Optimal Contracts with Shirking

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Abstract

I explicitly derive the optimal dynamic incentive contract in a standard continuous-time agency setting where the agent has a shirking action. My solution generates two dynamic contracts new to the literature. Both contracts include phases when the agent frequently shirks. In one contract, the shirking phases are relaxation periods rewarding the agent for good performance. In the other, the shirking phases are suspension-type arrangements punishing the agent for poor performance. In addition, I also explore the relationships between optimal contracting and taxes, bargaining and renegotiation.

Keywords: Principal-agent problem, optimal contracts, shirking, heterogeneous discounting, frequent shirking, hidden actions, hidden compensation, continuous-time, agency, quiet-life, suspension, renegotiation.
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1 Introduction

The paper explicitly solves the optimal contracting problem in a standard Brownian framework. This continuous-time setting models a dynamic principal-agent relationship where some asset (a firm, a project etc.) owned by a risk-neutral principal is contracted out to a risk-neutral agent to manage. The asset's variable cash flow is driven by Brownian motion. The profitability of the asset is influenced by the hidden actions of the agent. At any moment in time, the agent can either choose the high action or the shirking action. Shirking hurts asset performance but gives the agent extra utility. The principal owns the cash flow. To properly motivate the agent, the principal writes a contract which stipulates a suggested course of action, a cash compensation plan and a termination clause.

Previous progress towards the resolution of the optimal contracting problem has been made by DeMarzo and Sannikov (2006). That paper derives the optimal contract subject to the agent never shirking and the conditions under which the no-shirking restriction is without loss of generality. I call this contract the optimal Baseline contract. The paper also describes conditions under which the optimal contract involves the agent shirking forever. In my paper, I show that, in addition to the two forms described by DeMarzo and Sannikov (2006), the optimal contract can take on two more forms, for a total of four optimal contractual forms. See Theorems 2 and 3. Unlike the Baseline form, the two new forms both include phases when the agent frequently shirks.

Theorems 1 and 2 characterize the optimal value function and formally solve the optimal contracting problem. Theorem 3 then interprets the results and shows how all four optimal contractual forms can be thought of as different realizations of a single general optimal contractual form. In this general form, the principal picks good and poor performance thresholds and there is a rating tracking the agent's performance. Over time, the rating moves according to the performance of the underlying asset. When the rating is strictly between the thresholds, the agent does not shirk. Whenever the rating reaches the good performance threshold, the agent is rewarded and possibly shirks, and whenever the rating drops to the poor performance threshold, the agent is punished and possibly shirks. I then argue there are four different ways to reward and punish the agent at these thresholds, giving the four different optimal contractual forms.

One of the two new forms is called the Quiet-Life form. In a Quiet-Life contract, after sustained good performance, a relaxed "Quiet-Life phase" is triggered. During this phase the incentives of the contract are frequently (though not always) unresponsive to asset performance. One can think of the principal as frequently ignoring the performance of the asset. This causes the agent to frequently shirk. After the Quiet-Life phase concludes, the incentives return to being sensitive to asset performance all the time, and the agent applies high action all the time again. Then either sustained good performance brings about another round of the Quiet-Life phase or sustained poor performance triggers termination. In a Quiet-Life contract, the Quiet-Life phases represent a form of hidden compensation rewarding the agent after he reaches the good performance threshold. In Section 6.1, I compare Quiet-Life contracts with Baseline contracts. I find that the hidden compensation packages of Quiet-Life contracts tend to be less lucrative than the cash compensation packages of Baseline contracts. Termination also tends to be delayed in Quiet-Life contracts. These advantages that come with letting the agent shirk imply that the optimal contract may take the Quiet-Life form even when shirking is inefficient. See Theorems 1 and 2.

The other new form is called the Renegotiating Baseline form. In this model, there is no renegotiation and the principal is fully committed. However, a contract like the optimal Baseline contract can be improved with a renegotiation that is unanticipated by the agent. Specifically, every time the optimal Baseline contract calls for termination, the principal can surprise the agent with an offer to forgive him for his poor performance and to move his performance rating up away from the poor performance threshold. Both parties would be better off under this *incentive-incompatible* arrangement. The Renegotiating Baseline contract is so named because it is an incentive-compatible approximation of that arrangement. A Renegotiating Baseline contract starts out just like a Baseline contract. However, sustained poor performance does not trigger termination, but instead, a suspension phase during which cash compensation is postponed, the principal frequently ignores asset performance and the agent frequently shirks. Afterwards, the agent is forgiven for his poor performance and the contract behaves like a Baseline contract again. Some Renegotiating Baseline contracts are renegotiation-proof. In certain cases, the optimal contract, which a priori need not be renegotiation-proof, is a renegotiation-proof Renegotiating Baseline contract. See last corollary in Section 6.2.

In related work, DeMarzo and Sannikov (2006) considers a continuous-time financial contracting model where the agent may divert cash for personal gain. The paper then derives the optimal contract subject to the agent never diverting cash. This restriction is without loss of generality: a constant dead-weight cost of diversion and the unlimited ability to divert imply that the revelation principle holds. Biais, Mariotti, Platin, and Rochet (2007) and DeMarzo and Fishman (2007) consider discrete-time versions of the cash diversion model and He (2009) considers the Geometric Brownian setting. There is a close connection between these cash diversion models and the effort model considered in my paper. As Biais, Mariotti, and Rochet (2011) notes, the models are isomorphic if the agent is asked to never shirk in the effort model. Indeed, DeMarzo and Sannikov (2006) Section III shows that the optimal contract in the cash-diversion model is also the optimal contract in the effort model subject to the agent never shirking. Biais, Mariotti, Rochet, and Villeneuve (2010) considers a continuous-time Poisson version of the effort model.

In general, the no-shirking condition is with loss of generality and the action process of the unconstrained optimal contract in the effort model will not be a priori known. Grossman and Hart (1983) demonstrates that simultaneously determining the optimal action level and the associated optimal incentive scheme is complex even in single period principal-agent models. My paper builds upon the martingale techniques introduced by Sannikov (2008) to deal with the technical problems of the effort model.

My paper is part of the literature on dynamic contracting using recursive methods. Early contributions include Green (1987) and Spear and Srivastava (1987). There is also a small but growing literature on continuous-time contracting. Holmstrom and Milgrom (1987) is an early example. More recent papers not previously mentioned include Sannikov (2005) and DeMarzo and Sannikov (2008). Also, some of my optimality results have connections with the hurdled-calibrated contracts of Chassang (2011).

2 Setting

The model is inherited from DeMarzo and Sannikov (2006) (from now on DS), Section III. The principal contracts an agent to manage an asset belonging to the principal. The agent affects asset performance by selecting a hidden costly action at each moment in time. Formally, there is a probability space Ω equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and an $\mathcal{F}_t$-measurable stochastic process $\{Z_t\}_{t\geq 0}$. Additionally, there is a binary agent's hidden action set $\{0, A > 0\}$ and a mapping from action processes[1] $a = \{a_t\}_{t\geq 0}$ to measures $P^{\mu-a}$ on Ω. Asset performance is then characterized by the following condition on the mapping: under $P^{\mu-a}$, the stochastic process $\{Z_t\}_{t\geq 0}$ representing asset cash flows is Brownian motion plus a drift term $(\mu - a_t)dt$. The μ is a constant parameter of the model.

A *contract* for the agent specifies a cash compensation $I = \{I_t\}_{t\geq 0}$ for the agent, a termination clause τ and an action process a that the principal recommends for the agent. The $\mathcal{F}_t$-measurable process I tracks the cumulative cash compensation received by the agent. It is assumed to be non-decreasing and $I_0 = 0$. The termination clause τ is a stopping time. The hidden action 0 is called the high action and the hidden action A is called the shirking action. Whenever the agent shirks, he receives an extra utility flow $\lambda A dt$ where λ is a positive constant.

Fix a contract (I, τ, a) and suppose the agent follows the principal's recommended action process a. The agent discounts at rate γ and receives an outside option worth $R \geq 0$ once the contract is terminated. Therefore, the agent's total expected payoff from the contract at date 0 is

$$W_0 = \mathbf{E}_{P^{\mu-a}}\left[\int_0^\tau e^{-\gamma s}\left(dI_s + \lambda a_s ds\right) + e^{-\gamma\tau}R\right].$$

More generally, define the contract's *promised value* W_t to the agent after a history $\mathcal{F}_t$ to be the random variable:

$$W_t = \mathbf{E}_{P^{\mu-a}}\left[\int_t^\tau e^{-\gamma(s-t)}\left(dI_s + \lambda a_s ds\right) + e^{-\gamma(\tau-t)}R \;\middle|\; \mathcal{F}_t\right] \quad t \leq \tau.$$

The principal discounts at rate $r < \gamma$. Once the contract is terminated, he receives a liquidation payoff $L < \frac{\mu}{r}$. The principal's expected profit at date 0 is

$$b_0 = \mathbf{E}_{P^{\mu-a}}\left[\int_0^\tau e^{-rs}\left(dZ_s - dI_s\right) + e^{-r\tau}L\right].$$

A contract (I, τ, a) with date 0 expected agent payoff W_0 is *incentive-compatible* if $W_t \geq R$ for all times $t \leq \tau$ and $W_0 \geq \mathbf{E}_{P^{\mu-\hat{a}}}\left[\int_0^\tau e^{-\gamma s}\left(dI_s + \lambda\hat{a}_s ds\right) + e^{-\gamma\tau}R\right]$ for all alternative action processes $\hat{a}$. The *optimal contracting problem* is to find an incentive-compatible contract that maximizes the principal's date 0 expected profit.

[1] An action process is an $\mathcal{F}_t$-measurable stochastic process taking values in the binary agent's hidden action set $\{0, A\}$.

3 Fundamentals

In this section I summarize the solution method employed up to the derivation of the fundamental HJB-equation characterizing optimality. I also discuss some partial optimality results. Much of the material in this section is taken from DS, sections I.B.1 and III. The analysis of the HJB-equation, which represents the main technical contribution of the present paper, is covered in the succeeding sections and the Appendix.

The agent's incentives can be captured by a single state variable: the contract's promised value W_t. Fix a contract (I, τ, a), Equation (1) below characterizes the evolution of the contract's promised value.

Lemma 3.1. *At any moment in time $t \leq \tau$, there is an $\mathcal{F}_t$-measurable sensitivity β_t of the contract's promised value to asset performance such that*

$$dW_t = \gamma W_t dt - dI_t - \lambda a_t dt + \beta_t(dZ_t - (\mu - a_t)dt) \tag{1}$$

Proof. See proof of DS Lemma 2. □

One would expect the agent to not shirk at some time t only if the contract is sufficiently sensitive to asset performance at t. This suggests that there may be a way to characterize incentive-compatibility by looking at the contract's underlying sensitivity process $\beta = \{\beta_t\}_{t \geq 0}$. The following lemma formalizes this intuition.

Lemma 3.2. *A contract (I, τ, a) with sensitivity process β is incentive-compatible if and only if for all $t \leq \tau$, $W_t \geq R$ and*

$$a_t = \begin{cases} 0 & \Rightarrow \beta_t \geq \lambda \\ A & \Rightarrow \beta_t \leq \lambda \end{cases} \tag{2}$$

Proof. See proof of DS Lemma 3. □

From now on, a contract is assumed to be incentive-compatible. Define $B(W)$ as the payoff to the principal of the optimal contract subject to delivering value W to the agent. The function B is defined on the domain $[R, \infty)$. The solution to the optimal contracting problem is closely related to the characterization of B around a sufficiently large neighborhood of the $\arg\max$ of B. I will use dynamic programming to prove that B is the solution to some HJB-equation. The arguments made here will be informal. For example, I will assume that B is concave rather than prove concavity. Formal justifications appear in the Appendix.

Since the principal always has the option of a lump sum transfer $dI > 0$, it must be that $B(W) \geq B(W - dI) - dI$. Therefore $B' \geq -1$ everywhere. Define ω^B as the lowest W such that $B'(W) = -1$. The concavity of B then implies that it is optimal to pay $dI = \max\{W - \omega^B, 0\}$ to the agent. As DS notes, "these transfers, and the option to terminate, keep the agent's promised value between R and ω^B." Between R and ω^B, Ito's Lemma implies that the sum total of the principal's expected cash flow and changes in contract value is given by

$$\mathbf{E}[dZ + dB(W)] = \left(\mu - a + (\gamma W - \lambda a)B'(W) + \frac{\beta^2}{2}B''(W)\right) dt \tag{3}$$

where the action a and the sensitivity β satisfy Equation (2). Dynamic programming then implies

$$rB(W)dt = \max_{a,\beta \text{ s.t. Eq. (2)}} \mathbf{E}[dZ + dB(W)] =$$

$$\max_{a,\beta \text{ s.t. Eq. (2)}} \mu - a + (\gamma W - \lambda a)B'(W) + \frac{\beta^2}{2}B''(W). \tag{4}$$

Since B is concave, $B'' \leq 0$. Therefore, whenever $a = 0$, it is optimal to set sensitivity to $\beta = \lambda$ and whenever $a = A$, it is optimal to shut down sensitivity: $\beta = 0$. The intuition is that volatility of the agent's incentives should be as low as possible while still maintaining incentive-compatibility since lowering the volatility delays or avoids inefficient termination. Therefore, between R and ω^B, the principal's value function satisfies the following HJB-equation:

$$rB(W) = \max\left\{\mu + \gamma W B'(W) + \frac{\lambda^2}{2}B''(W),\ \mu - A + (\gamma W - \lambda A)B'(W)\right\} \tag{5}$$

The quantity $\mu + \gamma W B'(W) + \frac{\lambda^2}{2}B''(W)$ (alternatively, $\mu - A + (\gamma W - \lambda A)B'(W)$) is the principal's normalized instantaneous return when the agent applies action $a = 0$ (alternatively, $a = A$). When $rB(W) = \mu + \gamma W B'(W) + \frac{\lambda^2}{2}B''(W) \geq \mu - A + (\gamma W - \lambda A)B'(W)$, it is optimal to induce action 0 and the principal cannot do better by inducing action A instead. Likewise, when $rB(W) = \mu - A + (\gamma W - \lambda A)B'(W) \geq \mu + \gamma W B'(W) + \frac{\lambda^2}{2}B''(W)$, it is optimal to induce shirking and the principal cannot do better by inducing high action instead.

Is there ever a situation where it is optimal to never induce shirking? Intuitively, if shirking is inefficient ($\lambda < 1$) and sufficiently strong ($A \gg 0$), it is optimal to never induce it. Define the agent's and principal's payoffs when the agent shirks forever:

$$w^s = \frac{\lambda A}{\gamma} \quad \text{and} \quad b^s = \frac{\mu - A}{r}$$

Call (w^s, b^s) the shirking payoff. The following result, taken from DS, characterizes precisely when it is optimal to never induce shirking and what the corresponding optimal value function looks like.

Lemma 3.3. ***DS Propositions 1 and 8.*** *There exists a unique function b defined over $[R, \infty)$ and a unique $\omega \geq R$ such that the following conditions are simultaneously satisfied:*

- $b(R) = L$, $rb(\omega) = \mu - \gamma\omega$, $rb(W) = \mu + \gamma W b'(W) + \frac{\lambda^2}{2}b''(W)$ *on* $(R, \omega]$, *and* $b'(W) = -1$ *on* $[\omega, \infty)$
- *b is globally concave and twice continuously differentiable.*

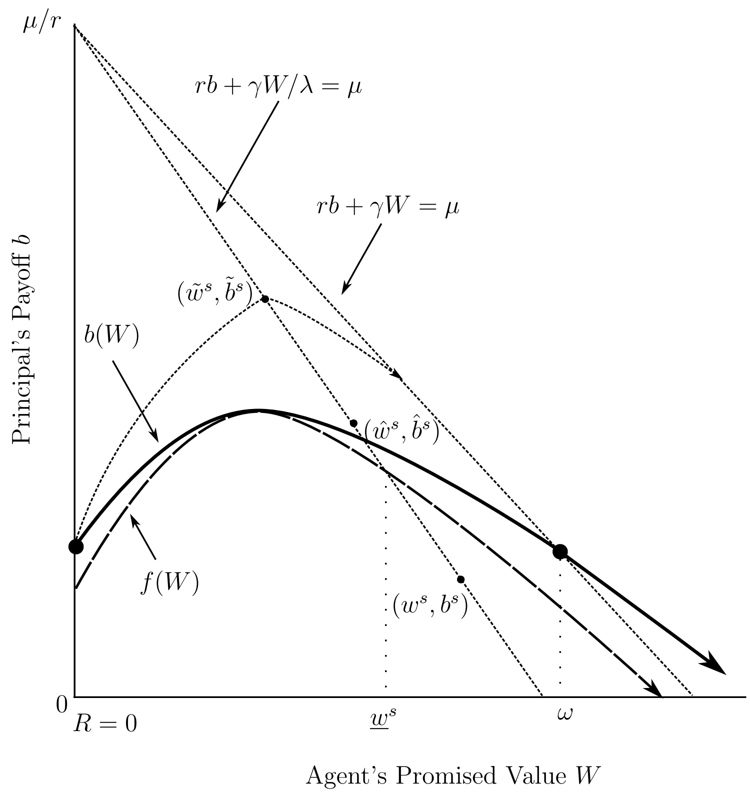

Figure 1: Taken from DS. When shirking is inefficient and sufficiently strong, the optimal contract never induces shirking.

The value $b(W)$ is the payoff to the principal of the optimal contract subject to delivering value W to the agent and subject to the agent never shirking.

Never inducing shirking is optimal (i.e. $B \equiv b$) if and only if

$$b^s \leq f(w^s) \quad \textit{where} \quad f(z) \equiv \min_{W \geq R} b(W) + \frac{\gamma}{r}(z - W)\, b'(W). \tag{6}$$

Given $\lambda < 1$, this condition implies a lower bound on A.

Figure 1 graphically summarizes the necessary and sufficient conditions for never inducing shirking to be optimal. The function f defined in Equation (6) is concave and lies strictly below b except at their common maximum. Fixing the degree of inefficiency, λ, of shirking fixes a linear track on which the shirking payoff (w^s, b^s) can be located. As the strength of the shirking action, A, increases, (w^s, b^s) slides further down the track. At some point, it will cross below the function f and the optimal contract will never induce shirking. For example, in Figure 1, this occurs when $w^s \geq \underline{w}^s$. Conversely, if $w^s < \underline{w}^s$, the optimal contract will involve shirking after some histories.

Indeed, for A sufficiently small, such as in the case of $(\tilde{w}^s, \tilde{b}^s)$ in Figure 1, the optimal contract is to simply let the agent shirk forever. The intuition is that the gain from never having to terminate more than makes up for any (small) loss in letting the agent shirk.

Definition. *The optimal contract subject to the agent never shirking will be called the optimal*

Baseline contract. The optimal contract subject to the agent shirking at all times will be called the optimal Static contract.

We now know that the optimal contract can take on at least two different forms, depending on the fundamentals: the Baseline and the Static forms. Are there any others? Consider the case when A takes an intermediate value, so that the shirking payoff is $(\hat{w}^s, \hat{b}^s)$ in Figure 1. Since $\hat{b}^s > f(\hat{w}^s)$, DS Proposition 8 implies that the optimal contract will employ shirking. However, $\hat{b}^s < \max b$. So letting the agent shirk forever certainly is not the optimal contract. Therefore, the optimal contract must employ shirking, but only temporarily.

When does the optimal contract employ shirking temporarily? How many of these optimal contractual forms are there? How are they designed? The rest of the paper answers these questions, solving the optimal contracting problem.

Sections 4 and 5 form the technical heart of the paper. Theorem 1 provides tight upper bounds on B and Theorem 2 uses these upper bounds to deduce the dynamics of the optimal diffusion W_t. This formally solves the optimal contracting problem. Section 6 then interprets these results. Readers have the option of going directly to Section 6.

4 Tight Upper Bounds on B

Overview of the Next Two Sections.
In this section I explain how the differential structure of B can be used to infer the incentive structure of the optimal contract. Thus, solving the optimal contracting problem reduces to solving for B. I then produce a function b^* that is a *tight upper bound* on B in the sense that there exists an interval E containing $\arg\max B$ such that $b^*|_E = B|_E$. In the next section, I show that E can be made large enough so that the optimal diffusion W_t stays within E. This allows me to deduce the optimal contract directly from b^* instead of B. Throughout this section and the next, I assume the optimal contract subject to the agent never shirking does not involve terminating immediately. This is equivalent to assuming $b'(R) > 0$. I will also assume the agent's outside option $R = 0$. The optimality results (Theorems 1 and 2) continue to hold when these two assumptions are removed. See Appendix, subsection 8.3. Also, see Appendix, subsection 8.2 for a complete characterization of B.

Recall, characterizing the optimal contract means specifying a cash compensation I, a termination clause τ and an action process a. In the previous section, I summarized DeMarzo and Sannikov's argument that the optimal cash compensation must be $dI = \max\{W_t - \omega^B, 0\}$ where W_t is the agent's promised value in the optimal contract. Such a compensation creates a reflecting boundary at ω^B and keeps $W_t \leq \omega^B$ for $t > 0$. Moreover, termination occurs only when W_t drops down to R.

Therefore, to characterize the optimal contract, it suffices to determine two processes between R and ω^B: 1) the optimal action process $a_t = a(W_t)$ and 2) the contract's promised value W_t. These two processes are simultaneously determined by the differential structure of B. In particular, no additional public randomization is needed.[2] On $[R, \omega^B)$, the function

[2]This is because all of my conjectured value functions are concave.

B is a solution to the HJB-equation of (5):

$$ry = \max\left\{\mu + \gamma x y'(x) + \frac{\lambda^2}{2} y''(x), \mu - A + (\gamma x - \lambda A) y'(x)\right\}$$

The form of the HJB-equation requires that solutions be constructed by pasting together solutions to two ODEs: the **high action ODE** $ry(x) = \mu + \gamma x y'(x) + \frac{\lambda^2}{2} y''(x)$ and the **shirking action ODE** $ry(x) = \mu - A + (\gamma x - \lambda A) y'(x)$. Once the optimal pasting of B is determined, the two processes of interest - a_t, W_t - can be simply read off:

$$a_t = a(W_t) = \begin{cases} 0 & \text{if } rB(W_t) = \mu + \gamma W_t B'(W_t) + \frac{\lambda^2}{2} B''(W_t) \\ A & \text{if } rB(W_t) = \mu - A + (\gamma W_t - \lambda A) B'(W_t) \end{cases} \tag{7}$$

and

$$dW_t = \begin{cases} \gamma W_t dt + \lambda (dZ_t - \mu dt) & \text{if } rB(W_t) = \mu + \gamma W_t B'(W_t) + \frac{\lambda^2}{2} B''(W_t) \\ \gamma W_t dt - \lambda A dt & \text{if } rB(W_t) = \mu - A + (\gamma W_t - \lambda A) B'(W_t) \end{cases} \tag{8}$$

I now describe some relevant properties of the HJB-equation's component ODEs. The high action ODE is a very well-behaved 2nd-order linear differential equation. All coefficients trivially admit globally convergent Taylor series expansions and therefore, so do all solutions to the high action ODE. Fix a point (X, Y), $X \geq 0$ and $Y < \frac{\mu}{r}$. Figure 2 shows some solutions to the high action ODE with starting point (X, Y) in the region below the line $y = \frac{\mu}{r}$. Solutions do not intersect beyond the starting point. The higher the initial slope, the greater the initial concavity. There exists a constant $M_{(X,Y)} > 0$ such that if the initial slope of the solution $\geq M_{(X,Y)}$ then the solution hits the line $y = \frac{\mu}{r}$.[3] If the initial slope is strictly between $M_{(X,Y)}$ and $\frac{rY - \mu}{\gamma X}$ then the solution starts out concave, stays below $y = \frac{\mu}{r}$, and eventually inflects and becomes permanently decreasing, convex.[4] Finally, if the initial slope $\leq \frac{rY - \mu}{\gamma X}$ the solutions starts out decreasing, convex and remains decreasing, convex.

Pick an arbitrary solution starting at some point (X, Y) with initial slope strictly between $M_{(X,Y)}$ and $\frac{rY - \mu}{\gamma X} < M_{(X,Y)}$, say g in Figure 2. Then g must inflect at some point. Call this point $(\omega^{\pi}_{(X,Y)}, g(\omega^{\pi}_{(X,Y)}))$ where $-\pi$ is the slope of g at its inflection. Next, attach the concave portion of g with the half-line starting at $(\omega^{\pi}_{(X,Y)}, g(\omega^{\pi}_{(X,Y)}))$ with slope $-\pi$ and call this engineered function $b^{\pi}_{(X,Y)}$. It is C^2 and concave. The higher the initial slope $g'(X)$, the lower is the π. By varying $g'(X)$ between $M_{(X,Y)}$ and $\frac{rY - \mu}{\gamma X} < M_{(X,Y)}$, $b^{\pi}_{(X,Y)}$ is defined for all $\pi \in (0, \frac{\mu - rY}{\gamma X})$. In addition, for $\pi \geq \frac{\mu - rY}{\gamma X}$, define $b^{\pi}_{(X,Y)}$ to simply be the half-line starting at (X, Y) with slope $-\pi$.

Each $b^{\pi}_{(X,Y)}$ admits a natural contractual interpretation. Suppose the model is altered so that there is a distortionary tax $\frac{1-\pi}{\pi}$ on cash. That is, for every dollar delivered to the

[3]Technically speaking, this is not quite true. The solution with initial slope $M_{(X,Y)}$ asymptotically approaches $y = \frac{\mu}{r}$. See Figure 2. However, I consider this function as hitting $y = \frac{\mu}{r}$ at infinity.

[4]In particular, the solution is already decreasing at the inflection point.

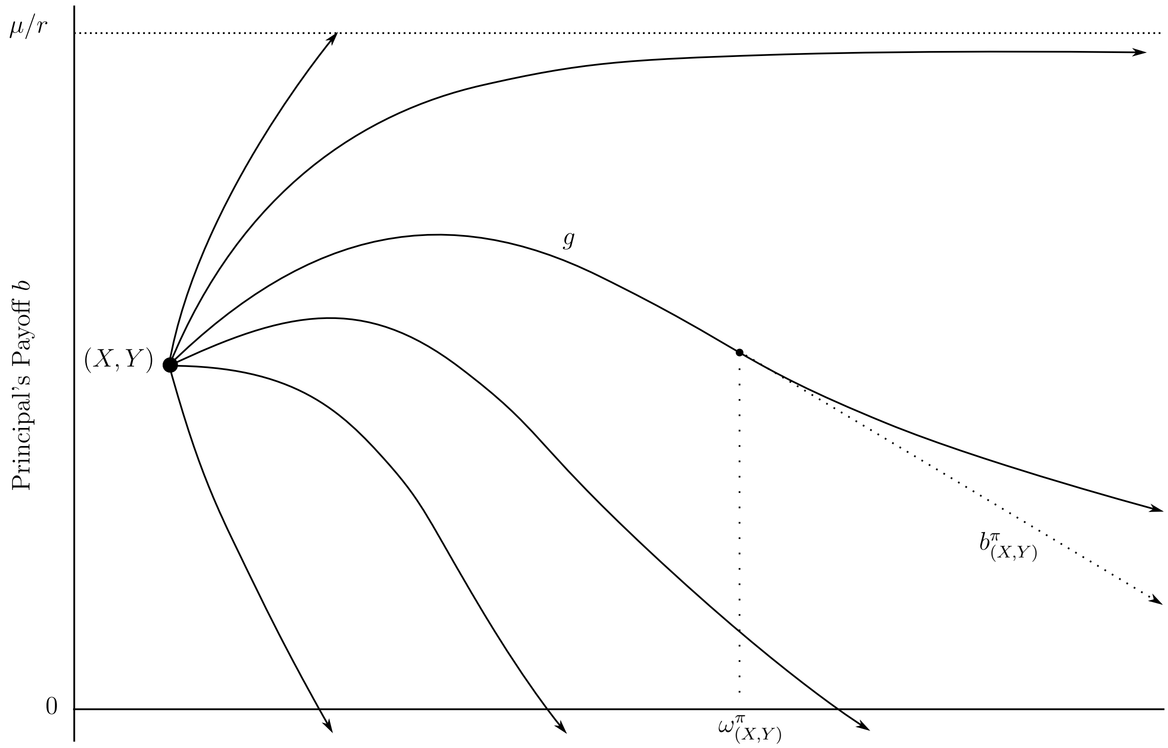

Figure 2: Solutions to the high action ODE.

agent, the principal must pay π dollars.[5] And suppose the outside options of the agent and principal are X and Y instead of R and L. Then $b^\pi_{(X,Y)}$ is the value function of the optimal Baseline contract in the altered model. That is, for every $W \geq X$, $b^\pi_{(X,Y)}(W)$ is the payoff to the principal from the the optimal contract subject to delivering W to the agent and subject to the agent never shirking. Given the differential structure of $b^\pi_{(X,Y)}$, it is clear that the optimal Baseline contract in the altered model and the optimal Baseline contract in the actual model share the same incentive scheme.[6] I can now state a generalization of DS Propositions 1 and 8. Define $B^\pi_{(X,Y)}(W)$ as the payoff to the principal of the optimal contract subject to delivering value $W \geq X$ to the agent in the altered model.

Lemma 4.1. *Fix any $\pi \in (0, \infty)$. The function $b^\pi_{(X,Y)}$ defined over $[X, \infty)$ and the value $\omega^\pi_{(X,Y)} \geq X$ are uniquely determined by the following conditions:*

- $b^\pi_{(X,Y)}(X) = Y$, $rb^\pi_{(X,Y)}(\omega^\pi_{(X,Y)}) = \mu - \pi\gamma\omega^\pi_{(X,Y)}$, $rb^\pi_{(X,Y)}(W) = \mu + \gamma W b^\pi_{(X,Y)}{}'(W) + \frac{\lambda^2}{2} b^\pi_{(X,Y)}{}''(W)$ *on* $(X, \omega^\pi_{(X,Y)}]$, *and* $b^\pi_{(X,Y)}{}'(W) = -\pi$ *on* $[\omega^\pi_{(X,Y)}, \infty)$
- $b^\pi_{(X,Y)}$ *is concave and twice continuously differentiable.*

Suppose the model is altered so that there is a distortionary tax $\frac{1-\pi}{\pi}$ on cash and the agent's and principal's outside options are X and Y respectively. Then the value $b^\pi_{(X,Y)}(W)$ is the

[5]The distortionary tax can be a subsidy: π can be less than 1.

[6]Formally, the diffusions W_t implied by the two value functions are identical up to changes in the terminating and cash compensation thresholds.

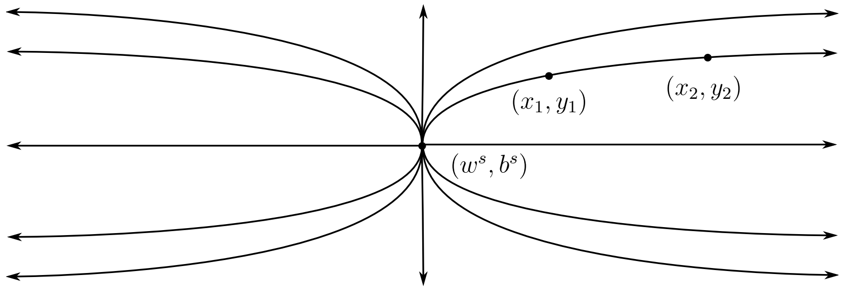

Figure 3: Solutions to the shirking action ODE.

payoff to the principal of the optimal contract subject to delivering value W to the agent and subject to the agent never shirking.

Never inducing shirking is optimal (i.e. $B^\pi_{(X,Y)} \equiv b^\pi_{(X,Y)}$) if and only if

$$b^s \leq f^\pi_{(X,Y)}(w^s) \quad \text{where} \quad f^\pi_{(X,Y)}(z) \equiv \min_{W \geq X} b^\pi_{(X,Y)}(W) + \frac{\gamma}{r}\,(z - W)\, b^\pi_{(X,Y)}{}'(W). \tag{9}$$

Given $\lambda < 1$, this condition implies a lower bound on A.

Proof. This is a straightforward extension of DS Propositions 1 and 8. □

Remark. *From now on, whenever $\pi = 1$, the superscript of $b^\pi_{(X,Y)}$ will be dropped; and whenever $(X,Y) = (R,L)$, the subscript of $b^\pi_{(X,Y)}$ will be dropped. The same convention will also be applied to $B^\pi_{(X,Y)}$, $\omega^\pi_{(X,Y)}$ and $f^\pi_{(X,Y)}$. This way the notation of Lemma 4.1 remains consistent with the earlier notation of Lemma 3.3. Furthermore, with a simple relabeling of objects, Figure 1 which summarized Lemma 3.3, can be reused to summarize Lemma 4.1.*

The shirking action ODE admits analytic solutions:

$$y = \frac{\mu - A}{r} + \alpha \left(x - \frac{\lambda A}{\gamma} \right)^{\frac{r}{\gamma}} \qquad \alpha \in \mathbb{R} \tag{10}$$

These solutions comprise a family of functions branching from the shirking payoff (w^s, b^s). Pick an arbitrary branch and two points (x_1, y_1), (x_2, y_2) on the branch. See Figure 3. Suppose (x_2, y_2) is further down along the branch and represents the payoff to the agent and principal from some contract C. Then (x_1, y_1) admits a contractual interpretation. It is the payoff to the agent and principal from the contract $\tilde{C}$ which, before enacting C, allows the agent to shirk for a period of length T where:

$$T = \frac{1}{\gamma} \log \left(\frac{x_2 - \frac{\lambda A}{\gamma}}{x_1 - \frac{\lambda A}{\gamma}} \right) = \frac{1}{r} \log \left(\frac{y_2 - \frac{\mu - A}{r}}{y_1 - \frac{\mu - A}{r}} \right)$$

During the shirking period, the agent's promised value is not sensitive to asset performance.

There are many ways to piece solutions to the two ODEs into a value function for the principal. In certain cases, a carefully put together value function using solutions to both the

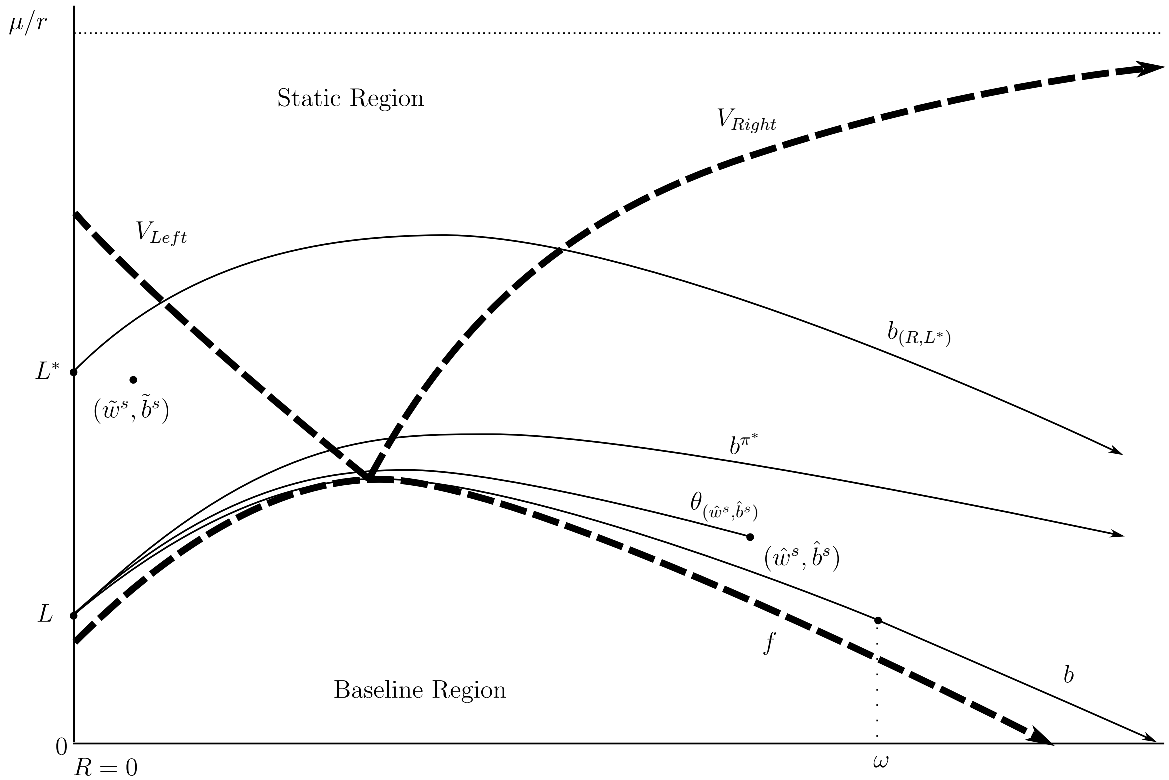

Figure 4: Graphical Representation of Theorem 1. The V-curve and f divide D, the domain of the shirking payoff, into four distinct regions. In the right region, $B \leq b^{\pi^*}$. In the left, $B \leq b_{(R,L^*)}$. The inequalities are tight and π^* and L^* are functions of the shirking payoff. The two remaining regions are when the optimal contract takes the Baseline and Static forms.

high action ODE and the shirking action ODE may imply the existence of a contract that dominates the optimal Baseline and Static contracts. These are precisely the cases when the optimal contract temporarily induces shirking. But before looking into these cases, I first characterize precisely when the optimal contract takes the Baseline and Static forms.

In general, the shirking payoff (w^s, b^s) can lie anywhere in $D = \{(x, y)|x > 0 \text{ and } y < \frac{\mu}{r}\}$. DS shows that the optimal contract takes the Baseline form if and only if $b^s \leq f(w^s)$. The optimal contract takes the Static form if and only if the shirking payoff is located on or inside a V-shaped curve above b. See Figure 4. The following lemma characterizes this V-curve:

Lemma 4.2. *Let V_{Left} denote the locus of points $(X \geq 0, Y < \frac{\mu}{r})$ such that $Y \geq b(X)$ and $b'_{(X,Y)}(X) = 0$. V_{Left} is a strictly decreasing, continuous curve starting on the vertical boundary $x = 0$ and ending at $(\arg\max b, \max b)$.*

Let V_{Right} denote the locus of points $\{(\arg\max b^\pi, \max b^\pi)|\pi \in (0, 1]\}$. Then V_{Right} is a strictly increasing, continuous curve starting at $(\arg\max b, \max b)$, with a horizontal asymptote $y = \frac{\mu}{r}$.

Definition. *The V-curve is defined to be $V_{Left} \cup V_{Right}$.*

Notice, f and the V-curve divide D into four regions. The bottom and top regions determine two forms of the optimal contract: Baseline and Static. As we shall see in the next section, the left and the right regions determine the two remaining and as yet uncharacterized forms of the optimal contract.

Suppose the shirking payoff is a point in the right region. For example, consider the case when the shirking payoff is $(\hat{w}^s, \hat{b}^s)$ in Figure 4. Pick the unique solution to the high action ODE going from (R, L) to $(\hat{w}^s, \hat{b}^s)$. Call this value function $\theta_{(\hat{w}^s,\hat{b}^s)}$. Any point $(W, \theta_{(\hat{w}^s,\hat{b}^s)}(W))$ on this value function is the payoff of a contract whose structure is implied by the construction of $\theta_{(\hat{w}^s,\hat{b}^s)}$. Each member of this particular family of contracts can be thought of as a "tenure" contract. If $(W, \theta_{(\hat{w}^s,\hat{b}^s)}(W)) = (\hat{w}^s, \hat{b}^s)$, the implied contract simply lets the agent shirk forever. Now suppose, $W < \hat{w}^s$. This portion of θ satisfies the high action ODE. Therefore, Equation (7) implies that the contract initially calls for high action from the agent and Equation (8) implies that the agent's promised value initially evolves according to $dW_t = \gamma W_t dt + \lambda(dZ_t - \mu dt)$. This incentive scheme is maintained until either W_t hits R and the contract is terminated, or W_t hits $\hat{w}^s$ and the contract enters the permanent shirking (or tenure) phase. Finally, since $\max \theta_{(\hat{w}^s,\hat{b}^s)} > \max b$, the contract $\mathcal{C}$ implied by $\theta_{(\hat{w}^s,\hat{b}^s)}$ delivering payoff $(\arg\max \theta_{(\hat{w}^s,\hat{b}^s)}, \max \theta_{(\hat{w}^s,\hat{b}^s)})$ dominates the optimal baseline contract. While this represents an improvement, $\theta_{(\hat{w}^s,\hat{b}^s)}$ lies below B and $\mathcal{C}$ is not the optimal contract.

Continuing with the example, consider the following counterfactual: suppose there was a distortionary tax subsidy $\frac{1-\pi}{\pi}$ on cash (that is, $\pi < 1$). One can show that for all sufficiently high subsidies, the optimal contract subject to delivering any value $W \geq R$ to the agent never induces shirking. That is, $B^\pi = b^\pi$ for al sufficiently low $\pi > 0$. Since increasing the tax subsidy makes cash compensation more attractive while leaving the effect of shirking unchanged, it must be that

$$B^{\pi_1} = b^{\pi_1} \Rightarrow B^{\pi_2} = b^{\pi_2} \quad \forall \pi_2 < \pi_1 \tag{11}$$

So let π^* be the largest π such that $B^\pi = b^\pi$. It must be a subsidy: $\pi^* < 1$. Otherwise, Equation (11) implies $B = b$, contradicting the assumption that the shirking payoff does not lie in the Baseline region. Moreover, since any decrease in tax increases overall efficiency in the model, it must be that

$$B^{\pi_1} \leq B^{\pi_2} \quad \forall \pi_2 < \pi_1$$

Therefore, $B \leq B^{\pi^*} = b^{\pi^*}$. In fact, this bound is tight and the process $(W_t, B(W_t))$ of the optimal contract will always lie on b^{π^*}.[7]

Now suppose the shirking payoff is a point in the left region. For example, consider the case when the shirking payoff is $(\tilde{w}^s, \tilde{b}^s)$ in Figure 4. Consider another counterfactual: this time, suppose the principal's outside option value $\tilde{L} > L$ is very high. Again, one can show that the for all sufficiently high principal outside options, the optimal contract subject to delivering any value $W \geq R$ to the agent never induces shirking: $B_{(R,\tilde{L})} = b_{(R,\tilde{L})}$. Let $L^* > L$ be the smallest such principal's outside option. Then $B \leq B_{(R,L^*)} = b_{(R,L^*)}$, and in fact, $b_{(R,L^*)}$ is a tight upper bound on B.

The following theorem summarizes the previous observations.

[7] Obviously, B cannot equal b^{π^*} everywhere. For example, $B'(W) = -1 < -\pi^* = b^{\pi^*}{}'(W)$, for all $W \geq \max\{\omega^B, \omega^{\pi^*}\}$.

Theorem 1. *The V-curve and f separate D into four regions. If the shirking payoff lies in the right region, there exists a maximal π^* (< 1) such that $B^* \leq b^* \equiv b^{\pi^*}$. If the shirking payoff lies in the left region, there exists a minimal L^* ($> L$) such that $B \leq b^* \equiv b_{(R,L^*)}$. In both cases, the concave optimal Baseline value function b^* is a tight upper bound on B. Finally, if the shirking payoff lies in the bottom and top regions, the optimal contract takes the Baseline and Static forms respectively.*

Proof. See Appendix, subsection 8.1. □

5 Resolution of the Optimal Contracting Problem

In the previous section I explained that the optimal contracting problem is solved once B is characterized, since B immediately implies the optimal contract's W_t and $a(W_t)$. I then proved Theorem 1, which provided a tight upper bound on B.

In this section, I show how the tight upper bound b^* of Theorem 1 is sufficient to imply the optimal contract's W_t and $a(W_t)$. This completes the solution to the optimal contracting problem. The optimal contracting result is then summarized in Theorem 2.

Recall, the optimal contract takes the baseline form if the shirking payoff lies below the graph of f. See Lemma 3.3 and Figure 1. More generally, if the agent and principal's outside options are (X, Y) and there is a distortionary tax $\frac{1-\pi}{\pi}$ on cash, then the optimal contract takes the baseline form if the shirking payoff lies on or below the graph of $f^{\pi}_{(X,Y)}$. See Lemma 4.1.

Now suppose the shirking payoff is some point in the right region of D, say, $(\hat{w}^s, \hat{b}^s)$. Then Theorem 1 implies the existence of a tight upper bound on B of the form b^{π^*}. The maximality of π^* implies that $\hat{b}^s = f^{\pi^*}(\hat{w}^s)$. So,

$$\hat{b}^s = b^{\pi^*}(W^*) + \frac{\gamma}{r}\left(\hat{w}^s - W^*\right) b^{\pi^*\,\prime}(W^*) \text{ for some } W^* > R, \tag{12}$$

and

$$\hat{b}^s \leq b^{\pi^*}(W) + \frac{\gamma}{r}\left(\hat{w}^s - W\right) b^{\pi^*\,\prime}(W) \text{ for all } W \geq R. \tag{13}$$

Let h be the solution to the shirking action ODE going through $(W^*, b^{\pi^*}(W^*))$. Together, Equations (12) and (13) imply

$$b^{\pi^*\,\prime}(W^*) = h'(W^*) \text{ and } \; b^{\pi^*\,\prime\prime}(W^*) = h''(W^*) \tag{14}$$

See Figure 5. Together, Equations (14) and the fact that $(\hat{w}^s, \hat{b}^s)$ is in the right region imply

$$\arg\max b^{\pi^*} < W^* < \min\{\omega^{\pi^*},\ \hat{w}^s\} \tag{15}$$

Now define the value function $B^{opt} \equiv b^{\pi^*}|_{[R,W^*)} \cup (W^*, h(W^*))$ over the interval $[R, W^*]$.[8]

[8] Set theoretically, $b^{\pi^*}|_{[R,W^*)} \cup (W^*, h(W^*))$ is clearly equivalent to $b^{\pi^*}|_{[R,W^*]}$. However, as a value functions that encodes the structure of its implied contract, $b^{\pi^*}|_{[R,W^*)} \cup (W^*, h(W^*))$ is distinct from $b^{\pi^*}|_{[R,W^*]}$. By separating $(W^*, h(W^*))$ from the rest of $b^{\pi^*}|_{[R,W^*]}$, the value function $b^{\pi^*}|_{[R,W^*)} \cup (W^*, h(W^*))$ explicitly

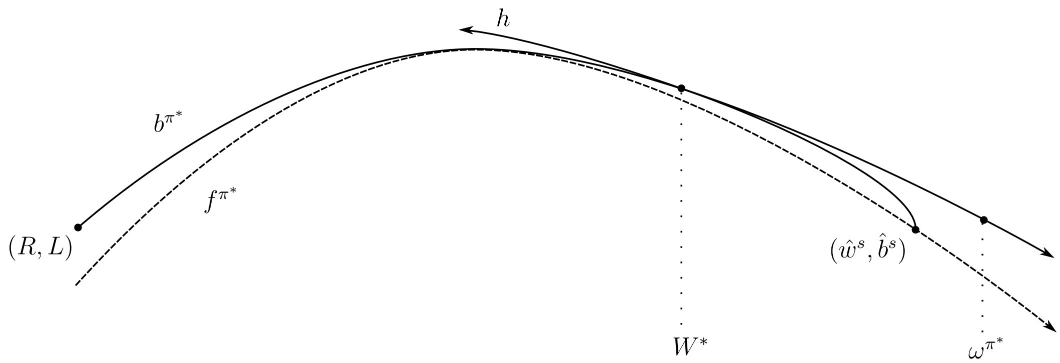

Figure 5: The smooth pasting and super contact at some W^* between b^{π^*} and a solution h to the shirking action ODE. Shirking payoff $(\hat{w}^s, \hat{b}^s)$ lies in the right region of D.

Any point $(W, B^{opt}(W))$ is the payoff of a contract whose structure is implied by the differential properties of B^{opt}. In particular, since $\max B^{opt} = \max b^{\pi^*} \geq \max B$, the structure of the optimal contract is implied.

Fix a point $(W_0, B^{opt}(W_0))$. To achieve this payoff, start the contract's promised value at W_0. When $W_t \in (R, W^*)$, the local differential structure of B^{opt} and Equation (7) imply that the contract induces high action from the agent. Equation (8) then implies that the contract's promised value evolves according to $dW_t = \gamma W_t dt + \lambda(dZ_t - \mu dt)$. When $W_t = W^*$, since $(W^*, h(W^*))$ represents the contribution of the shirking action ODE to B^{opt}, Equation (7) implies that the contract induces shirking from the agent. Together, Equations (8) and (15) then imply that the contract's promised value evolves according to

$$dW_t = (\gamma W^* - \lambda A)dt < 0dt \tag{16}$$

Lastly, the contract is terminated when $W_t = R$.

The dynamics of the diffusion W_t merit discussion. On (R, W^*), W_t is stochastic with volatility λ. But whenever $W_t = W^*$, the volatility disappears, and W_t deterministically moves downwards for an instant (see Equation (16)), after which, it enters (R, W^*) again. The implied motion is Brownian with a slow reflection at the reflecting boundary W^*. Formally, this type of diffusion is well-defined and is termed Sticky Brownian Motion.[9] See Lemma 6.1. The stickiness captures the property that, conditioned on W_t reaching W^*, the set of times when $W_t = W^*$ (almost surely) has positive measure. This is in direct contrast to regular reflecting Brownian motion. However, like regular reflecting Brownian motion, the set is nowhere dense and perfect.

Now suppose the shirking payoff is in the left region of D. By mimicking the arguments made previously, the optimal contract can again be deduced from the tight upper bound on B provided by Theorem 1. Specifically, the minimality of L^* implies the existence of a value W^* and a solution h to the shirking action ODE satisfying:

$$b_{(R,L^*)}(W^*) = h(W^*) \text{ and } b'_{(R,L^*)}(W^*) = h'(W^*) \text{ and } b''_{(R,L^*)}(W^*) = h''(W^*) \tag{17}$$

instructs the implied contract to switch to the shirking when $W_t = W^*$.

[9] For a thorough discussion of Sticky Brownian Motion, see Harrison and Lemoine (1981).

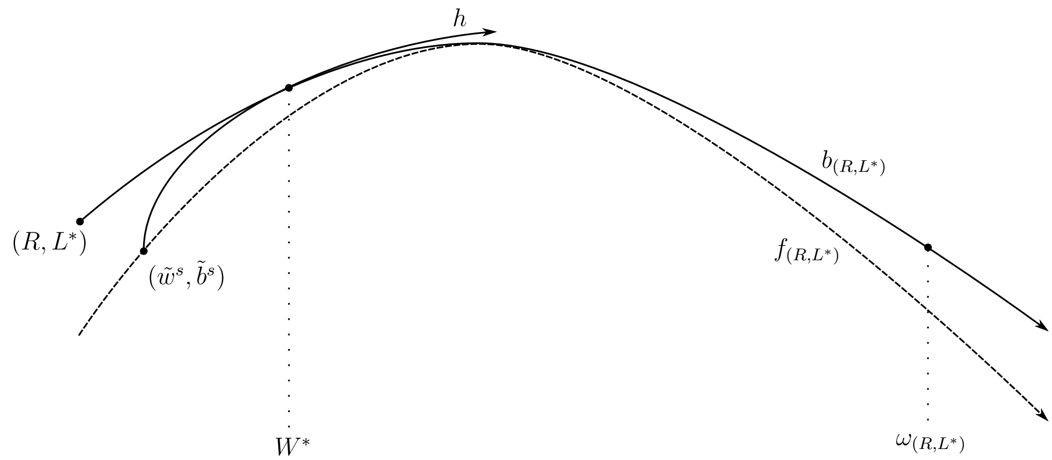

Figure 6: The smooth pasting and super contact at some W^* between $b_{(R,L^*)}$ and a solution h to the shirking action ODE. Shirking payoff $(\tilde{w}^s, \tilde{b}^s)$ lies in the left region of D.

Together, Equation (17) and the fact that $(\tilde{w}^s, \tilde{b}^s)$ is in the left region imply

$$\tilde{w}^s < W^* < \arg\max b_{(R,L^*)} < \omega_{(R,L^*)} \tag{18}$$

B^{opt} is now defined to be $(W^*, h(W^*)) \cup b_{(R,L^*)}|_{(W^*, \omega_{(R,L^*)}]}$. When $W_t \in (W^*, \omega_{(R,L^*)})$, the agent applies high action, $dW_t = \gamma W_t + \lambda(dZ_t - \mu dt)$, and cash compensation is triggered whenever $W_t = \omega_{(R,L^*)}$ causing W_t to reflect downwards. When $W_t = W^*$, the agent shirks, $dW_t = (\gamma W^* - \lambda A)dt > 0dt$, and volatility disappears causing W_t to deterministically move upwards for an instant, after which, it enters $(W^*, \omega_{(R,L^*)}]$ again. Therefore, W_t remains in the interval $[W^*, \omega_{(R,L^*)}]$ permanently, slowly reflecting upwards at the sticky boundary W^* and reflecting downwards at the non-sticky boundary $\omega_{(R,L^*)}$.

The optimal contracting problem is now solved and the following theorem summarizes the results.

Theorem 2. *Suppose the shirking payoff is in either the right or the left region of D and let $b^* = b^{\pi^*}$ or $b_{(R,L^*)}$ be the tight upper bound of B from Theorem 1. Then there exists a unique $W^* \in (R,\ \omega^* = \omega^{\pi^*}$ or $\omega_{(R,L^*)})$ and a unique solution h to the shirking action ODE such that the following smooth pasting and super-contact conditions are satisfied:*

$$h(W^*) = b^*(W^*) \quad \textit{and} \quad h'(W^*) = b^{*\,\prime}(W^*) \quad \textit{and} \quad h''(W^*) = b^{*\,\prime\prime}(W^*)$$

Suppose the shirking payoff is in the right region of D. Then $\arg\max b^{\pi*} < W^*$ and the contract that maximizes the principal's profit and delivers the value $W_0 \in [R, W^*]$ to the agent takes the following form:

- *When $W_t \in (R, W^*)$, the agent applies high action and $dW_t = \gamma W_t + \lambda(dZ_t - \mu dt)$.*
- *When $W_t = W^*$, the agent shirks and $dW_t = (\gamma W^* - \lambda A)dt < 0dt$.*

The utility the agent receives during the contract comes exclusively from shirking when $W_t = W^*$. This implies that $I \equiv 0$ and that W_t slowly reflects downwards at W^*. Since W_t is Sticky

Brownian motion near W^*, the contract spends a non-trivial expected amount of time at W^*. The contract is terminated when W_t reaches R. The principal's expected payoff at any point is given by a concave function defined over $[R, W^*]$: $B^{opt} \equiv b^{\pi^*}|_{[R,W^*)} \cup (W^*, h(W^*)) = B|_{[R,W^*]}$.

Now suppose the shirking payoff is in the left region of D. Then $W^* < \arg\max b_{(R,L^*)}$ and the contract that maximizes the principal's profit and delivers the value $W_0 \in [W^*, \omega_{(R,L^*)}]$ to the agent takes the following form:

- *When $W_t \in (W^*, \omega_{(R,L^*)})$, the agent applies high action and $dW_t = \gamma W_t + \lambda(dZ_t - \mu dt)$.*
- *When $W_t = W^*$, the agent shirks and $dW_t = (\gamma W^* - \lambda A)dt > 0dt$.*

The utility the agent receives during the contract comes from both shirking when $W_t = W^*$ and from receiving cash payments dI_t when $W_t = \omega_{(R,L^*)}$. This implies W_t slowly reflects upwards at W^* and reflects downwards at $\omega_{(R,L^*)}$. Since W_t is Sticky Brownian motion near W^*, the contract spends a non-trivial amount of time at W^*. Since W_t stays bounded between W^* and $\omega_{(R,L^*)}$, termination never occurs. The principal's expected payoff at any point is given by a concave function defined over $[W^*, \omega_{(R,L^*)}]$: $B^{opt} \equiv (W^*, h(W^*)) \cup b_{(R,L^*)}|_{(W^*,\omega_{(R,L^*)}]} = B|_{[W^*,\omega_{(R,L^*)}]}$.

In both cases, B^{opt} contains the maximum of B. Therefore, the optimal contracting problem is solved.

Theorem 2 only characterizes the optimal contract subject to delivering value W to the agent when W is in the domain of B^{opt}. While this is sufficient to solve the optimal contracting problem, one may be interested in optimal contracts that deliver higher payoffs to the agent. The Appendix provides a full characterization of B. The next section interprets the new optimal contractual forms and compares them with the Baseline form.

6 The Four Forms of the Optimal Contract

Fix any contract (I, τ, a). In Section 3, I defined the contract's promised value process W_t. This process captures the contract's incentives. Only when W_t is sufficiently sensitive to the performance dZ_t of the underlying asset will the agent be induced to apply high action. The contract's level of sensitivity at any time t can be deduced using the Martingale Representation Theorem and is called β_t. The sensitivity threshold for inducing high action is λ. I then argued whenever the principal wants to induce high action, β_t should be set to λ; and whenever the principal wants to induce shirking, β_t should be set to 0. This effectively pins down the two stochastic laws that will govern the incentives of the optimal contract:

Definition. *Suppose at some time t, the principal is not paying the agent.*[10] *If the optimal contract stipulates high action, the contract's promised value follows the* ***high action law:***

$$dW_t = \gamma W_t dt + \lambda(dZ_t - \mu dt)$$

[10] Cash compensation times will be dealt with separately.

which says to induce high action, the contract's promised value needs to be sensitive to asset performance, and in expectation, needs to compound at the agent's discount rate.

Similarly, when the optimal contract stipulates shirking, the contract's promised value follows the ***shirking law****:*

$$dW_t = \gamma W_t dt - \lambda A dt$$

which says the contract's promised value is not sensitive to asset performance, and in expectation, compounds at the agent's discount rate less the utility $\lambda A dt$ the agent automatically receives from shirking.

I can now state the main theorem characterizing the four possible forms of the optimal contract. It is an interpretation of Theorem 2.

Theorem 3. *The optimal contract has the following structure. The principal selects two thresholds: a poor performance threshold W^{poor} and a good performance threshold W^{good}. The contract's promised value W_t is started between the two performance thresholds. Whenever W_t is strictly between the two performance thresholds, it follows the high action law*

$$dW_t = \gamma W_t dt + \lambda(dZ_t - \mu dt)$$

While following this law, W_t is sensitive to asset performance and serves as a ***dynamic rating*** *of the agent's managerial performance. When good performance pushes the rating up to W^{good} the principal rewards the agent. The two options are cash compensation or hidden compensation by inducing shirking. When poor performance pushes the rating down to W^{poor} the principal punishes the agent. The two options are termination or suspension of the agent which leads to shirking.*

The four different ways to reward and punishment the agent produce the four possible forms of the optimal contract. They are summarized in the table below:

U^{poor}/U^{good}	*cash compensation*	*shirking*
termination	*Baseline*	***Quiet-Life***
shirking	***Renegotiating Baseline***	*Static*

Baseline contracts never induces shirking. Static contracts always induce shirking. The Quiet-Life and Renegotiating Baseline contracts both induce shirking non-permanently in between periods of high action.

Static contracts induce shirking forever and may supplement the agent with a fixed salary sdt. Consequently, there is no performance evaluation, the good and poor performance thresholds always coincide, and the agent's continuation payoff is permanently fixed at this value: $W^{poor} = W_t = W^{good} = \frac{\lambda A + s}{\gamma}$. The optimal static contract supplements the agent with a salary sdt just enough to prevent him from quitting: $s = \max\{0, \gamma R - \lambda A\}$. The payoff to the agent is $\max\{\frac{\lambda A}{\gamma}, R\}$ and the payoff to the principal is $\min\{\frac{\mu - A}{r}, \frac{\mu - A - (\gamma R - \lambda A)}{r}\}$.

The cash payments dI_t of Baseline and Renegotiating Baseline contracts are designed to reflect W_t downwards at W^{good}. Quiet-Life contracts do not use cash compensation: all utility received by the agent comes from shirking. The rest of this section is concerned with the Quiet-Life form and the Renegotiating Baseline form.

6.1 Interpreting Quiet-Life Contracts

All Quiet-Life contracts satisfy $W^{poor} = R$ and $W^{good} < \frac{\lambda A}{\gamma}$. The equality holds because termination occurs at W^{poor}. The strict inequality comes from two observations. First, W^{good} cannot be greater than $\frac{\lambda A}{\gamma}$ because such a threshold would constitute a promised payoff greater than what shirking alone can deliver. Second, W^{good} cannot be equal to $\frac{\lambda A}{\gamma}$ since that would imply permanent shirking when W_t reaches W^{good}, contradicting the property that Quiet-Life contracts induce shirking non-permanently.

What $W_t \leq W^{good} < \frac{\lambda A}{\gamma}$ implies is that when the agent finally reaches the good performance threshold, he receives a utility flow from shirking, which, if extended indefinitely, would represent a payoff greater than anything the contract actually promises. Thus shirking in Quiet-Life contracts serves to reward the agent, as a form of hidden compensation.

What does a typical hidden compensation package look like? Let H denote the nondecreasing hidden compensation process (similar to the cash compensation process I), where H_t is the amount of shirking utility received by the agent up to time t. At every moment when $W_t = W^{good}$ the agent receives a fixed shirking utility flow $\lambda A dt$. This implies:

$$dH_t = \begin{cases} 0dt & W_t < W^{good} \\ \lambda A dt & W_t = W^{good} \end{cases}$$

Thus to characterize H it suffices to characterize the random set of hidden compensation times $\mathscr{T}(W^{good}) = \{t | W_t = W^{good}\}$.

Away from the good performance threshold W^{good}, the promised value and performance rating W_t of the agent follows the high action law:

$$dW_t|_{W_t < W^{good}} = \gamma W_t + \lambda(dZ_t - \mu dt)$$

which is sensitive to asset performance. At the good performance threshold W^{good}, W_t follows the shirking action law:

$$dW_t|_{W_t = W^{good}} = (\gamma W^{good} - \lambda A)dt$$

and is no longer sensitive to asset performance. Consequently, the principal can ignore the agent's performance. The following lemma characterizes the implied incentive scheme:

Lemma 6.1. *Consider the following SDE:*

$$dW_t = \begin{cases} 0dt & \textit{if } \ W_t = R \\ \gamma W_t + \lambda(dZ_t - \mu dt) & \textit{if } \ W_t \in (R, W^{good}) \\ (\gamma W^{good} - \lambda A)dt & \textit{if } \ W_t = W^{good} < \frac{\lambda A}{\gamma} \end{cases}$$

The solution is well defined and is a member of a class of diffusions called ***Sticky Brownian***

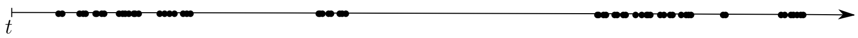

Figure 7: A sample sequence of hidden compensation times.

Motion *or slowly reflecting Brownian motion. Its infinitesimal generator is*

$$A = \begin{cases} 0\frac{d}{dx} & \text{if } x = R \\ \gamma x \frac{d}{dx} + \frac{\lambda^2}{2}\frac{d^2}{dx^2} & \text{if } x \in (R, W^{good}) \\ (\gamma W^{good} - \lambda A)\frac{d}{dx} & \text{if } x = W^{good} \end{cases}$$

Proof. See Harrison and Lemoine (1981). □

The implied mixture of sensitivity and no sensitivity of W_t to asset performance near W^{good} produces a set of hidden compensation times with the following properties:

Corollary 6.2. *Conditional on W_t reaching W^{good}, the set of hidden compensation times $\mathscr{T}(U^{good})$ (almost surely) has positive measure. Moreover, $\mathscr{T}(U^{good})$ is nowhere dense and perfect.*[11] *See Figure 7.*

The positive measure property conforms with economic intuition. A fixed utility flow over a set of times of measure 0 amounts to no utility at all. If a Quiet-Life contract's hidden compensation times were actually trivial then it would not be incentive-compatible. The nowhere dense and perfect properties of the hitting times of W^{good} are standard symptoms of W_t being driven by an underlying Brownian motion.

Definition. *Fix $\epsilon > 0$ and let $t_0^s = t_0^e = 0$. Then recursively define the random times $t_n^s = \inf\{t > t_{n-1}^e | W_t = W^{good}\}$ and $t_n^e = \inf\{t > t_n^s | W_t \leq W^{good} - \epsilon\}$.*[12] *Each interval $[t_n^s, t_n^e)$ where $n \geq 1$ and $t_n^s < \tau$ is defined to be a Quiet-Life phase.*

A Formal Description of Quiet-Life Contracts.
I can now give a more precise description of the dynamics of a Quiet-Life contract. In a Quiet-Life contract the promised value and performance rating W_t initially follows the high action law and is sensitive to asset performance. As a result, the agent initially applies high action all the time. Sustained good performance then pushes the performance rating up to the good performance threshold W^{good}. At this point the contract enters a Quiet-Life phase. Shirking is now permitted as a form of hidden compensation, and the agent frequently shirks. Here, *frequent* shirking means that the agent spends a non-trivial though not full portion of the Quiet-Life phase shirking. It also means that both immediately before and after a shirking time, (almost surely) there are infinitely many other times when the agent is shirking and infinitely many other times when the agent is not shirking. Therefore, during the Quiet-Life phase the principal frequently ignores the agent's performance. Also, during each Quiet-Life phase, the rating stays close to W^{good} and is frequently at W^{good}.

[11] The random set of cash compensation times of a Baseline contract is also nowhere dense and perfect, but has zero measure. A set is *perfect* if it contains all of its limit points, and has no isolated points.

[12] Since the contract almost surely terminates, define $\inf\{\emptyset\} = \infty$.

Eventually, poor performance brings the rating back down and the contract exits the Quiet-Life phase. The principal no longer ignores the agent's performance and the agent resumes applying high action at all times. This dynamic remains until sustained good performance triggers another round of the Quiet-Life phase or sustained poor performance finally triggers termination.

Theorem 2 implies that it is possible for the optimal contract to take the Quiet-Life form even when shirking is inefficient ($\lambda < 1$). The implication is that frequently ignoring the agent's performance and letting him shirk has intrinsic advantages. I now highlight this advantage by comparing Quiet-Life contracts against Baseline contracts, which never induce shirking.

Fix a Quiet-Life contract $\mathscr{Q}$ with some good performance threshold W^{good} and agent payoff W_0. I can design the companion baseline contract $\mathscr{B}$ with the same threshold W^{good} and the same agent payoff W_0. Let $W_t^{\mathscr{Q}}$ denote $\mathscr{Q}$'s promised value process and define $W_t^{\mathscr{B}}$ similarly. Obviously, these two contracts exhibit a large amount of structural similarity: they have the same performance thresholds,[13] the promised value process of the two contracts follow the same high action law on the open interval $(W^{poor} = R, W^{good})$, and both contracts terminate at $W^{poor} = R$. The only structural difference is at W^{good} where $\mathscr{Q}$ induces shirking as a form of hidden compensation and $\mathscr{B}$ delivers cash.

Formally, the contractual similarity of $\mathscr{Q}$ and $\mathscr{B}$ is captured by the fact that the infinitesimal generators of $W_t^{\mathscr{Q}}$ and $W_t^{\mathscr{B}}$ are identical except at W^{good}. DS shows that $W_t^{\mathscr{B}}$ is a reflecting Brownian motion.[14] Therefore, Lemma 6.1, which says that $W_t^{\mathscr{Q}}$ is a Sticky Brownian motion, implies the following result:

Corollary 6.3. *The incentive scheme of the Quiet-Life contract $\mathscr{Q}$ is slower than that of the Baseline contract $\mathscr{B}$. Call this slower incentive scheme of $\mathscr{Q}$* ***sticky incentives****. Formally, Let $S(t)$ be the non-decreasing process that keeps track of the total amount of time that the agent has shirked in $\mathscr{Q}$ up to time t. Let Z be the unique process satisfying $Z(t) - S(Z(t)) = t$. Then $Z(t) - t$ is non-negative, non-decreasing, and*

$$W^{\mathscr{Q}}_{Z(t)} =_d W^{\mathscr{B}}_t$$

Moreover, $Z(t) - t$ increases if and only if the agent shirks.

Let $H_t^{\mathscr{Q}} = \int_0^t \lambda A dS(t)$ denote the hidden compensation process of $\mathscr{Q}$, and $\tau^{\mathscr{Q}}$ denote the termination time of $\mathscr{Q}$. Similarly, let $I_t^{\mathscr{B}}$ denote the cash compensation process of $\mathscr{B}$, and $\tau^{\mathscr{B}}$ denote the termination time of $\mathscr{B}$. The following formalizes the value of sticky incentives:

Corollary 6.4. *A sticky incentive scheme implies that hidden compensation is more modest than cash compensation:*

$$H^{\mathscr{Q}}_{Z(t)} =_d I^{\mathscr{B}}_t \Rightarrow \mathbf{E}[H^{\mathscr{Q}}_t] < \mathbf{E}[I^{\mathscr{B}}_t] \quad \textit{for all } t > 0.$$

[13] Their poor performance thresholds are both R by assumption.

[14] Also see Biais, Mariotti, Plantin, and Rochet (2007), which gets a reflected Brownian motion in the continuous-time limit.

Furthermore, inefficient termination is delayed:

$$\tau^{\mathscr{Q}} =_d Z(\tau^{\mathscr{B}}) \geq \tau^{\mathscr{B}}$$

When the optimal contract takes the Quiet-Life form, the aforementioned advantages of sticky incentives outweigh possible inefficiency concerns with regards to shirking. In this case, Theorem 1 provides a useful quantification of how much the principal values having the freedom to allow the agent to shirk.

Corollary 6.5. *Suppose the optimal contract is a Quiet-Life contract. There exists a unique distortionary tax subsidy on cash such that at any time during the optimal contract, the principal is indifferent between continuing with the contract in the current setting and switching to a setting where he is restricted to induce the agent to never shirk but is compensated with the tax subsidy.*[15]

6.2 Interpreting Renegotiating Baseline Contracts

Definition. *The underlying Baseline contract of a Renegotiating Baseline contract $\mathscr{R}$ is the Baseline contract with the same good performance threshold as $\mathscr{R}$.*

In this model, the principal is fully committed, and there is no renegotiation. The name Renegotiating Baseline comes from the following observation, which I will show shortly: a Renegotiating Baseline contract's promised value behaves like that of its underlying Baseline contract under repeated renegotiations that are unexpected by the agent. Contracting with a naive agent that does not expect renegotiations can certainly lead to arrangements that improve the principal's payoff. A Renegotiating Baseline contract represents an *incentive-compatible* approximation of such an (incentive-incompatible) arrangement.

In a Renegotiating Baseline contract, shirking is induced at some poor performance threshold W^{poor} which need not equal R. The dynamics of a Renegotiating Baseline contract's promised value at W^{poor} is the mirror image of the dynamics of a Quiet-Life contract's promised value at W^{good}. Therefore, Quiet-Life concepts like frequent shirking and sticky incentives translate over. However, the role of shirking is different in Renegotiating Baseline contracts. Unlike in Quiet-Life contracts, in Renegotiating Baseline contracts $\frac{\lambda A}{\gamma} < W^{poor} \leq W_t$. This means that when the promised value and performance rating W_t drops down to the poor performance threshold W^{poor}, the agent receives a utility flow, which if extended indefinitely, would represent a value strictly less than anything the contract actually promises. Thus shirking times in Renegotiating Baseline contracts serve to punish the agent.

This is not to say the agent dislikes shirking. On the contrary, shirking is simply the best the agent can do for himself in this arrested phase of the contract. The canonical example of this phenomenon is *suspension*. During a suspension, the agent's compensation is frozen, and he does not work. Despite the agent's fondness for not working, he would rather be working hard and receiving cash compensation then be stuck in this low state.

[15] Recall from Section 4, a distortionary tax subsidy means for every dollar received by the agent, the principal pays less than one dollar.

The idea of contractual punishment is not new. A termination clause serves the same purpose. So why not just terminate like in a Baseline contract?

In many Baseline contracts (including the optimal one), when the performance rating is near the poor performance threshold W^{poor} and termination is probabilistically imminent, the principal is better off giving the agent some more slack. The principal can achieve this by breaking the terms of the contract and simply shifting the performance rating upwards, removing it from the vicinity of W^{poor}. This renegotiated Baseline contract effectively forgives the agent for his poor performance. Each time this is done the principal increases his own payoff as well as that of the agent. However, the value of this renegotiation is predicated on the agent not expecting to be forgiven and applying high action throughout. Unfortunately, if the agent expects that the principal will renege on termination, then the incentives to apply high action will be destroyed. Thus such a renegotiation is not incentive-compatible, and it is imperative that the principal commits to terminate as the contract originally dictates.

However, the potential losses due to a premature end to the principal-agent relationship may be great. Thus it is potentially profitable for the principal to find an incentive-compatible contract that mimics the above incentive-incompatible arrangement: a contract that induces high action most of the time but still is able to back out of termination during periods of poor performance. The Renegotiating Baseline contract achieves this by picking a poor performance threshold and inducing shirking there as a suspension phase.

From our discussion of Quiet-Life phases we know two things will happen when the principal induces shirking at W^{poor}:

1) W_t will eventually leave the vicinity of W^{poor} after the end of the suspension phase.

2) The contract will spend a nontrivial amount of time at W^{poor}.

That the Renegotiating Baseline contract pushes the rating upwards after poor performance means that its incentive scheme is qualitatively similar to that of a Baseline contract under repeated renegotiations that are unexpected by the agent. But since this push happens only after a suspension phase, the high action incentives of the underlying baseline contract are not compromised. The agent doesn't get the extra slack for free. By having to first suffer through suspension every time his performance rating drops to W^{poor}, the agent effectively buys the principal's forgiveness through the postponement of the cash compensation promised by the underlying Baseline contract.

A Formal Description of Renegotiating Baseline Contracts.

A Renegotiating Baseline contract begins as its underlying baseline contract, inducing high action and paying cash whenever the promised value and performance rating W_t hits the good performance threshold W^{good}.

However, when poor performance pushes the rating down to the poor performance threshold W^{poor}, a suspension phase - similar to a Quiet-Life phase - is triggered. During suspension, the principal frequently ignores the agent and the agent, lacking proper incentives to work, frequently exerts low effort. As a result, the rating sticks or is "pegged" around W^{poor} for a period of time, following the dynamics of Sticky Brownian motion. In particular, the rating is frequently at W^{poor}.

Eventually, suspension ends, the agent is forgiven for some of his poor performance, and W_t is allowed to float again as the agent receives more slack. The contract restarts the high action incentives of the underlying Baseline contract which induces the agent to never shirk and rewards sustained good performance with cash. This dynamic remains until sustained poor performance triggers suspension again.

The avoidance of termination through shirking/suspension phases is a clear advantage that Renegotiating Baseline contracts have over Baseline contracts. It is a primary reason why the optimal contract is sometimes a Renegotiating Baseline contract. In these cases, Theorem 1 provides a useful quantification of how much the principal values having the freedom to allow the agent to shirk. This quantification is different than the one given for optimal Quiet-Life contracts.

Corollary 6.6. *Suppose the optimal contract is a Renegotiating Baseline contract. There exists a unique upgraded principal's outside option $L^* > L$, such that at any time during the optimal contract, the principal is indifferent between continuing with the contract in the current setting and switching to a setting where he is restricted to induce the agent to never shirk but is compensated with the higher outside option L^* upon termination.*

Moreover, since a Renegotiating Baseline contracts "embeds" a portion of the renegotiableness of its underlying Baseline contract, it is not surprising that some Renegotiating Baseline contracts are renegotiation-proof. In fact:

Corollary 6.7. *Under some realizations of the model's parameters, the optimal contract, which a priori need not be renegotiation-proof, is a renegotiation-proof Renegotiating Baseline contract. Moreover, unlike optimal renegotiation-proof contracts that never induce shirking, all Renegotiating Baseline contracts never terminate and do not use public randomization.*

Proof. See proof of Theorem 2, general case, in the Appendix. □

7 Conclusion

In this paper I explicitly solve for the optimal contract in a standard dynamic agency model where the agent can shirk. While inducing the agent to never shirk is a good benchmark, in general, the optimal contract may involve shirking. Indeed, many real life arrangements do not induce high action at all times.

I find that the optimal contract takes on one of four forms depending on fundamentals, including two that involve temporary shirking: the Quiet-Life form and the Renegotiating Baseline form. A Quiet-Life contract induces shirking as a form of hidden compensation. During Quiet-Life phases, the agent is frequently shirks and the agent's performance rating sticks around the contract's good performance threshold. A Renegotiating Baseline contract mostly induces high action but periodically triggers suspension phases as a form of punishment. During these phases, cash compensation is postponed, the agent frequently shirks and the agent's performance rating sticks around the contract's poor performance threshold.

A common theme shared by the two new optimal contractual forms is the value of slowing down incentives. By slowing down incentives, I show how the Quiet-Life contract can delay termination and how the Renegotiating Baseline contract can mimic the dynamics of a beneficial but incentive-incompatible renegotiation arrangement in an incentive-compatible way. I also investigate connections between taxes, optimal contracting and renegotiation. Lastly, I solve the optimal contracting problem when the agent can bargain for higher payoffs.

8 Appendix

Lemmas 8.1 and 8.2 describe basic regularity properties of solutions to the high action ODE and are stated without proof.

Lemma 8.1. *Let f_1 and f_2 be two distinct solutions to the high action ODE and $x^* \geq 0$. Then*

$$f_1(x^*) \leq f_2(x^*) \text{ and } f_1''(x^*) \geq f_2''(x^*) \Longrightarrow f_1''(x) > f_2''(x) \text{ for all } x \in (x^*, \infty)$$

and

$$f_1(x^*) \leq f_2(x^*) \text{ and } f_1''(x^*) \leq f_2''(x^*) \Longrightarrow f_1''(x) < f_2''(x) \text{ for all } x \in [0, x^*)$$

Proof. The straightforward, albeit tedious, proof of this lemma involves Euler's Method. Fix a set of initial conditions for the first-best action ODE: $(x^*, f(x^*), f'(x^*))$ with $x^* \geq 0$. Then

$$f''(x^*) = \frac{rf(x^*) - \mu - \gamma x^* f'(x^*)}{\phi^2/2}$$

and by Euler's Method, we have

$$f(x^* + \Delta x) \approx f(x^*) + f'(x^*)\Delta x$$

$$f'(x^* + \Delta x) \approx f'(x^*) + \frac{rf(x^*) - \mu - \gamma x^* f'(x^*)}{\phi^2/2}\Delta x$$

$$f''(x^* + \Delta x) \approx \frac{rf(x^* + \Delta x) - \mu - \gamma(x^* + \Delta x)f'(x^* + \Delta x)}{\phi^2/2}$$

$$= \frac{r(f(x^*) + f'(x^*)\Delta x) - \mu - \gamma(x^* + \Delta x)(f'(x^*) + \frac{rf(x^*) - \mu - \gamma x^* f'(x^*)}{\phi^2/2}\Delta x)}{\phi^2/2}$$

$$= \frac{(1 - \gamma(x^* + \Delta x)\frac{\Delta x}{\phi^2/2})[rf(x^*) - \mu - \gamma x^* f'(x^*)] - (\gamma - r)\Delta x f'(x^*)}{\phi^2/2}$$

$$= \left[1 - \gamma(x^* + \Delta x)\frac{\Delta x}{\phi^2/2}\right] f''(x^*) - \frac{(\gamma - r)\Delta x f'(x^*)}{\phi^2/2}$$

Now let f_1 and f_2 satisfy the hypothesis of the first half of the lemma at x^* and fix an arbitrary upper bound D with $x^* < D$. Let Δx be small enough so that $1 - \gamma(D + \Delta x)\frac{\Delta x}{\phi^2/2} > 0$. The assumptions imply $f_2'(x^*) > f_1'(x^*)$, and then it is easy to see that the Euler approximations of f_1 and f_2 satisfy the hypothesis of the first half of the lemma at $x^* + \Delta x$ as well. In fact, the second derivative of the Euler approximation of f_2 is now *strictly* less than that of the Euler approximation of f_1 at $x^* + \Delta x$. Then induction shows that the second derivative of the Euler approximation of f_2 is strictly less than that of the Euler approximation of f_1 at $x^* + n\Delta x$, so long as $x^* + n\Delta x \in (x^*, D]$. Letting $\Delta x \to 0$, we have

$$f_1''(x) > f_2''(x) \text{ for all } x \in (x^*, D]$$

Since D was arbitrary, the first half of the lemma holds.

Now suppose f_1 and f_2 satisfy the hypothesis of the second half of the lemma. If $f_1 < f_2$ on $[0, x^*)$ then the second half of the lemma must hold. Suppose not, then there is some $\tilde{x} \in [0, x^*)$ such that $f_1''(\tilde{x}) \geq f_2''(\tilde{x})$. But then the first half of the lemma implies that $f_1''(x^*) > f_2''(x^*)$. Contradiction.

So it suffices to prove $f_1 < f_2$ on $[0, x^*)$. The hypothesis of the second half of the lemma immediately implies that f_1 lies below f_2 in a left neighborhood of x^*. This means that if it is not true that $f_1 < f_2$ on $[0, x^*)$ then there must be some point $\tilde{x}$ such that $f_1(\tilde{x}) = f_2(\tilde{x})$ and $f_1'(\tilde{x}) < f_2'(\tilde{x})$. But then this implies that $f_1''(\tilde{x}) > f_2''(\tilde{x})$ and once again the first half of the lemma implies a contradiction. □

Lemma 8.2. *Let f_1 and f_2 be two distinct solutions to the high action ODE and $x^* \geq 0$. Then*

$$f_1(x^*) \leq f_2(x^*) \text{ and } f_1'(x^*) \leq f_2'(x^*) \Longrightarrow f_1'(x) < f_2'(x) \text{ for all } x \in (x^*, \infty)$$

and

$$f_1(x^*) \leq f_2(x^*) \text{ and } f_1'(x^*) \geq f_2'(x^*) \Longrightarrow f_1'(x) > f_2'(x) \text{ for all } x \in [0, x^*)$$

Proof. If f_1 and f_2 satisfy the assumptions of the second half, then $f_1''(x^*) < f_2''(x^*)$. Then the second half result follows from the second half of Lemma 8.1.

Now suppose $f_1(x^*) \leq f_2(x^*)$ and $f_1'(x^*) \leq f_2'(x^*)$ and there exists an $x \in (x^*, \infty)$ such that $f_1'(x) \geq f_2'(x)$. Without loss of generality, we may choose x so that $f_1(x) < f_2(x)$. But then the second half of the lemma implies a contradiction. □

Corollary 8.3. *Fix $X \geq 0$, $\frac{\mu}{r} > Y_1 \geq Y_2$, $\pi_1 \geq \pi_2 > 0$ such that $(Y_1, \pi_1) \neq (Y_2, \pi_2)$ and $\omega^{\pi_2}_{(X,Y_2)} > X$. Then $b^{\pi_1}_{(X,Y_1)}{}'(W) < b^{\pi_2}_{(X,Y_2)}{}'(W)$ for all $W \in [X, \omega^{\pi_2}_{(X,Y_2)})$ and $\omega^{\pi_1}_{(X,Y_1)} < \omega^{\pi_2}_{(X,Y_2)}$.*

Proof. First suppose $\pi_1 > \pi_2$ and $Y_1 = Y_2 = Y$. Since the principal is worse off with π_1 than with π_2, $b^{\pi_1}_{(X,Y)}(W) < b^{\pi_2}_{(X,Y)}(W)$ for all $W > X$. Therefore, $b^{\pi_1}_{(X,Y)}{}'(X) < b^{\pi_2}_{(X,Y)}{}'(X)$. Lemma 8.2 and the inequality $-\pi_1 < -\pi_2$ then imply $b^{\pi_1}_{(X,Y)}{}'(W) < b^{\pi_2}_{(X,Y)}{}'(W)$ for all $W \in [X, \infty)$ and the high action ODE implies $b^{\pi_1}_{(X,Y)}{}''(X) > b^{\pi_2}_{(X,Y)}{}''(X)$. Now suppose $\omega^{\pi_1}_{(X,Y)} \geq \omega^{\pi_2}_{(X,Y)}$. Lemma 8.1 implies

$$b^{\pi_1}_{(X,Y)}{}''(\omega^{\pi_2}_{(X,Y)}) > b^{\pi_2}_{(X,Y)}{}''(\omega^{\pi_2}_{(X,Y)}) = 0$$

Contradiction, since $b^{\pi_1}_{(X,Y)}$ is concave. Therefore, $\omega^{\pi_1}_{(X,Y)} < \omega^{\pi_2}_{(X,Y)}$.

Next, suppose $\pi_1 = \pi_2 = \pi$ and $Y_1 > Y_2$. If $\omega^{\pi}_{(X,Y_1)} = X$ then clearly, $\omega^{\pi}_{(X,Y_1)} < \omega^{\pi}_{(X,Y_2)}$ and $b^{\pi}_{(X,Y_1)}{}'(W) = -\pi < b^{\pi}_{(X,Y_2)}{}'(W)$ for all $W \in [X, \omega^{\pi}_{(X,Y_2)})$. So suppose $\omega^{\pi}_{(X,Y_1)} > X$. DS shows that whenever $\omega^{\pi}_{(X,Y)} > X$ the point $(\omega^{\pi}_{(X,Y)}, b^{\pi}_{(X,Y)}(\omega^{\pi}_{(X,Y)}))$ lies on the line $ry + \pi\gamma x = \mu$. Since the principal is better off with Y_1 than with Y_2, $b^{\pi}_{(X,Y_1)}(W) > b^{\pi}_{(X,Y_2)}(W)$ for all $W \geq X$. This means $b^{\pi}_{(X,Y_1)}$ reaches the line $ry + \pi\gamma x = \mu$ first, and therefore, $\omega^{\pi}_{(X,Y_1)} < \omega^{\pi}_{(X,Y_2)}$. Since

$$-\pi = b^{\pi}_{(X,Y_1)}{}'(W) < b^{\pi}_{(X,Y_1)}{}'(W) \text{ for all } W \in [\omega^{\pi}_{(X,Y_1)}, \omega^{\pi}_{(X,Y_2)})$$

Lemma 8.2 implies $b^{\pi_1}_{(X,Y_1)}{}'(W) < b^{\pi_1}_{(X,Y_2)}{}'(W)$ for all $W \in [X, \omega^{\pi}_{(X,Y_2)})$. □

Corollary 8.4. $\lim_{\pi\downarrow 0} \omega^{\pi}_{(X,Y)} = \infty$ *and* $b^0_{(X,Y)} \equiv \lim_{\pi\downarrow 0} b^{\pi}_{(X,Y)}$ *is well-defined.* $b^0_{(X,Y)} < \frac{\mu}{r}$, *is everywhere increasing, solves the high action ODE and asymptotically approaches* $\frac{\mu}{r}$.

Proof. The previous corollary implies $\omega^0_{(X,Y)} \equiv \lim_{\pi\downarrow 0} \omega^{\pi}_{(X,Y)}$ is well defined. Suppose $\omega^0_{(X,Y)} < \infty$. Let g be a solution to the high action ODE starting at (X, Y) that intersects $y = \frac{\mu}{r}$. For each $W > X$, as π decreases, $b^{\pi}_{(X,Y)}$ monotonically increases and remains bounded above by $\frac{\mu}{r}$, $b^{\pi}_{(X,Y)}{}'(W)$ monotonically increases and remains bounded by $g'(X)$, and $b^{\pi}_{(X,Y)}{}''(W)$ monotonically decreases and remains bounded below by $g''(X)$. Therefore $b^0_{(X,Y)} \leq \frac{\mu}{r}$ is a well-defined function that solves the high action ODE on $[X, \omega^0_{(X,Y)}]$. Moreover,

$b^0_{(X,Y)}\ '(\omega^0_{(X,Y)}) = \lim_{\pi\downarrow 0} b^\pi_{(X,Y)}\ '(\omega^\pi_{(X,Y)}) = 0$ and $b^0_{(X,Y)}\ ''(\omega^0_{(X,Y)}) = \lim_{\pi\downarrow 0} b^\pi_{(X,Y)}\ ''(\omega^\pi_{(X,Y)}) = 0$. This forces $b^0_{(X,Y)}(\omega^0_{(X,Y)}) = \frac{\mu}{r}$. But the only solution g to the high action ODE satisfying initial conditions $g(x) = \frac{\mu}{r}$ and $g''(x) = 0$ is $g \equiv \frac{\mu}{r}$. Contradiction. Therefore $\omega^0_{(X,Y)} = \infty$ and $b^0_{(X,Y)} < \frac{\mu}{r}$ and solves the high action ODE. $b^0_{(X,Y)}$ is everywhere non-decreasing. However, if there exists a W such that $b^0_{(X,Y)}\ '(W) = 0$, then since $b^0_{(X,Y)}\ ''(W) < 0$, $b^0_{(X,Y)}\ '(W') < 0$ for all $W' > W$. Contradiction. So $b^0_{(X,Y)}$ is everywhere increasing. Since it is bounded above, it must asymptotically approach some value $\alpha \leq \frac{\mu}{r}$. Since $\lim_{W\to\infty} b^0_{(X,Y)}\ '(W) = 0$, the high action ODE implies that $0 = \lim_{W\to\infty} b^0_{(X,Y)}\ ''(W) = \frac{r\alpha-\mu}{\lambda^2/2}$. Therefore, $\alpha = \frac{\mu}{r}$. □

8.1 Proofs of Theorems 1 and 2

In this subsection, I assume $b'(R) > 0$.

Proof of Lemma 4.2. Recall I assume $R = 0$. The $R > 0$ case is proved separately.

Basic regularity properties of the high action ODE imply that V_{Left} is closed and connected. Notice $b'(0) > 0$ and from DS, we know $\lim_{Y\uparrow\frac{\mu}{r}} b'_{(0,Y)}(0) = -1$. By the Intermediate Value Theorem, there exists an $L' \in (L, \frac{\mu}{r})$ such that $b'_{(0,L')}(0) = 0$ and therefore, $(0, L') \in V_{Left}$. Clearly $(\arg\max b, \max b) \in V_{Left}$. Let $(X,Y) \in V_{Left}$. Then Corollary 8.3 implies $b'_{(X,\tilde{Y})}(X) \gtrless b'_{(X,Y)}(X)$ if $\tilde{Y} \lessgtr Y$. Thus $(X,\tilde{Y}) \notin V_{Left}$. Now suppose that $b'_{(\tilde{X},\tilde{Y})}(\tilde{X}) = 0$ and $\tilde{X} > X$. Since $b'_{(\tilde{X},b_{(X,Y)}(\tilde{X}))}(\tilde{X}) = b'_{(X,Y)}(\tilde{X}) < 0$, Corollary 8.3 implies $\tilde{Y} < b_{(X,Y)}(\tilde{X}) < Y$. This simultaneously implies that V_{Left} is a strictly decreasing curve and that it ends at $(\arg\max b, \max b)$.

Basic regularity properties of the high action ODE imply that V_{Right} is connected. Clearly $(\arg\max b, \max b) \in V_{Right}$. Let $(\arg\max b^{\pi_1}, \max b^{\pi_1})$ and $(\arg\max b^{\pi_2}, \max b^{\pi_2}) \in V_{Right}$. Suppose $x^* = \arg\max b^{\pi_1} = \arg\max b^{\pi_2}$. Since $b^{\pi_1}\ '(x^*) = b^{\pi_2}(x^*) = 0$, if $b^{\pi_1}(x^*) \neq b^{\pi_2}(x^*)$ then Lemma 8.2 implies that $b^{\pi_1}(0) \neq b^{\pi_2}(0)$. Contradiction. Now suppose $x^* = \arg\max b^{\pi_1} < \arg\max b^{\pi_2}$. Then $b^{\pi_1}\ '(x^*) = 0 < b^{\pi_2}\ '(x^*)$. If $b^{\pi_1}(x^*) \geq b^{\pi_2}(x^*)$ then Lemma 8.2 implies that $b^{\pi_1}(0) > b^{\pi_2}(0)$. Contradiction. Therefore, $\max b^{\pi_1} = b^{\pi_1}(x^*) < b^{\pi_2}(x^*) < \max b^{\pi_2}$. This implies that $\pi_1 > \pi_2$ and V_{Right} is strictly increasing. It remains to be shown that V_{Right} has a horizontal asymptote $y = \frac{\mu}{r}$. Since we now know $\arg\max b^\pi$ and $\max b^\pi$ are both decreasing functions of π, define $m = \lim_{\pi\downarrow 0} \arg\max b^\pi$ and $M = \lim_{\pi\downarrow 0} \max b^\pi$. If $m < \infty$, then $b^0\ '(m) = 0$. Contradiction. Thus V_{Right} is defined over $[\arg\max b, \infty)$. Finally, note $\frac{\mu}{r} \geq M \geq \sup b^0 = \frac{\mu}{r}$. □

Proof of Theorem 1. Recall, I assume $R = 0$. The $R > 0$ case is proved separately.

Suppose the shirking payoff (w^s, b^s) is in the right region. Then by assumption, (w^s, b^s) lies strictly below some point $(\arg\max b^{\pi'}, \max b^{\pi'})$. Thus $f^{\pi'}(w^s = \arg\max b^{\pi'}) = \max b^{\pi'} > b^s$. Now Lemma 4.1 and $\pi' < 1$ imply $B < B^{\pi'} = b^{\pi'}$. Let π^* be the largest π such $B \leq b^\pi$ and define $b^* = b^{\pi^*}$.

Suppose the shirking payoff (w^s, b^s) is in the left region. Then by assumption, (w^s, b^s) lies strictly below some point $(w^s, V_{Left}(w^s))$. Extend $b_{(w^s, V_{Left}(w^s))}$ leftwards following the high action ODE to some point $(0, Y')$. Since $b_{(w^s, V_{Left}(w^s))}\ '(w^s) = 0 < b'(w^s)$, Lemma 8.2 implies that $Y' > L$. Then $f_{(0,Y')}(w^s) = f_{(w^s, V_{Left}(w^s))}(w^s) = V_{Left}(w^s) > b^s$. Now Lemma 4.1 and $Y' > L$ imply $B < B_{(0,Y')} = b_{(0,Y')}$. Let L^* be the smallest Y such $B \leq b_{(0,Y)}$ and define $b^* = b_{(0,L^*)}$.

By definition $b^* = b^{\pi^*}$ or $b_{(0,L^*)}$ is an upper bound of B. Tightness will follow from the proof of Theorem 2.

When the shirking payoff lies on or below f, Lemma 3.3 implies that the optimal contract takes the Baseline form.

Finally, it needs to be shown when the shirking payoff (w^s, b^s) lies inside the V-curve then the optimal contract is Static. DS has shown that if $b'_{(w^s,b^s)}(w^s) \leq 0$ and $g'(w^s) \geq 0$ where g is the unique solution to the high action ODE going from $(0, L)$ to (w^s, b^s), then $B = g|_{[0,w^s)} \cup (w^s, b^s) \cup b_{(w^s,b^s)}|_{(w^s,\infty)}$. Consequently, the optimal contract is Static. See $(\tilde{w^s}, \tilde{b^s})$ in Figure 1. I now show that if the shirking payoff (w^s, b^s) lies on or inside the V-curve then it satisfies the aforementioned two differential properties. Suppose $w^s \leq \arg\max b$. If $g'(w^s) < 0$, then since $b'(w^s) \geq 0$, Lemma 8.2 implies that $g(0) > b(0)$. Contradiction. Therefore, $g'(w^s) \geq 0$. Also, since $b^s \geq V_{Left}(w^s)$, Corollary 8.3 implies that $b'_{(w^s,b^s)}(w^s) \leq b'_{(w^s,V_{Left}(w^s))}(w^s) = 0$.

Now suppose $w^s > \arg\max b$. Using similar arguments, one can again conclude that the two differential properties are satisfied. $\square$

Proof of Theorem 2. Recall, I assume $R = 0$. The $R > 0$ case is proved separately.

Suppose (w^s, b^s) is in the right region and $b^* = b^{\pi^*}$. If $b^s < f^{\pi^*}(w^s)$, then the basic regularity properties of the high action ODE imply that one can increase π^* slightly to some π' and still have $b^s < f^{\pi'}(w^s)$, contradicting the maximality of π^*. Therefore, $b^s = f^{\pi^*}(w^s)$. This then immediately implies the existence of $W^* > R$ and h satisfying $h(W^*) = b^*(W^*)$ and $h'(W^*) = b^{*\prime}(W^*)$. Now suppose $h''(W^*) < b^{*\prime\prime}(W^*)$. Then pick a slightly higher solution $\tilde{h}$ to the shirking action ODE. $\tilde{h}$ crosses b^* transversally exactly twice in a small neighborhood of W^*, once from below and once from above. Let $\tilde{W}$ be the point at which $\tilde{h}$ crosses b^* from above. Then $b^s > b^*(\tilde{W}) + \frac{\gamma}{r}(w^s - \tilde{W})b^{*\prime}(\tilde{W})$. Contradiction. Similar arguments rule out $h''(W^*) > b^{*\prime\prime}(W^*)$. Therefore, $h''(W^*) = b^{*\prime\prime}(W^*)$. Since $b^{*\prime\prime}(\omega^*) = 0$, $W^* < \omega^*$. Similar arguments work when (w^s, b^s) is in the left region and $b^* = b_{(0,L^*)}$.

Again, suppose (w^s, b^s) is in the right region. Since $b^s = f^{\pi^*}(w^s)$, π^* is strictly greater than the π' considered in the proof of Theorem 1. Now Corollary 8.3 implies that $\arg\max b^{\pi^*} < w^s$. Since $h'(W^*) = b^{\pi^*\prime}(W^*)$ implies W^* must lie between $\arg\max b^{\pi^*}$ and w^s, it is also the case $dW_t|_{W^*} = (\gamma W^* - \lambda A)dt < 0dt$. Similar arguments imply that when (w^s, b^s) is in the left region, $w^s < W^* < \arg\max b_{(0,L^*)}$ and $dW_t|_{W^*} = (\gamma W^* - \lambda A)dt > 0dt$.

The only thing left to show is that the contracts defined in the theorem imply B^{opt} is attainable.

Define $G_t = \int_0^t e^{-rs}(dZ_s - dI_s) + e^{-rt}b^*(W_t)$. From Ito's lemma,

$$e^{rt}dG_t = \left(\mu - a_t + (\gamma W_t - \lambda a_t)b^{*'}(W_t) + \frac{1}{2}\beta_t^2 b^{*''}(W_t) - rb^*(W_t)\right)dt$$

$$-(1 + b^{*'}(W_t))dI_t + (1 + \beta_t b^{*'}(W_t))\,(dZ_t - (\mu - a_t)dt)$$

At any moment in time, either $b^{*'}(W_t) = -1$ or $dI_t = 0$. Moreover, $a_t = A \Rightarrow \beta_t = 0$ and $a_t = 0 \Rightarrow \beta_t = \lambda$. Thus,

$$e^{rt}dG_t = \begin{cases} (\mu + \gamma W_t b^{*'}(W_t) + \frac{1}{2}\lambda^2 b^{*''}(W_t) - rb^*(W_t))dt + (1 + \lambda b^{*'}(W_t))(dZ_t - \mu dt) & \text{if } W_t \neq W^* \\ (\mu - A + (\gamma W^* - \lambda A)b^{*'}(W_t) - rb^*(W_t))dt + (dZ_t - (\mu - A)dt) & \text{if } W_t = W^* \end{cases}$$

In each case, the drift term is 0 and so G_t is a semimartingale. For all $t < \infty$,

$$\mathbf{E}\left[\int_0^\tau e^{-rs}(dZ_s - dI_s) + e^{-r\tau}L\right] = \mathbf{E}\left[G_{t\wedge\tau} + 1_{t\leq\tau}\left(\int_t^\tau e^{-rs}(dZ_s - dI_s) + e^{-r\tau}L - e^{-rt}b^*(W_t)\right)\right]$$

$$= \mathbf{E}[G_{t\wedge\tau}] + e^{-rt}\mathbf{E}\left[1_{t\leq\tau}\left(\int_t^\tau e^{-r(s-t)}(dZ_s - dI_s) + e^{-r(\tau-t)}L - b^*(W_t)\right)\right]$$

Rearranging,

$$\mathbf{E}\left[\int_0^\tau e^{-rs}(dZ_s - dI_s) + e^{-r\tau}L\right] - \mathbf{E}[G_{t\wedge\tau}] =$$

$$e^{-rt}\mathbf{E}\left[1_{t\leq\tau}\left(\int_t^\tau e^{-r(s-t)}(dZ_s - dI_s) + e^{-r(\tau-t)}L - b^*(W_t)\right)\right] \geq e^{-rt}1_{t\leq\tau}\left[L + R - W_t - b^*(W_t)\right]$$

The last inequality holds because the principal can always guarantee himself a payoff of $L + R - W_t$ subject to delivering W_t to the agent by paying a lump sum $W_t - R$ to the agent and terminating immediately. Since $t \wedge \tau$ is a bounded stopping time, $\mathbf{E}[G_{t\wedge\tau}] = G_0 = b^*(W_0)$ for all t. Since W_t is bounded for all t, so is $L + R - W_t - b^*(W_t)$. Taking $t \to \infty$ gives $\mathbf{E}\left[\int_0^\tau e^{-rs}(dZ_s - dI_s) + e^{-r\tau}L\right] \geq b^*(W_0)$. $\square$

Proof of Theorems 1 and 2 along with characterization of V-Curve when $R > 0$. When $R > 0$, Theorems 1 and 2 both hold upon a slight redrawing of the boundaries between the regions of D. Clearly, any such redrawing of boundaries will only involve the area of D strictly to the left of the vertical line $X = R$. In particular, the two boundaries that the Quiet-Life region shares with the Static and Baseline regions remain unchanged.

So suppose $w^s \leq R$. If $b^s \leq f(w^s)$, then by Lemma 3.3 the optimal contract takes the Baseline form. So suppose $b^s > f(w^s)$. Introduce the Baseline value function $b_{(R,\max\{\frac{\mu-\gamma R}{r},b^s\})}$. It is a straight line:

$$b_{(R,\max\{\frac{\mu-\gamma R}{r},b^s\})}(W) = \max\left\{\frac{\mu-\gamma R}{r}, b^s\right\} - (W-R) \text{ for all } W \geq R$$

It is straightforward to check that $b^s \leq f_{(R,\max\{\frac{\mu-\gamma R}{r},b^s\})}(w^s)$. Therefore, $b_{(R,\max\{\frac{\mu-\gamma R}{r},b^s\})}$ is an upper bound on B. Let $L^* > L$ be the lowest upgraded principal's outside option such that $b_{(R,L^*)}$ is an upper bound on B. When $b^s \geq \frac{\mu-\gamma w^s}{r}$ (equivalent to when $\lambda \geq 1$), one can explicitly verify that $L^* = \frac{\mu-A-(\gamma R-\lambda A)}{r} \geq \frac{\mu-\gamma R}{r}$. The Static contract with salary $\gamma R - \lambda A$ delivers payoff R to the agent and payoff $\frac{\mu-A-(\gamma R-\lambda A)}{r}$ to the principal. Therefore $B = b_{(R,\frac{\mu-A-(\gamma R-\lambda A)}{r})}$ and the optimal contract is the Static contract with salary $\gamma R - \lambda A$.

Finally, suppose $b^s \in (f(w^s), \frac{\mu-\gamma w^s}{r})$. One can explicitly verify that $b^s < f_{(R,\frac{\mu-\gamma R}{r})}(w^s)$. So let L^* be the lowest Y such that $b_{(R,Y)}$ is an upper bound on B. Then $L^* \in (L, \frac{\mu-\gamma R}{r})$ and $\omega_{(R,L^*)} > R$. The minimality of L^* implies the existence of a value $W^* \in [R, \omega_{(R,L^*)})$ and a solution h to the shirking action ODE satisfying:

$$b_{(R,L^*)}(W^*) = h(W^*) \text{ and } b'_{(R,L^*)}(W^*) = h'(W^*) \text{ and} \tag{19}$$
$$b''_{(R,L^*)}(W^*) = h''(W^*) \text{ if } W^* > R$$

When $R > 0$, it is possible for $W^* = R$. Such a smooth pasting at the boundary need not satisfy the super-contact condition. However, if $W^* > R$, then the same arguments used in the proof of the $R = 0$ case apply and $b''_{(R,L^*)}(W^*) = h''(W^*)$. In either case, $w^s < W^* < \arg\max b_{(R,L^*)}$, so $dW_t|_{W^*} = (\gamma W^* - \lambda A)dt > 0dt$. Using the same arguments as in the $R = 0$ case, $B^{opt} = h(W^*) \cup b_{(R,L^*)}|_{(W^*,\omega_{(R,L^*)}]}$ is attainable and the optimal contract is a Renegotiating Baseline contract. □

8.2 Characterization of B

In this subsection, I continue to assume $b'(R) > 0$.

Definition. *The high action inequality and shirking action inequality are defined to be*

$$ry \geq \mu + \gamma x y' + \frac{\lambda^2}{2}y'' \text{ and } ry \geq \mu - A + (\gamma x - \lambda A)y'$$

respectively.

Clearly, a function satisfies the HJB equation if and only if it satisfies both the high action and shirking action inequalities and satisfies at least one of them with equality.

Definition. *Let (u,v) be a point not equal to the shirking payoff. Let $h_{(u,v)}$ denote the unique solution to the shirking action ODE which goes through (u,v).*

Lemma 8.5. *Suppose g and h are solutions to the high action and shirking action ODEs respectively, satisfying*

$$g(u) = h(u) \text{ and } g'(u) = h'(u) \text{ and } g''(u) = h''(u) < 0 \text{ for some } u$$

Then

$$g(u) = h(u) = S(u) \equiv \frac{\mu - A}{r} + \frac{\gamma}{r\lambda}\frac{(u-\frac{\lambda A}{\gamma})^2}{\frac{\lambda}{2A}(1-\frac{r}{\gamma}) - (u - \frac{\lambda A}{\gamma})} \text{ and } u \in \left(-\infty, \frac{\lambda}{2A}(1-\frac{r}{\gamma}) + \frac{\lambda A}{\gamma}\right)$$

*The function S is a differentiable, convex function with unique interior minimum point equal to the shirking payoff $(\frac{\lambda A}{\gamma}, \frac{\mu-A}{r})$. Call the graph of this function the **U-curve**.*

Corollary 8.6. *Fix a point (X, Y) satisfying $X \geq 0$, $X \neq \frac{\lambda A}{\gamma}$ and $Y < \frac{\mu}{r}$. Let g be the unique solution to the high action ODE satisfying $g(X) = Y$ and $g'(X) = h'_{(X,Y)}(X)$. Then $g''(X)$ is strictly greater/lesser than $h''(X)$ if and only if (X, Y) lies strictly inside/outside the U-curve.*

Corollary 8.7. *Let $(X, S(X)) \neq (\frac{\lambda A}{\gamma}, \frac{\mu - A}{r})$ be a point on the U curve. Then between X and $\frac{\lambda A}{\gamma}$, $h_{(X,S(X))}$ satisfies the high action inequality.*

Lemma 8.8. *Let $0 \leq X < \frac{\lambda A}{\gamma}$ and $Y < \frac{\mu}{r}$. Then there exists at most one point $(W^*, S(W^*))$ satisfying $W^* \in (X, \frac{\lambda A}{\gamma})$ and $g'_{(W^*,S(W^*))}(W^*) = h'_{(W^*,S(W^*))}(W^*)$. Here, $g_{(W^*,S(W^*))}$ denotes the unique solution to the high action ODE starting at (X, Y) and going through $(W^*, S(W^*))$.*

Proof. Suppose there were two $W_1^* < W_2^*$. The convexity of the U-curve and the shirking action ODE imply $h'_{(W_2^*,S(W_2^*))}(W_2^*) > h'_{(W_1^*,S(W_1^*))}(W_1^*)$. Then by assumption, $g'_{(W_2^*,S(W_2^*))}(W_2^*) > g'_{(W_1^*,S(W_1^*))}(W_1^*)$. Moreover, by definition of the U-curve, $g''_{(W_2^*,S(W_2^*))}(W_2^*)$, $g''_{(W_1^*,S(W_1^*))}(W_1^*) < 0$. So certainly, $g_{(W_2^*,S(W_2^*))}(W_1^*) < S(W_1^*) = g_{(W_1^*,S(W_1^*))}(W_1^*)$. Moreover, $g'_{(W_2^*,S(W_2^*))}(W_1^*) > g'_{(W_2^*,S(W_2^*))}(W_2^*) > g'_{(W_1^*,S(W_1^*))}(W_1^*)$. But then Lemma 8.2 implies that $g_{(W_2^*,S(W_2^*))}(X) < g_{(W_1^*,S(W_1^*))}(X)$. Contradiction. □

Lemma 8.9. *Let g and h be solutions to the high action and shirking action ODEs respectively, satisfying*

$$g(u) = h(u) > S(u) \text{ and } g'(u) = h'(u) \text{ and } u < \frac{\lambda A}{\gamma}$$

Suppose $g(x) > S(x)$ and $x < \frac{\lambda A}{\gamma}$. Then

$$x \gtrless u \Rightarrow g'(x) \gtrless h'_{(x,g(x))}(x) \tag{20}$$

Proof. Since Corollary 8.6 implies $g''(u) > h''(u)$, Equation (20) holds for all x sufficiently close to u. Suppose the lemma is false for some value to the right of u. Let $x \in (u, \frac{\lambda A}{\gamma})$ be the smallest such value. The minimality of x implies both $g'(x) = h'_{(x,g(x))}(x)$ and $g(\tilde{x}) < h_{(x,g(x))}(\tilde{x})$ for all $\tilde{x} \in (u, x)$. These two observations then imply that $g''(x) \leq h''_{(x,g(x))}(x)$. Contradicting Corollary 8.6. Using similar arguments, one can prove the lemma for when $x < u$. □

Corollary 8.10. *Let $(X, S(X))$ be a point on the left arm of the U-curve (i.e. $X < \frac{\lambda A}{\gamma}$). Suppose $b'_{(X,S(X))}(X) < h'_{(X,S(X))}(X)$. Then there exists a unique $R^* \in (X, \frac{\lambda A}{\gamma})$ satisfying $b'_{(R^*,h_{(X,S(X))}(R^*))}(R^*) = h'_{(X,S(X))}(R^*)$.*

Proof. Existence follows from the Intermediate Value Theorem and the fact that

$$\lim_{x \uparrow \frac{\lambda A}{\gamma}} h'_{(X,S(X))}(x) = -\infty < -1 \leq b'_{(x,h_{(X,S(X))}(x))}(x).$$

Suppose there were two: $R_1^* < R_2^*$. Lemma 8.9 implies that $b_{(R_1^*,h_{(X,S(X))}(R_1^*))}(R_2^*) > h_{(X,S(X))}(R_2^*) = b_{(R_2^*,h_{(X,S(X))}(R_2^*))}(R_2^*)$. Now, extend $b_{(R_2^*,h_{(X,S(X))}(R_2^*))}$ leftwards following the high action ODE. Call the extended function g. Then Lemma 8.9 implies that $g(R_1^*) > h_{(X,S(X))}(R_1^*) = b_{(R_1^*,h_{(X,S(X))}(R_1^*))}(R_1^*)$. This implies that g and $b_{(R_1^*,h_{(X,S(X))}(R_1^*))}$ must have intersected at some point $x \in (R_1^*, R_2^*)$. Contradiction. □

Corollary 8.11. *Let (X, Y) be a point on or inside the left arm of the U-curve and let π be the unique tax such that $b^{\pi}_{(X,Y)}{}'(X) = h'_{(X,Y)}(X)$. Then $b^{\pi}_{(X,Y)}$ satisfies the shirking action ODE.*

Proof. Lemma 8.9 implies that the concave portion of $b^{\pi}_{(X,Y)}$ that lies to the left of $\frac{\lambda A}{\gamma}$ satisfies the shirking action ODE. Moreover, on $b^{\pi}_{(X,Y)}|_{[\frac{\lambda A}{\gamma},\infty)}$ is a decreasing, concave function with starting point $(\frac{\lambda A}{\gamma}, Y)$ where $Y > \frac{\mu - A}{r}$. It is straightforward to check that any such function defined over $[\frac{\lambda A}{\gamma}, \infty)$ satisfies the shirking action ODE. Therefore, if $\omega^{\pi}_{(X,Y)} > \frac{\lambda A}{\gamma}$, all parts of $b^{\pi}_{(X,Y)}$ have been accounted for and $b^{\pi}_{(X,Y)}$ satisfies the shirking action ODE. If $\omega^{\pi}_{(X,Y)} < \frac{\lambda A}{\gamma}$, then the linear part of $b^{\pi}_{(X,Y)}$ starts at some (x,y) with slope greater than $h'_{(x,y)}(x)$ and with $x < \frac{\lambda A}{\gamma}$. Again, it is straightforward to check that any such half-line satisfies the shirking action ODE. $\square$

Theorem 4. *Suppose the shirking payoff lies in the right region of D. Then there exists a tax subsidy $\pi^* \in (0,1)$ and two agent promised values $R < W^* < R^* < \frac{\lambda A}{\gamma}$ that are uniquely determined by the following conditions:*

- $b^{\pi^*}(W^*) = S(W^*)$, $b^{\pi^*\prime}(W^*) = h'_{(W^*,S(W^*))}(W^*)$ *and* $b^{\pi^*\prime\prime}(W^*) = h''_{(W^*,S(W^*))}(W^*)$
- $h'_{(W^*,S(W^*))}(R^*) = b'_{(R^*,h_{(W^*,S(W^*))}(R^*))}(R^*)$.

$B = b^{\pi^*}|_{[R,W^*)} \cup h_{(W^*,S(W^*))}|_{[W^*,R^*]} \cup b_{(R^*,h_{(W^*,S(W^*))}(R^*))}|_{(R^*,\infty)}$. *This function is concave since each component is concave and the pastings are all smooth. However, the pasting at R^* does not satisfy the super-contact condition and so B is C^1 but not C^2.*

Proof. Existence of π^* and W^* are provided by Theorems 1 and 2. Uniqueness of π^* and W^* follow from Lemma 8.8. Existence and Uniqueness of R^* is provided by Corollary 8.10. Denote by $\tilde{B}$, the conjectured $\tilde{B}$ of the Theorem. The first component of $\tilde{B}$ satisfies HJB-equation by construction. The second component of $\tilde{B}$ satisfies the HJB-equation because of Corollary 8.7. The third and final component of $\tilde{B}$ satisfies the HJB-equation because of Corollary 8.11. Clearly, $\tilde{B}$ is C^1 and is C^2 everywhere except R^*. Moreover, fix any starting point $W_0 \geq R$, the SDE implied by $\tilde{B}$ keeps W_t bounded. Specifically, $W_t \leq \max\{W_0, \omega_{(R^*,h_{(W^*,S(W^*))}(R^*))}\}$ for all $t \leq \tau$. Therefore, one can recycle the martingale technique used in the proof of Theorem 2 to prove that every point $(W_0, \tilde{B}(W_0))$ is in fact the payoff of a contract. Finally, since $\tilde{B} < \frac{\mu}{r}$, a slight modification of the same martingale technique can also be used to prove that $\tilde{B}$ is an upper bound on B. Therefore $B = \tilde{B}$. $\square$

When the shirking payoff is in the left region, it is very straightforward to describe B beyond $\omega_{(R,L^*)}$. Simply extend $b_{(R,L^*)}|_{(W^*,\omega_{(R,L^*)}]}$ to $b_{(R,L^*)}|_{(W^*,\infty)}$. Extending B leftwards from W^* to R is more complicated. However, these payoffs are all Pareto-dominated by the payoff to the optimal contract. An approach similar to the one used in the proof of Theorem 4 can be used to prove the following complete characterization of B when the shirking payoff is in the left region.

Theorem 5. *Suppose the shirking payoff lies in the left region of D. Then there exists a unique $\tilde{W}^*$ such that*

$$h'_{(\tilde{W}^*,S(\tilde{W}^*)}(\tilde{W}^*) = b'_{(\tilde{W}^*,S(\tilde{W}^*))}(\tilde{W}^*) \text{ and } h''_{(\tilde{W}^*,S(\tilde{W}^*)}(\tilde{W}^*) = b''_{(\tilde{W}^*,S(\tilde{W}^*))}(\tilde{W}^*)$$

Suppose $W^* > R$. If $\frac{\lambda A}{\gamma} > R$ or if $L > h_{(\tilde{W}^*,S(\tilde{W}^*)}(R)$, then there exists a unique $X^* \in (R, \tilde{W}^*)$ such that

$$h'_{(\tilde{W}^*,S(\tilde{W}^*))}(X^*) = g'_{(X^*,h_{(\tilde{W}^*,S(\tilde{W}^*))}(X^*))}(X^*).$$

Here, $g_{(X,Y)}$ denotes the unique solution to the high action ODE starting at (R,L) and going through (X,Y). Then $B = g_{(X^*,h_{(\tilde{W}^*,S(\tilde{W}^*))}(X^*))}|_{[R,X^*)} \cup h_{(\tilde{W}^*,S(\tilde{W}^*))}|_{[X^*,W^*]} \cup b_{(W^*,S(W^*))}|_{(W^*,\infty)}$.

Suppose $W^* > R$. If $L \leq h_{(\tilde{W}^*,S(\tilde{W}^*)}(R)$, then $B = h_{(\tilde{W}^*,S(\tilde{W}^*))}|_{[R,W^*]} \cup b_{(W^*,S(W^*))}|_{(W^*,\infty)}$.

Suppose $W^* \leq R$. Then there exists a unique $L^* > L$ such that

$$h'_{(R,L^*)}(R) = b'_{(R,L^*)}(R)$$

And $B = (R, L^* = h_{(R,L^*)}(R)) \cup b_{(R,L^*)}|_{(R,\infty)}$.

8.3 Characterizing B when $b'(R) \leq 0$

Recall V_{left} is characterized by Lemma 4.2 on $[R,\infty)$ and is the straight line segment $\{(x, \frac{\mu-\gamma x}{r})\}$ on $(0,R)$. When $b'(R) \leq 0$, the portion characterized by Lemma 4.2 collapses to the single point $(\arg\max b, \max b) = (R, L)$. So, $V_{Left} = (R,L) \cup \{(x, \frac{\mu-\gamma x}{r}) | x \in (0,R)\}$.

The shape of V_{Right} is not much different than before - strictly increasing, continuous curve starting at $(\arg\max b, \max b) = (R, L)$, with a horizontal asymptote $y = \frac{\mu}{r}$. However, there is still some degeneracy. Specifically, there is a $\tilde{\pi} < 1$ such that $(\arg\max b^{\pi}, \max^{\pi}) = (R, L)$ for all $\pi \in [\tilde{\pi}, 1]$.

The function f is still well defined and like before, it lies below b on $[R, \infty]$ and meets b at a single point: $(\arg\max b, \max b) = (R, L)$.

The four regions of D are now well-defined. The Baseline and Static domains require no comment. When the shirking payoff lies in the right region, Theorem 4 holds completely. Specifically, both forms of B are possible. When the shirking payoff lies in the left region, only the last form of B in Theorem 5 is possible.

References

[1] Biais, B., T. Mariotti, G. Plantin and J.-C. Rochet (2007) "Dynamic Security Design: Convergence to Continuous-Time and Asset Pricing Implications," *Review of Economic Studies* Vol. 74, pp. 345 - 390

[2] Biais, B., T. Mariotti and J.-C. Rochet (2011) "Dynamic Financial Contracting," prepared for the *Tenth World Congress of the Econometric Society.* Shanghai, 2011

[3] Biais, B., T. Mariotti, J.-C. Rochet and S. Villeneuve (2010) "Large Risks, Limited Liability, and Dynamic Moral Hazard," *Econometrica* Vol. 78, pp. 73 - 118

[4] Chassang, S. (2011) "Calibrated Incentive Contracts," Working Paper, Princeton

[5] DeMarzo, P. M. and M.J. Fishman (2007) "Optimal Long-Term Financial Contracting," *The Review of Financial Studies* Vol. 20, pp. 2079 - 2128

[6] DeMarzo, P. M. and Y. Sannikov (2006) Optimal Security Design and Dynamic Capital Structure in a Continuous-Time Agency Model," *Journal of Finance* Vol. 61, pp. 2681 - 2724

[7] DeMarzo, P.M. and Y. Sannikov (2008) "Learning in Dynamic Incentive Contracts," Unpublished Manuscript, University of California, Berkeley/NYU

[8] Green, E. (1987) "Lending and the Smoothing of Uninsurable Income," in *Contractual Arrangements for Intertemporal Trade,* ed. by E.C. Prescott and N. Wallace. Minneapolis: University of Minnesota Press pp. 3 - 25

[9] Grossman, S. J. and O. Hart (1983) "An Analysis of the Principal-Agent Problem," *Econometrica* Vol. 51, pp. 7 - 45

[10] He, Z. (2009) "Optimal Executive Compensation when Firm Size Follows Geometric Brownian Motion," *Review of Financial Studies* Vol. 22, pp. 859 - 892

[11] Harrison, M. and A. Lemoine (1981) "Sticky Brownian Motion as the Limit of Storage Processes," *Journal of Applied Probability* Vol. 18, pp. 216 - 226

[12] Holmstrom, B. and P. Milgrom (1987) "Aggregation and Linearity in the Provision of Intertemporal Incentives," *Econometrica* Vol. 55, pp. 303 - 328

[13] Karatzas, I. and S. Shreve *Brownian Motion and Stochastic Calculus.* Springer-Verlag, New York, 1991

[14] Sannikov, Y. (2005) "Agency Problems, Screening and Increasing Credit Lines," Unpublished Manuscript, University of California, Berkeley

[15] Sannikov, Y. (2008) "A Continuous-Time Version of the Principal-Agent Problem," *Review of Economic Studies* Vol. 75, pp. 957 - 984

[16] Spear, S.E. and S. Srivastava (1987) "On Repeated Moral Hazard with Discounting," *Review of Economic Studies* Vol. 54, pp. 599 - 617