J Optim Theory Appl (2012) 154:713–758
DOI 10.1007/s10957-012-0050-5

Optimal Control and Applications to Aerospace: Some Results and Challenges

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Received: 7 March 2012 / Accepted: 17 March 2012 / Published online: 3 April 2012

Abstract This article surveys the usual techniques of nonlinear optimal control such as the Pontryagin Maximum Principle and the conjugate point theory, and how they can be implemented numerically, with a special focus on applications to aerospace problems. In practice the knowledge resulting from the maximum principle is often insufficient for solving the problem, in particular because of the well-known problem of initializing adequately the shooting method. In this survey article it is explained how the usual tools of optimal control can be combined with other mathematical techniques to improve significantly their performances and widen their domain of application. The focus is put onto three important issues. The first is geometric optimal control, which is a theory that has emerged in the 1980s and is combining optimal control with various concepts of differential geometry, the ultimate objective being to derive optimal synthesis results for general classes of control systems. Its applicability and relevance is demonstrated on the problem of atmospheric reentry of a space shuttle. The second is the powerful continuation or homotopy method, consisting of deforming continuously a problem toward a simpler one and then of solving a series of parameterized problems to end up with the solution of the initial problem. After having recalled its mathematical foundations, it is shown how to combine successfully this method with the shooting method on several aerospace problems such as the orbit transfer problem. The third one consists of concepts of dynamical system theory, providing evidence of nice properties of the celestial dynamics that are of great interest for future mission design such as low-cost interplanetary space missions. The article ends with open problems and perspectives.

Keywords Optimal control · Pontryagin maximum principle · Second-order conditions · Conjugate point · Numerical methods · Shooting method ·

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Orbit transfer · Atmospheric reentry · Geometric optimal control · Optimal synthesis · Continuation/homotopy method · Dynamical systems · Mission design

1 Introduction: Optimal Control Problems in Aerospace

The purpose of this article is to provide a survey of the main issues of optimal control theory and of some geometric results of modern geometric nonlinear optimal control, with a specific focus on applications to aerospace problems. The goal is, here, not only to report on some usual techniques of optimal control theory (in particular, Pontryagin Maximum Principle, conjugate point theory, associated numerical methods) but also to show how this theory and these methods can be significantly improved by combining them with powerful modern techniques of geometric optimal control, of the theory of numerical continuation, or of dynamical system theory. I will illustrate the different approaches under consideration with different standard but nontrivial aerospace problems: the minimal time or minimal consumption orbit transfer problem with strong or low thrust, the minimal total thermal flux atmospheric reentry problem of a space shuttle, and space mission design using the dynamics around Lagrange points. Through these examples, I will attempt to put in evidence the limits of the standard techniques of optimal control, which are in general not sufficient to provide an adequate solution to the problem, and I will show how these techniques can be considerably enhanced by combining them with some mathematical considerations that are sometimes quite deep but are an efficient (and most of the times, superior) alternative to numerical refinement procedures or other computational efforts. In particular, I will focus on three approaches that have been successfully combined with optimal control and that are in my opinion of primary importance in aerospace applications. The first one is geometric optimal control, which started to be developed in the early 1980s and has widely proved its superiority over the classical theory of the 1960s. The main objective of geometric optimal control is to develop general techniques for general classes of nonlinear optimal control problems, using in particular the concept of Lie bracket to analyze the controllability properties of nonlinear control systems and the regularity properties of optimal trajectories, and to provide optimal synthesis results. I will show how recent results of geometric optimal control can be used to provide a deep geometric insight into the atmospheric reentry problem and lead to an efficient solution. The second technique focused on is the numerical continuation procedure, which is far from new but has been quite neglected until recently in optimal control probably because of its difficulty to be implemented efficiently and quite systematically. In the last ten years however much progress has been done that permits to understand better how this powerful procedure can be successfully applied, and I will show its particular relevance on the orbit transfer problem. The third technique mentioned, that I believe to be of particular interest for future aerospace applications, is the combination with dynamical system theory. Deep mathematical results from this theory permit to put in evidence nice properties of the celestial dynamics due to Lagrange points and gravitational effects, which are particularly interesting in view of designing low-cost space missions for future interplanetary exploration.

This article is addressed not only to mathematicians wanting to know more about these geometric or mathematical issues associated with concrete applications, but also to engineers already acquainted with usual techniques of optimal control, wishing to get more familiar with the more modern approaches of geometric control and other mathematical notions that have demonstrated significant enhancements in aerospace problems, or to experts of nonlinear control wishing to learn about applications of this discipline to nontrivial examples in aerospace.

The mathematical notions whose combination to optimal control has proved its relevance are mainly from (elementary) differential geometry; they are introduced and explained step by step, although they are mostly known by many readers, and can be skipped at the first reading.

The article is structured as follows. Section 2 surveys the most well-known theoretical and numerical aspects of optimal control. It is recalled how first-order necessary conditions lead to the famous Pontryagin Maximum Principle and how it can be used in practice. Second-order optimality conditions, leading to the conjugate point theory, are then briefly surveyed. The possible numerical approaches are then described and discussed, and their limits are underlined. Section 3 is devoted to show how techniques of geometric optimal control can be used in order to provide an efficient solution to the atmospheric reentry problem. In Sect. 4, the continuation method is first described, and a theoretical foundation is provided in terms of sensitivity analysis. It is then shown how it can be combined with a shooting method in order to solve different problems, such as the orbit transfer problem. In Sect. 5 the focus is made on the properties of the dynamics around Lagrange points and on potential applications to mission design. Throughout the article, and in the conclusion (Sect. 6), open problems and challenges are described.

2 A Short Survey on Optimal Control: Theory and Numerics

Let n and m be nonzero integers. Consider on $\mathbb{R}^n$ the control system

$$\dot{x}(t) = f\big(t, x(t), u(t)\big), \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is C^1, and where the controls are bounded and measurable functions, defined on intervals $[0, T(u)]$ of $\mathbb{R}^+$ and taking their values in a subset U of $\mathbb{R}^m$. Let M_0 and M_1 be two subsets of $\mathbb{R}^n$. Denote by $\mathcal{U}$ the set of *admissible controls* such that the corresponding trajectories steer the system from an initial point of M_0 to a final point in M_1. For such a control u, the *cost* of the corresponding trajectory $x_u(\cdot)$ is defined by

$$C(t_f, u) := \int_0^{t_f} f^0\big(t, x_u(t), u(t)\big)\, dt + g\big(t_f, x_u(t_f)\big), \tag{2}$$

where $f^0 : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are C^1. We investigate the *optimal control problem* of determining a trajectory $x_u(\cdot)$ solution of (1), associated with a control u on $[0, t_f]$, such that $x_u(0) \in M_0$, $x_u(t_f) \in M_1$, and minimizing the cost C. The final time t_f can be fixed or not. If one considers the minimal time problem, then one can choose $f^0 = 1$ and $g = 0$.

When the optimal control problem has a solution, we say that the corresponding control (or the corresponding trajectory) is minimizing or optimal.

2.1 Existence Results

This is not our aim to provide elaborate existence results for optimal controls. It should just be noted that usual existence results require some convexity on the dynamics since their proof usually relies on weak compactness properties. We refer to [1] for a survey of existence results in optimal control. The result given below, whose early version was obtained by Filippov [2], is standard.

Theorem 2.1 *Assume that U is compact, that there may be state constraint $c_1(x) \leqslant 0, \dots, c_r(x) \leqslant 0$, where $c_1, \dots, c_r$ are continuous functions on $\mathbb{R}^n$, and that M_0 and M_1 are compact subsets of $\mathbb{R}^n$ such that M_1 is accessible from M_0. Assume that there exists $b > 0$ such that every trajectory steering the system from M_0 to M_1 is bounded by b on $[0, t(u)]$ in C^0 norm and that the set*

$$\tilde{V}(t,x) := \big\{\big(f^0(t,x,u) + \gamma, f(t,x,u)\big) \mid u \in U,\ \gamma \geqslant 0\big\}$$

is convex for every $t \in \mathbb{R}$ and every $x \in \mathbb{R}^n$. Then there exists an optimal control u defined on $[0, t(u)]$ such that the corresponding trajectory steers the system from M_0 to M_1 in time $t(u)$ and minimizing the cost.

Even though existence results would certainly deserve many further interesting discussions, this is not the objective of this paper to report on that subject. It can however be noted that, if the set U is unbounded, then in general existence results lead to optimal controls that are not necessarily in $L^\infty([0, t(u)], U)$. This leads to a possible gap with the usual necessary conditions reported hereafter, which assume that the optimal control is essentially bounded. This gap may cause the so-called *Lavrentiev phenomenon* and raises the question of studying the regularity of the optimal control (see [3, 4] where such issues are investigated).

2.2 First-Order Optimality Conditions

The set of admissible controls on $[0, t_f]$ is denoted $\mathcal{U}_{t_f,\mathbb{R}^m}$, and the set of admissible controls on $[0, t_f]$ taking their values in U is denoted $\mathcal{U}_{t_f,U}$.

Definition 2.1 The *end-point mapping* $E : \mathbb{R}^n \times \mathbb{R}^+ \times \mathcal{U} \to \mathbb{R}^n$ of the system is defined by $E(x_0, T, u) = x(x_0, T, u)$, where $t \mapsto x(x_0, t, u)$ is the trajectory solution of (1), corresponding to the control u, such that $x(x_0, 0, u) = x_0$.

The set $\mathcal{U}_{t_f,\mathbb{R}^m}$, endowed with the standard topology of $L^\infty([0, t_f], \mathbb{R}^m)$, is open, and the end-point mapping is C^1 on $\mathcal{U}_{t_f,\mathbb{R}^m}$ (and C^k whenever the dynamics are C^k).

Note that, in terms of the end-point mapping, the optimal control problem under consideration can be written as the infinite-dimensional minimization problem

$$\min\big\{C(t_f, u) \mid x_0 \in M_0,\ E(x_0, t_f, u) \in M_1, u \in L^\infty(0, t_f; U)\big\}. \tag{3}$$

Definition 2.2 Assume that $M_0 = \{x_0\}$. A control u defined on $[0, t_f]$ is said to be singular iff the differential $\frac{\partial E}{\partial u}(x_0, t_f, u)$ is not of full rank.

Singular controls are of high importance in optimal control theory. At this step their potential influence can be stressed by noting that, in the above constrained minimization problem, the set of constraints is a local manifold around a given control u, provided that u is nonsingular. It is well known in constrained optimization theory that, in order to derive the usual necessary conditions exposed hereafter, it is in general needed that the set of constraints is (at least locally) a manifold. Hence at this step it can be easily understood that the existence of minimizing singular controls is a potential source of problems.

2.2.1 Lagrange Multipliers

Assume for one moment that we are in the simplified situation where $M_0 = \{x_0\}$, $M_1 = \{x_1\}$, T is fixed, and $U = \mathbb{R}^m$. That is, we consider the optimal control problem of steering system (1) from the initial point x_0 to the final point x_1 in time T and minimizing the cost (2) among controls $u \in L^\infty([0,T],\mathbb{R}^m)$. In that case, the optimization problem (3) reduces to

$$\min_{E(x_0,T,u)=x_1} C(T,u). \tag{4}$$

According to the well-known Lagrange multipliers rule (and using the C^1 regularity of our data), if u is optimal, then there exists $(\psi,\psi^0) \in \mathbb{R}^n \times \mathbb{R} \setminus \{0\}$ such that

$$\psi \cdot dE_{x_0,T}(u) = -\psi^0 dC_T(u). \tag{5}$$

Note that, if one defines the Lagrangian $L_T(u,\psi,\psi^0) := \psi E_{x_0,T}(u) + \psi^0 C_T(u)$, then this first-order necessary condition for optimality is written in the usual form as

$$\frac{\partial L_T}{\partial u}\big(u,\psi,\psi^0\big) = 0. \tag{6}$$

Here, the main simplification was $U = \mathbb{R}^m$, that is, we considered the case without control constraints. In the general case the situation is more intricate to deal with control constraints and even more when there are state constraints. When there are some control constraints, one possibility could be taking these constraints into account directly in the Lagrangian, with some additional Lagrange multiplier, as it is done, e.g., in [5]. This method leads however to weaker results than the Pontryagin Maximum Principle stated hereafter. Actually the method used by Pontryagin in order to take into account control constraints is stronger and consists of considering needle-like variations (see also Remark 2.2).

When there are some state constraints, it is also still possible to express a first-order necessary condition in terms of Lagrange multipliers as above but this has to be done in distributions spaces and the Lagrange multipliers must be expressed as Radon measures (see e.g. [6–9]).

Whatever simplified or general case we may consider, the first-order condition (6) is in this form not much tractable for practical purposes. This first-order condition can be in some sense parametrized along the trajectory, and this leads to the famous Pontryagin Maximum Principle.

2.2.2 Pontryagin Maximum Principle

The following statement is the most usual Pontryagin Maximum Principle, valuable for general nonlinear optimal control problems (1)–(2), with control constraints but without state constraint. Usual proofs rely on a fixed-point argument and on the use of Pontryagin cones (see, e.g., [9–12]).

Theorem 2.2 *If the trajectory $x(\cdot)$, associated to the optimal control u on $[0,t_f]$, is optimal, then it is the projection of an extremal $(x(\cdot),p(\cdot),p^0,u(\cdot))$ (called the extremal lift), where $p^0 \leqslant 0$, and $p(\cdot):[0,t_f]\to\mathbb{R}^n$ is an absolutely continuous mapping, called the adjoint vector, with $(p(\cdot),p^0)\neq(0,0)$, such that*

$$\dot{x}(t)=\frac{\partial H}{\partial p}\big(t,x(t),p(t),p^0,u(t)\big),\qquad \dot{p}(t)=-\frac{\partial H}{\partial x}\big(t,x(t),p(t),p^0,u(t)\big)$$

almost everywhere on $[0,t_f]$, where

$$H\big(t,x,p,p^0,u\big):=\big\langle p,f(t,x,u)\big\rangle+p^0f^0(t,x,u)$$

is the Hamiltonian, and there holds

$$H\big(t,x(t),p(t),p^0,u(t)\big)=\max_{v\in U}H\big(t,x(t),p(t),p^0,v\big) \tag{7}$$

almost everywhere on $[0,t_f]$. If moreover the final time t_f to reach the target M_1 is not fixed, then one has the following condition at the final time t_f:

$$\max_{v\in U}H\big(t_f,x(t_f),p(t_f),p^0,v\big)=-p^0\frac{\partial g}{\partial t}\big(t_f,x(t_f)\big). \tag{8}$$

Additionally, if M_0 and M_1 (or just one of them) are submanifolds of $\mathbb{R}^n$ locally around $x(0)\in M_0$ and $x(t_f)\in M_1$, then the adjoint vector can be built in order to satisfy the transversality conditions at both extremities (or just one of them)

$$p(0)\perp T_{x(0)}M_0,\quad p(t_f)-p^0\frac{\partial g}{\partial x}\big(t_f,x(t_f)\big)\perp T_{x(t_f)}M_1, \tag{9}$$

where T_xM_i denotes the tangent space to M_i at the point x.

The relation between the Lagrange multipliers of the previous section and $(p(\cdot),p^0)$ is that the adjoint vector can be constructed so that $(\psi,\psi^0)=(p(t_f),p^0)$ up to some multiplicative scalar. In particular, the Lagrange multiplier ψ is unique (up to a multiplicative scalar) iff the trajectory $x(\cdot)$ admits a unique extremal lift (up to a multiplicative scalar).

If $p^0<0$, then the extremal is said to be *normal*, and in this case, since the Lagrange multiplier is defined up to a multiplicative scalar, it is usual to normalize it so that $p^0=-1$. If $p^0=0$, then the extremal is said *abnormal*.

It can be also noted that, in the absence of control constraints, abnormal extremals project exactly onto singular trajectories (it is evident using (6)).

Remark 2.1 With respect to this relation, it can be noted that in the normal case the Lagrange multiplier ψ (or the adjoint vector $p(t_f)$ at the final time) coincides up to some multiplicative scalar with the gradient of the value function (solution of the Hamilton–Jacobi equation); see, e.g., [13] for precise results.

Remark 2.2 In the case $U = \mathbb{R}^m$ (no control constraints), the maximization condition (7) implies in particular that $\frac{\partial H}{\partial u}(t, x(t), p(t), p^0, u(t)) = 0$ almost everywhere on $[0, t_f]$. In this form, the Pontryagin Maximum Principle is exactly the parameterized version of the first-order necessary condition (6) of the simplified case. Note that, in the absence of control constraints, the proof is quite obvious and can be found, e.g., in [14, 15]. Note also that the maximization condition implies as well that the quadratic form $\frac{\partial^2 H}{\partial u^2}(t, x(t), p(t), p^0, u(t))$ is nonpositive almost everywhere on $[0, t_f]$. These two conditions remain however weaker than the maximization condition (7). Indeed, those two conditions are local, whereas the maximization condition (7) is global. In the proof of the general version of the Pontryagin Maximum Principle, needle-like variations of the control are the main tool in order to derive the strong condition (7) (note that a short proof of the Pontryagin Maximum Principle is provided in the general case, with needle-like variations and with a conic implicit function theorem, in [16]).

Remark 2.3 The scalar p^0 is a Lagrange multiplier associated with the instantaneous cost. It may happen that it is equal to 0, and these cases are said abnormal. Abnormal extremals are not detected with the usual Calculus of Variations approach,[1] because this approach postulates at the very beginning that, in a neighborhood of some given reference trajectory, there are other trajectories having the same terminal points, whose respective costs can be compared (and this leads to the Euler–Lagrange equations). But this postulate fails whenever the reference trajectory is isolated: it may indeed happen that there is only one trajectory joining the terminal points under consideration (see [17]). In this case, in some sense there is no optimization to do any more. Indeed since the trajectory joining the desired extremities is unique, then obviously it will be optimal for every optimization criterion we may consider. These cases may appear to be quite trivial, but actually in practice this issue is far from being obvious because a priori, given some extremities, we are not able to say if the resulting problem can be solved with a normal extremal (that is, with a $p^0 = -1$). It could happen that it is not: this is the case, for instance, for certain initial and final conditions in the well-known minimal-time attitude control problem (see [18], where such abnormals are referred to as exceptional singular trajectories).

Hence, when applying the Pontryagin Maximum Principle, we must distinguish between two extremal flows: the normal flow, with $p^0 < 0$ (and in general we normalize to $p^0 = -1$), and the abnormal one, for which $p^0 = 0$.

In many situations, where some qualification conditions hold, abnormal extremals do not exist in the problem under consideration, but in general it is impossible to say

[1]The usual framework of calculus of variations consists of solving the problem of minimizing the action $\int_0^1 L(t, x(t), \dot{x}(t))\,dt$ among all possible curves $x(\cdot)$. With respect to optimal control, this corresponds to the trivial control system $\dot{x}(t) = u(t)$, where all moving directions are authorized.

whether, given some initial and final conditions, these qualification conditions hold or not. Furthermore it can be noted that, when they exist, extremities of projections of abnormal extremals do not fill much of the state space (see [19, 20] for precise statements).

Remark 2.4 An ultimate remark on the multiplier p^0 is about its sign. According to the convention chosen by Pontryagin, we consider $p^0 \leqslant 0$. If, instead, we adopt the convention $p^0 \geqslant 0$, then we have to replace the maximization condition (7) with a minimization condition. This just consists in considering the opposite of the adjoint vector.

If there is no integral term in (2) (that is, $f^0 = 0$), this does not change anything for the considerations on p^0. The p^0 still appears in the transversality condition (9). Note that, if $M_1 = \mathbb{R}^n$ (no condition on the final point), then this condition leads to $p(t_f) = p^0 \frac{\partial g}{\partial x}(t_f, x(t_f))$, and then necessarily $p^0 \neq 0$ (otherwise the adjoint vector $(p(t_f), p^0)$ would be zero, and this would contradict the assertion of the Pontryagin Maximum Principle), and in that case we can normalize to $p^0 = -1$.

2.2.3 Generalizations

The Pontryagin Maximum Principle withstands many possible generalizations.

First, it can be expressed for control systems evolving on a manifold (see, e.g., [12]), that is, control systems of the form (1) with dynamics $f : M \times N \longrightarrow TM$, where M (resp., N) is a smooth manifold of dimension n (resp., m). This situation can be of interest if, for instance, the system is evolving on an energy surface (it is often the case in aerospace) and/or if the controls take their values in the unit sphere (this situation occurs often in aerospace as well, for instance, when the control models a thrust of constant modulus).

The Pontryagin Maximum Principle can be generalized to wider classes of functionals and boundary conditions; for instance, periodic boundary conditions (see [12]), systems involving some delays (see [9], intermediate conditions (see [5, 21], or, more generally, hybrid systems where the dynamics may change along the trajectory, accordingly to time and/or state conditions (see [16, 22, 23]). In particular in this last case, when the system crosses a given boundary, then a jump condition must hold on the adjoint vector, which means that the adjoint vector is no more continuous (but is however piecewise absolutely continuous). It can be also generalized to nonsmooth systems (see [6]).

Probably the most difficult generalization is when there are some state constraints. In that case, we impose to the trajectories to lie in a given part of the space. From the mathematical point of view the situation is more intricate since the adjoint vector becomes a measure, evolving in some distribution space. What can be probably considered as the most general Maximum Principle statement has been derived in [6], which can be applied to very general (possibly nonsmooth) control systems with state/control constraints. Note that a version of that result for smooth systems (with state and control constraints) has been written in [24]. As explained in [25], the difficulty in practice is that since the adjoint vector is a vectorial measure, it may admit an accumulation of atoms; in other words, the measure does not admit necessarily piecewise a density; for instance, there may occur an accumulation of touching points with the boundary of the state domain. To overcome this mathematical difficulty, the usual argument consists of assuming that, in practice, such an accumulation does not occur, and the trajectory is a "regular" succession of free arcs (i.e., arcs inside the authorized state domain) and of boundary arcs (i.e., arcs that are at the boundary of the authorized state domain), with possibly some isolated touching points (at which the trajectory touches the boundary). This regular structure was already assumed by Pontryagin and his co-authors in their seminal book [9]; however in this book they moreover restricted to so-called first-order constraints. In a few words, and roughly speaking, the order of a state constraint is the number of times one must differentiate the equality constraint along the trajectory with respect to time in order to make appear the control. The general case has been treated in [26] in the case where the Hessian of the Hamiltonian is nondegenerate and in [27] for the case where the controls appear linearly in the system (note that the jump conditions have been clarified in [28–31]). Note that a nice survey on the Pontryagin Maximum Principle for control systems with general state constraints has been written in [7]. In any case, under this regular structure assumption, as in the hybrid situation the adjoint vector is absolutely continuous by parts, with jump conditions at the boundary (the discontinuity of the adjoint vector is along the gradient of the frontier). It can be noted however that for state constraints of order greater than or equal to three, the bad chattering phenomenon of accumulation of touching points may occur in a typical way (see [32]).

2.2.4 Practical Use of the Pontryagin Maximum Principle

In practice in order to compute optimal trajectories with the Pontryagin Maximum Principle, the first step is to make explicit the maximization condition. A usual assumption to make this step feasible is to assume the so-called strict Legendre assumption, that is, to assume that the Hessian $\frac{\partial^2 H}{\partial u^2}(t, x, p, p^0, u)$ is negative definite. Under that assumption, a standard implicit function argument permits to end up, at least locally, with a control u expressed as a function of x and p. This assumption is, for instance, obviously satisfied for normal extremals if one considers control affine systems with a cost that is quadratic in u. Assume, for example, that we are in the normal case ($p^0 = -1$). Then, plugging the resulting expression of the control into the Hamiltonian equations and defining the *reduced (normal) Hamiltonian* by $H_r(t, x, p) := H(t, x, p, -1, u(x, p))$, it follows that every normal extremal is a solution of

$$\begin{aligned} \dot{x}(t) &= \frac{\partial H_r}{\partial p}\big(t, x(t), p(t)\big), \\ \dot{p}(t) &= -\frac{\partial H_r}{\partial x}\big(t, x(t), p(t)\big), \end{aligned} \tag{10}$$

and this leads to definition of the (normal) exponential mapping.

Definition 2.3 The exponential mapping is defined by $\exp_{x_0}(t, p_0) := x(t, x_0, p_0)$, where the solution of (10) starting from (x_0, p_0) at $t = 0$ is denoted as $(x(t, x_0, p_0), p(t, x_0, p_0))$.

In other words, the exponential mapping parameterizes the (normal) extremal flow. The abnormal extremal flow can be parameterized as well, provided that there holds such a kind of Legendre assumption in the abnormal case.

When the Hessian of the Hamiltonian considered above is degenerate, the situation can be far more intricate. A typical example is when one considers the minimal time problem for single-input control affine systems $\dot{x}(t) = f_0(x(t)) + u(t) f_1(x(t))$ without control constraints. In that case, the maximization condition leads to $\frac{\partial H}{\partial u} = 0$, that is, there must hold $\langle p(t), f_1(x(t))\rangle = 0$ along the corresponding extremal. By the way, note that, since the optimal control takes its values in the interior of the domain of constraints, it is necessarily singular. To compute the control, the method consists of differentiating two times this relation with respect to t, which first leads to $\langle p(t), [f_0, f_1](x(t))\rangle = 0$ and then to $\langle p(t), [f_0, [f_0, f_1]](x(t))\rangle + u(t)\langle p(t), [f_1, [f_0, f_1]](x(t))\rangle = 0$, where $[\cdot, \cdot]$ denotes the Lie bracket of vector fields. This permits as well to express the optimal control $u(t)$ as a function of $x(t)$ and $p(t)$, provided that the quantity $\langle p(t), [f_1, [f_0, f_1]](x(t))\rangle$ does not vanish along the extremal. The latter condition is called the strong generalized Legendre–Clebsch condition. We refer the reader to [14] for more details on this theory. It can also be shown that this kind of computation is valid in a "generic" situation (see [33–36]).

Remark 2.5 Note the important fact that the normal extremals are distinguished from the abnormal ones by a binary variable, namely, the variable $p^0 \in \{0, -1\}$. In the case where an abnormal flow is well defined, we then have to deal with two extremal flows. Intuitively, however, it is expected that the abnormal flow fills less space than the normal flow, in the sense that almost every point of the accessible set should be reached by a normal extremal. This kind of statement is however difficult to derive. There exist some results for control-affine systems and for control-affine systems without drift that assert that the end-points of projections of abnormal extremals fill only a negligible part of the state space (see [19, 20, 37] for details).

Remark 2.6 Note that the Pontryagin Maximum Principle is nothing else but a far-reaching version of the Lagrange multipliers necessary condition derived formerly. It is thus only a first-order necessary condition for optimality, asserting that if a trajectory is optimal, then it should be sought among projections of extremals joining the initial set to the final target. Conversely, the projection of a given extremal is not necessarily (locally or globally) optimal. This motivates the next section on second-order optimality conditions.

2.3 Second-Order Optimality Conditions

Throughout this section we assume that we are in the simplified situation where $M_0 = \{x_0\}$, $M_1 = \{x_1\}$, and $U = \mathbb{R}^m$. Also, in order to consider second-order derivatives, we assume that the dynamics are at least C^2.

2.3.1 Abstract Conditions

In this simplified situation, we have seen that the usual first-order necessary condition for optimality is (6), that is, the vanishing of the differential of the Lagrangian.

In this simplified situation where there is no constraint on the control, conditions of order two are also standard in terms of the Lagrangian. Defining as usual the intrinsic second-order derivative Q_T of the Lagrangian as the Hessian $\frac{\partial^2 L_T}{\partial^2 u}(u, \psi, \psi^0)$ restricted to the subspace ker $\frac{\partial L_T}{\partial u}$, it is well known that a second-order necessary condition for optimality is the nonpositivity of Q_T (recall the agreement $\psi^0 \leqslant 0$), and a second-order sufficient condition for *local* optimality is the negative definiteness of Q_T. In this form, these conditions are not convenient for practical purposes. Fortunately, in the same way that Lagrange multipliers conditions can be parameterized into the Pontryagin Maximum Principle, the above second-order conditions can be parameterized as well along the extremals, and this leads to the theory of conjugate points, briefly sketched next.

Remark 2.7 The above quadratic form is the one considered in the simplified situation. Such abstract conditions have been widely generalized in the literature (see, e.g., [38]).

2.3.2 *Conjugate Points*

Under the strict Legendre assumption mentioned previously, the quadratic form Q_T is negative definite whenever $T > 0$ is small enough. This leads naturally to the following definition.

Definition 2.4 The first conjugate time t_c along $x(\cdot)$ is defined as the infimum of times $t > 0$ such that Q_t has a nontrivial kernel.

Under the strict Legendre assumption, there holds $t_c > 0$, and this first conjugate time characterizes the (local) optimality status of the trajectory. Hereafter the local optimality is the sense of the L^∞ topology on the controls, but it can be improved (see [12, 14, 18, 39, 40]).

Theorem 2.3 *The trajectory* $x(\cdot)$ *is locally optimal on* $[0, t]$ *iff* $t < t_c$.

The following result is crucial for practical computations of conjugate times.

Theorem 2.4 *The time* t_c *is a conjugate time along* $x(\cdot)$ *iff the mapping* $\exp_{x_0}(t_c, \cdot)$ *is not an immersion at* p_0 (*that is*, *its differential is not injective*).

Its proof can be found in [12, 18, 25]. Essentially it states that computing a first conjugate time reduces to compute the vanishing of some determinant along the extremal. Indeed, the fact that the exponential mapping is not an immersion can be translated in terms of so-called *vertical Jacobi fields*. Note however that the domain of definition of the exponential mapping requires a particular attention in order to define properly these Jacobi fields according to the context: normal or abnormal extremal, final time fixed or not. A more complete exposition can be found in the survey article [18], which provides also some algorithms to compute first conjugate times in various contexts (however, always in the case where the control can be expressed as a smooth function of x and p) and some practical hints for algorithmic purposes (see also Sect. 2.4.2).

2.3.3 Generalizations, Open problems and Challenges

A first remark is that the conjugate point theory sketched previously can be generalized in the case where the initial and final sets M_0 and M_1 are not necessarily restricted to a single point. In that case, the notion of conjugate point must be replaced with the notion of *focal point*. The theory and the resulting algorithms remain however similar (see [18, 25]).

It should be also stressed that the above conjugate point theory holds only in the "smooth case," that is, whenever the optimal controls under consideration can be expressed as smooth functions of x and p (thus, essentially, when there is no constraint on the control, although this assumption involves some possible cases where there are some control constraints) and without any state constraint. In this theory, the definition and the computation of conjugate points are based on second-order conditions which do not involve in particular bang-bang situations where the control is discontinuous and consists of successive arcs saturating the constraints.

In the case where the extremal controls are continuous, the literature on first- and/or second-order sufficient conditions is vast (see also [41] and references therein), and there exist numerical procedures to test second-order sufficient conditions that are based on the Riccati equation (see, e.g., [42]); of course, these procedures are equivalent to the one described previously in terms of the exponential mapping. We refer also the reader to [17, 28, 40, 43] (and references therein) for extensions of such theories to the abnormal case.

A conjugate time theory has been developed in the bang-bang case, and we refer the reader to [44] whose introduction contains a brief survey unifying all (apparently different) approaches that have been developed, such as envelope theory, extremal fields, second-order conditions, and numerical approaches. Although this theory has been partially extended to the case of state constraints, up to now the concept of conjugate point in that case has not been defined. The situation is similar for trajectories involving both bang and singular arcs. At the moment there is no general conjugate point theory which would involve such cases. This is therefore an open (and important) problem to derive a complete conjugate point theory that would consist of any possible smooth, bang, singular, or boundary arcs.

From the algorithmic point of view, note that, although the theory of conjugate times in the bang-bang case has been well developed, the computation of conjugate times in the bang-bang case is difficult in practice (see, e.g., [45] and references therein). Besides, in the smooth case, as explained in the previous section, efficient tools are available (see [18]). In [44, 46] a regularization procedure is proposed which allows the use of these tools for the computation of the first conjugate time of a bang-bang situation for a single-input control-affine system, by showing the convergence of the conjugate time of the regularized system to the conjugate time of the initial bang-bang system. This result is of interest because it provides an efficient way to compute conjugate points in the bang-bang case. It is an open problem to extend that kind of result to more general systems and more general situations.

2.4 Numerical Methods in Optimal Control

It is usual to distinguish between two kinds of numerical approaches in optimal control: direct and indirect methods. Roughly speaking, direct methods consist in discretizing the state and the control and thus reduce the problem to a nonlinear optimization problem with constraints. Indirect methods consist of solving numerically the boundary-value problem derived from the application of the Pontryagin Maximum Principle and lead to the shooting methods.

2.4.1 Direct Methods

There exist many possible direct methods. In any case, one has to choose finite-dimensional representations of the control and of the state, and then express in a discrete way the differential equation representing the system, the minimization criterion, and all constraints under consideration. Once all static or dynamic constraints have been transcribed into a problem with a finite number of variables, one is ought to solve the resulting optimization problem with constraints, using some adapted optimization method.

Let us first explain hereafter one possible very simple way of such a discretization. Consider a subdivision $0 = t_0 < t_1 < \cdots < t_N = t_f$ of the interval $[0, t_f]$. Controls are discretized in such a way that they are piecewise constant on this subdivision (with values in U). Moreover, we choose a discretization process of ordinary differential equations; for instance, let us choose (to simplify) the standard explicit Euler method. Setting $h_i = t_{i+1} - t_i$, we obtain $x_{i+1} = x_i + h_i f(t_i, x_i, u_i)$. There exist of course an infinite number of possible variants. On one hand, one may discretize the set of admissible controls by piecewise constant, or piecewise affine controls, or splines, etc. On the other hand, there exist many methods in order to discretize ODE's, such as Euler methods (implicit or explicit), middle point, Heun, Runge–Kutta, Adams–Moulton, etc. (see, for instance, [47]). The choice of the method is guided by the problem under consideration. Here, we choose the Euler method for the simplicity of its writing, but in practice it should be avoided because it is too much rough, and at least an RK2 method should be chosen. Finally, we also discretize the cost by choosing a quadrature procedure. Then these discretization processes reduce the optimal control problem to the problem of minimizing $C(x_0, \ldots, x_N, u_0, \ldots, u_N)$, where the unknowns x_i and u_i are submitted to the constraints resulting from the differential system, the terminal conditions, the state and/or control constraints. In brief, in such a way we end up with a problem of the form $\min\{F(Z) \mid g(Z) = 0,\ h(Z) \leqslant 0\}$, which is a finite-dimensional (nonlinear) optimization problem with constraints. The dimension is of course larger as the discretization is finer.

There exist many numerical approaches to solve this kind of problem, such as gradient methods, quasi-Newton, penalization, dual methods, etc. (see, e.g., [47]). Note that in the last years much progress has been done in the direction of combining automatic differentiation softwares (such as the modeling language `AMPL`; see [48]) with expert optimization routines (such as the open-source package `IPOPT`; see [49], carrying out an interior point optimization algorithm for large-scale differential algebraic systems, combined with a filter line-search method). With such tools it has become very simple to implement with only few lines of code difficult (nonacademic) optimal control problems, with success and within a reasonable time of computation. Even more, web sites such as NEOS (http://neos-server.org/neos/) permit to launch online such kinds of computation: codes can be written in a modeling language such as [48] (or others) and can be combined with many optimization routines (specialized either for linear problems, nonlinear, mixed, discrete, etc). Note that there exists a large number (open-source or not) of automatic differentiation softwares and of optimization routines; it is however not our aim to provide a list of them. They are easy to find on the web.

There exist many possible variants of direct methods, and for an excellent survey on direct methods with a special interest to applications in aerospace, we refer the reader to [50, 51] (in this survey book sparsity issues, very important in practice, are also discussed). Among these different approaches, we quote the following.

Speaking in a general way, collocation methods consist of choosing specific points or nodes on every subinterval of a given subdivision of the time interval. From the point of view of the discretization spaces, these methods consist of approximating the trajectories (and/or the control functions) by polynomials on each subinterval. Then the collocation conditions state that the derivatives of the approximated state match exactly with the dynamics at the nodes mentioned previously. Note that Runge–Kutta discretizations are a particular case.

In spectral or pseudospectral methods, the above nodes are chosen as the zeros of special polynomials, such as the Gauss–Legendre or Gauss–Lobatto polynomials. Equivalently, these polynomials serve as a basis of approximation spaces for the trajectories and the controls. Since they share nice orthogonality properties, the collocation conditions turn into constraints that are easily tractable for numerical purposes. We refer the reader to [52, 53] and to the references therein for more details on these approaches.

There exist also some probabilistic approaches, such as the method described in [54], which consists of expressing the optimal control problem in measure spaces and then of seeking the optimal control as an occupation measure, which is approximated by a finite number of its moments.

Remark 2.8 Another approach to optimal control problems, which can be considered (although it can be discussed) as a direct method, consists of solving the Hamilton–Jacobi equation satisfied by the value function, that is, the optimal cost for the optimal control problem of reaching a given point, which is of the form $\frac{\partial S}{\partial t} + H_r(x, \frac{\partial S}{\partial x}) = 0$ (see, e.g., [55] for some numerical methods).

2.4.2 *Indirect Methods*

In the indirect approaches, instead of discretizing first, as in direct methods, we first apply the Pontryagin Maximum Principle as a first-order condition to the optimal control problem. It states that the optimal trajectory should be sought among the extremals that achieve the required terminal and/or transversality conditions. Therefore, denoting $z := (x, p)$, it reduces the problem to the boundary-value problem of determining an extremal solution of the extremal system $\dot{z}(t) = F(t, z(t))$ and satisfying boundary-value conditions of the form $R(z(0), z(t_f)) = 0$. Denoting by $z(t, z_0)$ the solution of the Cauchy problem $\dot{z}(t) = F(t, z(t))$, $z(0) = z_0$, and setting $G(z_0) := R(z_0, z(t_f, z_0))$, this boundary-value problem is then equivalent to solving $G(z_0) = 0$, that is, finding a zero of G. Solving such a nonlinear system of n equations with n unknowns can be achieved in practice by using a Newton-like method.

This method is called the *shooting method*, and G is called the *shooting function*. It has many possible refinements, in particular the multiple shooting method, which consists of subdividing the time interval $[0, t_f]$ into N intervals $[t_i, t_{i+1}]$, of considering as unknowns the values $z(t_i)$, and then of imposing continuity conditions by defining adequately the shooting function. Its interest is an improvement of the stability of the method (see [47]).

From the practical implementation point of view, note on one hand that there exist many variants of Newton methods, among which the Broyden method or the Powell hybrid method are quite competitive. On the other hand, note that, as for direct methods, the shooting methods can be combined with automatic differentiation. Here, the use of automatic differentiation can help to generate the Hamiltonian equations of extremals. This is particularly useful when one works on a problem whose model is not completely fixed. In [18] the authors provide the description for the package COTCOT (Conditions of Order Two and COnjugate Times), available for free on the web, achieving the automatic generation of the equations of the Pontryagin Maximum Principle and implementing codes for the numerical integration, the shooting method, and the computation of conjugate times.

Remark 2.9 It must be noted that, when implementing a shooting method, the structure of the trajectory should be known in advance, particularly in the case where the trajectory involves singular arcs (see, e.g., [56, 57]). This remark shows the importance of being able to determine at least locally the structure of optimal trajectories: this is one of the main issues of geometric optimal control theory, as explained further in this article (see Sects. 3.2 and 3.3).

Remark 2.10 Proving that the shooting method is feasible amounts to proving that the Jacobian of the shooting function is nonzero. In the simplified situation of Sect. 2.3, the shooting method is well posed at time t, locally around p_0 iff the exponential mapping $\exp_{x_0}(t, \cdot)$ is an immersion at p_0. In other words, according to Theorem 2.4, the shooting method is feasible (well posed) iff the final time under consideration is not a conjugate time.

This simple argument can be generalized to far more general situations. First of all, if the initial and final sets are not restricted to single points, the above argument still holds except that the notion of focal point has to be used instead of conjugate point (see Sect. 2.3.3). Note that a modification of the shooting method is proposed in [58], which consists of adding unknowns to the method (so that there are more unknowns than equations) to overcome partially the problem of a priori structure determination, and then the Newton method must be adapted with the use of a pseudo-inverse. In [59] it is shown that the shooting method is well posed also in the presence of control and state constraints, provided that a certain second-order coercivity holds; this second-order condition is not translated in terms of conjugate points, but this could be probably done if the corresponding conjugate point theory would exist (see Sect. 2.3.3).

2.4.3 An Open Problem

If we summarize the main issues of the previous direct and indirect approaches, we realize that direct methods consist of first discretizing the optimal control problem in order to reduce it to an usual nonlinear minimization problem with constraints (the dimension being as larger as the discretization is finer), and second of dualizing, by applying, e.g., a usual Lagrange–Newton method to the nonlinear minimization problem (applying the Kuhn–Tucker and then Newton method to solve the resulting optimality system); whereas indirect methods consist of first dualizing the optimal control problem, by applying the Pontryagin Maximum Principle (or, equivalently, the Lagrange multipliers necessary condition for optimality in infinite dimension), and second discretizing the resulting boundary-value problem, by applying a shooting method (that is, the Newton method composed with a numerical integration method). In shorter words, direct methods consist of (1) discretizing and (2) dualizing, and indirect methods consist of the converse: (1) dualizing and (2) discretizing. It is natural to wonder whether this diagram is commutative or not under usual approximation assumptions.

It happens that, even under usual assumptions of consistency and stability (Lax scheme), it is not. Although it is very simple to see that, under these usual assumptions, the indirect approach is convergent, the direct method may diverge whenever it was not conveniently designed. That is, although one chooses a convergent method in order to integrate the system, a convergent method in order to discretize the cost, the consistency and stability properties of the numerical schemes are not sufficient to ensure the convergence of the resulting direct method. Very simple counterexamples are provided in [60].

It is not obvious to obtain simple conditions on the schemes ensuring the convergence of the resulting direct method, and up to now there exist only few positive results. The results of [60] assert the convergence for "smooth" problems, provided that the underlying discretization method is based on a Runge–Kutta method whose all coefficients are positive. The smoothness assumptions mean that the optimal controls under consideration take their value in the interior of the authorized domain of control (so that the maximization condition of the Pontryagin Maximum Principle reduces to $\frac{\partial H}{\partial u} = 0$) and that coercivity second-order conditions hold, ensuring the smoothness of the optimal controls (as in Sect. 2.3), in both continuous and discrete cases. This is, for instance, the case for linear quadratic problems. We refer also the reader to [61] for further comments on this result and for other considerations on symplectic integrators. The class of Legendre pseudospectral methods is up to now the other one for which the commutation issues have been proved (see [52, 53] and also [62] for a detailed discussion on the commutation properties).

Apart from those few results, up to our knowledge, the situation is still open in the general case, and there do not exist any simple criteria or any systematic method to build adapted numerical schemes for discretizing the differential system and the cost in order to ensure the convergence of the resulting direct method. As explained above, since in the smooth case the conditions ensuring the commutation of the diagram rely on second-order conditions, the problem is clearly related to the theory of conjugate points, in the sense that, in order to handle the general case, there is need for a general conjugate point theory involving all possible situations (smooth, bang, singular arcs, state constraints). The, numerical schemes should be designed to ensure coercivity properties in the discretized second-order conditions under consideration.

It can be noted that this discrepancy in the dualization–discretization diagram arises as well in the infinite-dimensional setting, e.g., when one is interested to carry out practically the so-called HUM method (Hilbert Uniqueness Method), which is roughly speaking the optimal control problem of steering an infinite-dimensional linear control system from a given point to a final point by minimizing the L^2 norm of the control (linear quadratic problem in a Hilbert space). In the case of the wave equation a phenomenon of interference of high frequencies with the mesh has been put in evidence, which causes the divergence of the method (see [63] and references therein for more details and more possible remedies; see also [64] for a general result of convergence in the parabolic case). The literature is quite abundant for this commutation problem in the infinite-dimensional framework; however the situation is still not well understood, in particular for hyperbolic equations where the question is raised as well of deriving a systematic way to build adapted schemes so that discretization and dualization commute.

2.4.4 Comparison Between Methods

We can sketch a brief comparison between both direct and indirect approaches, although such comments are a bit of caricatural. Anyway, it can be said that direct methods have the following advantages on indirect methods: they do not require any a priori theoretical study, in particular, one does not have to know a priori the structure of switchings; they are more robust, the model can be easily modified, and they are less sensitive to the choice of the initial condition. Moreover, it is easy to take into account some constraints of any possible kind. However, it is difficult to reach with direct methods the precision provided by indirect methods. A direct discretization of an optimal control problem often causes several local minima. Direct methods require a large amount of memory and thus may become inefficient if the dimension of the space is too large or if the problem cannot be easily parallelized or does not have an evident sparse structure.

The advantages of indirect methods are their extremely good numerical accuracy. Indeed since they rely on the Newton method, they inherit of the very quick convergence properties of the Newton method. Moreover the shooting methods can, by construction, be parallelized, and their implementation can thus be achieved on a cluster of parallel computers. They however suffer from the following drawbacks: the optimal controls are computed in an open-loop form; they are based on the maximum principle, which gives a necessary condition for optimality only, and thus one should be able to check, a posteriori, the optimal status of the computed trajectory (with conjugate point theory); the method is not soft, in the sense that, for instance, the structure of switchings has to be known a priori. Furthermore, it is not easy to introduce state constraints because, on one hand, this requires to apply a maximum principle with state constraints and, on the other hand, the presence of state constraints may imply a very intricate structure of the optimal trajectory, in particular the structure of switchings. The main drawback of the shooting methods is that they are difficult to make converge. Indeed, since they are based on the Newton method, they suffer from the usual drawback of the Newton method, that is, they may be very difficult to initialize properly. In other words, to make a shooting method converge, one should be able to guess good initial conditions for the adjoint vector. Indeed, the domain of convergence of the Newton method may happen to be very small, depending on the optimal control problem.

There exist many solutions to overcome the different flaws of both approaches. There is however no universal answer, and the choice of the method should be guided by the practical problem under consideration and by the experience (see the excellent surveys [50, 51, 65, 66]). Speaking however in a general way, a first idea for a reasonable solution consists of combining both direct and indirect approaches, thus obtaining a so-called hybrid method. When one addresses an optimal control problem, one could indeed try at first to implement a direct method. In such a way, one can hope to get a first (maybe rough) approximation of the optimal trajectory and a good idea of the structure of switchings and of the associated adjoint vector. If one wishes more numerical accuracy, one can then carry out an indirect method, hoping that the result provided by the direct method gives a sufficient approximation, thus providing an initial point hopefully belonging to the domain of convergence of the shooting method. Combining in such a way both direct and indirect methods, one can take benefit of the extremely good accuracy provided by the shooting method, reducing considerably the drawback due to the smallness of the domain of convergence. Applying first a direct method, one can obtain an approximation of the adjoint vector. Indeed, the total discretization method consists of solving a nonlinear programming problem with constraints. The Lagrange multipliers associated to this problem give an approximation of the adjoint vector (see [5, 65, 67]).

By the way, among the many variants of direct and indirect approaches, we mention here the possibility of designing hybrid methods, neither direct or indirect, consisting of solving the boundary-value problem resulting from the application of the PMP, not by the Newton method, but by an optimization method, in which the unknowns may, for instance, only consist of the initial adjoint vector, and the minimization functional is the cost seen as a function of the initial adjoint vector (there are many possible various formulations for such problems). Furthermore, we quote the so-called *direct multiple shooting method* (see [68, 69]), based on constrained nonlinear programming, where the optimization variables are, similarly to the multiple shooting method, the states at some nodes, and where the controls are parameterized over the intervals between the nodes by well-chosen functions. The advantage of such an approach is that it can be efficiently parallelized, and it has nice sparsity features (see [51] for variants).

In the present article our aim is to focus on applications of optimal control to aerospace, and in such problems indirect methods are often privileged because, although they are difficult to make converge, they offer a very good numerical accuracy. Hence, in the sequel of this article we will describe several optimal control problems in aerospace, providing some methods in order to make converge the shooting method:

- a geometric insight (geometric optimal control tools) for the problem of atmospheric reentry of a space shuttle (Sect. 3),
- the continuation method for orbit transfer problems (Sect. 4),
- dynamical systems theory for interplanetary mission design (Sect. 5).

3 Geometric Optimal Control and Applications to the Atmospheric Reentry Problem

In this section we focus on the problem of the atmospheric reentry of a space shuttle controlled by its bank angle, where the cost to minimize is the total thermal flux. The engine is moreover submitted to state constraints on the thermal flux, the normal acceleration, and the dynamic pressure. It is our aim to show how results of geometric optimal control can help to make the shooting method converge.

3.1 The Atmospheric Reentry Problem

The precise problem under consideration is the following. We call atmospheric phase the period of time in which the altitude of the engine is between around 20 and 120 kilometers. It is indeed in this range that, in the absence of any motor thrust, the aerodynamic forces (friction with the atmosphere) can be employed to adequately control the space shuttle to as to steer it to a desired final point and meanwhile satisfying the state constraints in particular on the thermal flux. Thus, during this phase, the shuttle can be considered as a glider, only submitted to the gravity force and the aerodynamic forces. The control is the bank angle, and the minimization criterion under consideration is the total thermal flux. The model of the control system is

$$
\begin{aligned}
\frac{dr}{dt} &= v \sin\gamma, \\
\frac{dv}{dt} &= -g\sin\gamma - \frac{1}{2}\rho\frac{SC_D}{m}v^2 + \Omega g_v(r, v, \gamma, L, l, \chi, \Omega), \\
\frac{d\gamma}{dt} &= \cos\gamma\left(-\frac{g}{v} + \frac{v}{r}\right) + \frac{1}{2}\rho\frac{SC_L}{m}v\cos\mu + \Omega g_\gamma(r, v, \gamma, L, l, \chi, \Omega), \\
\frac{dL}{dt} &= \frac{v}{r}\cos\gamma\cos\chi, \\
\frac{dl}{dt} &= \frac{v}{r}\frac{\cos\gamma\sin\chi}{\cos L}, \\
\frac{d\chi}{dt} &= \frac{1}{2}\rho\frac{SC_L}{m}\frac{v}{\cos\gamma}\sin\mu + \frac{v}{r}\cos\gamma\tan L\sin\chi + \Omega g_\chi(r, v, \gamma, L, l, \chi, \Omega).
\end{aligned}
\tag{11}
$$

Here, r denotes the distance of the center of gravity of the shuttle to the center of the Earth, v is the modulus of its relative velocity, γ is the flight angle, L is the latitude, l is the longitude, and χ is the azimuth. The scalar Ω is the angular rotation speed of the planet. In the above model, the terms g_v, g_γ, and g_χ represent the Coriolis

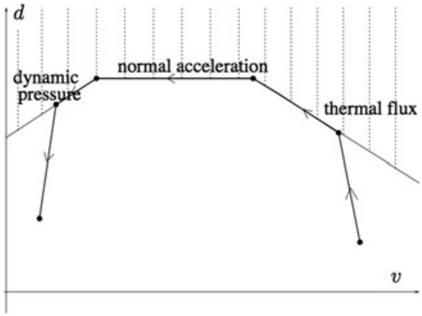

Fig. 1 Constraints, and Harpold-Graves strategy

and centripetal forces. The gravitational force is $g(r) := \frac{\mu_0}{r^2}$, where μ_0 is the gravitational constant. The aerodynamic forces consist of the drag force, whose modulus is $\frac{1}{2}\rho S C_D v^2$, which is opposite to the velocity vector, and of the lift force, whose modulus is $\frac{1}{2}\rho S C_L v^2$, which is perpendicular to the velocity vector. Here, $\rho(r) := \rho_0 e^{-\beta r}$ is the air density, S is some positive coefficient, and C_D and C_L are the drag and the lift coefficients; they depend on the angle of attack and on the Mach number of the shuttle. Note that more specific models can be used and that in general the gravity, the air density and the aerodynamic coefficients are tabulated (we refer to [31] for precise tabulations used in the study and for more details).

The control is the bank angle μ; it acts on the orientation of the lift force and thus makes the shuttle turn left or right but also acts on the altitude. It is a scalar control that is assumed to take values in $[0, \pi]$. Note that the mass m of the engine is constant along this atmospheric phase since it is assumed that there is no thrust. The optimal control problem under consideration is to steer the vehicle from precise initial conditions (with free initial longitude and azimuth) to some precise final conditions (with free flight angle and azimuth); see [31] for numerical values. Moreover, the system is submitted to three state constraints: the (instantaneous) thermal flux $\varphi := C_q\sqrt{\rho}v^3$, the normal acceleration $\gamma_n := \gamma_{n_0}\rho v^2$, and the dynamic pressure $P := \frac{1}{2}\rho v^2$ have to remain less than prescribed maximal values. They are drawn in Fig. 1 in the flight domain, in terms of the drag $d = \frac{1}{2}\frac{SC_D}{m}\rho v^2$ and of v. The minimization criterion is the total thermal flux along the flight, $J(\mu) := \int_0^{t_f} C_q\sqrt{\rho}v^3\,dt$.

Note that, if we approximate $\dot{v} \simeq -d$, then $J(\mu) = K\int_{v_0}^{v_f}\frac{v^2}{\sqrt{d}}\,dv$ (with $K > 0$), and hence for this approximated criterion, the optimal strategy is to maximize the drag d all along the flight. This strategy, described in [70] and usually employed, reduces the problem to the problem of finding a trajectory tracking the boundary of the authorized domain in the following order: thermal flux, normal acceleration, and dynamic pressure. The advantage of this method is that along the boundary arcs the control can be easily expressed in closed-loop (feedback), which is very convenient in view of stabilization issues and of real-time embarked implementation.

Anyway, this strategy is not optimal for the minimization criterion under consideration, and it was the aim of [30, 71, 72] to solve this optimal control problem with geometric considerations. A version of the Pontryagin Maximum Principle can be applied to that problem, but it is then difficult to make the resulting shooting method converge, due to the fact that the domain of convergence is very small, and getting a good initial condition of the adjoint vector is a real challenge. Of course, many numerical refinements can be proposed to overcome this initialization problem, and similar optimal control problems have been considered in a number of articles (see, e.g., [50, 51, 66] with various approaches (direct or indirect). This is indeed a standard problem, but we insist on the fact that our objective is to show how a result of geometric optimal control can be of some help in order to guess a good initial condition to make the shooting method converge (rather than making it converge through numerical refinements). Note that, without the aid of such a tool, solving this problem with a shooting method is nearly intractable.

3.2 Geometric Optimal Control Results and Application to the Problem

In this section, instead of providing a solution with computational or numerical refinements, our goal is to provide a rough analysis of the control system and show how geometric control can be of some help in order to provide a better understanding of the structure of the system and finally lead to a precise description of the optimal trajectories, then reducing the application of the shooting method to an easy exercise.

Before that, let us first explain what is geometric control. As mentioned in the introduction of this article, modern optimal control theory combines classical techniques developed in the 1960s, typically the Pontryagin Maximum Principle, with other powerful mathematical techniques in order to provide results on the structure of optimal trajectories for general classes of nonlinear control systems. Literally, geometric optimal control is the combination of classical optimal control with geometric methods in system theory. More precisely, it can be described as the combination of the knowledge inferred from the Pontryagin Maximum Principle with geometric considerations such as the use of Lie brackets, of subanalytic sets, of differential geometry on manifolds, and of symplectic geometry and Hamiltonian systems, with the ultimate objective of deriving *optimal synthesis* results, permitting to describe in a precise way the structure of optimal trajectories. In other words, the objective is to derive results saying that, according to the class of control systems we are considering, the optimal trajectories have a precise structure and are of such or such a kind. Geometric optimal control has furnished a modern and uniform framework to realize this objective.

The foundations of geometric control can be dated back, first, to the important Chow's theorem (see [73]) on reachable sets of integral curves of families of vector fields, which was not part of the Calculus of Variations theory, and second, to the articles [74, 75], where Brunovsky discovered that it was possible to derive regular synthesis results using geometric considerations for a large class of control systems, yielding a precise description of the structure of optimal trajectories. Since then, many different tools from differential geometry have been introduced in optimal control, forming gradually a package of techniques and knowledge now identified as *geometric optimal control*: the use of differentiable manifolds, extending the field of applications of the PMP to optimal control problems naturally posed on a manifold or on a Lie group (see, e.g., [12, 76]), very much encountered in mechanics, robotics, aerospace, quantum control theory, etc; Lie brackets and Lie algebras, used to derive important results on accessibility and controllability properties, to derive higher-order optimality conditions (see [77–79]) allowing one, as mentioned above, to restrict the set of optimal candidates and to derive local regularity and optimal synthesis results (see [80–82]); stratification and real analyticity theory considerations (see [83, 84]), used as well for regularity and optimal synthesis issues; singularity theory, providing a starting point to the classification of extremals (see [14, 30, 85–87]) and allowing one to study how trajectories may lose optimality (see [88–90]); the use of the Miele clock form (see [91]), widely generalized with the framework of symplectic geometry and methods, the latter being used to provide sufficient conditions for optimality in terms either of conjugate points, Maslov index, extremal field theory, or of optimal synthesis (see [12, 14]); fine considerations from differential geometry, e.g., the concepts of conjugate or cut locus, of Jacobi curves or of curvature, used to provide global optimality results (see, e.g., [92, 93]); sub-Riemannian metrics (see, e.g., [94]), much used for applications to robotics and more recently to aerospace problems; and many other notions and mathematical concepts, borrowed from differential geometry and related areas.

Typically, one should keep in mind the following idea. The aim of using these geometric tools is to provide a complement to the PMP whenever its application alone happens to be insufficient to adequately solve an optimal control problem, due to a lack of information. As explained in details in Sect. 2, the PMP is a first-order condition for optimality, and its aim is to select a set of trajectories that are candidates to be optimal. Apart from the ultimate goal of providing a complete optimal synthesis, one of the objectives of geometric control is to derive higher-order optimality conditions in order to restrict more the set of candidate optimal trajectories. Second-order conditions have been briefly reviewed in Sect. 2.3, and their connection to conjugate point theory has been put in evidence. More generally, the objective of higher-order conditions is to select among the extremals derived from the PMP those who are candidates to be indeed optimal. When this selection is achieved in a so nice way that there exists only one possible way to steer the system from the initial to the final prescribed conditions, one speaks of an optimal synthesis, although this wording underlies some more regularity properties, in particular regularity properties of the selected extremals ensuring that they form an extremal field (see, e.g., [84]). We refer the reader interested in a deeper insight on geometric control issues to the textbooks [12, 14, 76, 92, 95–97] and references therein. Note that we do not mention here the many geometric issues related with stabilization that are outside of the scope of this article.

Let us now show how, starting from a simple remark on the structure of the control system (11), results from geometric control theory can be applied and then help to guess the precise structure of optimal trajectories, and ultimately make the application of a shooting method much easier. In a first step, let us assume that the rotation of the planet can be neglected, that is, $\Omega = 0$. Note that, at the end, this is not the case and the effect of the Coriolis force happens to be necessary to reach the desired final conditions. Anyway, assuming that $\Omega = 0$, the control system (11) gets simpler, and in particular the three first differential equations can be written as the single-input control-affine system in $\mathbb{R}^3$

$$\dot{x}(t) = f_0\big(x(t)\big) + u(t) f_1\big(x(t)\big), \quad \big|u(t)\big| \leqslant 1, \tag{12}$$

where $u := \cos(\mu)$, x:=(r, v, γ), and

$$f_0(x) := v \sin\gamma \frac{\partial}{\partial r} - \left(g \sin\gamma + \frac{1}{2}\rho \frac{SC_D}{m} v^2\right)\frac{\partial}{\partial v} + \cos\gamma\left(-\frac{g}{v} + \frac{v}{r}\right)\frac{\partial}{\partial \gamma},$$

$$f_1(x) := \frac{1}{2}\rho \frac{SC_L}{m} v \frac{\partial}{\partial \gamma}.$$

Ignoring temporarily the coordinates (L, l, χ), the induced optimal control problem is to steer the above three-dimensional control system, from a given $x(0)$ to a final target ($\gamma(t_f)$ is free, but the two first coordinates are fixed), with a control satisfying the constraint $|u| \leqslant 1$, and moreover, under the three state constraints on the thermal flux, normal acceleration, and dynamic pressure (which depend only on x), by minimizing the cost J.

Reparameterizing by the instantaneous cost (dividing the equations by $\varphi := C_q\sqrt{\rho}v^3$ and setting $s = \varphi t$ as a new time), we end up with the minimal time problem for a single-input control-affine system, with the constraint $|u| \leqslant 1$ on the control, and with pure state constraints of the form $c_i(x) \leqslant 0$, $i = 1, 2, 3$.

Besides, there exist results coming from geometric optimal control theory, providing a qualitative description of minimal time trajectories for control systems of the form (12), within small time, in small dimension (two and three), and under generic assumptions. We refer the reader to [80, 81, 92, 98] for precise results. For instance, in dimension three, in the absence of state constraint, it is proved in [80] that, if the vector fields of the system are such that f_0, f_1, and their Lie bracket $[f_0, f_1]$ are linearly independent at x_0, then minimal-time trajectories starting from x_0 are locally bang-bang with at most two switchings. Moreover, denoting by x_+ (resp., x_-) an arc corresponding to the control $u = 1$ (resp., $u = -1$), the small time accessible set is homeomorphic to a cone whose boundary consists of all trajectories of the form x_+x_- and x_-x_+ (i.e., concatenations of two bang arcs). Furthermore, using the Miele clock form (see [31, 91, 99] for the use of this form in the plane, and see [12, 14] for a generalization in a symplectic context), according to the sign of some coefficient only depending on the Lie structure of the vector fields at x_0, it is shown that, locally around x_0, trajectories $x_+x_-x_+$ (starting from x_0) are of minimal time, whereas trajectories $x_-x_+x_-$ are of maximal time (or, conversely, according to that sign).

Motivated by the atmospheric reentry problem, this kind of result has been extended in [30] to the case where there are state constraints. Actually in this reference local optimal syntheses are derived for systems in dimension two and three, with one or several state constraints. This classification involves many cases, depending on the order of the state constraints under consideration and cannot be sketched in few words. In the case of the atmospheric reentry problem, all state constraints are of order two, since one has to differentiate two times with respect to t the relations characterizing a boundary arc to make appear the control. The results of [30], combined with numerical simulations, then lead to the following result.

Proposition 3.1 *The optimal trajectory for the simplified three-dimensional model* (12) *is of the kind* $x_-x_+x_{\text{flux}}x_+x_{\text{acc}}x_+$, *where* x_+ (*resp.*, x_-) *is an arc corresponding to the control* $u = 1$ (*resp.*, $u = -1$), *and* x_{flux} (*resp.*, x_{acc}) *denotes a boundary arc saturating the constraint on the thermal flux* (*resp. on the normal acceleration*).

Note that, since the above-mentioned geometric results are of local nature, to make them global, they must be combined with numerical simulations and possibly with conjugate point arguments, as it was done for the atmospheric reentry problem in [71]. In this latter reference, it was also shown, using perturbation arguments, how this result in dimension three could be used in order to provide an approximation of the optimal trajectory for the true problem in dimension six. In this perturbation argument, the parameter Ω is in some sense viewed as a small parameter, but to justify properly the argument, it must also be observed that the simplified three-dimensional system is almost a projection onto $\mathbb{R}^3$ of the complete system in dimension six. Anyway, what is important is that the strategies announced in Proposition 3.1 provide a good approximation of the optimal trajectories of the complete problem in dimension six.

Now, the point is that it is very easy to make a shooting method converge for the simplified problem in dimension three. Indeed, since one knows precisely the structure of the optimal trajectory, the trajectory to be determined can be parameterized only with its switching times, and hence the shooting problem reduces to a problem with only five unknowns (which are the switching times). The resulting optimal trajectory can then serve as a good initial guess for seeking the optimal trajectory of the complete problem in dimension six. Moreover, it is possible to derive as well a good approximation of the initial adjoint vector in dimension six, by completing the Lagrange multiplier of the optimal solution in dimension three with zeros (it is shown in [71, 72] that it is indeed a good approximation because for $\Omega = 0$, the optimal trajectory of the three-dimensional problem can be viewed as a singular trajectory of the six-dimensional problem with corank three). This approximation is accurate enough to make converge a shooting method in dimension six.

3.3 Open Challenges

It has been shown previously how a result of geometric optimal control theory on local optimal syntheses can help to make a shooting method converge, or at least can simplify its implementation by describing precisely the structure of the optimal trajectory (e.g., as a succession of bang, singular, or boundary arcs, in a precise order). As briefly surveyed previously, these results exist only for control-affine systems in small dimension (essentially, two and three). Note that, in dimension three, more general situations have been considered in [81] for single-input control-affine systems, providing a precise local structure of optimal trajectories having a finite number of switching times and involving possible singular arcs. These results have been generalized in the deep article [100], in which the author studies the local structure of minimal-time trajectories for single-input control-affine systems with a starting point in a submanifold S. It is shown that, if the codimension of S is less than or equal to four, then generic minimal time trajectories starting from S are concatenations of at most seven between bang and singular arcs. This result can be applied to four-dimensional systems when S is a point. For larger dimensions, the situation is far more intricate, not only due to the possible occurrence of singular trajectories, but also to the generic occurrence of *Fuller phenomena* (see [32, 86, 101]), in which case an infinite number of switchings may occur in a compact time interval.

It would be however useful to derive such local optimal synthesis results for systems in larger dimensions, however necessarily under strong assumptions in particular to avoid the Fuller phenomenon, for instance, in view of providing alternative ways of making converge the shooting method for the orbit transfer problem. Concerning the latter problem, note that, in order to generate the accessible set for the orbit transfer problems, the authors of [102] have used tools of Riemannian geometry to determine the cut and conjugate loci on a complete two-surface of revolution in order to infer the global structure of the extremals of the problem.

Note that the results of geometric optimal control mentioned in the previous section essentially rely on a careful analysis of the extremal flow using second-order conditions or a Hamiltonian approach and hence are strongly related to the concept of conjugate time (surveyed previously in this paper). These methods should permit to derive local optimal syntheses in larger dimension under additional assumptions and as well for control-affine systems with more than one control (although it can be expected that the situation is much more complicated). Note however that, according to the results of [34–36], generic (in the Whitney sense) control-affine systems do not admit any minimizing singular trajectory whenever the number of controls is more than two (more precisely, it is shown in these references that such generic control-affine systems do not admit any trajectories satisfying the Goh necessary condition derived in [103]). For a first result concerning the classification of extremals for control-affine systems with two controls, we quote the recent article [87], with an application to the minimum time control of the restricted three-body problem.

4 The Continuation Method and Applications

4.1 The Continuation Method

The objective of continuation or homotopy methods is to solve a problem step by step from a simpler one by parameter deformation. There exists a well-developed theory and many algorithms and numerical methods implementing these ideas, and the field of applications encompasses Brouwer fixed-point problems, polynomial and nonlinear systems of equations, boundary-value problems in many diverse forms, etc. We refer the reader to the textbook [104] or to the survey articles [105, 106] for a complete report on these theories and methods.

Here, we will use the continuation or homotopy approach in order to solve the shooting problem resulting from the application of the Pontryagin Maximum Principle to an optimal control problem. More precisely, the method consists of deforming the problem into a simpler one that we are able to solve (without any careful initialization of the shooting method) and then of solving a series of shooting problems, step by step, to come back to the original problem. In practice the choice of an adapted parameter deformation of the problem is done according to an intuition or a heuristics with respect to the physical meaning of the different parameters entering into the problem and thus may require considerable physical insight into the problem. The homotopic parameter λ can be a physical parameter (or several) of the problem, or an artificial one. Some examples are provided in the sequel. The deformation should also be chosen to enjoy sufficient regularity conditions, making the homotopy method converge. Notice that not only the simpler problem should be chosen according to a heuristics, but also the path between the simpler problem and the original problem. When the homotopic parameter λ is a real number and when the path is linear in λ (meaning that, in some coordinates, the path consists of a convex combination of the simpler and of the original problem, with $\lambda \in [0, 1]$), the homotopy method is rather called a continuation method in the literature. The continuation method consists then of tracking a set of zeros as the parameter λ is increased monotonically from 0 to 1 (starting from the simpler known solution). Numerical continuation is well known in numerical analysis and has been applied to a wide field of various problems. It can fail whenever the path of zeros which is tracked has bifurcation points or more generally singularities, or whenever this path fails to exist globally and does not reach $\lambda = 1$. Homotopy methods generalize continuation methods, in the sense that the parameter λ is not necessarily increased monotonically from 0 to 1, dealing with the possible occurrence of bifurcations or singularities, and in the sense that the parameter λ is not necessarily a real number but can be considered in more general spaces (it can be a real number, or a vectorial number, or even a parameter evolving in some general Banach space); furthermore, in general homotopy methods the path can be nonlinear and considered in various spaces.

For the moment, for the sake of simplicity, we focus on the continuation method and consider a real parameter $\lambda \in [0, 1]$ (we comment further on homotopy methods). Let us provide shortly the basic arguments showing the feasibility of the continuation method. From the theoretical point of view, regularity properties require at least that the optimal solution is continuous, or differentiable, with respect to the parameter λ that is expected to increase monotonically in [0, 1]. This kind of property is usually derived using an implicit function argument, which is encountered in the literature as *sensitivity analysis*. Let us explain what is the general reasoning of sensitivity analysis, in the simplified framework of Sect. 2.2.1, that is, assuming that $M_0 = \{x_0\}$, $M_1 = \{x_1\}$, and $U = \mathbb{R}^m$. We are faced with a family of optimal control problems, parameterized by λ, that can be as in (4) written in the form of

$$\min_{E_{x_0,T,\lambda}(u_\lambda)=x_1} C_{T,\lambda}(u). \tag{13}$$

According to the Lagrange multipliers rule, if u_λ is optimal, then there exists $(\psi_\lambda, \psi_\lambda^0) \in \mathbb{R}^n \times \mathbb{R} \setminus \{(0, 0)\}$ such that $\psi_\lambda \, dE_{x_0,T,\lambda}(u_\lambda) + \psi_\lambda^0 \, dC_{T,\lambda}(u) = 0$. Assume that there are no minimizing abnormal extremals in the problem. Under this assumption, since the Lagrange multiplier $(\psi_\lambda, \psi_\lambda^0)$ is defined up to a multiplicative scalar, we can definitely assume that $\psi_\lambda^0 = -1$. Then, we are seeking $(u_\lambda, \psi_\lambda)$ such that $F(\lambda, u_\lambda, \psi_\lambda) = 0$, where the function F is defined by

$$F(\lambda, u, \psi) := \begin{pmatrix} \psi \, dE_{x_0,T,\lambda}(u) - dC_{T,\lambda}(u) \\ E_{x_0,T,\lambda}(u) - x_1 \end{pmatrix} = \begin{pmatrix} \frac{\partial L_{T,\lambda}}{\partial u}(u, \psi) \\ E_{x_0,T,\lambda}(u) - x_1 \end{pmatrix},$$

where $L_{T,\lambda}(u, \psi) := \psi E_{x_0,T,\lambda}(u) - C_{T,\lambda}(u)$ is the Lagrangian, as defined in Sect. 2.2.1. Let $(\bar{\lambda}, u_{\bar{\lambda}}, \psi_{\bar{\lambda}})$ be a zero of F. Assume that F is of class C^1. If the Jacobian of F with respect to (u, ψ), taken at the point $(\bar{\lambda}, u_{\bar{\lambda}}, \psi_{\bar{\lambda}})$, is invertible, then according to a usual implicit function argument, one can solve the equation $F(\lambda, u_\lambda, \psi_\lambda) = 0$, and the solution $(u_\lambda, \psi_\lambda)$ depends in a C^1 way on the parameter λ. Note that this standard argument from sensitivity analysis is at the basis of the well-known Lagrange–Newton method in optimization.

Let us now analyze the invertibility condition of the Jacobian of F with respect to (u, ψ). This Jacobian matrix is

$$\begin{pmatrix} Q_{T,\lambda} & dE_{x_0,T,\lambda}(u)^* \\ dE_{x_0,T,\lambda}(u) & 0 \end{pmatrix}, \tag{14}$$

where $dE_{x_0,T,\lambda}(u)^*$ is the transpose of $dE_{x_0,T,\lambda}(u)$, and where $Q_{T,\lambda}$ is the quadratic form considered in Sect. 2.3.1, defined as the Hessian $\frac{\partial^2 L_{T,\lambda}}{\partial^2 u}(u, \psi, \psi^0)$ restricted to $\ker \frac{\partial L_{T,\lambda}}{\partial u}$. The matrix (14) (which is a matrix of operators) is called *sensitivity matrix* in sensitivity analysis. It is easy to prove that this sensitivity matrix is invertible iff the linear mapping $dE_{x_0,T,\lambda}(u)$ is surjective and the quadratic form Q_T is nondegenerate. Having in mind the definitions given previously in this article, the meaning of these assumptions is the following. The surjectivity of $dE_{x_0,T,\lambda}(u)$ exactly means that the control u is not singular (see Definition 2.2). The nondegeneracy of $Q_{T,\lambda}$ is exactly related with the concept of conjugate point (see Definition 2.4). Note that, as long as we do not encounter any conjugate time along the continuation path, the extremals that are computed are locally optimal. It follows that, to ensure the surjectivity of $dE_{x_0,T,\lambda}(u)$ along the continuation process, it suffices to assume the absence of singular minimizing trajectory. Note that, in the simplified problem that we considered, where the controls are not constrained, singular trajectories are exactly the projections of abnormal extremals.

Therefore, we conclude that, as long as we do not encounter any minimizing singular control nor conjugate point along the continuation procedure, the continuation method is *locally* feasible, and the extremal solution $(u_\lambda, \psi_\lambda)$ which is locally computed as above is of class C^1 with respect to the parameter λ. These two sufficient assumptions are the basic ones ensuring the existence of a local solution in the continuation procedure and thus its local feasibility.

Before going to global considerations, let us make an ultimate comment on these two assumptions. The absence of conjugate point can be tested numerically: as explained in Sect. 2.3.2, it suffices to test the vanishing of some determinant along the extremal flow (see Sect. 2.4.2). As long as this test does not detect any conjugate point along the continuation process, the extremals that are computed are locally optimal. The assumption of the absence of minimizing singular trajectories is of a much more geometric nature. Such results exist for some classes of control-affine systems under some strong Lie bracket assumptions (see [12, 14, 107, 108]). Moreover, as mentioned in Sect. 3.3, generic (in the Whitney sense) control-affine systems with more than two controls have no minimizing singular trajectory; hence, for such kinds of systems, the assumption of the absence of minimizing singular trajectory is automatically satisfied.

Remark 4.1 The implicit function argument given above is on the control, but the continuation method is usually implemented on the exponential mapping (see Definition 2.3) and consists of tracking a path of initial adjoint vectors doing the job. More precisely, instead of (13), one has to solve

$$\exp_{x_{0,\lambda}}(T, p_{0,\lambda}) = x_1, \tag{15}$$

where the exponential mapping is parameterized by λ. This is the shooting method in the simplified case, and the sufficient conditions above ensure the local feasibility.

The previous implicit function arguments permit to ensure the local feasibility of the continuation procedure, locally around a given solution (that is, locally around a given parameter λ). Now to make it global over [0, 1], we ought to ensure that the path of zeros $\lambda \mapsto p_{0,\lambda}$ is globally defined on [0, 1] and joins $p_{0,0}$ to $p_{0,1}$. It could indeed happen that the path is not globally defined and either reaches some singularity or wanders off to infinity before reaching $\lambda = 1$. To eliminate the first possibility, since a limit of optimal controls is optimal as well (see, e.g., [16, 20]), we can make the assumption of the absence of minimizing singular trajectory and of conjugate point over all the domain under consideration (not only along the continuation path), and for every $\lambda \in [0, 1]$. As said before, the absence of singular minimizing trajectory over the whole space is generic for large classes of systems, and hence this is a reasonable assumption; however, the global absence of conjugate point is a strong assumption. There exist however some other possibilities to tackle singularities.[2] To eliminate the second possibility, we ought to provide sufficient conditions ensuring that the tracked paths remain bounded. In other words, considering (15), we have to ensure that the initial adjoint vectors $p_{0,\lambda}$ that are computed along the continuation procedure remain bounded, uniformly with respect to the homotopic parameter λ. This means that we have to ensure that the exponential mapping is proper, uniformly with respect to λ. The properness of the exponential mapping is studied in [20], where it is proved that, if the exponential mapping is not proper, then there exists an abnormal minimizer (see also [109] and [16, Lemma 2.16] for a more general statement). By contraposition, if one assumes the absence of minimizing abnormal extremals, then the required boundedness follows.

Note again that, in the simplified problem that we considered, where the controls are unconstrained, singular trajectories are exactly the projections of abnormal extremals. Hence, we have obtained the following result.

Proposition 4.1 *In the simplified case where* $M_0 = \{x_0\}$, $M_1 = \{x_1\}$, *and* $U = \mathbb{R}^m$, *if, for every* $\lambda \in [0, 1]$, *there is no minimizing singular trajectory nor conjugate points over all the domain, then the continuation procedure* (15) *is globally feasible on* [0, 1].

It is not easy to weaken the assumption of the absence of conjugate point over all the domain. One way is to consider a smaller domain, covering the continuation paths under consideration; in this case however one ought to ensure that the tracked

[2] Singularities due to conjugate points may be either detected and then handled with specific methods, or can be removed generically by Sard arguments (see comments further, on homotopy methods).

continuation path remains in the domain, that is, remains far from its boundary. There does not seem to exist any simple and tractable condition ensuring this fact in general. Note that in [110] the authors use the concept of *injectivity radius* in order to provide estimations of domains in which the continuation method is globally feasible, on an example which is however specific to Riemannian geometry.

This simple Proposition 4.1 withstands many possible generalizations. For more general optimal control problems, Proposition 4.1 can be extended quite easily, by adapting the above arguments, and in particular the implicit function argument (although this may be a bit technical, for instance, whenever there are some state constraints; see [111]). In any case, this kind of result provides the mathematical foundations ensuring the global feasibility of the continuation method in optimal control. It can be noted that the feasibility of the continuation method has been much studied for other less specific issues in numerical analysis (see [104, 105] and references therein).

In the more general case of homotopies, the parameter λ is not necessarily increasing monotonically from 0 to 1, and we can encounter turning points (see [106]). One of the methods, known as differential homotopy (or differential pathfollowing), consists of tracking a path of zeros $s \mapsto (\lambda(s), p_{0,\lambda(s)})$ satisfying (15) for every s. It is then usual to assume that the mapping F has maximal rank (more precisely, that 0 is a regular value of F) so that the path of zeros evolves on a submanifold (see, e.g., [104] for the details): this kind of implicit function argument permits to establish, as before, the local feasibility of the method; but now the difference is that turning points[3] are allowed: the zero $p_{0,\lambda}$ is not necessarily a local function of λ. The global feasibility issues require topological considerations such as connectedness features. Note that, if one does not make this assumption that the mapping F has maximal rank, then one is faced with the possible occurrence of singularities. As explained previously for the continuation method, assuming the absence of singularities is a too strong assumption in general. In the existing literature there are essentially two approaches to tackle this difficulty. The first approach, of local type, consists of detecting the possible singularities or bifurcations along the zero path. There is a huge literature on this problem, and we refer to [104] for a survey on these methods applied to homotopy procedures. The second approach, of global type, consists of considering a global perturbation of the homotopy function, more precisely, of the simpler problem under consideration, in order to ensure that, with probability one, 0 is a regular value of F. This variant of the method that can be proved to be globally convergent is known as globally convergent probability-one homotopy method. It is based on nontrivial transversality arguments, combined in [112] with Sard's theorem and yielding to homotopy methods with a guarantee of success of probability one with respect to the choice of the simpler problem (see [106] for a nice survey discussion and the statement of a general result of global convergence). The "almost everywhere" statement of such a result is used to avoid the possible singularities of the curves to be tracked in the homotopy procedure. The last crucial requirement to ensure global feasibility is as before that the tracked paths remain bounded, in order to ensure that the zero paths are globally

[3] A turning point is a point of the path of zeros at which $\lambda(s)$ has a local extremum.

well defined and do not wander off to infinity. This properness can be handled as before by assuming the absence of abnormal minimizers (see arguments of the previous discussion). Having in mind these issues, it is then possible to derive results similar to Proposition 4.1, according to the specific homotopy method under consideration.

Note that the mathematical foundations of the differential homotopy method applied to optimal control are studied in [113] and more deeply in [87], where the relation between turning points of the path and conjugate points is clearly elucidated. Note that the authors of [87], studying by homotopy a three-body problem, recommend to stop following a path in case a conjugate point (resulting into a turning point) appears, and provide some hints to jump to another path (these hints are however specific to their problem).

From the numerical point of view, there exist many methods and strategies in order to implement continuation or homotopy methods, and one has to distinguish between differential pathfollowing (see, e.g., [114] for applications to orbit transfer problems), simplicial methods (see, e.g., [115] for similar applications), predictor–corrector methods, piecewise-linear methods, etc. Extensive documentation about path following methods with theoretical and algorithmic issues can be found in [104]. Also, many codes can be found on the web, such as the well-known `Hompack90` (see [116]) or the recent `Hampath` (see [113]), just to name a few.

4.2 Application to the Orbit Transfer Problem with Low Thrust

In this section we focus on the orbit transfer problem with low thrust, where the system under consideration consists of the controlled Kepler equations

$$\ddot{q}(t) = -q(t)\frac{\mu}{r(t)^3} + \frac{T(t)}{m(t)}, \qquad \dot{m}(t) = -\beta\|T(t)\|, \tag{16}$$

where $q(t) \in \mathbb{R}^3$ is the position of the engine at time t, $r(t) := \|q(t)\|$, $T(t)$ is the thrust at time t, and $m(t)$ is the mass, with $\beta := 1/I_{\mathrm{sp}} g_0$. Here g_0 is the usual gravitational constant, and I_{sp} is the specific impulsion of the engine. The thrust is submitted to the constraint $\|T(t)\| \leqslant T_{\max}$, where the typical value of the maximal thrust $T_{\max}$ is around 0.1 N for low-thrust engines. The orbit transfer problem consists of steering the engine from a given initial orbit (e.g., an initial excentric inclined orbit) to a final one (e.g., the geostationary orbit). Controllability properties, ensuring the feasibility of the problem, have been studied in [25, 117], based on the analysis of the Lie algebra generated by the vector fields of the system. For this control problem, one is interested in realizing this transfer by minimizing the transfer time or the fuel consumption.

Let us first show how the minimal time problem of steering this control system from any initial position to some final orbit can be solved by combining a shooting method with a continuation procedure. On this problem one immediately realizes that the main difficulty is the fact that the maximal thrust is very low. It is then not surprising to observe numerically that the lower is the maximal thrust, the smaller is the domain of convergence of the Newton method in the shooting problem. In these conditions it is natural to carry out a continuation on the value of the maximal thrust, starting with larger values of the maximal thrust (for which the problem is no more realistic, but for which the shooting method is by far easier to make converge), and then decreasing step by step the value of the maximal thrust, down to low realistic values.

This strategy was implemented in [118] in order to realize the minimal time 3D transfer of a satellite from a low and eccentric inclined initial orbit toward the geostationary orbit, for an engine of around 1500 kg. Their continuation procedure starts with the orbit transfer problem with the value $T_{\max} = 60$ N, for which the domain of convergence of the shooting function is large enough so that the shooting method can be initialized easily. Then they decrease the value of $T_{\max}$ step by step in order to reach down the value $T_{\max} = 0.14$ N. Along this continuation procedure, the authors prove that the minimal time t_f is right-continuous with respect to the maximal thrust $T_{\max}$; hence, in theory, it could be expected that the minimal time t_f obtained at the step k of the continuation procedure is a good initial guess for the step $k+1$. However, they note that this strategy is not so much efficient numerically for low thrusts, in the sense that, for low values of $T_{\max}$, the value of $T_{\max}$ has to be decreased with very small steps to ensure convergence. The authors then use the remarkable heuristics $t_f T_{\max} \simeq \text{Cst}$, which allows them to significantly improve the efficiency of their continuation procedure and to reach down the low value of $T_{\max} = 0.14$ (for which the resulting time of transfer is more than six months).

The minimal fuel consumption orbit transfer problem has also been solved in [119, 120]. It consists of minimizing the cost $\int_0^{t_f} \|T(t)\|\,dt$, and the problem is more difficult than the minimal time problem, because the optimal control derived from the PMP is no more continuous. This lack of continuity implies difficulties to apply the shooting method. To overcome this problem, the authors propose to implement a continuation on the cost functional, parameterized by $\lambda \in [0,1]$. More precisely, they propose to minimize the cost $\int_0^{t_f} ((1-\lambda)\|T(t)\|^2 + \lambda\|T(t)\|)\,dt$. The case $\lambda = 0$ corresponds to the minimization of the energy, while $\lambda = 1$ corresponds to the original problem (minimization of the consumption). For every $\lambda < 1$, the application of the PMP leads to smooth controls, for which the shooting method can be applied successfully. Also, for $\lambda = 0$, the shooting problem is easier to initialize. The authors prove that it is possible to follow a path of solutions starting from $\lambda = 0$ and reaching a value of λ very close to 1, which permits then to initialize successfully the shooting method with $\lambda = 1$.

It can be noted that the heuristics $t_f T_{\max} \simeq \text{Cst}$ has been understood and clearly explained in [102]. In this work, based on preliminary results of [121], where the optimal trajectories of the energy minimization problem are approximated using averaging techniques, the averaged Hamiltonian system is explicitly computed and is shown to be a Riemannian problem. The geodesics and their integrability properties are investigated and deeply analyzed, as well as the Riemannian metrics of the averaged system. Since the averaged system is Riemannian, this means, roughly speaking, that optimal trajectories are straight lines up to a change of coordinates. Since the averaged system can serve as a good approximation of the initial system for low values of the maximal thrust (this fact is proved in these references), the heuristics follows. This is one more example where a geometric insight provides a good understanding of the problem, leading to an efficient numerical solving.

Remark 4.2 In [16] it is shown how one can moreover take into account a shadow cone (eclipse) constraint in the orbit transfer problem. The approach is based on an hybridization of the problem, considering that the controlled vector fields are zero when crossing the shadow cone. A regularization procedure consisting of smoothing the system, combined with a continuation, is also implemented (it is actually the objective of the article to derive convergence properties of smoothing procedures).

4.3 A Continuation Approach to the Strong Thrust Orbit Transfer Problem by Flattening the Earth

In this section we describe an alternative approach to the strong thrust minimal consumption orbit transfer planification problem developed in [122], consisting of considering at first the problem for a flat model of the Earth with constant gravity and then of introducing step by step (by continuation) the variable gravity and the curvature of the Earth, in order to end up with the true model.

Of course, the fuel efficient orbit transfer of a satellite has been widely studied with many possible approaches such as impulsive orbit transfers and direct or shooting methods (see [51], and see [122] for a list of methods and references). Here we described shortly an unusual approach, based on the remark that the problem is extremely easy to solve whenever the Earth is flat with a constant gravity. Then we pass continuously to the initial model. We restrict to the two-dimensional case and consider the coplanar orbit transfer problem with a spherical Earth and a central gravitational field $g(r) = \frac{\mu}{r^2}$. Written in cylindrical coordinates, the control system under consideration is

$$
\begin{aligned}
&\dot r(t) = v(t)\sin\gamma(t), \qquad \dot\varphi(t) = \frac{v(t)}{r(t)}\cos\gamma(t),\\
&\dot v(t) = -g\big(r(t)\big)\sin\gamma(t) + \frac{T_{\max}}{m(t)}u_1(t),\\
&\dot\gamma(t) = \left(\frac{v(t)}{r(t)} - \frac{g(r(t))}{v(t)}\right)\cos\gamma(t) + \frac{T_{\max}}{m(t)v(t)}u_2(t),\\
&\dot m(t) = -\beta T_{\max}\sqrt{u_1(t)^2 + u_2(t)^2},
\end{aligned}
\tag{17}
$$

where the thrust is $T(t) := u(t)T_{\max}$ (here in the application, $T_{\max}$ is large since we consider a strong thrust), and the control is $u(t) = (u_1(t), u_2(t))$ satisfying the constraint $u_1(t)^2 + u_2(t)^2 \leqslant 1$. The optimal control problem under consideration consists of steering the above system from a given initial configuration $r(0) = r_0$, $\varphi(0) = \varphi_0$, $v(0) = v_0$, $\gamma(0) = \gamma_0$, $m(0) = m_0$ to some point of a specified final orbit $r(t_f) = r_f$, $v(t_f) = v_f$, $\gamma(t_f) = \gamma_f$ by maximizing the final mass $m(t_f)$. Note that the final time t_f must be fixed in this problem, otherwise the optimal control problem would not have any solution (see [119]) since it is always better in terms of consumption to let the engine turn more around the planet with shorter thrust arcs. The application of the PMP to this problem leads to a shooting problem with discontinuous controls (consisting of thrust and balistic arcs) that is not easy to solve directly because it is difficult to initialize adequately. In contrast, consider the very simple flat Earth model

$$
\begin{aligned}
&\dot{x}(t)=v_x(t), \qquad \dot{h}(t)=v_h(t),\\
&\dot{v}_x(t)=\frac{T_{\max}}{m(t)}u_x(t), \qquad \dot{v}_h(t)=\frac{T_{\max}}{m(t)}u_h(t)-g_0,\\
&\dot{m}(t)=-\beta T_{\max}\sqrt{u_x(t)^2+u_h(t)^2},
\end{aligned}
\tag{18}
$$

where x denotes the horizontal variable, h is the altitude, and v_x and v_h are the corresponding components of the velocity. The control $(u_x(\cdot), u_h(\cdot))$ must satisfy the constraint $u_x(\cdot)^2 + u_h(\cdot)^2 \leqslant 1$. It happens that the problem of maximizing the final mass $m(t_f)$ (here, it makes sense to consider a free final time), with initial conditions $x(0) = x_0$, $h(0) = h_0$, $v_x(0) = v_{x0}$, $v_h(0) = v_{h0}$, $m(0) = m_0$, and final conditions $h(t_f) = h_f$, $v_x(t_f) = v_{xf}$, $v_h(t_f) = 0$, is extremely simple to solve. It can even be solved explicitly, analytically, and the shooting method can be simplified in order to converge automatically and instantaneously, without any careful initialization (see [122] for details). In view of that, it is tempting to try to pass continuously from this simple model to the initial one by acting on the gravity and on the curvature of the planet. Note that, since the coordinates used in (18) are Cartesian whereas those in (17) are polar, at the end of the continuation procedure a change of coordinates will be required. Evidently, this change of coordinates is $x = r\varphi$, $h = r - r_T$ (where r_T is the radius of the Earth), $v_x = v\cos\gamma$, $v_h = v\sin\gamma$, and for the control, $u_x = u_1\cos\gamma - u_2\sin\gamma$, $u_h = u_1\sin\gamma + u_2\cos\gamma$. When passing from polar to Cartesian coordinates, note however that we not take into account an obvious physical feature: in the absence of control ($u = 0$), in the flat Earth model (18) there do not exist any horizontal trajectories (for which $h(t)$ is constant), whereas the round Earth model (17) does admit round (Keplerian) orbits (for which $r(t)$ is constant). This still holds even though we transform the flat Earth model with a variable gravity. This is of course due to the model that is too much simplist, and we are going to modify this model accordingly by introducing some new terms into the dynamics of the flat Earth model, so that there may exist such horizontal trajectories with null thrust. First of all, let us apply the above change of coordinates to the control system (17). This leads to

$$
\begin{aligned}
&\dot{x}(t)=v_x(t)+v_h(t)\frac{x(t)}{r_T+h(t)}, \qquad \dot{h}(t)=v_h(t),\\
&\dot{v}_x(t)=\frac{T_{\max}}{m(t)}u_x(t)-\frac{v_x(t)v_h(t)}{r_T+h(t)},\\
&\dot{v}_h(t)=\frac{T_{\max}}{m(t)}u_h(t)-\frac{\mu}{(r_T+h(t))^2}+\frac{v_x(t)^2}{r_T+h(t)},\\
&\dot{m}(t)=-\beta T_{\max}\sqrt{u_x(t)^2+u_h(t)^2}.
\end{aligned}
\tag{19}
$$

This control system is exactly system (17) expressed in cylindrical coordinates. With respect to the flat Earth model (18), except the fact that the gravity term is variable, we observe the presence of additional terms in the dynamics of x, v_x and v_h, which can be viewed for the flat Earth model as kinds of correcting terms that permit the possible occurrence of horizontal trajectories. In view of that, in order to pass continuously from the flat Earth model (18) to the (actually round Earth) model (19), we introduce two parameters λ_1 and λ_2, the first of which is acting on the gravity, and the second of which permits to introduce the correcting terms. Finally, we consider the family of control systems

$$\begin{aligned}
\dot{x}(t) &= v_x(t) + \lambda_2 v_h(t)\frac{x(t)}{r_T + h(t)}, \qquad \dot{h}(t) = v_h(t), \\
\dot{v}_x(t) &= \frac{T_{\max}}{m(t)}u_x(t) - \lambda_2 \frac{v_x(t)v_h(t)}{r_T + h(t)}, \\
\dot{v}_h(t) &= \frac{T_{\max}}{m(t)}u_h(t) - \frac{\mu}{(r_T + \lambda_1 h(t))^2} + \lambda_2 \frac{v_x(t)^2}{r_T + h(t)}, \\
\dot{m}(t) &= -\beta T_{\max}\sqrt{u_x(t)^2 + u_h(t)^2},
\end{aligned} \tag{20}$$

parameterized by $0 \leqslant \lambda_1 \leqslant 1$ and $0 \leqslant \lambda_2 \leqslant 1$. Then, we implement the following continuation procedure on the resulting family of optimal control problems. Implementing a continuation on λ_1, keeping $\lambda_2 = 0$, we first pass from the simplified flat Earth model (18) with constant gravity (for $\lambda_1 = \lambda_2 = 0$) to the intermediate flat Earth model with variable gravity (for $\lambda_1 = 1$ and $\lambda_2 = 0$). Along this first continuation it makes sense to consider free final times. Then, we implement a second continuation on the parameter λ_2, keeping $\lambda_1 = 1$, to pass continuously to the initial model (for $\lambda_1 = \lambda_2 = 1$). Along this second continuation, we fix the final time for the optimal control problems under consideration to the value obtained at the end of the first continuation.

The details of the procedure and numerical simulations are provided in [122], and comparisons are led with usual direct methods.

To end this section, it remains to explain how the change of coordinates acts onto the adjoint vector, in order to come back to the initial cylindrical coordinates after the continuation procedure. Denoting by F the change of variables from Cartesian to cylindrical coordinates, one passes from the adjoint vector in Cartesian coordinates to cylindrical coordinates by applying the transpose of the inverse of the differential of F. This is indeed a general geometric result, whose proof is provided in the appendix of [122].

4.4 Other Applications

We mention shortly two other applications of continuation methods.

Solving the Atmospheric Reentry Problem by Continuation Another approach to solve the atmospheric reentry problem of Sect. 3.1 by a shooting method, implemented in [123], consists of carrying out a continuation on the maximal value of the state constraint on the thermal flux in order to introduce this constraint step by step. The procedure automatically determines the structure of the optimal trajectory and allows one to start from the easier problem without state constraint and to introduce the constraints progressively. The theoretical foundations which allow one to take into account the change of the structure of the trajectory (and hence the number of unknowns in the shooting method) along the continuation procedure were derived in [111] for first-order state constraints and in [124] for second-order state constraints, and allow one to prove that, under some appropriate assumptions, the change in the structure of the trajectory is regular is the sense that, when a constraint becomes active along the continuation, only one boundary arc appears. Note indeed that it can happen that infinitely many boundary arcs appear; see, for instance, [32], where this phenomenon is shown to be typical for constraints of order more than or equal to three. Here however in the problem under consideration the state constraints are of order two. To take into account this change of structure along the continuation, the usual continuation procedure must be modified accordingly. For the atmospheric reentry problem with a constraint on the thermal flux, this procedure is described in details in [123] and allows one to recover in a nice way the results of [71].

General Goddard's Problem and Singular Trajectories Variants of Goddard's problems are investigated in [57, 125] for nonvertical trajectories. The control is the thrust force, and the objective is to maximize a certain final cost, typically, the final mass. Performing an analysis based on the PMP, it is proved that optimal trajectories may involve singular arcs (along which the norm of the thrust is neither zero nor maximal) that are computed and characterized. Numerical simulations are carried out, both with direct and indirect methods, demonstrating the relevance of taking into account singular arcs in the control strategy. The indirect method combines a shooting method with a continuation method. The continuation approach leads to a quadratic regularization of the problem similar to the one presented in Sect. 4.2 and is a way to tackle with the problem of nonsmoothness of the optimal control. Note that this quadratic regularization has also been used in [119]. To tackle the lack of continuity of the optimal control u, which makes difficult the application of a shooting method, the authors consider a family of optimal control problems indexed by a continuation parameter $\lambda \in [0, 1]$ with minimization criterion $\int_0^{t_f} ((1-\lambda)\|u(t)\|^2 + \lambda\|u(t)\|)\, dt$, so that the case $\lambda = 0$ corresponds to the minimization of the energy, and $\lambda = 1$ to the original problem (minimization of the consumption) under consideration in their articles.

5 Dynamical Systems Theory and Mission Design

5.1 Dynamics Around Lagrange Points

Consider the so-called circular restricted three-body problem, in which a body with negligible mass evolves in the gravitational field of two masses m_1 and m_2 (primaries) and is assumed to have circular coplanar orbits with the same period around their center of mass. The gravitational forces exerted by any other body are neglected. In the solar system this problem provides a good approximation for studying many problems. In a rotating frame the equations are

$$\ddot{x} - 2\dot{y} = \frac{\partial \Phi}{\partial x}, \qquad \ddot{y} + 2\dot{x} = \frac{\partial \Phi}{\partial y}, \qquad \ddot{z} = \frac{\partial \Phi}{\partial z}$$

with

$$\Phi(x,y,z) := \frac{x^2+y^2}{2} + (1-\mu)\big((x+\mu)^2+y^2+z^2\big)^{-1/2} + \mu\big((x-1+\mu)^2+y^2+z^2\big)^{-1/2} + \frac{\mu(1-\mu)}{2}.$$

These equations have the Jacobi first integral $J := 2\Phi - (\dot{x}^2+\dot{y}^2+\dot{z}^2)$, and hence the solutions evolve on a five-dimensional energy manifold, the topology of which determines the so-called Hill's region of possible motions (see, e.g., [126]).

It is well known that the above dynamics admit five equilibrium points called Lagrange points, the three first of which, denoted L_1, L_2, and L_3, being collinear points on the axis joining the centers of the two primaries, and the two last of which, denoted L_4 and L_5, located in an equilateral way with respect to the primaries. It must be noted that the linearized system around these equilibrium points admits eigenvalues with zero real part, and hence the study of their stability is not obvious. It follows from a generalization of a theorem of Lyapunov (due to Moser [127]) that, for a value of the Jacobi integral a bit less than the one of the Lagrange points, the solutions have the same qualitative behavior as the solutions of the linearized system around the Lagrange points. It was then established in [128] that the three collinear Lagrange points are always unstable, whereas L_4 and L_5 are stable under some conditions (that are satisfied in the solar system, for instance, for the Earth–Moon system, or for the system formed by the Sun and any other planet).

The dynamics around these Lagrange points have particularly interesting features for space mission design. Using Lyapunov–Poincaré's theorem, it is shown that there exists a two-parameter family of periodic trajectories around every Lagrange point (see [25, 128]), among which the well-known halo orbits are periodic orbits that are diffeomorphic to circles whose interest for mission design was put in evidence by Farquhar (see [129]). There exist many other families of periodic orbits (called Lissajous orbits) and quasi-periodic orbits around Lagrange points (see [130]). The invariant (stable and unstable) manifolds of these periodic orbits, consisting of all trajectories converging to the orbit (as the time tends to $\pm\infty$), are four-dimensional tubes, topologically equivalent to $S^3 \times \mathbb{R}$, in the five-dimensional energy manifold (see [131]). Hence they play the role of separatrices. Therefore they can be used for mission design and space exploration, since a trajectory starting inside such a tube (called transit orbit) stays inside this tube. It can be noted however that the invariant manifolds of halo orbits (which can be really seen as tubes) are chaotic in large time: they do not keep their aspect of tube and behave in a chaotic way, far from the halo orbit (see [126]). In contrast, the invariant manifolds of eight-shaped Lissajous orbits[4] (which are eight-shaped tubes) are numerically shown in [132] to keep their regular structure globally in time. Moreover, in the Earth–Moon system, it is shown that they permit to fly over almost all the surface of the Moon, very close to the surface (between 1500 and 5000 kilometers). These features are of particular

[4]Eight-shaped Lissajous orbits are the Lissajous orbits of the second kind, in the sense that they are diffeomorphic to a curve having the shape of an eight. They are chiefly investigated in [132].

interest in view of designing low-cost space missions to the Moon. Indeed, in the future space exploration, the Moon could serve as an intermediate point (with a lunar space station) for farther space missions.

5.2 Applications to Mission Design and Challenges

The idea of using the specific properties of the dynamics around Lagrange points in order to explore lunar regions is far from new but has recently received a renewal of interest (see, e.g., [133, 134]). In [126, 135], the authors combine the use of low-thrust propulsion with the use of the nice properties of invariant manifolds of periodic orbits around Lagrange points in order to design low-cost trajectories for space exploration. Their techniques consist of stating an optimal control problem that is numerically solved using either a direct or an indirect transcription, carefully initialized with the trajectories of the previously studied system (with no thrust). They are able to realize in such a way a reasonable compromise between fuel consumption and time of transfer, and design trajectories requiring moderate propellant mass and reaching the target within reasonable time. In these works the previously studied circular restricted three-body problem approximation is used to provide an appropriate first guess for carefully initializing an optimal control method (for instance, a shooting method) applied to a more precise model. In view of that, and having in mind the previous methodology based on continuation, it is natural to develop an optimal planification method based on the combination of the dynamics of the three-body problem with a continuation on the value of the maximal authorized thrust. This idea is used in the recent article [87], where a homotopy procedure is carried out on the maximal value of the thrust, starting from a zero value (natural dynamics) and ending with a low value of the thrust. The authors are then able to design minimal time trajectories with low thrust passing from a geostationary orbit around the Earth to a circular lunar one.

This idea opens new directions for future investigations and is a promising method for designing efficiently fuel low-consumption space missions. Although the properties of the dynamics around Lagrange points have been widely used for developing planification strategies, up to now, and up to our knowledge they have not been combined with continuation procedures that would allow one to introduce, for instance, the gravitational effects of other bodies, or values of the maximal thrust that are low or mild, or other more complex models. This is a challenge for potential future studies. Note that, in [136], the author implements a numerical continuation procedure to compute minimal-energy trajectories with low thrust steering the engine from the Earth to the Lagrange point L_1 in the Earth–Moon system, by making a continuation on the gravitational constant of the Moon. The continuation procedure is initialized with the usual Kepler transfer, in which the Moon coincides with the point L_1 and ends up with a trajectory reaching the point L_1 with a realistic gravitational effect of the Moon.

Another challenge, which is imperative to be solved within next years, is the problem of space debris cleaning. Indeed, recently it was observed a drastic growth of space debris in the space around the Earth, in particular near the SSO orbit and polar orbits with altitude between 600 and 1200 km (indeed these orbits are intensively used for Earth observation). These debris are due to former satellites that were abandoned and now cause high collision risks for future space flights. It has become an urgent challenge to clean the space at least from its biggest debris in order to stabilize the debris population; otherwise it will soon become almost impossible to launch new satellites. At present, all space agencies in the world are aware of that problem and are currently working to provide efficient solutions for designing space debris collecting missions. One of them, currently led at EADS (see [137]), consists of deorbiting five heavy debris per year, selected in a list of debris (in the LEO region) so that the required fuel consumption for the mission is minimized. The problem to be solved turns into a global optimization problem consisting of several continuous transfer problems and of a combinatorial path problem (selection of the debris and of the collecting order). It is not obvious to solve since it must combine continuous optimal control methods with combinatorial optimization and other considerations that are specific to the problem. The results of [137] (which are valuable for high-thrust engines) provide first solutions in this direction and open new problems for further investigation. For instance, it is an open problem to design efficient space cleaning missions for low-thrust engines, taking benefit of the gravitational effects due to Lagrange points and to invariant manifolds associated with their periodic orbits. Such studies can probably be carried out with appropriate continuation procedures, carefully initialized with trajectories computed from the natural dynamics of the three-body problem.

6 Conclusion and Final Remarks

Optimal Control and Trajectory Optimization Although the techniques of optimal control surveyed in this article provide a nice way to design efficient trajectories, in particular in aerospace problems, their applications require a reasonably simple model. In practice many problems remain difficult due to the complexity of real-life models. For instance, in the problem of low-thrust orbit transfer, many problems remain such as the one of taking into account the gravitational perturbations due to the Earth or the Moon, the atmospheric drag, the constraint of crossing the Van Allen barrier as quickly as possible, cone constraints on the control, eclipse constraints, taking into account the launching phase, and the insertion of the problem in a more global one, using multidisciplinary optimization. The eclipse constraint in particular may be viewed as a state constraint and can be handled by modelizing the system as a hybrid system. This problem is called the shadow cone problem. The objective is to develop necessary optimality conditions leading to efficient computation algorithms. Usual approaches are based on penalization methods, and there is a challenging problem to use rather shooting methods, based on the Pontryagin approach, which are potentially more efficient from the point of view of the convergence. Of course, all the constraints mentioned above are of different nature. Some of them can probably be treated, for instance, by using some continuation procedures, but some others are not so well adapted to the use of indirect methods, and then, according to the problem, one should then rather use direct methods (see discussions in [50, 51]). There is a compromise to be found between the complexity of the model under consideration, the robustness of that model, and the choice of an adapted numerical method to treat the problem.

The presence of numerous constraints on the controls or on the state makes the trajectography problems difficult, and in certain situations as, for instance, in the problem of atmospheric reentry, a preliminary step to optimization is an estimation of the accessible set. A challenging question is then to combine the tools of numerical optimal control and of stabilization with fine geometric techniques developed recently in nonlinear control in order to design an estimation tool of the accessible set.

In many situations the system under consideration is modeled by infinite-dimensional systems (PDEs). From the mathematical point of view, many difficulties arise, and from the numerical point of view, one has to use sophisticated techniques of numerical analysis (on meshes in particular). Applicative issues concern, for instance, motor outflows, fuel optimal management, the minimization of electromagnetic interferences or other perturbations (acoustic, thermic, etc). The modelization of these problems uses optimal control of PDEs. The numerical implementation of necessary optimality conditions (of PMP type) causes numerous difficulties, as put in evidence, e.g., in [63]. For hyperbolic equations interference phenomena appear between the mesh and high-frequency modes. Some remedies do exist, such as, for instance, high-frequency filtering or the use of multigrid methods, and the objective is to adapt and apply them to the complex systems stemming from aerospace. Note that in certain cases the system in consideration is made of a finite-dimensional system coupled with a "quite simple" partial differential equation, for instance, the problem of optimizing the trajectories of planes in order to minimize noise pollution. The model should take into account sound propagation, shock waves, and thus, wave-like PDEs. Nowadays the control of such coupled systems is a challenging problem in mathematics, and recently some first results have been published, which show how nonlinear couplings may help to recover controllability properties for the system (see [138]). Many other problems require complex models based on nonlinear PDEs: propulsion problems, thermics, aerodynamics, etc. Due to the complexity of these problems, every optimization procedure is in general impossible, and the questions that arise are in general on nonlinear control, in view of applications concerning in particular the design and dimensioning of space engines.

Pluridisciplinary Optimization In celestial mechanics many issues are still to be investigated in the very interesting field of the dynamics around Lagrange points; in particular, it should be done a precise cartography of all invariant manifolds generated by all periodic orbits (not only halo or eight-shaped orbits) around Lagrange points in view of mission design. The existence of such invariant manifolds indeed makes possible the design of low-cost interplanetary missions. The design of trajectories taking advantage of these corridors, of gravitational effects of celestial bodies of the solar system, of "swing-by" strategies, is a difficult problem related to techniques of continuous and discrete optimization (multidisciplinary optimization). It is an open challenge to design a tool combining refined techniques of nonlinear optimal control, continuation procedures, mixed optimization, and global optimization procedures.

Many problems are modeled by hybrid systems, that is, systems whose dynamics may evolve with the time and contain discrete variables. An example is the problem of shadow cone constraint, and another one is the global launcher problem in which the dynamics change whenever modules fall down. A certain number of deep theoretical results do exist on Pontryagin Maximum Principle in the hybrid case (see [6, 7]), but the question of an efficient numerical implementation is still open in general (see [16]); indeed, when one implements a version of hybrid maximum principle, one is then immediately faced with a combinatorial explosion. It is not clear how to adapt efficiently tools of pluridisciplinary optimization to that difficult problem. Besides, not so far from hybrid, the question is still open to derive a version of the PMP for general systems that are both discrete and continuous in time (for instance, control systems on time scales).

Another optimization problem is to determine the optimal placement of actuators, controls, in order to minimize or maximize a certain criterion: for instance, where the retrorockets should be optimally placed for the attitude control of a satellite, what should be the optimal shape of tailpipes in order to guarantee the best possible fluid outflow, where should the injectors be positioned in a motor to maximize combustion, etc. This kind of problem is part of the thematic of optimal design, in which the unknown is no more a vector but a domain. The problems of optimal locations of sensors or actuators in linear partial differential equations have been widely considered in engineering problems (see, e.g., [139] and references therein). Usual popular approaches consist of recasting the optimal location problem for distributed systems as an optimal control problem with an infinite-dimensional Riccati equation and then of computing approximations with optimization techniques. These techniques rely however on an exhaustive search over a predefined set of possible candidates and are limited with combinatorial difficulties due to the selection problem. We thus recover the usual flaws of combinatorial optimization methods. Many challenging problems fall into this category.

An important problem in aerospace is the optimal design of launchers. The objective is to optimize both the trajectory and the engine (launcher). In the optimal design of a launcher one may seek to optimize thrust levels, the number of floors, of tanks, or to know what type of propellant should be used. Methods usually employed consist of splitting the global problem into subproblems handled with specific methods like genetic algorithms. The development of pluridisciplinary optimization tools should provide some breakthroughs in this domain. Another very important problem, which could be treated efficiently with this kind of approach, is the problem of space cleaning mentioned previously. We indeed have at our disposal a precise catalog of fragments, wreckage, and scraps, and one of the top priorities in the next years is to clean the space from big fragments (essentially coming from old satellites). The problem is to design optimally a space vehicle able to collect in minimal time a certain number of fragments, themselves being chosen in advance in the catalog in an optimal way. This problem combines techniques of continuous optimal control in order to determine a minimal time trajectory between two successive fragments, and techniques of discrete optimization for the best possible choice of the fragments to be collected.

Inverse Problems Having available a certain number of measures, one aims at detecting flaws in the structure of an engine, which are due to shocks, thermic or electromagnetic problems. This is an inverse problem which requires a mesh adapted to the engine, adequately placed sensors providing the measures, and this problem is thus related to the aforementioned ones.

The objective may be also to reconstruct the electromagnetic, thermic, or acoustic environment of a launcher (after take-off or along the flight) from measures, in order to protect efficiently the fragile, delicate components of the launcher like computers for instance. This is a difficult inverse problem modeled with nonlinear PDEs. The challenge is to develop a tool permitting to design efficiently a launcher in order to make it more robust, less sensitive to environment perturbations. From the point of view of numerical analysis, this requires the development of mesh methods or spectral methods that are adapted to this specific problem. It can be noted that since fractional derivatives appear naturally in fluid mechanics problems (acoustic in particular), for instance, when computing a heat flux getting out from the side of a fluid outflow in function of the time evolution of the internal source, or in the modeling of viscoelastic materials, it is important to develop efficient numerical approximation schemes of fractional derivatives. It is a challenge to improve fractional methods with optimal inverse reconstruction procedures and with optimal design issues.

Finally, a last inverse problem is the one of optimal design of sensors. The problem is to determine where the sensors should be placed in order to ensure an optimal observation of the system, for instance, in view of ensuring the success of online guidance processes. The applications are numerous in aerospace, and this problem is connected to the previous ones, the measures serving also, for instance, to reconstruct the electromagnetic or thermic environment of an engine, or to detect flaws. This problem enters into the category of shape optimization problems. In a general way measures are taken to reconstruct an environment. A difficult question is to determine which measures are required in order to optimize the reconstruction and to be able to approximate in the best possible way an inverse problem.

Combined with guidance objectives, these inverse problems may probably be recast in terms of pluridisciplinary optimization, as discussed previously. This results into difficult, complex problems and raises very interesting challenges for the future.

Acknowledgements I am grateful to B. Bonnard, U. Boscain, J.-B. Caillau, J. Gergaud, F. Giannessi, D.G. Hull, A. Rapaport, and M. Sigalotti for discussions and advice.

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