author (to appear).

PROOF. Consider the problem

$$\text{(26)}\qquad \begin{cases} y_1'' + \dfrac{N-1}{\rho} y_1' + |y_2|^{p-1} y_2 = 0, & \rho > 0, \\ y_2'' + \dfrac{N-1}{\rho} y_2' + |y_1|^{q-1} y_1 = 0, & \rho > 0, \\ y_1'(0) = y_2'(0) = 0, \quad y_1(0) = \alpha > 0, \quad y_2(0) = \beta > 0. \end{cases}$$

(We denote $y_i(.) = y_i(\alpha, \beta; .)$, $i = 1, 2$, the unique maximal solution of (26) when no confusion arises.) Under the hypothesis (11), it is known from [**17**, Theorem 3.2, p. 138] that there exists a (first) $R_{\alpha,\beta} > 0$ such that $y_1 y_2(R_{\alpha,\beta}) = 0$. Note that (26) implies that $y_1', y_2' < 0$ on $(0, R_{\alpha,\beta}]$.

We claim that the map $(\alpha, \beta) \mapsto R_{\alpha,\beta}$ is continuous for $\alpha, \beta > 0$.

To prove this, fix (α_0, β_0) and $i \in \{1, 2\}$ and assume that

$$\text{(27)}\qquad y_i(\alpha_0, \beta_0; R_{\alpha_0,\beta_0}) = 0.$$

Then there exists $\varepsilon_0 \in (0, R_{\alpha_0,\beta_0})$ such that $y_i'(\alpha_0, \beta_0, .) < 0$ on $(0, R_{\alpha_0,\beta_0} + \varepsilon_0]$. By continuous dependence, for all $\varepsilon \in (0, \varepsilon_0]$, there is $\delta_\varepsilon > 0$ such that $|\alpha - \alpha_0| + |\beta - \beta_0| < \delta_\varepsilon$ implies

$$y_i'(\alpha, \beta; .) < 0 \quad \text{on } (0, R_{\alpha_0,\beta_0} + \varepsilon_0],$$

$$y_i(\alpha, \beta; R_{\alpha_0,\beta_0} - \varepsilon) > 0 \quad \text{and} \quad y_i(\alpha, \beta; R_{\alpha_0,\beta_0} + \varepsilon) < 0.$$

It follows that for all (α, β) close to (α_0, β_0), $y_i(\alpha, \beta; .)$ possesses a first zero $R^{(i)}_{\alpha,\beta}$ and that the function $(\alpha, \beta) \mapsto R^{(i)}_{\alpha,\beta}$ is continuous at (α_0, β_0). If (27) is true for a single $i \in \{1, 2\}$, then $R_{\alpha,\beta} = R^{(i)}_{\alpha,\beta}$ in the neighborhood of (α_0, β_0). Otherwise, if (27) is true for both $i = 1, 2$, then $R_{\alpha,\beta} = R^{(1)}_{\alpha,\beta} \wedge R^{(2)}_{\alpha,\beta}$ in the neighborhood of (α_0, β_0). In both cases, $R_{\alpha,\beta}$ is continuous at (α_0, β_0) and the claim is proved.

Since $K := \{\alpha > 0, \beta > 0, \alpha^a + \beta^b = 1\}$ is compact, we deduce that $R_0 := \max_K R_{\alpha,\beta} < \infty$ and $\max_K y_1' \vee y_2'(R_{\alpha,\beta}) =: -k < 0$. Moreover, since $y_1^a + y_2^b \leq 1$ on $[0, R_{\alpha,\beta}]$, standard arguments show that, for some $\delta > 0$ and $M \geq 1$ independent of $(\alpha, \beta) \in K$, the solution (y_1, y_2) exists and satisfies $|y_1| + |y_2| \leq M$ on $[0, R_{\alpha,\beta} + \delta]$ and $y_1' \vee y_2' \leq -k/2$ on $[R_{\alpha,\beta}, R_{\alpha,\beta} + \delta]$. Therefore,

$$\text{(28)}\qquad y_1 \wedge y_2(R_{\alpha,\beta} + \delta) \leq -k\delta/2.$$

Now, let $R = R_0 + \delta$ and assume that (z_1, z_2) is a solution of (24). Using the integral form of (24), (26), and subtracting, we obtain, for all $r \in [0, R_{\alpha,\beta} + \delta]$,

$$(z_1 - y_1)(r) = \int_0^r \int_0^s \left(\frac{\sigma}{s}\right)^{N-1} (\eta_1 + |y_2|^{p-1} y_2 - |z_2|^{p-1} z_2)(\sigma)\, d\sigma ds$$

hence, using $|y_2| \leq M$,

$$\begin{aligned} |z_1 - y_1|(r) &\leq R \int_0^r (|\eta_1| + p|y_2 - z_2|(|y_2|^{p-1} + |z_2|^{p-1}))(s)\, ds \\ &\leq R \int_0^R |\eta_1(s)|\, ds + RC(p) \int_0^r (M^{p-1}|y_2 - z_2| + |y_2 - z_2|^p)(s)\, ds. \end{aligned}$$

Let $w(r) = \max_{s \in [0,r]}(|z_1 - y_1| + |z_2 - y_2|)(s)$ and $\gamma = p \vee q$. Using the similar inequality for $|z_2 - y_2|$, the fact that w is nondecreasing and $M \geq 1$, it follows that

$$w(r) \leq R\int_0^R (|\eta_1(s)| + |\eta_2(s)|)\,ds + RC(\gamma)(M^{\gamma-1} + w^{\gamma-1}(r))\int_0^r w(s)\,ds.$$

By Gronwall's inequality and (25), we deduce

$$w(r) \leq R\eta \exp[R^2C(\gamma)(M^{\gamma-1} + w^{\gamma-1}(r))] \quad \text{on } [0, R_{\alpha,\beta} + \delta]. \tag{29}$$

Assuming $\eta < R^{-1}\exp[-R^2C(\gamma)(M^{\gamma-1} + 1)]$, this implies that $w(r) < 1$, hence

$$w(r) \leq R\eta \exp[R^2C(\gamma)(M^{\gamma-1} + 1)] \quad \text{on } [0, R_{\alpha,\beta} + \delta]. \tag{30}$$

(Indeed, otherwise, since $w(0) = 0$, there is r such that $w(r) = 1$ and (29) yields a contradiction.) Finally, for η small enough (depending only on γ, k, δ, M, R), (28) and (30) imply that z_1 or z_2 achieves some negative values on $[0, R_{\alpha,\beta} + \delta]$. The Lemma is proved. □

PROOF OF THEOREM 2 (i). We first prove the upper estimates. Let

$$\alpha(t) = [u(t,0)]^{(pq-1)/2(p+1)}, \quad \beta(t) = [v(t,0)]^{(pq-1)/2(q+1)} \quad \text{and}$$

$$\gamma(t) = \alpha(t) + \beta(t), \quad 0 < t < T.$$

Define

$$w_1(t,\rho) = \frac{u(t,\gamma(t)^{-1}\rho)}{\gamma(t)^{2(p+1)/(pq-1)}}, \quad w_2(t,\rho) = \frac{v(t,\gamma(t)^{-1}\rho)}{\gamma(t)^{2(q+1)/(pq-1)}}, \quad 0 < t < T,\ \rho \geq 0.$$

By hypotheses we have $\lim_{t\to T}\gamma(t) = \infty$,

$$0 \leq w_1, w_2 \leq 1, \quad \partial_\rho w_1, \partial_\rho w_2 \leq 0, \quad 0 < t < T,\ \rho \geq 0,$$

and

$$\partial_\rho w_1(t,0) = \partial_\rho w_2(t,0) = 0, \quad [w_1(t,0)]^{(pq-1)/2(p+1)} + [w_2(t,0)]^{(pq-1)/2(q+1)} = 1.$$

Also,

$$A(t,\rho) \equiv \Delta w_1 + w_2^p + \gamma(t)^{-2(r-1)((p+1)/(pq-1)-1/(r-1))}w_1^r = \frac{\partial_t u(t,\gamma(t)^{-1}\rho)}{\gamma(t)^{2p((q+1)/(pq-1))}} \tag{31}$$

$$B(t,\rho) \equiv \Delta w_2 + w_1^q + \gamma(t)^{-2(s-1)((q+1)/(pq-1)-1/(s-1))}w_2^s = \frac{\partial_t v(t,\gamma(t)^{-1}\rho)}{\gamma(t)^{2q((p+1)/(pq-1))}}. \tag{32}$$

Following [**25**, Lemma 3.1], we observe that $\lambda(t) := u(t,\gamma(t)^{-1}\rho)$ satisfies

$$\lambda'(t) \geq \partial_t u(t,\gamma(t)^{-1}\rho)$$

thanks to (7), and we integrate by parts. For all $0 < t < \tau < T$ this yields

$$\begin{aligned}\int_t^\tau \frac{\partial_t u(s,\gamma(s)^{-1}\rho)}{u^{p(q+1)/(p+1)}(s,0)}\,ds &\leq \left[\frac{\lambda(s)}{u^{p(q+1)/(p+1)}(s,0)}\right]_t^\tau + \frac{p(q+1)}{p+1}\int_t^\tau \frac{\lambda(s)\partial_t u(s,0)}{u^{1+p(q+1)/(p+1)}(s,0)}\,ds \\ &\leq \frac{u(\tau,0)}{u^{p(q+1)/(p+1)}(\tau,0)} + \frac{p(q+1)}{p+1}\int_t^\tau \frac{\partial_t u(s,0)}{u^{p(q+1)/(p+1)}(s,0)}\,ds\end{aligned}$$

hence,

$$\frac{1}{T-t}\int_t^T \frac{\partial_t u(s,\gamma(s)^{-1}\rho)}{\gamma(s)^{2p((q+1)/(pq-1))}}\,ds \tag{33}$$

$$\leq p\frac{q+1}{pq-1}\frac{u^{-(pq-1)/(p+1)}(t,0)}{T-t} \equiv g_1(t), \quad 0<t<T,\ \rho\geq 0,$$

and similarly

$$\frac{1}{T-t}\int_t^T \frac{\partial_t v(s,\gamma(s)^{-1}\rho)}{\gamma(s)^{2q((p+1)/(pq-1))}}\,ds \tag{34}$$

$$\leq q\frac{p+1}{pq-1}\frac{v^{-(pq-1)/(q+1)}(t,0)}{T-t} \equiv g_2(t), \quad 0<t<T,\ \rho\geq 0.$$

Let $R>0$ and $t\in(0,T)$. Integrating (31)–(32) on $(t,T)\times(0,R)$ and using (33)–(34), it follows from the mean-value theorem that there exists $t'\in(t,T)$ such that $\omega_1(\rho)=w_1(t',\rho)$ and $\omega_2(\rho)=w_2(t',\rho)$ satisfy

$$\begin{cases} \Delta\omega_1+\omega_2^p+\varepsilon_1(t',\rho)=h_1(\rho), & \rho\in(0,R),\\ \Delta\omega_2+\omega_1^q+\varepsilon_2(t',\rho)=h_2(\rho), & \rho\in(0,R), \end{cases}$$

with

$$0\leq\omega_1,\omega_2\leq 1, \quad \omega_1',\omega_2'\leq 0, \quad \omega_1,\omega_2\in C^2([0,R])$$

$$\omega_1'(0)=\omega_2'(0)=0, \quad [\omega_1(0)]^{(pq-1)/2(p+1)}+[\omega_2(0)]^{(pq-1)/2(q+1)}=1,$$

where

$$\varepsilon_1(t',\rho)=\gamma(t')^{-2(r-1)((p+1)/(pq-1)-1/(r-1))}w_1^r(t',\rho),$$

$$\varepsilon_2(t',\rho)=\gamma(t')^{-2(s-1)((q+1)/(pq-1)-1/(s-1))}w_2^s(t',\rho),$$

and where $h_1(\rho):=A(t',\rho)\geq 0$ and $h_2(\rho):=B(t',\rho)\geq 0$ verify

$$\int_0^R (h_1(\rho)+h_2(\rho))\,d\rho \leq R(g_1(t)+g_2(t)), \quad 0<t<T. \tag{35}$$

Now choose R given by Lemma 3.1. Since $\|\varepsilon_1(t')\|_{L^\infty(0,\infty)}+\|\varepsilon_2(t')\|_{L^\infty(0,\infty)}\to 0$ as $t'\to T$, we deduce from the Lemma that $R(g_1(t)+g_2(t))\geq \eta/2$ for all t close enough to T. Since (cf. Proposition 2.1)

$$0<C_1\leq \frac{w_1^{q+1}(t,0)}{w_2^{p+1}(t,0)}=\frac{u^{q+1}(t,0)}{v^{p+1}(t,0)}\leq C_2<\infty, \quad T/2<t<T$$

under the current assumptions on $p,q,r,s,$ the upper estimates in (9)–(10) follow.

The lower estimates are much easier. They are actually particular cases of the following result.

Proposition 3.2. *Let* (u,v) *be a blowing-up solution of* (1)–(2) *satisfying* (7). *Let* α *and* β *be as in Proposition* 2.1. *Then we have the following*:

(i) *If* $s < p+1$, *then there exists a constant* $C > 0$ *such that*

$$\|u(t)\|_\infty \geq C(T-t)^{-1/((p/\beta)-1)}, \quad 0 < t < T. \tag{36}$$

(ii) *If* $r < q+1$, *then there exists a constant* $C > 0$ *such that*

$$\|v(t)\|_\infty \geq C(T-t)^{-1/((q/\alpha)-1)}, \quad 0 < t < T. \tag{37}$$

Proof. The assumptions (7) on the solution (u, v) imply that $\Delta u(t,0) \leq 0$ and $\Delta v(t,0) \leq 0$. Therefore,

$$\begin{cases} u_t(t,0) \leq v^p(t,0) + u^r(t,0), \\ v_t(t,0) \leq u^q(t,0) + v^s(t,0), \end{cases} \tag{38}$$

for $t \in (0,T)$.

We first prove (i). By the first equation of (38) and the comparison result of Proposition 2.1 we have

$$\begin{aligned} u_t(t,0) &\leq v^{\beta(p/\beta)}(t,0) + u^r(t,0) \\ &\leq C u^{p/\beta}(t,0) + u^r(t,0) \\ &\leq C u^{p/\beta}(t,0), \end{aligned}$$

for t close to T. Here we have used the fact that $r \leq p/\beta$ and that u blows up. Then, by integrating the last differential inequality we obtain (36). (37) follows similarly by using the second equation of (38). □

We now turn to prove parts (ii) and (iii) of Theorem 2.

Proof of Theorem 2 (ii). Assume for instance that u blows up and v remains bounded. Therefore, $u \geq 0$ satisfies

$$u_t - \Delta u = F(t,x,u), \quad 0 < t < T,\ x \in \boldsymbol{R}^N,$$

where $F(t,x,u) := u^r + b(t,x)$ with $|b(t,x)| \leq C$ on $(0,T) \times \boldsymbol{R}^N$. Since $(N-2)r < N+2$, the upper estimate in (12) follows from an obvious adaptation of the arguments in [**13**, Section 6]. The lower estimate is obtained by standard arguments using the variation of constants formula (cf. the proof of part (iii)). □

Proof of Theorem 2 (iii). Since u satisfies (21) and (7), the upper estimate in (12) follows from [**25**, Theorem 4 (i)], and the upper estimate in (13) follows similarly.

To establish the lower estimates, we use the upper estimates obtained in (12)–(13). Clearly, by the first equation in (38) and by using the fact that $p < s-1$, the fact that u blows up and the upper estimate in (13) we have

$$\begin{aligned} \|u(t)\|_\infty &\leq \|u_0\|_\infty + \int_0^t \|u(\tau)\|_\infty^r\,d\tau + \int_0^t \|v(\tau)\|_\infty^p\,d\tau \\ &\leq \|u_0\|_\infty + \int_0^t \|u(\tau)\|_\infty^r\,d\tau + C\int_0^t (T-\tau)^{-p/(s-1)}\,d\tau \\ &\leq \|u_0\|_\infty + C + \int_0^t \|u(\tau)\|_\infty^r\,d\tau. \end{aligned}$$

The lower estimate in (12) follows by integration. The lower estimate in (13) follows similarly. □

REMARK 3.1. Part (iii) of Theorem 2 holds for simultaneous and non-simultaneous blow-up as well. A simple example of simultaneous blow-up in this context is obtained by taking $r = s$, $p = q$, $r > q + 1$ and $u_0 = v_0 = \lambda\phi$, λ large. We do not know whether simultaneous blow-up occurs for all $r > q + 1$ or $s > p + 1$. We suspect that such behavior should occur as a limiting case which separates non-simultaneous blow-up of either component and that it should be rather unstable.(*) Indeed, when $r > q + 1$, $s > p + 1$ and $N = 1$ for instance, it follows from Proposition 2.2 that u blows up with v bounded if u_0 is "very large" as compared with v_0, and vice-versa.

Finally, in the limiting case of Theorem 2 (i), we have the following result.

THEOREM 3. *Let* (u, v) *be a blowing-up solution of* (1)–(2) *satisfying* (7). *Assume that*

$$r = p\frac{q+1}{p+1} \quad \text{and} \quad s = q\frac{p+1}{q+1}. \tag{39}$$

Then (9) *and* (10) *hold provided that*

$$\frac{N-2}{2} \le \max\left\{\frac{p+1}{pq-1}, \frac{q+1}{pq-1}\right\}.$$

PROOF. The proof follows along the lines of the proof of Theorem 2 (i). Using the same notation as in the last proof, ω_1 and ω_2 will satisfy the elliptic system

$$\begin{cases} \Delta\omega_1 + \omega_2^p + \omega_1^r = h_1(\rho), & \rho \in (0, R), \\ \Delta\omega_2 + \omega_1^q + \omega_1^s = h_2(\rho), & \rho \in (0, R), \end{cases}$$

with same additional properties. By hypotheses and [**18**, Theorem 2.1, p. 466] there must exist $C > 0$ a constant such that $R(g_1(t) + g_2(t)) \ge C$ for t close to T. The conclusion now follows by definition (33)–(34) and Proposition 2.1. □

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References

[1] D. Andreucci, M. A. Herrero and J. J. L. Velázquez, Liouville theorems and blow up behaviour in semilinear reaction diffusion systems, Ann. Inst. H. Poincaré, Anal. Non Linéaire, **14** (1997), 1–53.

[2] J. Beberness and D. Eberly, Mathematical Problems from Combustion Theory, Springer-Verlag, New York, 1989.

[3] N. Bedjaoui and Ph. Souplet, Critical blowup exponents for a system of reaction-diffusion equations with absorption, Z. Angew. Math. Phys., **53** (2002), 197–210.

[4] G. Caristi and E. Mitidieri, Blow-up estimates of positive solutions of a parabolic system, J. Differential Equations, **113** (1994), 265–271.

(*) *Note added in proof:* results in that direction have been obtained recently by J. Rossi and the first author (to appear).

[5] M. Chlebík and M. Fila, From critical exponents to blow-up rates for parabolic problems, Rend. Mat. Appl. Ser VII, **19** (1999), 449–470.

[6] K. Deng, Blow-up rates for parabolic systems, Z. Angew. Math. Phys., **47** (1996), 132–143.

[7] M. Escobedo and M. A. Herrero, Boundedness and blow up for a semilinear reaction diffusion system, J. Differential Equations, **89** (1991), 176–202.

[8] M. Escobedo and H. A. Levine, Critical blow up and global existence numbers for a weakly coupled system of reaction-diffusion equations, Arch. Ration. Mech. Anal., **129** (1995), 47–100.

[9] M. Fila and P. Quittner, The blow-up rate for a semilinear parabolic system, J. Math. Anal. Appl., **238** (1999), 468–476.

[10] M. Fila and Ph. Souplet, The blow-up rate for a semilinear parabolic problems on general domains, Nonlinear Differ. Eq. and Appl. NoDEA, **8** (2001), 473–480.

[11] A. Friedman and J. B. McLeod, Blow-up of positive solutions of semilinear heat equations, Indiana Univ. Math. J., **34** (1985), 425–447.

[12] B. Gidas and J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, Comm. Pure Appl. Math., **34** (1981), 525–598.

[13] Y. Giga and R. V. Kohn, Characterizing blowup using similarity variables, Indiana Univ. Math. J., **36** (1987), 1–40.

[14] A. Haraux and F. B. Weissler, Nonuniqueness for a semilinear initial value problem, Indiana Univ. Math. J., **32** (1982), 167–189.

[15] D. D. Joseph and T. S. Lundgren, Quasilinear Dirichlet problems driven by positive sources, Arch. Ration. Mech. Anal., **49** (1973), 241–269.

[16] O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural'ceva, Linear and Quasilinear Equations of Parabolic Type, Amer. Math. Soc. Translation of Math., Monographs, **23**, Providence, RI, 1968.

[17] E. Mitidieri, A Rellich type identity and applications, Comm. Partial Differential Equations, **18** (1993), 125–151.

[18] E. Mitidieri, Non existence of positive solutions of semilinear elliptic systems in $\boldsymbol{R}^N$, Differential and Integral Equations, **9** (1996), 465–479.

[19] C. V. Pao, Nonlinear Parabolic and Elliptic Equations, Plenum Press, New York, 1992.

[20] F. Quirós and J. D. Rossi, Non-simultaneous blow-up in a semilinear parabolic system, Z. Angew. Math. Phys., **52** (2001), 342–346.

[21] F. Quirós and J. D. Rossi, Non-simultaneous blow-up in a nonlinear parabolic system, Adv. Nonlinear Stud., **3** (2003), 397–418.

[22] F. Rothe, Global Solutions of Reaction-Diffusion Systems, Lecture Notes in Math., No. 1072, Springer-Verlag, Berlin, 1984.

[23] A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov and A. P. Mikhailov, Blow-up in Quasilinear Parabolic Equations, Walter de Gruyter, Berlin, New York, 1995.

[24] S. Snoussi and S. Tayachi, Asymptotic self-similar behavior for a semilinear parabolic system, Commun. Contemp. Math., **3** (2001), 363–392.

[25] Ph. Souplet and S. Tayachi, Blowup rates for nonlinear heat equations with gradient terms and for parabolic inequalities, Colloq. Math., **88** (2001), 135–154.

[26] F. B. Weissler, An L^∞ blow-up estimate for a nonlinear heat equation, Comm. Pure Appl. Math., **38** (1985), 291–295.

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