Optimal consumption and portfolio with both fixed and proportional transaction costs

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Abstract

We consider a market model with one riskfree and one risky asset, in which the dynamics of the risky asset is governed by a geometric Brownian motion. In this market we consider an investor who consumes from the bank account and who has the opportunity at any time to transfer funds between the two assets. We suppose that these transfers involve a fixed transaction cost $k > 0$, independent of the size of the transaction, plus a cost proportional to the size of the transaction.

The objective is to maximize the cumulative expected utility of consumption over a planning horizon. We formulate this problem as a combined stochastic control/impulse control problem, which in turn leads to a (nonlinear) quasivariational Hamilton-Jacobi-Bellman inequality (QVHJBI). We prove that the value function is the unique viscosity solution of this QVHJBI. Finally numerical results are presented.

1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a given filtration $\{\mathcal{F}_t\}_{t\geq 0}$. We denote by $X(t)$ the amount of money the investor has in the bank at time t and by $Y(t)$ the amount of money invested in the risky asset at time t. We assume that in the absence of consumption and transactions the process $X(t)$ grows deterministically at exponential rate r, while $Y(t)$ is

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a geometric Brownian motion, i.e.

$$dX(t) = rX(t)dt; \qquad X(0) = x \tag{1.1}$$

$$dY(t) = \alpha Y(t)dt + \sigma Y(t)dW(t)\ ; \qquad Y(0) = y \tag{1.2}$$

where $W(t)$ is a 1-dimensional $\mathcal{F}_t$-Brownian motion and $\alpha > r > 0$ and $\sigma \neq 0$ are constants.

Suppose that at any time t the investor is free to choose a *consumption rate* $c(t) \geq 0$. This consumption is automatically drawn from the bank account holding with no extra costs. Moreover, at any time the investor can decide to transfer money from the bank account to the stock and conversely. Assume that a purchase of size ℓ of stocks incurs a transaction cost consisting of a sum of a fixed cost $k > 0$ (independent of the size of the transaction) plus a cost $\lambda\ \ell$ proportional to the transaction ($\lambda \geq 0$). These costs are drawn from the bank account. Similarly a sale of size m of stocks incures the fixed cost $K > 0$ plus the proportional cost μm ($\mu \geq 0$). For simplicity we will assume that $K = k$ and $\mu = \lambda$. In this context the investor will only change his portfolio finitely many times in any finite time interval. The control of the investor will consist of a combination of a regular *stochastic control* $c(t)$ and an *impulse control* $v = (\tau_1, \tau_2, \ldots; \xi_1, \xi_2, \ldots)$. Here $0 \leq \tau_1 < \tau_2 < \ldots$ are $\mathcal{F}_t$-stopping times giving the times when the investor decides to change his portfolio and $\{\xi_j \in \mathbf{R}; j = 1, 2, \ldots\}$ are $\mathcal{F}_{\tau_j}$-measurable random variables giving the sizes of the transactions at these times. We assume that

$$c(t) \text{ is } \mathcal{F}_t\text{-adapted, } c(t,\omega) \geq 0 \text{ and } \lim_{j\to\infty} \tau_j = \infty \quad \text{a.s.} \tag{1.3}$$

(possibly $\tau_n = \infty$ a.s. for some $n < \infty$).

If such a control $w = (c, v)$ is applied to the system $(X(t), Y(t))$ it gets the form

$$dX(t) = (rX(t) - c(t))dt\ ; \qquad \tau_i \leq t < \tau_{i+1} \tag{1.4}$$

$$dY(t) = \alpha Y(t)dt + \sigma Y(t)dW(t)\ ; \qquad \tau_i \leq t < \tau_{i+1} \tag{1.5}$$

$$X(\tau_{i+1}) = X(\tau_{i+1}^-) - k - \xi_{i+1} - \lambda|\xi_{i+1}|\ , \tag{1.6}$$

$$Y(\tau_{i+1}) = Y(\tau_{i+1}^-) + \xi_{i+1}\ . \tag{1.7}$$

Thus a positive value of ξ_{i+1} indicates that money is being taken from the bank account at time τ_{i+1} to buy stocks, and conversely if ξ_{i+1} is negative.

If our agent has the amount x on the bank account and y in stocks, his *net wealth* is given by

$$H(x, y) = \max\{x + y - \lambda|y| - k, \min\{x, y\}\}\ . \tag{1.8}$$

Therefore it is natural to define the *solvency region* $\mathcal{S}$ by

$$\mathcal{S} = \{(x, y) \in \mathbf{R}^2; H(x, y) \geq 0\} \tag{1.9}$$

and we put

$$\tilde{\mathcal{S}} = \mathbf{R}^+ \times \mathcal{S} .$$

(See Figure 1 and also Remark 2.2.)

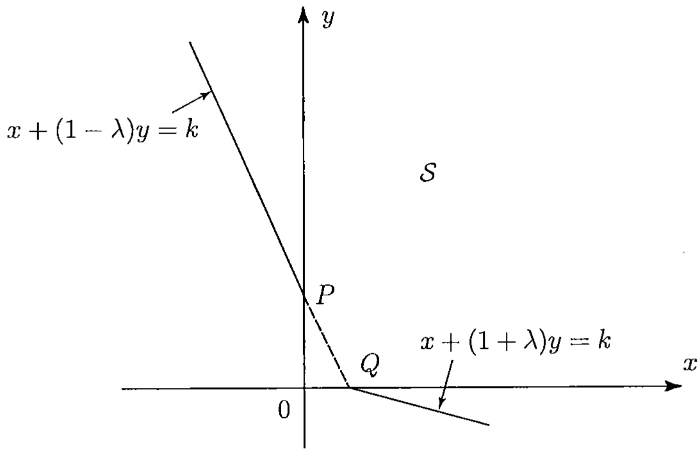

Figure 1

The investor's objective is to maximize over all combined controls $w = (c, v)$ the expression

(1.10) $$J^w(s,x,y) = E^{s,x,y}\Big[\int_0^\infty e^{-\delta(s+t)}\frac{c^\gamma(t)}{\gamma}dt\Big] = e^{-\delta s}E^{x,y}\Big[\int_0^\infty e^{-\delta t}\frac{c^\gamma(t)}{\gamma}dt\Big]$$

where $\delta > 0$, $0 < \gamma < 1$ are constants ($1-\gamma$ is the relative risk aversion coefficient), $E^{s,x,y}$ denotes the expectation with respect to the probability law $P^{s,x,y}$ of

(1.11) $$Z(t) = Z^w(t) := (s + t, X(t), Y(t)) ; \qquad t \geq 0$$

starting at $z = (s, x, y)$.

We seek the value function(s)

(1.12) $$\Phi(s,x,y) = \sup_{w \in \mathcal{W}} J^w(s,x,y) , \qquad \Psi(x,y) = \Phi(0,x,y)$$

where $\mathcal{W} = \mathcal{W}(x,y)$ is the set of all *admissible* controls, i.e. all combined controls which satisfy (1.3) and which do not cause the process $Z(t)$ to exit from $\mathcal{S}$. Note that

(1.13) $$J^w(s,x,y) = e^{-\delta s}J^w(0,x,y) \quad \text{and} \quad \Phi(s,x,y) = e^{-\delta s}\Phi(0,x,y) = e^{-\delta s}\Psi(x,y) ,$$

so the introduction of the s-variable is not really necessary. However, it turns out to be convenient in order to simplify the notation and the arguments in some of the proofs later, as in Theorem 3.5 and Theorem 4.2.

We also seek a corresponding *optimal* control, i.e. a combined control w^* such that

$$\Phi(s,x,y) = J^{w^*}(s,x,y) \; . \tag{1.14}$$

This problem may be regarded as a generalization of optimal consumption and portfolio problems studied by Merton [M] and Davis & Norman [DN]. See also Shreve and Soner [SS]. [M] considers the case with no transaction costs ($\lambda = k = 0$), in which case the problem is no longer a combined control problem but a pure stochastic control problem. In this case it is proved in [M] that it is optimal to choose the portfolio such that

$$\frac{Y(t)}{X(t)} = \frac{\pi^*}{1-\pi^*} \qquad \text{for all } t \; , \tag{1.15}$$

(the Merton line) where

$$\pi^* = \frac{\alpha - r}{(1-\gamma)\sigma^2} \; . \tag{1.16}$$

Moreover, the corresponding value function in the Merton case $\lambda = k = 0$ is given by

$$\Phi_0(s,x,y) = e^{-\delta s} C_1 (x+y)^\gamma \tag{1.17}$$

where

$$C_1 = \frac{1}{\gamma} C_0^{\gamma-1}, \qquad \text{with} \qquad C_0 = \frac{1}{1-\gamma}\Big[\delta - \gamma r - \frac{\gamma(\alpha-r)^2}{2\sigma^2(1-\gamma)}\Big] \tag{1.18}$$

provided that

$$\delta > \gamma\Big[r + \frac{(\alpha-r)^2}{2\sigma^2(1-\gamma)}\Big]. \tag{1.19}$$

See e.g. [DN, Section 2].

From now on we assume that (1.19) holds.

It is intuitively obvious that

$$\Phi(s,x,y) \le \Phi_0(s,x,y) \; . \tag{1.20}$$

We will prove this rigorously later (Corollary 2.2).

[DN] and [SS] consider the case with proportional transaction costs only ($k = 0$), in which case the problem can be formulated as a singular stochastic control problem. It is proved in [DN] and [SS] that under some conditions there exist two straight lines Γ_1, Γ_2 through the origin, bounding a cone NT, such that it is optimal to make no transactions if $(X(t), Y(t)) \in NT$ and make transactions corresponding to local time at $\partial(NT)$, resulting in reflections back to NT every time $(X(t), Y(t)) \in \partial(NT)$. Depending on the parameters the Merton line may or may not go between the lines Γ_1, Γ_2 (see the discussion in [AMS, Section 7.2]). For an extension of these results to markets with jumps see [FØS1] and [FØS2].

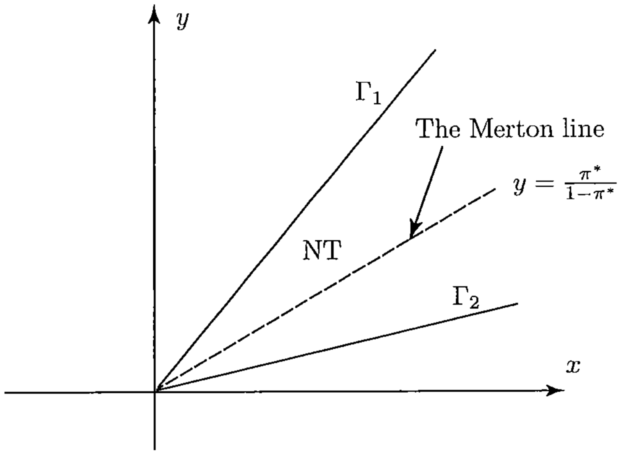

Figure 2

The first paper to model markets with fixed transaction costs $k > 0$ by impulse control theory seems to be [EH], but they do not consider optimal consumption.

Perhaps the paper which is closest to ours is [K]. Here optimal consumption in markets with fixed transaction costs are considered, but consumption is only allowed at the discrete times of the transactions. This makes it possible to put the problem within the framework of impulse control and quasi-variational inequalities.

In our paper we allow consumption to take place at any time, independent of the (discrete) times chosen for the transactions. As explained above we model this as a combined stochastic control and impulse control problem, or a *combined control* problem, for short. To the best of our knowledge this is the first time such a combined control model has been studied.

In Section 2 we introduce quasi-variational Hamilton-Jacobi-Bellman inequalities (QVHJBI) associated to this combined control problem. We point out that if a function $\phi(s, x, y)$ satisfies these QVHJBI (and some additional smoothness conditions), then ϕ coincides with the value function Φ, defined by (1.12). (See Theorem 2.1).

In Section 3 we prove that the value function Φ is the unique viscosity solution of the QVHJBI formulated in Section 2.

Finally in Section 4 we present some numerical estimates for Φ and the optimal consumption-investment policy $w^* = (c^*, v^*)$.

Remark 1.1 Another natural choice of solvency region would be the set

(1.21) $$\mathcal{S}_+ := [0, \infty) \times [0, \infty) .$$

This choice models a situation where no borrowing or shortselling is allowed. We will mostly use the choice $\mathcal{S}$ given by (1.9) in this paper, but we point out that the proofs carry over to the $\mathcal{S}_+$ case with only minor modifications. (Usually the $\mathcal{S}_+$ case is simpler.)

2 Quasi-variational Hamilton-Jacobi-Bellman inequalities (QVHJBI)

Let A^c be the generator of the process $Z^c(t) = (s+t, X^c(t), Y^c(t))$ when there are no transactions, i.e. A^c is the partial differential operator given by

$$(2.1) \qquad (A^c f)(s,x,y) = \frac{\partial f}{\partial s} + (rx-c)\frac{\partial f}{\partial x} + \alpha y \frac{\partial f}{\partial y} + \tfrac{1}{2}\sigma^2 y^2 \frac{\partial^2 f}{\partial y^2}$$

for any $f : \mathbf{R}^3 \to \mathbf{R}$ and (s,x,y) such that the derivatives exist. In particular, if $f(s,x,y) = e^{-\delta s} g(x,y)$ then

$$(A^c f)(s,x,y) = e^{-\delta s} L^c g(x,y) ,$$

where

$$(2.2) \qquad L^c g(x,y) = -\delta g + (rx-c)\frac{\partial g}{\partial x} + \alpha y \frac{\partial g}{\partial y} + \tfrac{1}{2}\sigma^2 y^2 \frac{\partial^2 g}{\partial y^2} .$$

For $(x,y) \in \mathcal{S}$ and $\xi \neq 0$ put

$$(2.3) \qquad x' = x'(\xi) = x - k - \xi - \lambda|\xi| , \qquad y' = y'(\xi) = y + \xi .$$

We define the *intervention operator* $\mathcal{M}$ by

$$(2.4) \qquad \mathcal{M}h(x,y) = \sup\{h(x',y'); \xi \in \mathbf{R}\setminus\{0\}, (x',y') \in \mathcal{S}\}$$

for all locally bounded $h : \mathcal{S} \to \mathbf{R}$, $(x,y) \in \mathcal{S}$.

If $(x',y') \notin \mathcal{S}$ for all $\xi \in \mathbf{R}\setminus\{0\}$ we put $\mathcal{M}h(x,y) = 0$. If for all $(x,y) \in \mathcal{S}$ there exists $(x',y') = (x'(\xi), y'(\xi)) \in \mathcal{S}$ such that

$$\mathcal{M}h(x,y) = h(x',y')$$

then we put

$$(2.5) \qquad \widehat{\xi}(x,y) = \widehat{\xi}_h(x,y) = (x',y')$$

(More precisely, we let $\widehat{\xi}(x,y)$ denote a measurable selection of the map $(x,y) \to (x',y')$.)

If Φ is the value function for our problem, then for each s we can interprete $\mathcal{M}\Phi(s,x,y)$ as the maximal value we can obtain by making an admissible transaction at (s,x,y).

Following [BØ] we call a locally bounded function $h : \widetilde{\mathcal{S}} \to \mathbf{R}^+$ *stochastically* C^2 with respect to Z^c if $(A^c h)(z)$ exists for a.a. $z = (s,x,y)$ with respect to *the Green measure* (expected occupation time measure) $G(z_0, \cdot)$, and the generalized Dynkin formula holds for h, i.e.

$$E^{z_0}[h(Z^c(\tau'))] = E^{z_0}[h(Z^c(\tau))] + E^{z_0}\Big[\int_{\tau}^{\tau'} (A^c h)(Z^c(t))dt\Big]$$

for all stopping times τ, τ' such that

(2.6) $$\tau \le \tau' \le T_R := \inf\{t > 0, |Z^c(t)| \ge R\} \wedge R \qquad \text{for some } R < \infty .$$

Recall that for each $z_0 \in \widetilde{\mathcal{S}}$ the *Green measure* $G(z_0, \cdot)$ of the process Z^c in $\widetilde{\mathcal{S}}$ is defined by

$$G(z_0, H) = E^{z_0}\Big[\int_0^\tau \mathcal{X}_H(Z^c(t))dt\Big] \qquad \text{for all Borel sets } H \subset \widetilde{\mathcal{S}}$$

where $\tau = \inf\{t > 0; Z^c(t) \notin \widetilde{\mathcal{S}}\}$ and $\mathcal{X}_H(z) = 1$ if $z \in H$, $\mathcal{X}_H(z) = 0$ if $z \notin H$.

If h is a function on $\mathcal{S}$ we define

(2.7) $$\mathcal{L}h(x,y) = \sup_{c \ge 0}\Big\{L^c h(x,y) + \frac{c^\gamma}{\gamma}\Big\}; \qquad (x,y) \in \mathcal{S}$$

and

(2.8) $$\mathcal{L}_0 h(x,y) = L^0 h(0,y) = -\delta h + \alpha y \frac{\partial h}{\partial y} + \tfrac{1}{2}\sigma^2 y^2 \frac{\partial^2 h}{\partial y^2}$$

for all points (x, y) where the partial derivatives of h involved in $L^c h$ exist.

We then put

(2.9) $$\mathcal{L}_1 h(x,y) = \begin{cases} \mathcal{L}h(x,y) & \text{for} \quad (x,y) \in \mathcal{S} \setminus (\ell_1 \cup \ell_2) \setminus [0,P] \\ \mathcal{L}_0 h(x,y) & \text{for} \quad (x,y) \in [0,P] \end{cases}$$

Note that at $[0, P]$ the only admissible consumption is $c = 0$.

By adapting Theorem 3.1 in [BØ] to our situation we get the following sufficient QVHJBI:

Theorem 2.1 *Let $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ be as defined in (1.9) and put $U = \mathcal{S} \setminus (\ell_1 \cup \ell_2)$, $\widetilde{U} = [0, \infty) \times U$.*

a) *Suppose we can find a locally bounded function $\psi : \mathcal{S} \to \mathbf{R}^+$ such that $\psi \in C^1(U)$ and*

(2.10) $$\varphi(s,x,y) := e^{-\delta s}\psi(x,y) \quad \textit{is stochastically } C^2 \textit{ with respect to } Z^c(t)$$
$$\textit{for all Markov controls } c = c(x,y)$$

(2.11) $$\mathcal{L}_1 \psi \le 0 \qquad \textit{a.e. with respect to } G(z_0, \cdot) \textit{ on } \widetilde{U} \textit{ for all } z_0 \in \widetilde{U}$$

(2.12) $$\psi(x,y) \ge \mathcal{M}\psi(x,y) \qquad \textit{for all } (x,y) \in U .$$

Then

(2.13) $$\psi(x,y) \ge \Psi(x,y) \qquad \textit{for all } (s,x,y) \in \widetilde{U} .$$

b) *Define the continuation region*

$$D = \{(x,y) \in U; \psi(x,y) > \mathcal{M}\psi(x,y)\}$$

Suppose

$$\mathcal{L}_1\psi(x,y) = 0 \qquad \text{on } D\,.$$

and that $\widehat{\xi}(x,y) = \widehat{\xi}_\psi(x,y)$ *(defined in (2.5)) exists for all* $(x,y) \in \mathcal{S}$. *Define*

$$c^*(x,y) = \begin{cases} (\frac{\partial\psi}{\partial x})^{\frac{1}{\gamma-1}} & \text{for} \quad (x,y) \in U \setminus [0,P] \\ 0 & \text{for} \quad (x,y) \in [0,P] \end{cases}$$

and define the impulse control

$$v^* := (\tau_1^*, \tau_2^*, \ldots; \xi_1^*, \xi_2^*, \ldots)$$

as follows:

Put $\tau_0^* = 0$ *and inductively*

(2.14) $$\tau_{k+1}^* = \inf\{t > \tau_k^*; (X^{(k)}(t), Y^{(k)}(t)) \notin D\}$$

(2.15) $$\xi_{k+1}^* = \widehat{\xi}(X^{(k)}(\tau_{k+1}^{*-}), Y^{(k)}(\tau_{k+1}^{*-}))$$

where $\widehat{\xi}$ *is as defined in (2.2) and* $(X^{(k)}, Y^{(k)})$ *is the process obtained by applying the combined control*

$$w^{(k)} := (c^*, (\tau_1^*, \ldots, \tau_k^*; \xi_1^*, \ldots, \xi_k^*))\;; \qquad k = 1, 2, \ldots$$

Suppose $w^* := (c^*, v^*) \in \mathcal{W}$ *and that*

(2.16) $$e^{-\delta t}\psi(X^{(w^*)}(t), Y^{(w^*)}(t)) \to 0 \qquad \text{as } t \to \infty \text{ a.s.}$$

and that the family

(2.17) $$\{e^{-\delta\tau}\psi(X^{(w^*)}(\tau), Y^{(w^*)}(\tau));\ \tau \ \text{ stopping time}\}$$

is uniformly integrable. Then

(2.18) $$\psi(x,y) = \Psi(x,y)$$

and w^* *is optimal.*

Proof. This follows by the proof of Theorem 3.1 in [BØ] with only minor modifications.

Note that the Hamilton-Jacobi-Bellman inequality (HJBI) (3.7) in [BØ] has the following form in our case, if $(x,y) \in U \setminus [0,P]$,

$$\mathcal{L}\psi(x,y) = \sup_{c \geq 0}\Big\{ -\delta\psi + (rx - c)\frac{\partial\psi}{\partial x} + \alpha y\frac{\partial\psi}{\partial y} + \frac{1}{2}\sigma^2 y^2 \frac{\partial^2\psi}{\partial y^2} + \frac{c^\gamma}{\gamma}\Big\} \leq 0\,.$$

This can only hold if $\frac{\partial\psi}{\partial x} > 0$ and then the supremum of this expression is obtained when

$$c = c^* = \left(\frac{\partial\psi}{\partial x}\right)^{\frac{1}{\gamma-1}}. \tag{2.19}$$

If $(x,y) \in [0,P]$ then only the zero consumption $c = c^* = 0$ is admissible so again by the HJBI we get $L^0\psi(0,y) = 0$. □

We can use this to prove the claim (1.20):

Corollary 2.2
a) *As in (1.17)-(1.18) let*

$$\Phi_0(s,x,y) = e^{-\delta s}C_1(x+y)^\gamma \tag{2.20}$$

be the value function for the Merton problem ($k = \lambda = 0$). *Then*

$$\Phi(s,x,y) \le \Phi_0(s,x,y) \qquad \text{for all } (s,x,y) \in \tilde{S}\ . \tag{2.21}$$

b) *Let b be a constant such that*

$$1 - \lambda \le b \le 1 + \lambda \tag{2.22}$$

Suppose

$$\delta > \gamma\alpha\ . \tag{2.23}$$

Then there exists $K < \infty$ such that

$$\Phi(s,x,y) \le e^{-\delta s}K(x+by)^\gamma \qquad \text{for all } (s,x,y) \in \tilde{S}\ . \tag{2.24}$$

Proof. a) We verify that $\varphi := \Phi_0$ satisfies the conditions of Theorem 2.1 a): First, φ is C^2 and therefore trivially stochastically C^2. Hence (2.10) holds. Second, $\psi := e^{rs}\varphi$ satisfies (2.11), because Φ_0 satisfies the Hamilton-Jacobi-Bellman equation. Third, if we put, as in (2.3)

$$x' = x'(\xi) = x - \xi - k - \lambda|\xi| \qquad \text{and} \qquad y' = y'(\xi) = y + \xi\ ,$$

then $x' + y' \le x + y$ for all x, y, ξ and therefore

$$\begin{aligned} \mathcal{M}\Phi_0(s,x,y) &= \sup_{\xi\neq 0} \Phi_0(s,x',y') = \sup_{\xi\neq 0}\{e^{-\delta s}C_1(x'+y')^\gamma\} \\ &\le e^{-\delta s}C_1(x+y)^\gamma = \Phi_0(s,x,y)\ , \end{aligned} \tag{2.25}$$

where C_1 is defined in (1.18). Therefore (2.12) also holds.

So by (2.13) we get (2.21).

b) We proceed as in a), except now we choose $K < \infty$ and define

$$u(x,y) = K(x+by)^\gamma \,. \tag{2.26}$$

Now we get

$$x' + by' = \begin{cases} x+by-k-\xi(1+\lambda-b) & \text{for } \xi > 0 \\ x+by-k-\xi(1-\lambda-b) & \text{for } \xi < 0 \end{cases}$$

Thus in any case we have, by (2.22),

$$x' + by' \leq x + by$$

and this proves that

$$u(x,y) \geq \mathcal{M}u(x,y) \,.$$

It remains to verify that $\psi := u$ satisfies (2.11): By (2.26) we get

$$\mathcal{L}u(x,y) = (x+by)^{\gamma-2}\Big[\Big(\frac{1-\gamma}{\gamma}(K\gamma)^{\frac{\gamma}{\gamma-1}} - \delta K\Big)(x+by)^2 + K\gamma(rx+\alpha by)(x+by) - \tfrac{1}{2}\sigma^2 K\gamma(1-\gamma)b^2y^2\Big].$$

Hence $\mathcal{L}u(x,y) \leq 0$ for all $(x,y) \in \mathcal{S}$ if and only if

$$\Big[\frac{1-\gamma}{\gamma}(K\gamma)^{\frac{\gamma}{\gamma-1}} - \delta K + K\gamma\alpha\Big](x+by)^2 \leq \tfrac{1}{2}\sigma^2 K\gamma(1-\gamma)b^2y^2$$

for all $(x,y) \in \mathcal{S}$. This holds if and only if

$$\delta > \gamma\alpha + (1-\gamma)(K\gamma)^{\frac{1}{\gamma-1}}. \tag{2.27}$$

If (2.23) holds, then (2.27) holds for K large enough. Thus (2.24) follows from Theorem 2.1a). □

Remark 2.3 Corollary 2.2 proves in particular that the value function Φ is *finite*. Moreover, $\Phi(s,x,y)$ is *bounded* on every straight line in $\mathcal{S}$ of the form

$$x + by = \text{constant},$$

for every constant $b \in [1-\lambda, 1+\lambda]$.

Remark 2.4 (Some comments on the boundary behaviour) Define the line segments ℓ_1, ℓ_2 by

$$\ell_1 = \{(x,y);\ x + (1-\lambda)y = k,\ x < 0\} \tag{2.28}$$

$$\ell_2 = \{(x,y);\ x + (1+\lambda)y = k,\ y < 0\} \tag{2.29}$$

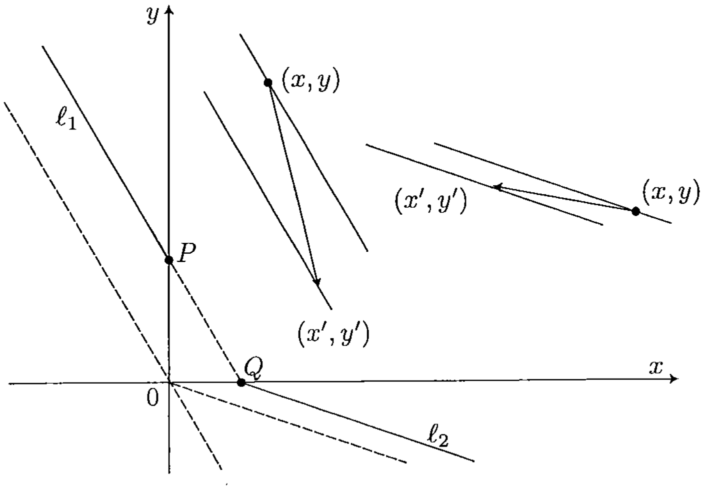

Figure 3

and let the points P, Q be the end points of these segments, i.e.

(2.30) $$P = \left(0, \frac{k}{1-\lambda}\right), \qquad Q = (k, 0) .$$

Suppose the current position of the investor is a point $(x, y) \in \mathcal{S}$. If we make a transaction of size ξ at that instant, then after the transaction the new position is given by

(2.31) $$\begin{cases} x' = x - \xi - \lambda|\xi| - k \\ y' = y + \xi \end{cases}$$

Hence

(2.32) $$x' + (1-\lambda)y' = x + (1-\lambda)y - k - \lambda(|\xi| + \xi)$$

and

(2.33) $$x' + (1+\lambda)y' = x + (1+\lambda)y - k - \lambda(|\xi| - \xi) .$$

In particular, if we *sell* stocks ($\xi < 0$) then $x' + (1-\lambda)y' = x + (1-\lambda)y - k$, so (x, y) will move to a point (x', y') on the line parallell to ℓ_1 lying $\frac{k}{1-\lambda}$ units below the parallell of ℓ_1 through (x, y). See Figure 3.

Similarly, if we *buy* stocks ($\xi > 0$) then $x' + (1+\lambda)y' = x + (1+\lambda)y - k$, so (x, y) will move to a point (x', y') on the line parallell to ℓ_2 lying $\frac{k}{1+\lambda}$ units below the parallell of ℓ_2 through (x, y).

We now use this to deduce the boundary behaviour of the value function Ψ on $\partial\mathcal{S}$.

(i) If $(x, y) \in \ell_1$ then we have to make an immediate transaction to avoid that the diffusion $Y(t)$ will take us out of $\mathcal{S}$. By the above we see that the only possibility is to *sell* stocks of such an amount that $(x', y') = (0, 0)$. We conclude that

$$\Psi(x,y) = \mathcal{M}\Psi(x,y) = 0 \qquad \text{for } (x,y) \in \ell_1\,. \tag{2.34}$$

(ii) If $(x,y) \in \ell_2$ we argue similarly: The only admissible action is to *buy* stocks immediately of such an amount that $(x', y') = (0,0)$. Hence

$$\Psi(x,y) = \mathcal{M}\Psi(x,y) = 0 \qquad \text{for } (x,y) \in \ell_2\,. \tag{2.35}$$

(iii) On the segment $0 < x < k$, $y = 0$ we are not allowed to make any transaction. There is no diffusion and all we can do is to consume optimally. Hence the HJB equation indicates that, with $\mathcal{L}$ as in (2.6) we should have

$$\mathcal{L}\Psi = -\delta\Psi + rx\frac{\partial\Psi}{\partial x} + \frac{1-\gamma}{\gamma}\Big(\frac{\partial\Psi}{\partial x}\Big)^{\frac{\gamma}{\gamma-1}} = 0 \qquad \text{for } x \in (0,k)\,, \tag{2.36}$$

provided that Ψ is smooth enough (see Section 4).

(iv) On the segment $x = 0$, $0 < y < \frac{k}{1-\lambda}$ we cannot consume because this would bring us outside $\mathcal{S}$. Hence the HJB equation indicates that

$$\mathcal{L}\Psi(0,y) = c^* = 0 \qquad \text{for } 0 < y < \tfrac{k}{1-\lambda}$$

and hence that

$$\mathcal{L}_0\Psi := -\delta\Psi + \alpha y\frac{\partial\Psi}{\partial y} + \tfrac{1}{2}\sigma^2 y^2\frac{\partial^2\Psi}{\partial y^2} = 0 \qquad \text{for } \ 0 < y < \tfrac{k}{1-\lambda}\,,$$

provided that Ψ is smooth enough (see Section 4).

Summarizing we see that the boundary behaviour of Ψ on $\partial\mathcal{S}$ can be described by

$$\begin{cases} \Psi(x,y) = \mathcal{M}\Psi(x,y) = 0 & \text{for } (x,y) \in \ell_1 \cup \ell_2 \\ \mathcal{L}\Psi(x,y) = 0 & \text{for } 0 \le x \le k,\ y = 0 \ \text{ i.e., } (x,y) \in [0,Q] \\ \mathcal{L}_0\Psi(x,y) = 0 & \text{for } x = 0,\ 0 \le y \le y_1 = \frac{k}{1-\lambda} \ \text{ i.e., } (x,y) \in [0,P]. \end{cases} \tag{2.37}$$

Note that Ψ is not continuous on $\partial\mathcal{S}$: The points $(0, \frac{k}{1-\lambda})$ and $(k,0)$ are points of discontinuity. However, Ψ is upper semicontinuous.

3 Viscosity solutions

Theorem 2.1 is a *verification theorem*, stating that if we can find a smooth enough function satisfying the required (quasi-) variational inequalities, then we have also found the value function of the problem. It is natural to ask if the converse is also true: Is the value function always a solution of the corresponding (quasi-) variational inequalities? The problem is that the value function need not be smooth enough for these inequalities to be welldefined in the strong sense. In fact, the value function is not even continuous at the points P and Q (see (2.35) and below). However, we shall see that the inequalities are satisfied in a weak sense: The value function is a *viscosity solution* of the (quasi-) variational inequalities.

We first recall the following concepts, which will be useful for us:

Definition 3.1 *If C is a topological space and $u: C \to \mathbf{R}$ is a function, then the* upper semi-continuous *(usc)* envelope $\overline{u}: C \to \mathbf{R}$ *and the* lower semi-continuous *(lsc)* envelope $\underline{u}: C \to \mathbf{R}$ *of u are defined by*

$$\overline{u}(x) = \limsup_{\substack{y \to x \\ y \in C}} u(y) , \qquad \underline{u}(x) = \liminf_{\substack{y \to x \\ y \in C}} u(y) , \quad \textit{respectively.}$$

We let USC(C) and LSC(C) denote the set of usc functions and lsc functions on C, respectively.

Note that in general we have

$$\underline{u} \leq u \leq \overline{u}$$

and that u is usc if and only if $u = \overline{u}$, u is lsc if and only if $u = \underline{u}$. In particular, u is continuous if and only if

$$\underline{u} = u = \overline{u} .$$

As in (2.7) we let $\mathcal{L}$ be the differential operator

$$\mathcal{L}h(x,y) = \sup_{c \geq 0} \left\{ -\delta h + (rx - c)\frac{\partial h}{\partial x} + \alpha y \frac{\partial h}{\partial y} + \frac{1}{2}\sigma^2 y^2 \frac{\partial^2 h}{\partial y^2} + \frac{c^\gamma}{\gamma} \right\} \tag{3.1}$$

and as in (2.2) we put

$$\mathcal{M}h(x,y) = \sup_{\xi \neq 0} \{ h(x', y'); (x', y') \in \mathcal{S} \} \qquad \text{for } (x,y) \in \mathcal{S} , \tag{3.2}$$

where

$$x' = x - k - \xi - \lambda|\xi| , \qquad y' = y + \xi . \tag{3.3}$$

The quasi-variational inequalities (2.9), (2.10) and (2.13) of Theorem 2.1 together with the boundary behaviour (2.37) can be combined into one equation as follows:

$$F(D^2\Psi(\zeta), D\Psi(\zeta), \Psi(\zeta), \Psi, \zeta) = 0 \qquad \text{for all } \zeta = (x,y) \in \mathcal{S} , \tag{3.4}$$

where

$$F: \mathbf{R}^{2\times 2} \times \mathbf{R}^2 \times \mathbf{R}^{\mathcal{S}} \times \mathbf{R}^2 \to \mathbf{R}$$

is defined by

$$F(P,p,f,\zeta) = \begin{cases} \max\{\Lambda(P,p,f,\zeta), (\mathcal{M}f - f)(\zeta)\} ; & \zeta \in \mathcal{S}^0 \\ \Lambda(P,p,f,\zeta) ; & \zeta \in [0,Q] \\ \Lambda_0(P,p,f,\zeta) ; & \zeta \in [0,P] \\ (\mathcal{M}f - f)(\zeta) ; & \zeta \in \ell_1 \cup \ell_2 \end{cases} \tag{3.5}$$

where

$$\Lambda(P,p,f,\zeta) = -\delta f + r\zeta_1 p_1 + \alpha \zeta_2 p_2 + \tfrac{1}{2}\sigma^2 \zeta_2^2 P_{22} + \max_{c \geq 0} \left(-cp_1 + \frac{c^\gamma}{\gamma} \right) \tag{3.6}$$

and

(3.7) $$\Lambda_0(P,p,f,\zeta) = -\delta f + \alpha\zeta_2 p_2 + \tfrac{1}{2}\sigma^2\zeta_2^2 P_{22}\,.$$

Note that F is not a local operator: The value of F at z depends on the value of g on the whole space $\mathcal{S}$.

Note that

(3.8) $$\overline{F}(P,p,f,\zeta) = \max\{\Lambda(P,p,f,\zeta), (\mathcal{M}f - f)(\zeta)\} \text{ for } all \quad \zeta \in \mathcal{S}$$

and that

(3.9) $$\underline{F}(P,p,f,\zeta) = F(P,p,f,\zeta) \qquad \text{(i.e. } F \text{ is lsc)}$$

Following Barles [B] we now give the definition of viscosity solution of equations of type (3.4):

Definition 3.2

a) *A function $u \in \mathrm{USC}(\mathcal{S})$ is a* viscosity subsolution *of*

(3.10) $$F(D^2u(\zeta), Du(\zeta), u(\zeta), u, \zeta) = 0 \qquad \textit{for all } \zeta = (x,y) \in \mathcal{S}$$

if for every function f which is C^2 in a neighboorhood of $\mathcal{S}$ and for every point $\zeta_0 \in \mathcal{S}$ such that $f \geq u$ on $\mathcal{S}$ and $f(\zeta_0) = u(\zeta_0)$ we have

(3.11) $$\overline{F}(D^2f(\zeta_0), Df(\zeta_0), u(\zeta_0), u, \zeta_0) \geq 0\,.$$

b) *A function $u \in \mathrm{LSC}(\mathcal{S})$ is a* viscosity supersolution *of (3.10) if for every function f which is C^2 in a neighboorhood of $\mathcal{S}$ and for every point $\zeta_0 \in \mathcal{S}$ such that $f \geq u$ on $\mathcal{S}$ and $f(\zeta_0) = u(\zeta_0)$ we have*

(3.12) $$\underline{F}(D^2f(\zeta_0), Df(\zeta_0), u(\zeta_0), u, \zeta_0) \leq 0\,.$$

c) *We say that a function $u: \mathcal{S} \to \mathbf{R}$ is a* viscosity solution *of (3.10) if u is locally bounded and $\overline{u}$ is a viscosity subsolution and $\underline{u}$ is a viscosity supersolution of (3.10).*

An equivalent definition of viscosity solutions which is useful for proving uniqueness results is the following (see [CIL, Section 2]):

Definition 3.3

a) *A function $u \in \mathrm{USC}(\mathcal{S})$ is a* viscosity subsolution *of (3.4) if*

(3.13) $$\overline{F}(P,p,u,\zeta) \geq 0 \qquad \textit{for all } (p,P) \in \bar{J}^{2,+}_{\mathcal{S}}u(\zeta),\ \zeta \in \mathcal{S}$$

b) *A function $u \in \mathrm{LSC}(\mathcal{S})$ is a* viscosity supersolution *of (3.4) if*

$$\underline{F}(P,p,u,\zeta) \leq 0 \qquad \textit{for all } (p,P) \in \bar{J}^{2,-}_{\mathcal{S}}u(\zeta),\ \zeta \in \mathcal{S}\,.$$

Here the second order "superjets" $J_{\mathcal{S}}^{2,+}$, $J_{\mathcal{S}}^{2,-}$ *and their "closures"* $\bar{J}_{\mathcal{S}}^{2,+}$, $\bar{J}_{\mathcal{S}}^{2,-}$ *are defined by*

$$J_{\mathcal{S}}^{2,+}u(\zeta) = \{(p,P) \in \mathbf{R}^2 \times \mathbf{R}^{2\times 2}\ ;$$

$$\limsup_{\substack{\eta\to\zeta \\ \eta\in\mathcal{S}}}\{[u(\eta) - u(\zeta) - p\cdot(\eta-\zeta) - \tfrac{1}{2}(\eta-\zeta)^T P(\eta-\zeta)]|\eta-\zeta|^{-2}\} \le 0\} \tag{3.14}$$

(where $(\ )^T$ *denotes matrix transposed)*

$$\bar{J}_{\mathcal{S}}^{2,+}u(\zeta) = \{(p,P) \in \mathbf{R}^2 \times \mathbf{R}^{2\times 2}\ ;\ \exists(\zeta_n, p_n, P_n) \in \mathcal{S} \times \mathbf{R}^2 \times \mathbf{R}^{2\times 2}\ ,$$

with $(p_n, P_n) \in J_{\mathcal{S}}^{2,+}u(\zeta_n)$ *and* $(\zeta_n, u(\zeta_n), p_n, P_n)$

$$\to (\zeta, u(\zeta), p, P) \quad \textit{when}\ \ n\to\infty\} \tag{3.15}$$

and

$$J_{\mathcal{S}}^{2,-}u = -J_{\mathcal{S}}^{2,+}(-u)\ , \qquad \bar{J}_{\mathcal{S}}^{2,-}u = -\bar{J}_{\mathcal{S}}^{2,+}(-u)\ . \tag{3.16}$$

We are now ready for the first main result of this section:

Theorem 3.4 *The value function* Ψ *is a viscosity solution of (3.4).*

Proof. We first make some useful observations:
Suppose $w = (c, v) \in \mathcal{W}$ is an admissible control with $v = (\tau_1, \tau_2, \ldots; \xi_1, \xi_2, \ldots)$ where $\tau_1 > 0$ a.s. Then by the *Markov property* we have, with J^w as in (1.10),

$$J^w(z) = E^z\Big[\int_0^\tau e^{-\delta(s+t)}\frac{c^\gamma(t)}{\gamma}dt + J^w(Z^{(w)}(\tau))\Big] \tag{3.17}$$

for all stopping times $\tau \le \tau_1$.

Note that

$$\Psi(\zeta) \ge \mathcal{M}\Psi(\zeta) \qquad \text{for all } \zeta \in \mathcal{S}\ . \tag{3.18}$$

To see this, suppose on the contrary that there exists ζ_1 such that

$$\Psi(\zeta_1) < \mathcal{M}\Psi(\zeta_1)\ .$$

This would mean that we could improve the performance at ζ_1 by making a transaction immediately. This contradicts that $\Psi(\zeta_1)$ is the optimal performance value at ζ_1.

Also note that since τ_1 is a stopping time, we know that $\{\omega; \tau_1(\omega) = 0\}$ is $\mathcal{F}_0$-measurable and hence this event has either probability 1 or 0. So we either have

$$\tau_1(\omega) = 0 \quad \text{a.s.} \quad \text{or} \qquad \tau_1(\omega) > 0 \quad \text{a.s.}$$

a) We prove that Ψ is a viscosity subsolution. To this end, let f be a C^2 function in a neighboorhood of $\mathcal{S}$ and let $\zeta_0 \in \mathcal{S}$ be such that $f \ge \Psi$ on $\mathcal{S}$ and $f(\zeta_0) = \Psi(\zeta_0)$. We consider two cases separately:

Case 1. $\Psi(\zeta_0) = \mathcal{M}\Psi(\zeta_0)$.
In this case ζ_0 cannot be a point in $[0,P] \cup [0,Q]$. Therefore $F(D^2 f(\zeta_0), Df(\zeta_0), f(\zeta_0), \Psi, \zeta_0) \geq (\Psi - \mathcal{M}\Psi)(\zeta_0) = 0$ and hence (3.11) holds at ζ_0 for $u = \Psi$.

Case 2. $\Psi(\zeta_0) > \mathcal{M}\Psi(\zeta_0)$.
In this case ζ_0 cannot be a point in $\ell_1 \cup \ell_2$. So it suffices to prove that $\mathcal{L}f(\zeta_0) \geq 0$ if $\zeta_0 \in \mathcal{S}^0 \cup [0,Q]$ and that $\mathcal{L}_0 f(\zeta_0) \geq 0$ if $\zeta_0 \in [0,P]$.

We argue by contradiction: *Suppose* $\zeta_0 \in \mathcal{S}^0 \cup [0,Q]$ *and* $\mathcal{L}f(\zeta_0) < 0$. Then from the definition of $\mathcal{L}$ we deduce that $\frac{\partial f}{\partial x}(\zeta_0) > 0$. Hence by continuity $\frac{\partial f}{\partial x}(\zeta) > 0$ in a neighboorhood G of ζ_0. But then, with $\zeta = (x,y)$,

$$\mathcal{L}f(\zeta) = -\delta f(\zeta) + (rx - \widehat{c})\frac{\partial f}{\partial x} + \alpha y \frac{\partial f}{\partial y} + \tfrac{1}{2}\sigma^2 y^2 \frac{\partial^2 f}{\partial y^2} + \frac{\widehat{c}^\gamma}{\gamma}$$

with $\widehat{c} = \widehat{c}(\zeta) = (\frac{\partial f}{\partial x})^{\frac{1}{\gamma-1}}$ for all $\zeta \in G \cap \mathcal{S}$.

Hence $\mathcal{L}f(\zeta)$ is continuous on $G \cap \mathcal{S}$ and so there exists a (bounded) neighboorhood G_0 of ζ_0 such that

$$\mathcal{L}f(\zeta) < \tfrac{1}{2}\mathcal{L}f(\zeta_0) < 0 \qquad \text{for all } \zeta \in G_0 \cap \mathcal{S}\,. \tag{3.19}$$

Now let ε be any number such that

$$0 < \varepsilon < (\Psi - \mathcal{M}\Psi)(\zeta_0)\,. \tag{3.20}$$

Let $\tilde{w} = (\tilde{c}, \tilde{v})$ with $\tilde{v} = (\tilde{\tau}_1, \tilde{\tau}_2, \ldots; \tilde{\xi}_1, \tilde{\xi}_2, \ldots)$ be an ε-optimal control, in the sense that

$$\Psi(x_0, y_0) \leq J^{\tilde{w}}(0, x_0, y_0) + \varepsilon\,; \qquad \zeta_0 = (x_0, y_0)\,.$$

If $\tilde{\tau}_1 = 0$ a.s. then $(X^{(\tilde{w})}, Y^{(\tilde{w})})$ makes an immediate jump from ζ_0 to some point $\zeta_0' \in \mathcal{S}$ and hence

$$J^{\tilde{w}}(0, \zeta_0) = J^{\tilde{w}}(0, \zeta_0')\,.$$

But then

$$\Psi(\zeta_0) \leq J^{\tilde{w}}(0,\zeta_0) + \varepsilon = J^{\tilde{w}}(0, \zeta_0') + \varepsilon \leq \Psi(\zeta_0') + \varepsilon \leq \mathcal{M}\Psi(\zeta_0) + \varepsilon$$

which contradicts (3.20). We conclude that $\tilde{\tau}_1 > 0$ a.s.

Fix $R < \infty$ and define τ to be the stopping time

$$\tau = \tilde{\tau}_1 \wedge R \wedge \inf\{t > 0; (X^{(\tilde{w})}(t), Y^{(\tilde{w})}(t)) \notin G_0\}\,. \tag{3.21}$$

Then by the Dynkin formula we have

$$E^{\zeta_0}[e^{-\delta\tau} f(X^{(\tilde{w})}(\tau), Y^{(\tilde{w})}(\tau))] = f(\zeta_0) + E^{\zeta_0}\Big[\int_0^\tau e^{-\delta t} L^{\tilde{c}} f(X^{(\tilde{w})}(t), Y^{(\tilde{w})}(t))dt\Big]. \tag{3.22}$$

Combining this with (3.17) we get

$$\begin{aligned}\Psi(\zeta_0) &\le J^{(\tilde w)}(0,\zeta_0)+\varepsilon\\ &= E^{0,\zeta_0}\Big[\int_0^{\tau} e^{-\delta t}\frac{\tilde c^{\gamma}(t)}{\gamma}dt + J^{\tilde w}(Z^{(\tilde w)}(\tau))\Big]+\varepsilon\\ &\le E^{0,\zeta_0}\Big[\int_0^{\tau} e^{-\delta t}\frac{\tilde c^{\gamma}(t)}{\gamma}dt + e^{-\delta\tau}\Psi(X^{(\tilde w)}(\tau),Y^{(\tilde w)}(\tau))\Big]+\varepsilon\\ &\le E^{0,\zeta_0}\Big[\int_0^{\tau} e^{-\delta t}\frac{\tilde c^{\gamma}(t)}{\gamma}dt + e^{-\delta\tau}f(X^{(\tilde w)}(\tau),Y^{(\tilde w)}(\tau))\Big]+\varepsilon\\ &= f(\zeta_0)+E^{0,\zeta_0}\Big[\int_0^{\tau} e^{-\delta t}\Big(L^{\tilde c}f(X^{(\tilde w)}(t),Y^{(\tilde w)}(t))+\frac{\tilde c^{\gamma}(t)}{\gamma}\Big)dt\Big]+\varepsilon\\ &\le \Psi(\zeta_0)+E^{0,\zeta_0}\Big[\int_0^{\tau} e^{-\delta t}\mathcal{L}f(X^{(\tilde w)}(t),Y^{(\tilde w)}(t))dt\Big]+\varepsilon\,.\end{aligned}$$

We conclude from this that

$$E^{0,\zeta_0}\Big[\int_0^{\tau} e^{-\delta t}\mathcal{L}f(X^{(\tilde w)}(t),Y^{(\tilde w)}(t)dt\Big]\ge -\varepsilon\,. \tag{3.23}$$

On the other hand, from (3.19) we deduce that

$$E^{0,\zeta_0}\Big[\int_0^{\tau} e^{-\delta t}\mathcal{L}f(X^{(\tilde w)}(t),Y^{(\tilde w)}(t))dt\Big]\le \frac{1}{2\delta}\mathcal{L}f(\zeta_0)(1-E^{0,\zeta_0}[e^{-\delta\tau}])\,. \tag{3.24}$$

Thus we see that (3.23) contradicts (3.24) if $\varepsilon>0$ is chosen small enough.

This contradiction shows that we must have

$$\mathcal{L}f(\zeta_0)\ge 0 \qquad \text{if } \zeta_0\in\mathcal{S}^0\cup[0,Q]\,.$$

A similar argument shows that $\mathcal{L}_0 f(\zeta_0)\ge 0$ if $\zeta_0\in[0,P]$. We conclude that $u=\Psi$ is a viscosity subsolution of (3.10).

b) Next we prove that Ψ is a viscosity supersolution. So we let f be a C^2 function in a neighbourhood of $\mathcal{S}$ and we let $\zeta_0\in\mathcal{S}$ be such that $f\le\Psi$ on $\mathcal{S}$ and $f(\zeta_0)=\Psi(\zeta_0)$. We want to show that

$$F(D^2f(\zeta_0),Df(\zeta_0),f(\zeta_0),\Psi,\zeta_0)\le 0\,.$$

Since $\mathcal{M}\Psi-\Psi\le 0$ everywhere we see from (3.5) that this holds for $\zeta_0\in\ell_1\cup\ell_2$ and it suffices to show that

$$\mathcal{L}f(\zeta_0)\le 0 \qquad \text{for } \zeta_0\in\mathcal{S}^0\cup[0,Q] \tag{3.25}$$

and

(3.26) $$\mathcal{L}_0 f(\zeta_0) \le 0 \qquad \text{for } \zeta_0 \in [0,P]\,.$$

For $\varepsilon > 0$ let $\widehat{w} = \widehat{w}_{\varepsilon,c}$ be an admissible control beginning with a constant consumption rate $c \ge 0$ and no transactions up to the first time $\tau = \tau_\varepsilon$ the process $Z^c(t)$ exits from

$$K_\varepsilon = \{(s,x,y);|(s,x,y)-(0,x_0,y_0)| < \varepsilon\} \cap \widetilde{\mathcal{S}}$$

where $\zeta_0 = (x_0, y_0)$.

Then by combining Dynkin's formula with the *dynamic programming principle* ([Kr, Th. 6, p. 150]) we get

$$\begin{aligned}\Psi(\zeta_0) &\ge E^{0,\zeta_0}\Big[\int_0^\tau e^{-\delta t}\frac{c^\gamma}{\gamma}dt + e^{-\delta\tau}\Psi(X^{(\hat{w})}(\tau), Y^{(\hat{w})}(\tau))\Big]\\ &\ge E^{0,\zeta_0}\Big[\int_0^\tau e^{-\delta t}\frac{c^\gamma}{\gamma}dt + e^{-\delta\tau}f(X^{(\hat{w})}(\tau), Y^{(\hat{w})}(\tau))\Big]\\ &= f(\zeta_0) + E^{0,\zeta_0}\Big[\int_0^\tau e^{-\delta t}\Big(L^c f(X^{(\hat{w})}(t), Y^{(\hat{w})}(t)) + \frac{c^\gamma}{\gamma}\Big)dt\Big].\end{aligned}$$

We conclude that

$$E^{0,\zeta_0}\Big[\int_0^{\tau_\varepsilon} e^{-\delta t}\Big(L^c f(X^{(\hat{w})}(t), Y^{(\hat{w})}(t)) + \frac{c^\gamma}{\gamma}\Big)dt\Big] \le 0\,.$$

By dividing by $E^{0,\zeta_0}[\tau_\varepsilon]$ and letting $\varepsilon \to 0$ we conclude that

(3.27) $$L^c f(\zeta_0) + \frac{c^\gamma}{\gamma} \le 0$$

for all $c \ge 0$ such that $\widehat{w}_{\varepsilon,c}$ is admissible for ε small enough. If $\zeta_0 \in \mathcal{S}^0 \cup [0,Q]$ this is clearly the case for all $c \ge 0$ and therefore (3.27) implies that $\mathcal{L}f(\zeta_0) \le 0$. If $\zeta_0 \in [0,P]$ the only such admissible c is $c = 0$. Therefore we get $\mathcal{L}_0 f(\zeta_0) \le 0$ in this case, as required. □

Next we turn to the question of uniqueness. Our second main result in the section is the following:

Theorem 3.5 (Comparison theorem)

a) *Suppose u is a viscosity subsolution and v is a viscosity supersolution of (3.4) and that u and v satisfy the estimates*

(3.28) $$-C|x+y|^\gamma \le u(x,y) \qquad \textit{for all} \quad (x,y) \in \mathcal{S}$$

(3.29) $$v(x,y) \le C|x+y|^\gamma \qquad \textit{for all} \quad (x,y) \in \mathcal{S}$$

for some constant $C < \infty$. Then

$$u \le v \qquad in \ \mathcal{S}^0 .$$

b) *Moreover, if in addition*

$$v(x,y) = \liminf_{\substack{(\xi,\eta)\to(x,y)\\ (\xi,\eta)\in\mathcal{S}^0}} v(\xi,\eta) \qquad \text{for all } (x,y)\in\partial\mathcal{S} \tag{3.30}$$

then

$$u \le v \qquad in \ \mathcal{S} .$$

Corollary 3.6 *Suppose u and v are two viscosity solutions of (3.4) satisfying (3.28) and (3.29). Then*

$$u = v \qquad in \ \mathcal{S}^0$$

and u is continuous in $\mathcal{S}^0$.

Proof of Corollary 3.6. Since u is a viscosity solution it follows that $\overline{u}$ is a viscosity subsolution and $\underline{u}$ is a viscosity supersolution and similarly for v. Hence, by Theorem 3.5,

$$\overline{u} \le \underline{v} \le \overline{v} \le \underline{u} \le \overline{u} \qquad \text{in } \ \mathcal{S}^0 .$$

This implies that

$$\underline{u} = \overline{u} = \underline{v} = \overline{v} .$$

□

Proof of Theorem 3.5. The proof is based on the technique of Ishii (see [B], [CIL] and [IL]) and on the proofs of Lemma 3.12 of [AMS] and Theorem 5.7 in [AST]. Consequently we shall not give here a detailed proof but rather point out the special treatment required to handle the non local intervention operator $\mathcal{M}$.

Let u and v be as in Theorem 3.5. We first construct a strict supersolution of (3.4) by making a perturbation of v:

Choose $\gamma' \in (\gamma, 1)$ such that (see (1.19))

$$\delta > \gamma'\Big[r + \frac{(\alpha-r)^2}{2\sigma^2(1-\gamma')}\Big]. \tag{3.31}$$

Put

$$g(x,y) = (x+y)^{\gamma'} \tag{3.32}$$

and choose $\varepsilon > 0$. Then

$$\mathcal{M}(v+\varepsilon g) \le \mathcal{M}v + \varepsilon\mathcal{M}g \tag{3.33}$$

and hence

$$\mathcal{M}(v+\varepsilon g) - (v+\varepsilon g) \le (\mathcal{M}v - v) + \varepsilon(\mathcal{M}g - g) . \tag{3.34}$$

Since v is a supersolution we have

$$\mathcal{M}v - v \leq 0 . \tag{3.35}$$

Moreover, with $\zeta = (x, y)$,

$$\begin{aligned}(\mathcal{M}g - g)(\zeta) &= \sup_{\xi\neq 0}\{g(x - k - \xi - \lambda|\xi|, y + \xi)\} - g(x, y)\\ &= \sup_{\xi\neq 0}\{(x + y - k - \lambda|\xi|)^{\gamma'}\} - (x + y)^{\gamma'}\\ &\leq (x + y)^{\gamma'}\Big[\Big(1 - \frac{k}{x_1 + x_2}\Big) - 1\Big] .\end{aligned} \tag{3.36}$$

Therefore, for each compact subset C of $\mathcal{S}\setminus\{0\}$ there exists $\rho_1 > 0$ such that $(\mathcal{M}g - g)(\zeta) \leq -\rho_1$ for all $\zeta \in C$. So from (3.34) and (3.35) we get

$$\mathcal{M}(v + \varepsilon g) - (v + \varepsilon g) \leq -\varepsilon\rho_1 \qquad \text{in } C . \tag{3.37}$$

Now if we define the operator L^0 by (see (2.2))

$$L^0 = -\delta I + rx\frac{\partial}{\partial x} + \alpha y\frac{\partial}{\partial y} + \tfrac{1}{2}\sigma^2 y^2\frac{\partial^2}{\partial y^2} \tag{3.38}$$

where I is the identity operator, then

$$\begin{aligned}L^0 g(x, y) &= -\delta(x + y)^{\gamma'} + (rx + \alpha y)\gamma'(x + y)^{\gamma'-1}\\ &\quad + \tfrac{1}{2}\sigma^2 y^2\gamma'(\gamma' - 1)(x + y)^{\gamma'-2}\\ &= (x + y)^{\gamma'}\Big[- \delta + \gamma'\frac{rx + \alpha y}{x + y} + \tfrac{1}{2}\sigma^2\gamma'(\gamma' - 1)\frac{y^2}{(x + y)^2}\Big].\end{aligned} \tag{3.39}$$

If we put

$$\eta = \frac{y}{x + y} \qquad \text{so that} \qquad \frac{x}{x + y} = 1 - \eta$$

we get

$$L^0 g(x, y) = (x + y)^{\gamma'}[-\delta + \gamma' r + \gamma'(\alpha - r)\eta + \tfrac{1}{2}\sigma^2\gamma'(\gamma' - 1)\eta^2] .$$

By (3.31) it follows that

$$L^0 g(x, y) < 0 \qquad \text{for all } (x, y) \neq (0, 0) .$$

Consequently, on every compact C of $\mathcal{S}\setminus\{0\}$ there exists $\rho_2 > 0$ such that

$$L^0 g(x, y) - \max_{c\geq 0}\Big(- c\frac{\partial g}{\partial x}\Big) \leq -\rho_2 \qquad \text{on } C .$$

Therefore, since v is a supersolution of (3.4) we conclude that on every compact C of $\mathcal{S} \setminus \{0\}$ there exists $\rho > 0$ such that

$$v_\varepsilon := v + \varepsilon g$$

is a viscosity supersolution of

$$F(D^2 v_\varepsilon(\zeta), Dv_\varepsilon(\zeta), v_\varepsilon(\zeta), v_\varepsilon, \zeta) = -\varepsilon\rho \qquad \text{for } \zeta \in C .$$

Let us now prove the theorem by contradition. Assume that

$$\sup_{\zeta \in \mathcal{S}} \{u(\zeta) - v(\zeta)\} > 0 . \tag{3.40}$$

Choose $\varepsilon > 0$ such that

$$\sup_{\zeta \in \mathcal{S}} \{u(\zeta) - v_\varepsilon(\zeta)\} > 0 . \tag{3.41}$$

Define

$$h(\zeta) := u(\zeta) - v_\varepsilon(\zeta) . \tag{3.42}$$

Since h is upper semicontinuous and tends to $-\infty$ when $|\zeta| \to \infty$, the set

$$\text{Argmax } h := \{\overline{\zeta}; h(\overline{\zeta}) = \sup\{h(\zeta); \zeta \in \mathcal{S}\}\}$$

is nonempty and compact in $\mathcal{S} \setminus \{0\}$. Choose an open set $G \subset \mathcal{S} \setminus \{0\}$ containing this compact (G open relative to $\mathcal{S}$) and with $\overline{G}$ compact. In order to get a contradition it suffices to prove that

$$u \le v_\varepsilon \qquad \text{in } G .$$

Thus we have reduced the problem to proving a comparison theorem for a strict supersolution v_ε and a subsolution u of (3.4) in an open subset G of $\mathcal{S} \setminus \{0\}$ with compact closure $\overline{G}$, when the supremum of $u - v_\varepsilon$ is attained in G only. This is proved by using Ishii's technique, adapted as in [B, Theorem 4.6] and in [AMS, Theorem 5.7] for the boundary conditions. We now explain this in more detail:

For $j = 1, 2, \ldots$ define, for $(\zeta, \eta) \in \mathcal{S} \times \mathcal{S}$,

$$H_j(\zeta, \eta) = u(\zeta) - v_\varepsilon(\eta) - \frac{j}{2}|\zeta - \eta|^2 \tag{3.43}$$

and put

$$m_j = \sup\{H_j(\zeta, \eta); (\zeta, \eta) \in \mathcal{S} \times \mathcal{S}\} \tag{3.44}$$

and

$$m = \sup\{h(\zeta); \zeta \in \mathcal{S}\} . \tag{3.45}$$

Proceeding exactly as in [AMS] we obtain that there exist ζ_j, η_j in $\mathcal{S}$ such that

(3.46) $$m_j = H_j(\zeta_j, \eta_j) < \infty \, .$$

Moreover,

(3.47) $$j|\zeta_j - \eta_j|^2 \to 0 \qquad \text{as } j \to \infty$$

and

(3.48) $$m_j \to m \qquad \text{as } j \to \infty \, .$$

Suppose

(3.49) $$\text{Argmax } h \text{ is contained in } \mathcal{S}^0$$

and choose $\hat{\zeta} \in \mathcal{S}^0$ such that

$$m = h(\hat{\zeta}) \, .$$

Then we get that

(3.50) $$(\zeta_j, \eta_j) \in \mathcal{S}^0 \times \mathcal{S}^0 \qquad \text{for } j \text{ large enough} \, .$$

By applying [CIL, Theorem 3.2] we now obtain that there exist 2×2 matrices P_j, Q_j such that

$$(p_j, P_j) \in \bar{J}^{2,+}u(\zeta_j) \qquad \text{and} \qquad (q_j, Q_j) \in \bar{J}^{2,-}v_\varepsilon(\eta_j)$$

and

(3.51) $$\begin{bmatrix} P_j & 0 \\ 0 & -Q_j \end{bmatrix} \leq 3j \begin{bmatrix} I & -I \\ -I & I \end{bmatrix} ,$$

where

$$p_j = j(\zeta_j - \eta_j) \qquad \text{and} \qquad q_j = p_j \, .$$

Since u is a subsolution and v_ε is a strict supersolution we obtain

(3.52) $$\overline{F}(P_j, p_j, u, \zeta_j) \geq 0$$

and

(3.53) $$\underline{F}(Q_j, q_j, v_\varepsilon, \eta_j) \leq -\varepsilon\rho \, .$$

From (3.53) it follows that

$$\Lambda(Q_j, q_j, v_\varepsilon, \eta_j) < 0$$

and proceeding as in [AMS] we obtain

(3.54) $$\Lambda(P_j, p_j, u, \zeta_j) - \Lambda(Q_j, q_j, v_\varepsilon, \eta_j) < 0 \, .$$

Consequently $\Lambda(P_j, p_j, u, \zeta_j) < 0$ and from (3.52) we obtain that

(3.55) $$(\mathcal{M}u - u)(\zeta_j) \geq 0 \, .$$

From (3.53) we have

(3.56) $$(\mathcal{M}v_\varepsilon - v_\varepsilon)(\eta_j) \leq -\varepsilon\rho < 0 \, .$$

Therefore, combining (3.55) and (3.56) we get

$$m_j < u(\zeta_j) - v_\varepsilon(\eta_j) < \mathcal{M}u(\zeta_j) - \mathcal{M}v_\varepsilon(\eta_j) - \varepsilon\rho \,. \tag{3.57}$$

Since $\zeta_j, \eta_j \to \hat{\zeta}$ and u is usc, we obtain

$$m < \liminf_{j\to\infty} [\mathcal{M}u(\zeta_j) - \mathcal{M}v_\varepsilon(\eta_j)] \,. \tag{3.58}$$

Since u is usc and v_ε is lsc we see, after some reflections, that

$$\limsup_{j\to\infty} \mathcal{M}u(\zeta_j) \le \mathcal{M}u(\hat{\zeta}) \tag{3.59}$$

and

$$\limsup_{j\to\infty} (-\mathcal{M}v_\varepsilon(\eta_j)) \le -\mathcal{M}v_\varepsilon(\hat{\zeta}) \,. \tag{3.60}$$

Hence we get the desired contradition

$$\begin{aligned} m &< \mathcal{M}u(\hat{\zeta}) - \mathcal{M}v_\varepsilon(\hat{\zeta}) \\ &= \sup_{\xi_1\neq 0}\{u(\hat{x}'(\xi_1), \hat{y}'(\xi_1))\} - \sup_{\xi_2\neq 0}\{v_\varepsilon(\hat{x}'(\xi_2), \hat{y}'(\xi_2))\} \\ &\le \sup_{\xi\neq 0}\{(u - v_\varepsilon)(\hat{x}'(\xi), \hat{y}'(\xi))\} \\ &\le \sup\{(u - v_\varepsilon)(\zeta); \zeta \in \mathcal{S}\} = m \,, \end{aligned}$$

where $\hat{\zeta} = (\hat{x}, \hat{y})$ and (see (2.3))

$$\hat{x}'(\xi) = \hat{x} - \xi - \lambda|\xi| - k \,, \qquad \hat{y}'(\xi) = \hat{y} + \xi \,.$$

This contradiction shows that assumption (3.49) cannot hold. Therefore we must have

$$\hat{\zeta} \in \text{Argmax}\, h \cap \partial\mathcal{S} \,. \tag{3.61}$$

To treat the boundary points we proceed exactly as in the proof of Theorem 5.7 of [AST], which itself is based on the proof of Theorem 4.6 in [B] (see Appendix, p. 166).

From (3.30) there exists a sequence $\{\zeta_j\} \subset \mathcal{S}^0 \cap G$ converging to $\hat{\zeta}$ such that $v_\varepsilon(\zeta_j) \to v_\varepsilon(\hat{\zeta})$ when $j \to \infty$. Let $\varepsilon_j = |\zeta_j - \hat{\zeta}|$. The idea is now to introduce the test function

$$w_j(\zeta, \eta) = u(\zeta) - v_\varepsilon(\eta) - \theta_j(\zeta, \eta) \,; \qquad (\zeta, \eta) \in \mathcal{S} \times \mathcal{S}$$

where

$$\theta_j(\zeta, \eta) = \frac{|\zeta - \eta|^2}{2\varepsilon_j} + \tfrac{1}{4}\Big(\frac{d(\eta) - d(\zeta)}{d(\zeta_j)} - 1\Big)^4 + \tfrac{1}{4}|\zeta - \hat{\zeta}|^4 \,.$$

Here $d(\eta)$ denotes the distance from η to $\partial\mathcal{S}$ and similarly for $d(\zeta), d(\zeta_j)$.

Following exactly the same steps as in [AST] and treating the term $\mathcal{M}u - u$ as we did before, we obtain a contradiction also in the case (3.61). This shows that (3.40) cannot hold and this completes the proof of Theorem 3.5. □

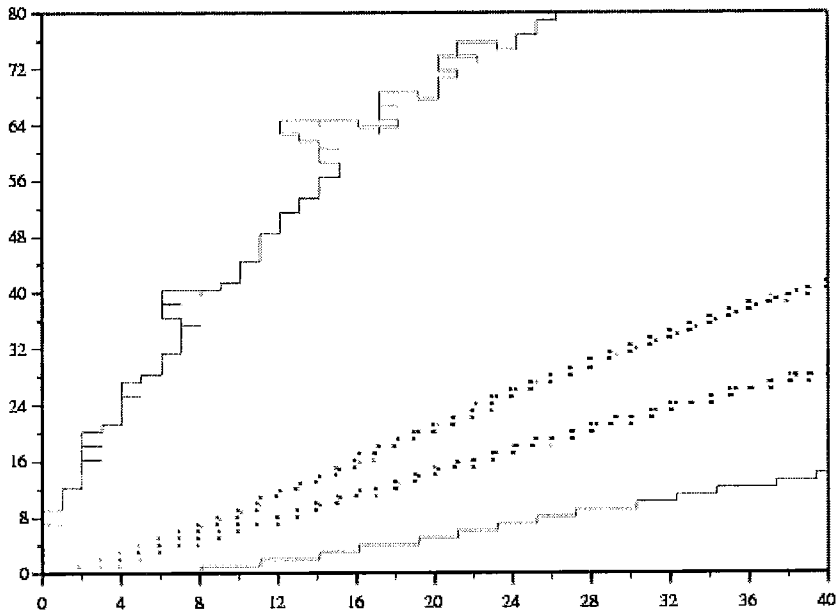

Figure 4: Optimal policy for $k = 0.1, \lambda = 0.01, \sigma = 0.3, r = 0.07, \delta = 0.1, \gamma = 0.3, \alpha = 0.11$

4 Numerical results

In this section we present the result of a numerical method used to approximate the viscosity solution of (3.4) in the case when the solvency region is $\mathcal{S}_+ = [0, \infty) \times [0, \infty)$ (see Remark 1.1). This method will be detailed in a forthcoming paper [CØS]. It is based on an iterative method which permits us to obtain the QVHJBI as a limit of variational HJB inequalities. Each variational inequality is approximated by a finite difference scheme and then solved by a Howard algorithm.

Figure 4 represents the boundary of the no-transaction region for the following values of the parameters: $k = 0.1, \lambda = 0.01, \sigma = 0.3, r = 0.07, \delta = 0.1, \gamma = 0.3, \alpha = 0.11$. The dotted lines (situated inside the no-transaction region) represent the set of positions reached after a transaction.

Remark For results on viscosity solutions of QVIs corresponding to impulse control problems (which, however, do not apply to our situation), see [I], [P] and [TY].

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