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OPTIMAL CONTROL OF A SEMIDISCRETE CAHN–HILLIARD–NAVIER–STOKES SYSTEM*

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Abstract. In this paper, the optimal boundary control of a time-discrete Cahn–Hilliard–Navier–Stokes system is studied. A general class of free energy potentials is considered which, in particular, includes the double-obstacle potential. The latter homogeneous free energy density yields an optimal control problem for a family of coupled systems, which result from a time discretization of a variational inequality of fourth order and the Navier–Stokes equation. The existence of an optimal solution to the time-discrete control problem as well as an approximate version is established. The latter approximation is obtained by mollifying the Moreau–Yosida approximation of the double-obstacle potential. First order optimality conditions for the mollified problems are given, and in addition to the convergence of optimal controls of the mollified problems to an optimal control of the original problem, first order optimality conditions for the original problem are derived through a limit process. The newly derived stationarity system is related to a function space version of C-stationarity.

Key words. Cahn–Hilliard–Navier–Stokes system, double-obstacle potential, mathematical programming with equilibrium constraints, optimal boundary control, Yosida regularization, C-stationarity

AMS subject classifications. 35J87, 49J20, 34G25

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1. Introduction.

The coupled Cahn–Hilliard–Navier–Stokes (CH-NS) system is a quantitative model which describes the hydrodynamics, such as demixing or phase separation, of multiphase fluids. While the Navier–Stokes part of such a system captures the fluid dynamics over time (see, e.g., [16]), the Cahn–Hilliard model is related to an H^{-1}-gradient flow for a Ginzburg–Landau free energy, which covers the phase separation behavior [15]. Mathematically, a CH-NS system describing the hydrodynamics of a two-phase fluid flow is given by

$$\partial_t v - \frac{1}{Re}\Delta v + v\cdot\nabla v + \nabla\pi + Kc\nabla w = 0 \quad \text{in}\,\Omega_T, \tag{1.1}$$

$$\operatorname{div} v = 0 \quad \text{in}\,\Omega_T, \tag{1.2}$$

$$\partial_t c - \frac{1}{Pe}\nabla\cdot(b(c)\nabla w) + v\cdot\nabla c = 0 \quad \text{in}\,\Omega_T, \tag{1.3}$$

$$w \in \partial\Phi(c) - \gamma^2\Delta c \quad \text{in}\,\Omega_T, \tag{1.4}$$

$$c(0) = c_a, \quad v(0) = v_a \quad \text{in}\,\Omega\,\text{at}\,t=0, \tag{1.5}$$

$$\nabla c\cdot\vec{n} = 0, \quad \nabla w\cdot\vec{n} = 0 \quad \text{on}\,\partial\Omega\times(0,T), \tag{1.6}$$

$$v = r \quad \text{on}\,\partial\Omega\times(0,T). \tag{1.7}$$

Here, v denotes the velocity of the fluid and π the related pressure; c, typically with values in $[-1,1]$, is the order parameter describing the mass concentrations c_1 and c_2

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of the fluid phases; and w is the associated chemical potential. The capillary number K, the Reynolds number Re, the Péclet number Pe, and the diffusivity parameter γ are given positive constants depending on material properties. The function $b(\cdot)$ represents the mobility involved in the phase separation process. Further, the space-time cylinder is given by $\Omega_T := \Omega \times (0,T)$, i.e., it is the Cartesian product of a spatial domain $\Omega \subset \mathbb{R}^n$, $n \in \{2,3\}$, and a time interval $(0,T)$ with $T > 0$ given. By $\partial\Omega$ we denote the boundary of Ω and by $\vec{n}$ the outward unit normal on $\partial\Omega$. By r we denote some prescribed boundary velocity, and c_a and v_a are given initial data.

In the above system, which is related to Model H in [29], the mapping Φ corresponds to the homogeneous free energy density contained in the Ginzburg–Landau energy model. It is usually nonconvex for capturing spinodal decomposition. Popular physically relevant choices are the logarithmic potential $\Phi(c)(x) := \Phi_\psi(c)(x) := \psi(c(x)) - \frac{1}{2}c^2(x)$ for $\psi(c) := (1+c)\ln(1+c) + (1-c)\ln(1-c)$, which can be found, e.g., in Cahn and Hilliard's seminal paper [15] but also in the Flory–Huggins theory for describing phase separation processes in the thermodynamics of polymer solutions, and the double-obstacle potential [8, 9, 21], i.e., Φ_ψ for $\psi(c) := 0$ if $|c| \leq 1$ and ∞ otherwise, which, in the context of polymer solutions, appears appropriate to model situations of rapid wall-hardening [37]. A frequently used but possibly less relevant choice for Φ in material science is given by the double-well potential [18], i.e., Φ_ψ for $\psi(c) := c^4$. While the logarithmic and double-well potentials enjoy differentiability properties (allowing one to replace $\partial\Phi$ by the Fréchet derivative Φ'), respectively, the double-obstacle potential has a possibly genuinely set-valued derivative $\partial\Phi(c)$ at c. The latter clearly complicates the situation, both analytically and numerically, and gives rise to a variational inequality in (1.4).

For the coupled CH-NS system, results on the existence of solutions were obtained in [39] in the case $\Omega = \mathbb{R}^2$, in [12] for a periodical channel, and in [3, 2] for the general case. There are various ways in which Model H can be generalized in order to allow for fluids with different densities. Some of these models together with related analytical and numerical results are discussed in [35, 13, 17, 2, 1, 23, 20]. On the numerical level, for the CH-NS system with the double-well potential we refer to [30, 31, 32] and we refer to [11] for three-phase flows; see also [12, 13, 10, 4, 34, 33]. In [25], an adaptive solver based on reliable and efficient residual a posteriori error analysis for the double-obstacle potential was developed.

In this paper we are interested in the optimal control of the coupled CH-NS system. In this context, an objective functional $\mathcal{J}$ is minimized subject the CH-NS system, i.e., we seek to solve the problem

$$\text{minimize } \mathcal{J}(c,v,u) \text{ subject to } (1.1)\text{–}(1.6), v = u \text{ on } \partial\Omega \times (0,T), \tag{1.8}$$

where the control u is an element of a closed, linear control space U. A particular instance of U is the closed, linear subspace of the trace space containing controls operating in the direction normal to a nonempty subset Γ_c of the boundary $\partial\Omega$ of the spatial domain Ω only. We also refer to Problem 3.2 below for a time-discrete version of this optimization task (1.8).

With respect to applications the study of the above optimization problem is relevant, for instance, in the formation of polymeric membranes in the context of an immersion precipitation process. In this context, a polymer solution is immersed in a coagulation bath which contains a nonsolvent. Due to the concentration difference between the polymeric solution and the nonsolvent, the polymeric solution decomposes into two phases, a polymer-rich one and a polymer-poor one. It is well known [43] that the performance of the resulting polymer membrane depends significantly on its morphology (i.e., the porosity structure), which is the result of the phase separation process.

Optimal control problems for phase separation modeled by either the Cahn–Hilliard or the Allen–Cahn system were previously studied in [22, 41, 42, 28, 19]. In these papers, however, no coupling with other physically relevant systems occur. Concerning research on the coupled CH-NS system we mention that while some work on the analysis and numerics for the coupled CH-NS system is available as discussed above, to the best of our knowledge the literature on the optimal control of the CH-NS system is essentially void. Hence, as a first step toward the optimal control of the CH-NS system with a rather general choice of the free energy, in this paper we study the optimal control of the time-discrete version of (1.8). We further note that semidiscretization in time is a common approach toward the numerical solution of time-dependent optimal control problems. In principle, however, we emphasize that the analysis and the derivation of stationarity conditions are also of interest in the time-continuous setting. This would require an analytic framework that extends the one by Abels [2, 3] for the CH-NS system to the optimal control setting including the corresponding adjoint system. This goes beyond the scope of the present paper and remains for future research.

The rest of the paper is organized as follows. In section 2 the time-discrete CH-NS system is stated and an appropriate solution concept is introduced. Further, energy estimates are derived which are then used to prove existence of a solution of the time-discrete CH-NS system for a given control action u. With respect to the choice of the free energy, smooth as well as nonsmooth homogeneous free energy densities are possible. This covers in particular the case of the double-obstacle potential giving rise to a variational inequality. The associated semidiscrete optimal control problem is studied in section 3, where, besides existence of an optimal solution, a first order optimality system for a smooth free energy is derived. The latter includes the case of a mollified version of the Moreau–Yosida approximation of the double-obstacle potential. Finally, in section 4 a stationarity system for the double-obstacle potential is derived through a limit process of the associated stationarity system of section 3. The resulting system is of so-called C-stationarity type and is suitable for numerical realization.

1.1. Notation. In order to simplify the notation and to ease the exposition, from now on we set $Re = Pe = K = 1$, and we consider the constant mobility case only, i.e., $b(\cdot) \equiv 1$ in (1.1)–(1.6).

Further, $\mathbb{N}$ denotes the positive integers, and $\mathbb{N}^* := \mathbb{N} \cup \{\infty\}$ and $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ are the extended positive natural and real numbers, respectively. The duality pairing between a Banach space X and its dual X^* is written as $\langle .,. \rangle_X : X^* \times X \to \mathbb{R}$. As usual, strong and weak convergence are denoted by $\to$ and $\rightharpoonup$, respectively. For a Hilbert space H its inner product is given by $(.|.)_H : H \times H \to \mathbb{R}$, whereas $J_H : H \to H^*$ denotes the canonic isomorphism due to the Riesz theorem. Let $N \in \{2, 3\}$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. We define $W_0 := \{c \in L^2(\Omega) : \int_\Omega c = 0\}$, $W_1 := H^1(\Omega) \cap W_0$ with the norm $\|c\|_{W_1} := \|\nabla c\|_{L^2(\Omega)}$, and $W_{-1} := W_1^*$ and $-\Delta : W_1 \to W_{-1}$ by $\langle -\Delta w, \hat{w} \rangle_{W_1} := \int_\Omega \nabla w \cdot \nabla \hat{w}$. Furthermore, the spaces W_i for $i = 2, 3$ are given by $W_i := -\Delta^{-1}(W_{i-2})$ with the norms $\| \, . \, \|_{W_i} := \| -\Delta(\, . \,) \|_{W_{i-2}}$ and $W_{-i} := W_i^*$. Then, there is a natural embedding $W_1 \hookrightarrow W_0 \cong W_0^* \hookrightarrow W_{-1}$. Since $-\Delta$ is injective on W_1, these indeed are norms. Moreover, we set $V_1 := \{v \in H^1(\Omega; \mathbb{R}^N) : \operatorname{div} v = 0\}$, $\widetilde{V}_1 := V_1 \cap H_0^1(\Omega; \mathbb{R}^N)$, $V_0 := \widetilde{V}_0 :=$ the closure of V_1 in $L^2(\Omega; \mathbb{R}^N)$,

$V_{-1} := V_1^*$, $\widetilde{V}_{-1} := \widetilde{V}_1^*$, and $Y_i := W_i \times V_i$, $\widetilde{Y}_i := W_i \times \widetilde{V}_i$ for $i \in \{-1, 0, 1\}$. On $\widetilde{V}_1$ we use the norm $\|v\|_{\widetilde{V}_1}$ given by the functional $|\widetilde{v}|_{\widetilde{V}_1} := \|\nabla \widetilde{v}\|_{V_0}$ that is a seminorm on V_1. The other spaces are equipped with their standard norms. The space W_1^M denotes the M-times product $W_1 \times \cdots \times W_1$. Other product spaces are denoted analogously. We define $C_P := \|\operatorname{Id}\|_{\mathcal{L}(\widetilde{V}_1; V_1)}$. The subspace S_1 of $S := H^{1/2}(\partial\Omega; \mathbb{R}^N)$ is given by $S_1 := \operatorname{Tr}(V_1)$, where Tr denotes the usual zero order trace operator $\operatorname{Tr} : H^1(\Omega; \mathbb{R}^N) \to H^{1/2}(\partial\Omega; \mathbb{R}^N)$. Moreover, $S_{-1} := S_1^*$.

We fix a time step size $\tau > 0$ and define for an arbitrary vector space X and $M \in \mathbb{N}$ the operators $D^+, S_+, S_- : X^{M+1} \to X^M$ by

$$\begin{aligned} D^+(x_0, \ldots, x_M) &:= \tfrac{1}{\tau}(x_1 - x_0, \ldots, x_M - x_{M-1}), \\ S_+(x_0, \ldots, x_M) &:= (x_1, \ldots, x_M), \\ S_-(x_0, \ldots, x_M) &:= (x_0, \ldots, x_{M-1}). \end{aligned}$$

For ease of notation, we use the following convention: Whenever we add $x \in X^{M_1}$ to $y \in X^{M_2}$ for $M_2 > M_1$, this is understood in the sense $(x_0 + y_0, \ldots, x_{M_1-1} + y_{M_1-1})$, i.e., we project y onto its first M_1 components.

Remark 1.1. For the proof of existence of a solution to the semidiscrete CH-NS system below, we reduce the inhomogeneous, discretized version of the Navier–Stokes equation (2.3) below to an equation satisfying homogeneous Dirichlet boundary conditions. This will be done with the help of the operator $F \in \mathcal{L}(S_1; V_1)$ satisfying $\operatorname{Tr} \circ F = \operatorname{Id}_{S_1}$, where Id_{S_1} denotes the identity operator on S_1. Observe that such an operator F always exists. Indeed, by a result due to Heron [24], the subspace S_1 is given by $\{u \in S : \int_{\partial\Omega} u \cdot \vec{n} = 0\}$. In particular, S_1 is a Hilbert space and the trace operator regarded from V_1 into S_1 is a linear, bounded, and surjective mapping between Hilbert spaces. Consequently, there exists a right inverse operator $F \in \mathcal{L}(S_1; V_1)$ to Tr, i.e., $\operatorname{Tr} \circ F = \operatorname{Id}_{S_1}$ (cf. Aubin [5]). Moreover, we henceforth use the notation F for a right inverse as defined above.

2. The semidiscrete CH-NS system. In our analysis we rely on the following discretization schemes and the pertinent notion of solution. For the corresponding mathematical description, below we use $Y_{-1}^M = Y_{-1} \times \cdots \times Y_{-1}$ (M-times) and analogously for the other relevant spaces. Further, the actions of operators like Δ and ∇ are understood in the componentwise sense whenever applied to elements of W_1^M and V_1^M, respectively, and analogously for the multivalued operator A defined below.

Definition 2.1 (solution of the semidiscrete CH-NS system). *Let $M \in \mathbb{N}$, initial data $(c_a, v_a) \in Y_1$, a right-hand side $(f^c, f^v) \in \widetilde{Y}_{-1}^M$, a boundary value $r \in S_1^{M+1}$, and a multivalued operator $A \subset W_1 \times W_{-1}$ be given. A pair $(c, v) \in Y_1^{M+1}$ is called a solution to the semidiscrete CH-NS system with data $\mathcal{I} := (M, \tau, (c_a, v_a), (f^c, f^v), r, A)$, if there exists $w \in W_1^{M+1}$ such that $Tr\, v = r \in S_1^{M+1}$, $c_0 = c_a$, $w_0 = 0$, $v_0 = v_a$, and*

$$D^+c - \Delta S_+ w + \nabla S_- c \cdot S_+ v = f^c \quad \in W_{-1}^M, \tag{2.1}$$

$$-S_+ w - \Delta(S_+ c) - I(S_- c) + A S_+ c \ni 0 \quad \in W_{-1}^M, \tag{2.2}$$

$$D^+ v - \Delta(S_+ v) + (S_- v \cdot \nabla) S_+ v - S_+ w \nabla S_- c = f^v \quad \in \widetilde{V}_{-1}^M. \tag{2.3}$$

In the definition above and depending on the context we consider the negative Laplace operator $-\Delta$ incorporating respective boundary conditions either as a mapping from W_1 into its dual W_{-1} or from V_1 into $\widetilde{V}_{-1}$. Similarly, I represents the canonical injection either of W_1 into W_{-1} or V_1 into $\widetilde{V}_{-1}$, again depending on the context. The operators $-\Delta : W_1 \to W_{-1}$ and $I : W_1 \to W_{-1}$ are self-adjoint, whereas the adjoints of $-\Delta : V_1 \to \widetilde{V}_{-1}$ and $I : V_1 \to \widetilde{V}_{-1}$ read as follows:

$$-\Delta^* : \widetilde{V}_1 \to V_{-1}, \quad \langle -\Delta^* \widetilde{v}, v\rangle_{V_1} := \langle -\Delta v, \widetilde{v}\rangle_{\widetilde{V}_1},$$
$$I^* : \widetilde{V}_1 \to V_{-1}, \quad \langle I^* \widetilde{v}, v\rangle_{V_1} := \langle v, \widetilde{v}\rangle_{\widetilde{V}_1}.$$

The evaluation of viscosity part $(u\cdot\nabla)v$ in q for $u, v, q \in V_1$ leads to the trilinear form $b(u, v, q)$, which, depending on the context, will be written in the following ways:

$$\langle (u\cdot\nabla)v, q\rangle_{V_1} := \langle b_1(v, q), u\rangle_{V_1} := \langle b_2(u, q), v\rangle_{V_1} := b(u, v, q) := \sum_{i,j=1}^{N} \int_\Omega u_i \partial_i v_j q_j.$$

It is well known that $\langle (u\cdot\nabla)\widetilde{v}, \widetilde{v}\rangle_{V_1} = 0$ for all $u \in V_1$ and $\widetilde{v} \in \widetilde{V}_1$; see, e.g., [40, Chapter 2, Lemma 1.3].

Remark 2.1. Let us consider $b(c(x)) := 1$ and $\Phi(c)(x) := \varphi(c)(x) - \frac{1}{2}c^2(x)$ in the system (1.1)–(1.7). If we replace $\partial_t c(t, x)$ by the forward difference $\frac{1}{\tau}(S_+c(x) - S_-c(x))$ and relate all other occurrences of $c(t, x)$ either to $S_-c(x)$ or to $S_+c(x)$ and do the same for v and w, then we obtain the weak formulation (2.1)–(2.3) with $A := \partial\varphi$. Here, we assume the mean value of c to remain constant and use the fact that by formulating the discrete version of (1.1) in the space $\widetilde{V}_1^M$, the pressure term drops out. Note that discretizing the original system in this way the quadratic terms $c\nabla w$, $v\cdot\nabla v$, and $v\cdot\nabla c$ become linear in an iterative forward solver. Moreover, the nonconvex part $-\frac{1}{2}c^2$ of $\Phi(c)$ leading to the term $-I(S_-c)$ in (2.2) only contributes to the values at the "old time slice" S_-c and not to those at S_+c.

Remark 2.2. Assume that (c, v) is a solution to the semidiscrete CH-NS system with data $(M, \tau, (c_a, v_a), (f^c, f^v), r, A)$. Then it holds that

$$S_+w = (-\Delta)^{-1}(f^c - D^+c - \nabla S_-c \cdot S_+v), \tag{2.4}$$

where $(-\Delta)^{-1}$ denotes the associated solution operator. Thus, the vector w is unique. Moreover, note that the assumption $\operatorname{Tr} v = r$ implies $r_0 = \operatorname{Tr} v_a$. This condition invokes compatibility of the boundary value r at initial time and the initial value v_a of the fluid velocity.

Remark 2.3. Let $(c, w, v) \in W_1^2 \times W_1^2 \times V_1^2$ with $c_0 \in W_2$ and $f^c \in W_{-1}$.

1. If (c, w, v) solves $D^+c - \Delta S_+w + \nabla S_-c \cdot S_+v = f^c$ and if $f^c \in W_0$, so is $\nabla S_-c \cdot S_+v$ and therefore $-\Delta S_+w = f^c - D^+c - \nabla S_-c \cdot S_+v$ as well. Consequently, $w_1 \in W_2$.
2. If $-S_+w - \Delta S_+c - IS_-c + a = 0$ with $a, S_+w, S_-c \in W_i$ for $i \in \{0, 1\}$, then $-\Delta c_1 \in W_i$ and therefore $c_1 \in W_{i+2}$.

The next reformulation of the problem proves to be useful for the following existence result.

LEMMA 2.2. *The system* (2.1)–(2.3) *is equivalent to the system*

$$(-\Delta)^{-1}D^+c + (-\Delta + A)S_+c - I(S_-c) + (-\Delta)^{-1}(\nabla S_-c \cdot S_+v) - (-\Delta)^{-1}f^c \ni 0 \quad \in W_{-1}^M, \tag{2.5}$$

$$-S_+w + (-\Delta)^{-1}\big[f^c - (D^+c + \nabla S_-c \cdot S_+v)\big] = 0 \quad \in W_1^M, \tag{2.6}$$

$$D^+v - \Delta(S_+v) + (S_-v\cdot\nabla)S_+v - S_+w\nabla S_-c - f^v = 0 \quad \in V_{-1}^M. \tag{2.7}$$

Proof. This result is easily seen by applying $(-\Delta)^{-1}$ to (2.1). ∎

Definition 2.3. *We define the solution sets $\mathcal{S}(\mathcal{I}) \subset Y_1^{M+1}$ and $\mathcal{S}^w(\mathcal{I}) \subset (W_1 \times W_1 \times V_1)^{M+1}$, respectively, as*

$$\begin{aligned}\mathcal{S}(\mathcal{I}) &:= \{(c,v) : (c,v) \text{ is a solution to the semidiscrete CH-NS system for} \\ &\qquad \text{given data } \mathcal{I} = (M, \tau, (c_a, v_a), (f^c, f^v), r, A)\}, \\ \mathcal{S}^w(\mathcal{I}) &:= \{(c,w,v) : (c,v) \in \mathcal{S}(\mathcal{I}),\ w_0 = 0,\ S_+ w \text{ satisfies } (2.4)\}.\end{aligned}$$

2.1. Energy estimates I. Before establishing the existence of solutions to the semidiscrete CH-NS system we study energy estimates. Such estimates are useful in the existence proof in order to show that solutions to suitable auxilary problems yield solutions to the original problem.

Definition 2.4 (energy functionals). *For a given potential $\varphi : W_1 \to \overline{\mathbb{R}}$ and for $(c,v) \in Y_1$ we define the free energy, the kinectic energy, and the (total) energy, respectively, according to*

$$\mathrm{E}_{\mathrm{free}}(c) := \frac{1}{2}\big[\|c\|_{W_1}^2 - \|c\|_{W_0}^2\big] + \varphi(c), \qquad \mathrm{E}_{\mathrm{kin}}(v) := \frac{1}{2}\|v\|_{V_0}^2,$$
$$\mathrm{E}(c,v) := \mathrm{E}_{\mathrm{free}}(c) + \mathrm{E}_{\mathrm{kin}}(v).$$

Here and below for a convex functional φ $\partial\varphi$ denotes the subdifferential of convex analysis and $D(A)$ denotes the domain of a given operator A.

Lemma 2.5. *Assume that $\tau > 0$, $(c, \widetilde{v}) \in \widetilde{Y}_1^2$, $w_1 \in W_1$, $(f^c, f^v) \in \widetilde{Y}_{-1}$, $\varphi : W_1 \to \overline{\mathbb{R}}$ convex, $\varphi(c_0) < \infty$, $A = \partial\varphi \subset W_1 \times W_{-1}$, $B \in \mathcal{L}(V_1; \widetilde{V}_{-1})$ with $\langle B\hat{v}, \hat{v}\rangle_{\widetilde{V}_1} = 0$ for all $\hat{v} \in \widetilde{V}_1$ and*

(2.8)
$$D^+c - \Delta w_1 = f^c, \quad w_1 \in -\Delta(S_+c) - I(S_-c) + AS_+c, \quad D^+\widetilde{v} - \Delta S_+\widetilde{v} + BS_+\widetilde{v} = f^v.$$

Then for $c = (c_0, c_1)$ and $\widetilde{v} = (\widetilde{v}_0, \widetilde{v}_1)$ it holds that

$$\mathrm{E}_{\mathrm{free}}(c_1) - \mathrm{E}_{\mathrm{free}}(c_0) \le \tau\big[\langle f^c, w_1\rangle_{W_1} - \|w_1\|_{W_1}^2\big],$$
$$\mathrm{E}_{\mathrm{kin}}(\widetilde{v}_1) - \mathrm{E}_{\mathrm{kin}}(\widetilde{v}_0) \le \tau\big[\langle f^v, S_+\widetilde{v}\rangle_{\widetilde{V}_1} - \|S_+\widetilde{v}\|_{\widetilde{V}_1}^2\big].$$

Proof. The inclusion for w_1 implies $c_1 \in D(A)$ and therefore $\varphi(c_1) < \infty$. We set $v^* := w_1 - (-\Delta(S_+c) - I(S_-c))$, hence $v^* \in Ac_1 = \partial\varphi(c_1)$. Using the latter equation, (2.8), $\langle Tx, x-y\rangle \ge \frac{1}{2}[\langle Tx, x\rangle - \langle Ty, y\rangle]$ for any symmetric and positive operator $T \in \mathcal{L}(X; X^*)$, and the definition of the free energy and the given equation, we find that

$$\begin{aligned}&\mathrm{E}_{\mathrm{free}}(c_1) - \mathrm{E}_{\mathrm{free}}(c_0) \\ &\quad = \frac{1}{2}\big[\|c_1\|_{W_1}^2 - \|c_0\|_{W_1}^2\big] - \frac{1}{2}\big[\|c_1\|_{W_0}^2 - \|c_0\|_{W_0}^2\big] + \big[\varphi(c_1) - \varphi(c_0)\big] \\ &\quad \le \langle -\Delta c_1, c_1 - c_0\rangle_{W_1} + \langle -Ic_0, c_1 - c_0\rangle_{W_1} + \langle v^*, c_1 - c_0\rangle_{W_1} \\ &\quad = \tau\left\langle w_1, D^+c\right\rangle_{W_1} = \tau\left\langle D^+c, w_1\right\rangle_{W_1} = \tau\left\langle f^c + \Delta w_1, w_1\right\rangle_{W_1} \\ &\quad = \tau\big[\langle f^c, w_1\rangle_{W_1} - \|w_1\|_{W_1}^2\big].\end{aligned}$$

Analogously, it follows that

$$\begin{aligned}&\mathrm{E}_{\mathrm{kin}}(\widetilde{v}_1) - \mathrm{E}_{\mathrm{kin}}(\widetilde{v}_0) \\ &\quad = \frac{1}{2}\|\widetilde{v}_1\|_{V_0}^2 - \frac{1}{2}\|\widetilde{v}_0\|_{V_0}^2 \le \langle \widetilde{v}_1, \widetilde{v}_1 - \widetilde{v}_0\rangle_{V_1} = \tau\left\langle D^+\widetilde{v}, S_+\widetilde{v}\right\rangle_{\widetilde{V}_1} \\ &\quad = \tau\left\langle f^v + \Delta S_+\widetilde{v} - BS_+\widetilde{v}, S_+\widetilde{v}\right\rangle_{\widetilde{V}_1} = \tau\big[\langle f^v, S_+\widetilde{v}\rangle_{\widetilde{V}_1} - \|S_+\widetilde{v}\|_{\widetilde{V}_1}^2\big],\end{aligned}$$

which completes the proof. ∎

Next we provide an estimate for the total energy related to a modified version of our original problem (2.1)–(2.3) which allows a specific nonlinear coupling.

LEMMA 2.6. *Let* $\tau > 0$, $\varphi : W_1 \to \overline{\mathbb{R}}$ *be convex,* $A = \partial\varphi$, $(c, w, v) \in W_1^2 \times W_1^2 \times V_1^2$, $(f^c, f^v) \in \widetilde{Y}_{-1}$, $r \in S_1^2$, *and* $\varphi(c_0) < \infty$. *Moreover, assume we are given operators* $B \in \mathcal{L}(V_1; \widetilde{V}_{-1})$, $Q = (Q_1, Q_2) : \widetilde{Y}_1 \to \widetilde{Y}_{-1}$ *such that for all* $\hat{v} \in \widetilde{V}_1$, $\hat{w} \in W_1$

$$\langle B\hat{v}, \hat{v}\rangle_{\widetilde{V}_1} = 0, \qquad \langle Q(\hat{w}, \hat{v}), (\hat{w}, \hat{v})\rangle_{\widetilde{Y}_1} = 0.$$

For $\widetilde{v} := v - Fr \in \widetilde{V}_1^2$ *(with an implicit requirement for* v*) assume that*

$$D^+c - \Delta w_1 + Q_1(w_1, S_+\widetilde{v}) + \nabla c_0 \cdot FS_+r = f^c, \tag{2.9}$$

$$w_1 \in -\Delta(S_+c) - I(S_-c) + AS_+c, \tag{2.10}$$

$$D^+v - \Delta(S_+v) + BS_+\widetilde{v} + Q_2(w_1, S_+\widetilde{v}) = f^v. \tag{2.11}$$

Then for a constant C *depending only on* τ, Ω *and* $\|Fr\|_{V_1^2}$, *with* $z_1 := f^c - \nabla c_0 \cdot FS_+r$ *and* $z_2 := f^v - [FD^+r + FS_+r + BFS_+r]$, *it holds that*

$$\mathrm{E}(c_1, v_1) - \mathrm{E}(c_0, v_0) + \frac{\tau}{2}\big[\|w_1\|_{W_1}^2 + \|S_+\widetilde{v}\|_{\widetilde{V}_1}^2\big] \le C(\|z_1\|_{W_{-1}}^2 + \|z_2\|_{\widetilde{V}_{-1}}^2 + \|\widetilde{v}_0\|_{V_0}^2 + 1).$$

Proof. Since $\widetilde{v} \in \widetilde{V}_1$ satisfies

$$D^+\widetilde{v} - \Delta S_+\widetilde{v} + BS_+\widetilde{v} = z_2 - Q_2(w_1, S_+\widetilde{v}),$$

it follows from Lemma 2.5 that

$$\begin{aligned}
\mathrm{E}_{\mathrm{free}}(c_1) - \mathrm{E}_{\mathrm{free}}(c_0) &\le \tau\big[\langle z_1, w_1\rangle_{W_1} - \langle Q_1(w_1, S_+\widetilde{v}), w_1\rangle_{W_1} - \|w_1\|_{W_1}^2\big],\\
\mathrm{E}_{\mathrm{kin}}(\widetilde{v}_1) - \mathrm{E}_{\mathrm{kin}}(\widetilde{v}_0) &= \tau\big[\langle z_2, S_+\widetilde{v}\rangle_{\widetilde{V}_1} - \langle Q_2(w_1, S_+\widetilde{v}), S_+\widetilde{v}\rangle_{\widetilde{V}_1} - \|S_+\widetilde{v}\|_{\widetilde{V}_1}^2\big].
\end{aligned}$$

In order to pass from $\mathrm{E}_{\mathrm{kin}}(\widetilde{v}_i)$ to $\mathrm{E}_{\mathrm{kin}}(v_i)$ we use

$$\begin{aligned}
&2\big[\mathrm{E}_{\mathrm{kin}}(v_1) - \mathrm{E}_{\mathrm{kin}}(v_0)\big]\\
&\quad= 2\big[\mathrm{E}_{\mathrm{kin}}(\widetilde{v}_1 + Fr_1) - \mathrm{E}_{\mathrm{kin}}(\widetilde{v}_0 + Fr_0)\big] = \|\widetilde{v}_1 + Fr_1\|_{V_0}^2 - \|\widetilde{v}_0 + Fr_0\|_{V_0}^2\\
&\quad= (\widetilde{v}_1 + \widetilde{v}_0 + F(r_0 + r_1)|\widetilde{v}_1 - \widetilde{v}_0 + F(r_1 - r_0))_{V_0}\\
&\quad\le \|\widetilde{v}_1\|_{V_0}^2 - \|\widetilde{v}_0\|_{V_0}^2 + 2(\|\widetilde{v}_1\|_{V_0} + \|\widetilde{v}_0\|_{V_0})(\|Fr_0\|_{V_0} + \|Fr_1\|_{V_0})\\
&\qquad + (\|Fr_0\|_{V_0} + \|Fr_1\|_{V_0})^2\\
&\quad\le 2\big[\mathrm{E}_{\mathrm{kin}}(\widetilde{v}_1) - \mathrm{E}_{\mathrm{kin}}(\widetilde{v}_0)\big] + C(\|\widetilde{v}_1\|_{V_0} + \|\widetilde{v}_0\|_{V_0} + 1)\\
&\quad\le 2\big[\mathrm{E}_{\mathrm{kin}}(\widetilde{v}_1) - \mathrm{E}_{\mathrm{kin}}(\widetilde{v}_0)\big] + \tfrac{\tau}{2}\|\widetilde{v}_1\|_{\widetilde{V}_1}^2 + C(\|\widetilde{v}_0\|_{V_0}^2 + 1)
\end{aligned}$$

for constants C depending only on Ω, τ and $\|Fr\|_{V_0^2}$. From this relation we obtain the estimate

$$\begin{aligned}
&\mathrm{E}(c_1, v_1) - \mathrm{E}(c_0, v_0)\\
&\quad\le \mathrm{E}_{\mathrm{free}}(c_1) - \mathrm{E}_{\mathrm{free}}(c_0) + \mathrm{E}_{\mathrm{kin}}(\widetilde{v}_1) - \mathrm{E}_{\mathrm{kin}}(\widetilde{v}_0) + \tfrac{\tau}{4}\|\widetilde{v}_1\|_{V_1}^2 + C(\|\widetilde{v}_0\|_{V_0}^2 + 1)\\
&\quad\le \tau\big[\langle z_1, w_1\rangle_{W_1} + \langle z_2, S_+\widetilde{v}\rangle_{\widetilde{V}_1} - \|w_1\|_{W_1}^2 - \tfrac{3}{4}\|S_+\widetilde{v}\|_{\widetilde{V}_1}^2\big] + C(\|\widetilde{v}_0\|_{V_0}^2 + 1)\\
&\quad\le -\tfrac{\tau}{2}\big[\|w_1\|_{W_1}^2 + \|\widetilde{v}_1\|_{\widetilde{V}_1}^2\big] + C(\|z_1\|_{W_{-1}}^2 + \|z_2\|_{\widetilde{V}_{-1}}^2 + \|\widetilde{v}_0\|_{V_0}^2 + 1).
\end{aligned}$$

This yields the assertion. ∎

COROLLARY 2.7. *Let the assumptions of Lemma* 2.6 *be satisfied. If the functional* $\varphi : W_1 \to \mathbb{R}$ *satisfies*

$$\varphi(f) - \tfrac{1}{2}\|f\|_{W_0}^2 \;\geq\; C_\varphi \tag{H$_1$}$$

for a constant $C_\varphi \in \mathbb{R}$ *and every* $f \in W_1$, *then there exists a constant* $m > 0$ *depending on* $\tau, \Omega, c_0, v_0, \varphi(c_0), C_\varphi$, *and* Fr *such that every solution* (c, w, v) *to* (2.9)–(2.11) *satisfies*

$$\|w_1\|_{W_1}^2 + \|\widetilde{v}_1\|_{\widetilde{V}_1}^2 \;\leq\; m^2.$$

Proof. This is a direct consequence of Lemma 2.6 and $\mathrm{E}(c_1, v_1) \geq C_\varphi$. ∎

2.2. Existence of solutions to the semidiscrete CH-NS system. The existence of a solution to the semidiscrete CH-NS system for one time step is studied next. For an arbitrary finite number of steps M, this result will be applied iteratively in the proof of Theorem 2.11 below.

In the existence proof we utilize results on several classes of operators. For the reader's convenience we briefly recall the definitions of these classes. A multivalued operator $A \subset X \times X^*$ mapping a Banach space X into its dual is called *(strongly) monotone* if there exists a constant $\alpha \geq 0$ ($\alpha > 0$) such that

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle_X \geq \alpha \|x_1 - x_2\|_X^2$$

for all $(x_1, x_1^*), (x_2, x_2^*) \in A$, and it is *maximal monotone* if it is maximal among all monotone operators. A single-valued operator $A : X \to X^*$ is *pseudomonotone* if and only if for every sequence (x_n) in X with $x_n \rightharpoonup x$ the implication

$$\overline{\lim}\, \langle Ax_n, x_n - x \rangle \leq 0 \qquad \Longrightarrow \qquad \langle Ax, x - v \rangle_X \leq \underline{\lim}\, \langle Ax_n, x_n - v \rangle_X$$

is satisfied for every $v \in X$. Here and below, $\underline{\lim}$ and $\overline{\lim}$ denote the limit inferior and the limit superior, respectively. Finally, $A : X \to X^*$ is said to be *totally continuous* if $x_n \rightharpoonup x$ in X implies $Ax_n \to Ax$ in X^*. Thus, every totally continuous operator is pseudomonotone.

PROPOSITION 2.8. *Let* $(c_a, v_a) \in Y_1$, $(\hat{g}^c, \hat{g}^v) \in \widetilde{Y}_{-1}$, *and* $r \in S_1^2$ *with* $Tr\, v_a = r_0$. *Assume that* $A \subset W_1 \times W_{-1}$ *is maximal monotone and* $R = (R_1, R_2) : \widetilde{Y}_1 \to \widetilde{Y}_{-1}$ *is pseudomonotone and bounded such that for some constant* C_1 *and all* $(\hat{c}, \hat{v}) \in \widetilde{Y}_1$ *it holds that*

$$\langle R(\hat{c}, \hat{v}), (\hat{c}, \hat{v}) \rangle_{\widetilde{Y}_1} \geq -C_1(1 + \|\hat{c}\|_{W_1} + \|\hat{v}\|_{\widetilde{V}_1}). \tag{2.12}$$

Moreover, suppose that the operators $\mathcal{A}_1, \mathcal{B}^{v_a} : Y_1 \to Y_{-1}$ *are defined for all* $(\hat{c}, \hat{v}) \in Y_1$ *by*

$$\mathcal{A}_1(\hat{c}, \hat{v}) := \big((\tfrac{1}{\tau}(-\Delta)^{-1} - \Delta)\hat{c}, (\tfrac{1}{\tau} - \Delta)\hat{v}\big), \quad \mathcal{B}^{v_a}(\hat{c}, \hat{v}) := \big(0, (v_a \cdot \nabla)\hat{v}\big).$$

Then there exists a pair $(c, v) \in Y_1^2$ *such that*

$$Tr\, v = r,\; c_0 = c_a,\; v_0 = v_a, \tag{2.13}$$

$$(-\Delta)^{-1} D^+ c + (-\Delta + A) S_+ c - I S_- c + R_1(S_+ c, S_+ v - F S_+ r) \ni \hat{g}^c, \tag{2.14}$$

$$D^+ v - \Delta(S_+ v) + (S_- v \cdot \nabla) S_+ v + R_2(S_+ c, S_+ v - F S_+ r) = \hat{g}^v. \tag{2.15}$$

Proof. Using $D^+c = \frac{1}{\tau}(S_+c - S_-c)$ and an analogue relation for v, the pair (c, v) solves (2.14)–(2.15) if and only if

$$(\tfrac{1}{\tau}(-\Delta)^{-1} - \Delta + A)S_+c + R_1(S_+c, S_+v - FS_+r) \ \ni\ \hat{g}^c + (\tfrac{1}{\tau}(-\Delta)^{-1} + I)S_-c,$$
$$(\tfrac{1}{\tau} - \Delta)S_+v + (S_-v \cdot \nabla)S_+v + R_2(S_+c, S_+v - FS_+r) = \hat{g}^v + \tfrac{1}{\tau}S_-v.$$

If we set $(\hat{c}, \hat{v}) := (S_+c, S_+v - FS_+r) \in \widetilde{Y}_1$ and $(g^c, g^v) := \big(\hat{g}^c + (\frac{1}{\tau}(-\Delta)^{-1} + I)S_-c,\ \hat{g}^v + \frac{1}{\tau}S_-v - \big[(\frac{1}{\tau} - \Delta)FS_+r + (S_-v \cdot \nabla)FS_+r\big]\big) \in Y_{-1}$ and define the operator $\mathcal{A}_2 \subset Y_1 \times Y_{-1}$ by $\mathcal{A}_2(\hat{c}, \hat{v}) := (A\hat{c}, 0)$, then $(c, v) \in Y_1^2$ is a solution of (2.13)–(2.15) if and only if $(\hat{c}, \hat{v}) \in Y_1$ satisfies $\hat{v} \in \widetilde{V}_1$ and

$$(\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{B}^{v_a} + R)(\hat{c}, \hat{v}) \ni (g^c, g^v),$$

which we regard as an equation in $\widetilde{Y}_{-1}$. The operator $\mathcal{A}_1$ is strongly monotone, $\mathcal{A}_2$ is maximal monotone, and the operators $\mathcal{B}^{v_a}$ and R are pseudomonotone and bounded. These properties remain valid if we restrict these operators and regard them as mappings from $\widetilde{Y}_1$ into $\widetilde{Y}_{-1}$. Moreover, $\mathcal{A}_1 + \mathcal{B}^{v_a} + R$ is coercive on $\widetilde{Y}_1$. Therefore, Browder's theorem [14] implies that $\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{B}^{v_a} + R \subset \widetilde{Y}_1 \times \widetilde{Y}_{-1}$ is surjective. This finishes the proof. ∎

In the following lemma we study properties of the terms coupling the Cahn–Hilliard and Navier–Stokes systems.

LEMMA 2.9. *Let $c_b \in W_1$ be given. Then the operator $P = (P_1, P_2) : Y_1 \to Y_{-1}$ defined by*

$$P_1(\hat{w}, \hat{v}) := \nabla c_b \cdot \hat{v}, \quad P_2(\hat{w}, \hat{v}) := -\hat{w}\nabla c_b$$

is bilinear, bounded, and totally continuous and satisfies $\langle P(\hat{w}, \hat{v}), (\hat{w}, \hat{v})\rangle_{Y_1} = 0$ for $(\hat{w}, \hat{v}) \in Y_1$.

Proof. By Sobolev's embedding theorem and since $N \le 3$, the mappings $(\hat{c}, \hat{v}) \mapsto \nabla\hat{c} \cdot \hat{v} : W_1 \times V_1 \to W_{-1}$ and $(\hat{c}, \hat{w}) \mapsto \hat{w}\nabla\hat{c} : W_1 \times W_1 \to V_{-1}$ are bilinear, bounded, and compact in both components. Thus, P is totally continuous and bounded and satisfies $\langle P(\hat{w}, \hat{v}), (\hat{w}, \hat{v})\rangle_{Y_1} = 0$. ∎

PROPOSITION 2.10. *Suppose we are given $0 < \tau$, $(c_a, v_a), (c_b, v_b) \in Y_1$, $(f^c, f^v) \in \widetilde{Y}_{-1}$, $r \in S_1^2$ with $Tr\, v_a = r_0$, a proper, convex, and lower-semicontinuous functional $\varphi : W_1 \to \mathbb{R}$ satisfying* (H$_1$) *for a constant $C_\varphi \in \mathbb{R}$ and $\varphi(c_a) < \infty$. Let $A := \partial\varphi \subset W_1 \times W_{-1}$. Then there exists a triple $(c, w, v) \in (W_1 \times W_1 \times V_1)^2$ such that $c_0 = c_a$, $w_0 = 0$, $v_0 = v_a$, $Tr\, v = r$, and*

$$D^+c - \Delta w_1 + \nabla c_b \cdot S_+v = f^c,$$
$$w_1 \in (-\Delta + A)S_+c - IS_-c,$$
$$D^+v - \Delta S_+v + (v_b \cdot \nabla)S_+v - w_1\nabla c_b = f^v.$$

In particular, if $(c_a, v_a) = (c_b, v_b)$, then $(c, w, v) \in \mathcal{S}^w(1, \tau, (c_a, v_a), (f^c, f^v), r, A)$.

Proof. The proof is decomposed into several steps. First we show the boundedness of solutions independently of the involved operators followed by the precise construction of the associated operators. In step 3 the total continuity of one of these operators is proved. The last two steps show applicability of Proposition 2.8 and the fact that we obtain a solution of the original problem by our proof technique.

1. Assume that we are given an operator $Q = (Q_1, Q_2) : \widetilde{Y}_1 \to \widetilde{Y}_{-1}$ satisfying $\langle Q(\hat{w}, \hat{v}), (\hat{w}, \hat{v})\rangle_{\widetilde{Y}_1} = 0$ for all $(\hat{w}, \hat{v}) \in \widetilde{Y}_1$. By Corollary 2.7 there exists a constant

$m > 0$ depending on $\tau, \Omega, Fr, c_a, v_a, f^c$, and f^v such that every solution $(c, w, v) \in (W_1 \times W_1 \times V_1)^2$, $\widetilde{v} := v - Fr \in \widetilde{V}_1$, to

$$\begin{gathered}\operatorname{Tr} v = r,\ c_0 = c_a,\ v_0 = v_a,\\ D^+c - \Delta w_1 + Q_1(w_1, S_+\widetilde{v}) + \nabla c_b \cdot FS_+r = f^c, \quad w_1 \in -\Delta(S_+c) - I(S_-c) + AS_+c\\ D^+v - \Delta(S_+v) + (v_b \cdot \nabla)S_+v + Q_2(w_1, S_+\widetilde{v}) = f^v,\end{gathered}$$

satisfies the estimate

$$\|w_1\|_{W_1}^2 + \|S_+\widetilde{v}\|_{\widetilde{V}_1}^2 \ \le\ m^2.$$

In particular, m does not depend on Q. Moreover, with $P_1(\hat{w}, \hat{v}) := \nabla c_b \cdot \hat{v}$ and $P_2(\hat{w}, \hat{v}) := -\hat{w}\nabla c_b$ according to Lemma 2.9, for $(\hat{w}, \hat{v}) \in \widetilde{Y}_1$ we have that

$$\begin{aligned}\|P_1(\hat{w}, \hat{v})\|_{W_{-1}} \ &=\ \|\nabla c_b \cdot \hat{v}\|_{W_{-1}} \ \le\ \beta_1\|\hat{v}\|_{\widetilde{V}_1},\\ \|P_2(\hat{w}, \hat{v})\|_{\widetilde{V}_{-1}} \ &=\ \|-\hat{w}\nabla c_b\|_{\widetilde{V}_{-1}} \ \le\ \beta_2\|\hat{w}\|_{W_1}\end{aligned}$$

for constants $\beta_1, \beta_2 > 0$ depending on Ω and c_b. The operator $-\Delta : W_1 \to W_{-1}$ is strongly monotone with constant 1.

2. We define a function $d : \mathbb{R} \to \mathbb{R}$ and a norm $|||.|||$ on $\widetilde{Y}_{-1}$ by

$$d(t) := \begin{cases} 1 & \text{if } t \le 1,\\ 2 - t & \text{if } 1 < t < 2,\\ 0 & \text{if } 2 \le t,\end{cases}$$

$$|||(w^*, v^*)||| := \frac{1}{2m}\left(\frac{1}{\beta_1}\|w^*\|_{W_{-1}} + \min\left(\frac{1}{\beta_2}, \frac{1}{4\beta_1\beta_2}\right)\|v^*\|_{\widetilde{V}_{-1}}\right).$$

Furthermore, we define the operators $Q = (Q_1, Q_2) : \widetilde{Y}_1 \to \widetilde{Y}_{-1}$ and $Q_1^{\hat{v}}, M_{\hat{v}} : W_1 \to W_{-1}$ for $\hat{v} \in \widetilde{V}_1$ as

$$\begin{aligned}Q(\hat{w}, \hat{v}) &:= d\big(|||P(\hat{w}, \hat{v})|||\big)P(\hat{w}, \hat{v}),\\ Q_1^{\hat{v}}\hat{w} &:= Q_1(\hat{w}, \hat{v}),\\ M_{\hat{v}}\hat{w} &:= -\Delta\hat{w} + Q_1^{\hat{v}}\hat{w}.\end{aligned}$$

The operator Q inherits the property $\langle Q(\hat{w}, \hat{v}), (\hat{w}, \hat{v})\rangle_{\widetilde{Y}_1} = 0$ from P. For $\hat{w}_1, \hat{w}_2 \in W_1$ and $\hat{v} \in \widetilde{V}_1$ we have

$$\begin{aligned}\|Q_1^{\hat{v}}\hat{w}_1 - Q_1^{\hat{v}}\hat{w}_2\|_{W_{-1}} &= \|d\big(|||P(\hat{w}_1, \hat{v})|||\big)P_1(\hat{w}_1, \hat{v}) - d\big(|||P(\hat{w}_2, \hat{v})|||\big)P_1(\hat{w}_2, \hat{v})\|_{W_{-1}}\\ &= \big|d\big(|||P(\hat{w}_1, \hat{v})|||\big) - d\big(|||P(\hat{w}_2, \hat{v})|||\big)\big|\, \|P_1(\hat{w}_1, \hat{v})\|_{W_{-1}},\end{aligned}$$

since $P_1(\hat{w}_1, \hat{v}) = P_1(\hat{w}_2, \hat{v}) = P_1(0, \hat{v})$. If $\|P_1(0, \hat{v})\|_{W_{-1}} \ge 4\beta_1 m$, then $|||P(\hat{w}_i, \hat{v})||| \ge |||P(0, \hat{v})||| \ge 2$ and therefore $d\big(|||P(\hat{w}_i, \hat{v})|||\big) = 0$. Consequently, we continue the above estimation by

$$\begin{aligned}&\|Q_1^{\hat{v}}\hat{w}_1 - Q_1^{\hat{v}}\hat{w}_2\|_{W_{-1}}\\ &\quad\le \Big|\,|||P(\hat{w}_1, \hat{v})||| - |||P(\hat{w}_2, \hat{v})|||\,\Big|\, 4\beta_1 m\\ &\quad= 4\beta_1 m\Big|\frac{1}{2m}\min\left(\frac{1}{\beta_2}, \frac{1}{4\beta_1\beta_2}\right)\big[\|P_2(\hat{w}_1, \hat{v})\|_{\widetilde{V}_{-1}} - \|P_2(\hat{w}_2, \hat{v})\|_{\widetilde{V}_{-1}}\big]\Big|\\ &\quad\le \frac{1}{2}\|\hat{w}_1 - \hat{w}_2\|_{W_1},\end{aligned} \tag{2.16}$$

where we use the Lipschitz continuity of d with modulus 1 for the first inequality and the reverse triangle inequality for (2.16). From this we infer that $Q_1^{\hat{v}} : W_1 \to W_{-1}$ is Lipschitz continuous with constant $\frac{1}{2}$. Thus $M_{\hat{v}} : W_1 \to W_{-1}$ is also Lipschitz continuous and strongly monotone with constant $\frac{1}{2}$. Therefore, the mapping $S_1 : \widetilde{Y}_1 \to W_1$ given by

$$S_1(\hat{c}, \hat{v}) \;:=\; M_{\hat{v}}^{-1}\big[f^c - \tfrac{1}{\tau}(\hat{c} - c_a) - \nabla c_b \cdot FS_+ r\big]$$

is well defined.

3. Now we show that S_1 is totally continuous. For this purpose assume that $(\hat{c}_n, \hat{v}_n) \rightharpoonup (\hat{c}, \hat{v})$ in $\widetilde{Y}_1$ as $n \to \infty$. Then $f_n := f^c - \frac{1}{\tau}(\hat{c}_n - c_a) - \nabla c_b \cdot FS_+ r$ converges strongly to $f := f^c - \frac{1}{\tau}(\hat{c} - c_a) - \nabla c_b \cdot FS_+ r$ in W_{-1} due to the compact embedding of W_1 into W_{-1}. With $\hat{w}_n := S_1(\hat{c}_n, \hat{v}_n) = M_{\hat{v}_n}^{-1} f_n$, which implies $-\Delta \hat{w}_n + Q_1^{\hat{v}_n} \hat{w}_n = f_n$, and

$$\begin{aligned}\|Q_1^{\hat{v}_n} \hat{w}_n\|_{W_{-1}} &= \|d\big(|||P(\hat{w}_n, \hat{v}_n)|||\big) P_1(\hat{w}_n, \hat{v}_n)\|_{W_{-1}} \;\le\; 1 \cdot \|P_1(\hat{w}_n, \hat{v}_n)\|_{W_{-1}} \\ &\le \beta_1 \|\hat{v}_n\|_{\widetilde{V}_1}\end{aligned}$$

we obtain that $(Q_1^{\hat{v}_n} \hat{w}_n)$ and $(-\Delta \hat{w}_n)$ are bounded in W_{-1} and $(\hat{w}_n)$ is bounded in W_1. For any weakly converging subsequence $(\hat{w}_m)$ of $(\hat{w}_n)$ with weak limit $\hat{w} \in W_1$ it holds that $Q_1^{\hat{v}_m} \hat{w}_m \to Q_1^{\hat{v}} \hat{w}$ since P is totally continuous by Lemma 2.9. Consequently,

$$\begin{aligned}\|\hat{w}_m - \hat{w}\|_{W_1}^2 &\le \langle -\Delta(\hat{w}_m - \hat{w}), \hat{w}_m - \hat{w}\rangle_{W_1} \\ &= \langle f_m - f, \hat{w}_m - \hat{w}\rangle_{W_1} - \Big\langle Q_1^{\hat{v}_m} \hat{w}_m - Q_1^{\hat{v}} \hat{w}, \hat{w}_m - \hat{w}\Big\rangle_{W_1} \\ &\to 0 \quad \text{as } m \to \infty.\end{aligned}$$

Since $\hat{w}$ is the unique solution to $-\Delta \hat{w} + Q_1^{\hat{v}} \hat{w} = f$, S_1 must be totally continuous.

4. Consider the operator $R = (R_1, R_2) : \widetilde{Y}_1 \to \widetilde{Y}_{-1}$ given by

$$\begin{aligned}R_1(\hat{c}, \hat{v}) &:= (-\Delta)^{-1} Q_1(S_1(\hat{c}, \hat{v}), \hat{v}), \\ R_2(\hat{c}, \hat{v}) &:= Q_2(S_1(\hat{c}, \hat{v}), \hat{v}).\end{aligned}$$

We show now that R satisfies the assumptions of Proposition 2.8. We start by establishing the boundedness of R and (2.12). We use $d\big(|||P(\hat{w}, \hat{v})|||\big) = 0$ if $|||P(\hat{w}, \hat{v})||| \ge 2$ and $|||P(\hat{w}, \hat{v})||| \ge \gamma \|P(\hat{w}, \hat{v})\|_{\widetilde{Y}_{-1}}$ for $\gamma := \frac{1}{2m} \min(\frac{1}{\beta_1}, \min(\frac{1}{\beta_2}, \frac{1}{4\beta_1\beta_2}))$ to estimate

$$\|Q(\hat{w}, \hat{v})\|_{\widetilde{Y}_{-1}} \;=\; \big|d\big(|||P(\hat{w}, \hat{v})|||\big)\big|\, \|P(\hat{w}, \hat{v})\|_{\widetilde{Y}_{-1}} \;\le\; \frac{2}{\gamma}.$$

Consequently, R is bounded and satisfies (2.12). In order to realize that R is pseudomonotone it suffices to note that with S_1 and P also Q and R are totally continuous, which implies that R is indeed pseudomonotone.

5. Note that A is maximal monotone by assumption. Therefore, by Proposition 2.8 there exists a pair $(c, v) \in Y_1^2$ satisfying for $\widetilde{v} := v - Fr$

$$\begin{gathered}\operatorname{Tr} v = r,\; c_0 = c_a,\; v_0 = v_a, \\ (-\Delta)^{-1} D^+ c + (-\Delta + A) S_+ c - I S_- c + R_1(S_+ c, S_+ \widetilde{v}) \ni (-\Delta)^{-1}(f^c - \nabla c_b \cdot FS_+ r), \\ D^+ v - \Delta(S_+ v) + (v_b \cdot \nabla) S_+ v + R_2(S_+ c, S_+ \widetilde{v}) = f^v.\end{gathered}$$

Let us define $w := S_1(S_+c, S_+\widetilde{v})$. Thus we obtain

$$D^+c - \Delta w + Q_1(w, S_+\widetilde{v}) + \nabla c_b \cdot FS_+r \;=\; \frac{1}{\tau}(S_+c - c_a) + M_{S_+\widetilde{v}}w + \nabla c_b \cdot FS_+r \;=\; f^c,$$

which furthermore shows that

$$\begin{aligned} w = (-\Delta)^{-1}(-\Delta w) \;&=\; (-\Delta)^{-1}(f^c - \nabla c_b \cdot FS_+r) - (-\Delta)^{-1}D^+c - R_1(\overline{c}, S_+\widetilde{v}) \\ &\in (-\Delta + A)S_+c - IS_-c. \end{aligned}$$

Therefore, by (2.9), (2.11), and Corollary 2.7 we infer

$$\|w\|^2_{W_1} + \|S_+\widetilde{v}\|^2_{\widetilde{V}_1} \;\le\; m^2.$$

This implies

$$|||P(w, S_+\widetilde{v})||| \le \frac{1}{2m}\left(\frac{1}{\beta_1}\beta_1\|S_+\widetilde{v}\|_{\widetilde{V}_1} + \frac{1}{\beta_2}\beta_2\|w\|_{W_1}\right) \le \frac{1}{2m}\sqrt{2(\|w\|^2_{W_1} + \|S_+\widetilde{v}\|^2_{\widetilde{V}_1})} \le 1.$$

Hence, from the definition of Q we conclude $d(|||P(w, S_+\widetilde{v})|||) = 1$ and thus $Q(w, S_+\widetilde{v}) = P(w, S_+\widetilde{v})$. This proves the assertion. ∎

THEOREM 2.11 (existence of solutions of the semidiscrete CH-NS system). *For every $(M, \tau, (c_a, v_a), (f^c, f^v), r) \in \mathbb{N}\times\mathbb{R}\times Y_1\times\widetilde{Y}^M_{-1}\times S^{M+1}_1$ with $\mathrm{Tr}\, v_a = r_0$ and $0 < \tau$, and for every proper, convex, and lower-semicontinuous functional $\varphi : W_1 \to \overline{\mathbb{R}}$ satisfying* (H_1) *for a constant $C_\varphi \in \mathbb{R}$ and $\varphi(c_a) < \infty$, the set $\mathcal{S}(M, \tau, (c_a, v_a), (f^c, f^v), r, \partial\varphi)$ is nonempty.*

Proof. We prove this theorem by induction over M. For $M = 0$, the assertion is immediate. Hence, we consider $(M+1, \tau, (c_a, v_a), (f^c, f^v), r) \in \mathbb{N}\times\mathbb{R}\times Y_1\times Y^{M+1}_{-1}\times S^{M+1}_1$ with $\mathrm{Tr}\, v_a = r_0$ and $0 < \tau$.

Assuming that we have $(c^1, v^1) \in \mathcal{S}(M, \tau, (c_a, v_a), (S_-f^c, S_-f^v), S_-r, \partial\varphi)$, by Proposition 2.10, there exists $(c^2, v^2) \in \mathcal{S}(1, \tau, (c^1_M, v^1_M), (f^c_{M+1}, f^v_{M+1}), (r_M, r_{M+1}), \partial\varphi)$. Therefore $(c, v) := ((c^1_0, \ldots, c^1_M, c^2_1), (v^1_0, \ldots, v^1_M, v^2_1))$ is an element of the solution set $\mathcal{S}(M+1, \tau, (c_a, v_a), (f^c, f^v), r, \partial\varphi)$. ∎

2.3. Energy estimates II. In order to pass to the limit in the semidiscrete CH-NS system with approximating sequences, we need some a priori estimates for the energy. These are proved next.

PROPOSITION 2.12. *Consider $M \in \mathbb{N}$, $\tau > 0$, the initial data $(c_a, v_a) \in Y_1$, as well as bounds $C_r, C_\varphi, C_{c_a} \in \mathbb{R}$. Then there exists a constant C depending only on $\Omega, \tau, M, (c_a, v_a), C_r, C_\varphi$, and C_{c_a} such that for all $r \in S^{M+1}_1$ with $\|Fr\|_{V_1^{M+1}} \le C_r$, and all convex functionals $\varphi : W_1 \to \overline{\mathbb{R}}$ satisfying* (H_1) *with constant C_φ, and $\varphi(c_a) \le C_{c_a}$, every solution $(c, w, v) \in \mathcal{S}^w(M, \tau, (c_a, v_a), 0, r, \partial\varphi)$ is bounded such that $\|(c, w, v)\|_{(W_1\times W_1\times V_1)^{M+1}} \le C$ and $\max\{\mathrm{E}(c_i, v_i) : i = 0, \ldots, M\} \le C$.*

Proof. We prove the proposition by induction over M. For $M = 0$, the assertion is obvious. Now, suppose it is valid for some $M \in \mathbb{N}$. Let a solution $(c, w, v) \in \mathcal{S}^w(M+1, \tau, (c_a, v_a), 0, r, \partial\varphi)$ be given with $r \in S^{M+2}$ and $\|Fr\|_{V_1^{M+2}} \le C_r$ and with $\varphi : W_1 \to \overline{\mathbb{R}}$ convex such that $\varphi(f) - \frac{1}{2}\|f\|^2_{W_0} \ge C_\varphi$ and $\varphi(c_a) \le C_{c_a}$. By our induction hypothesis there exists a constant C_1 depending only on $\Omega, \tau, M, (c_a, v_a), C_r, C_\varphi$, and C_{c_a} such that $\|(c, w, v)\|_{X^{M+1}} \le C_1$ and $\max\{\mathrm{E}(c_i, v_i) : i = 0, \ldots, M\} \le C_1$. Here, we use the definition $X := W_1 \times W_1 \times V_1$ and the notation $\|(c, w, v)\|_{X^{M+1}} = \|((c_0, \ldots, c_M), (w_0, \ldots, w_M), (v_0, \ldots, v_M))\|_{X^M}$.

By Lemma 2.6 and with $\widetilde{v} := v - Fr$, $Bv := (v_M \cdot \nabla)v$, and $Q := P$ given in Lemma 2.9 (with $c_b := c_M$) there exists a constant C_2 depending only on Ω, τ and C_r with

$$\begin{aligned}\mathrm{E}(c_{M+1}, v_{M+1}) &+ \tfrac{\tau}{2}(\|w_{M+1}\|_{W_1}^2 + \|\overline{\widetilde{v}}_M\|_{\widetilde{V}_1}^2)\\ &\le \mathrm{E}(c_M, v_M) + C_2(\|(c_M, v_M)\|_{Y_1}^2 + 1)\\ &\le C_1 + C_1^2 C_2 + C_2 \;=:\; C_3,\end{aligned}$$

where we use that $\|BFr_{M+1}\|_{\widetilde{V}_{-1}} = \|(v_M \cdot \nabla)Fr_{M+1}\|_{\widetilde{V}_{-1}} \le C\|v_M\|_{V_1}$ and an analog estimate for $\nabla c_M \cdot Fr_{M+1}$. In particular, $\mathrm{E}(c_{M+1}, v_{M+1}) \le C_3$. Because of $\mathrm{E}(c_{M+1}, v_{M+1}) \ge \mathrm{E}_{\mathrm{free}}(c_{M+1}) \ge \frac{1}{2}\|c_{M+1}\|_{W_1}^2 + C_\varphi \ge C_\varphi$, we have

$$\|c_{M+1}\|_{W_1}^2 \;\le\; 2(C_3 - C_\varphi)$$

as well as

$$\|w_{M+1}\|_{W_1}^2 + \|\overline{\widetilde{v}}_M\|_{\widetilde{V}_1}^2 \;\le\; \tfrac{2}{\tau}(C_3 - \mathrm{E}(c_{M+1}, v_{M+1})) \;\le\; \tfrac{2}{\tau}(C_3 - C_\varphi).$$

Finally, we note that

$$\begin{aligned}\|v_{M+1}\|_{V_1}^2 &= \|\widetilde{v}_{M+1} + Fr_{M+1}\|_{V_1}^2 \;\le\; 2\big(\|\widetilde{v}_{M+1}\|_{V_1}^2 + \|Fr_{M+1}\|_{V_1}^2\big)\\ &\le 2C_P\|\widetilde{v}_{M+1}\|_{\widetilde{V}_1}^2 + 2\|Fr_{M+1}\|_{V_1}^2.\end{aligned}$$

This completes the proof. ∎

3. Optimal control of the semidiscrete CH-NS system.

Now we are ready to state the optimization problem under investigation, prove existence of a solution, and establish a suitable stationary characterization. For the application of the theory developed in section 2 we invoke the following assumption.

Assumption 3.1. The space U_1 is a closed, linear subspace of S_1 with induced inner product and with its dual denoted by $U_{-1} := U_1^*$. We fix $M \in \mathbb{N}$, $M \ge 2$, initial values $(c_a, v_a) \in Y_1$ with $c_a \in H^2(\Omega)$ and $\mathrm{Tr}\, v_a \in U_1$, right-hand sides $(f^c, f^v) \in Y_{-1}$, $\gamma \ge 0$, and a desired concentration $c_e \in W_1$. Moreover, let $0 < \tau_0 \le 2$ be given.

PROBLEM 3.2. *Given a proper, convex, and lower-semicontinuous functional $\varphi : W_1 \to \overline{\mathbb{R}}$ we consider the problem* (P_φ)

$$(\mathrm{P}_\varphi) \qquad \inf\{J(c,u) \mid (c,v,u) \in W_1^{M+1} \times V_1^{M+1} \times U_1^{M+1},\ (c,v) \in \mathcal{S}_\varphi u\},$$

where the functional $J : W_1^{M+1} \times U_1^{M+1} \to \mathbb{R}$ and the set $\mathcal{S}_\varphi u$ for $u \in U_1^{M+1}$ are given by

$$\begin{aligned}J(c,u) &:= \frac{1}{2}\Big(\gamma\|c_M - c_e\|_{W_0}^2 + \|u\|_{U_1^{M+1}}^2\Big),\\ \mathcal{S}_\varphi u &:= \mathcal{S}(M, \tau, (c_a, v_a), (f^c, f^v), u, \partial\varphi).\end{aligned}$$

Moreover, we write $\mathcal{S}_\varphi^w u := \mathcal{S}^w(M, \tau, (c_a, v_a), (f^c, f^v), u, \partial\varphi)$.

Before we address the existence of minimizers of problem (P_φ) we need a closedness result for the solution sets along a sequence $(u^{(n)})$ and for a sequence $(\varphi^{(n)})$ approximating $\varphi^{(\infty)} := \varphi$.

PROPOSITION 3.1. *Assume we are given a bounded sequence $(u^{(n)})_{n\in\mathbb{N}}$ in U_1^{M+1} and a sequence $(\varphi^{(n)})_{n\in\mathbb{N}^*}$ of proper, convex, and lower-semicontinuous functionals from W_1 into $\overline{\mathbb{R}}$ satisfying* (H_1) *of Corollary 2.7 for a common constant $C = C_{\varphi^{(n)}}$ for all $n \in \mathbb{N}$. In addition, suppose that sequence $(A^{(n)})$ with $A^{(n)} := \partial\varphi^{(n)} \subset W_1 \times W_{-1}$ for $n \in \mathbb{N}^*$ fulfills one of the following conditions:*

(H$_2$) *Whenever* $A^{(m)} \ni (\hat{c}^{(m)}, \hat{a}^{(m)}) \rightharpoonup (\hat{c}^{(\infty)}, \hat{a}^{(\infty)})$ *in* $W_1 \times W_{-1}$ *for a subsequence* $m \in M \subset \mathbb{N}$ *with*

$$\varliminf_{M \ni m \to \infty} \left\langle \hat{a}^{(m)} - \hat{a}^{(\infty)}, \hat{c}^{(m)} - \hat{c}^{(\infty)} \right\rangle_{W_1} \leq 0,$$

then $(\hat{c}^{(\infty)}, \hat{a}^{(\infty)}) \in A^{(\infty)}$ *and* $\langle \hat{a}^{(m)} - \hat{a}^{(\infty)}, \hat{c}^{(m)} - \hat{c}^{(\infty)} \rangle_{W_1} \to 0$.

(H$_3$) *It holds that* $A^{(n)}(W_1) \subset W_0$ *and* $\left(-\Delta \hat{c}^{(n)} \middle| \hat{a}^{(n)}\right)_{W_0} \geq 0$ *for all* $n \in \mathbb{N}$ *and all* $(\hat{c}^{(n)}, \hat{a}^{(n)}) \in A^{(n)}$ *with* $-\Delta \hat{c}^{(n)} \in W_0$. *Moreover, whenever* $A^{(m)} \ni (\hat{c}^{(m)}, \hat{a}^{(m)}) \rightharpoonup (\hat{c}^{(\infty)}, \hat{a}^{(\infty)})$ *in* $W_1 \times W_0$ *for a subsequence* $m \in M \subset \mathbb{N}$, *then* $(\hat{c}^{(\infty)}, \hat{a}^{(\infty)}) \in A^{(\infty)}$.

Then there exists a subsequence $(u^{(m)})$ *of* $(u^{(n)})$ *converging weakly to* $u^{(\infty)}$ *in* U_1^{M+1} *and a sequence of solutions* $(c^{(n)}, w^{(n)}, v^{(n)}) \in \mathcal{S}^w_{\varphi^{(n)}} u^{(n)}$ *for* $n \in \mathbb{N}^*$ *such that* $(c^{(m)})$ *converges strongly in* W_1^{M+1} *to* $c^{(\infty)}$ *and* $(w^{(m)}, v^{(m)})$ *converges weakly in* $(W_1 \times V_1)^{M+1}$ *to* $(w^{(\infty)}, v^{(\infty)})$. *Moreover, if* (H$_3$) *holds true, then the sequences* $(c^{(n)})$ *and* $(S_+ a^{(n)})$ *are bounded in* $H^2(\Omega)^{M+1}$, *respectively,* W_0^M, *for* $a^{(n)} \in A^{(n)} c^{(n)}$ *given by*

$$\begin{aligned} S_+ a^{(n)} &= (-\Delta)^{-1} f^c \\ &\quad - [(-\Delta)^{-1} D^+ c^{(n)} + (-\Delta + A) S_+ c^{(n)} - I S_- c^{(n)} + (-\Delta)^{-1} (\nabla S_- c^{(n)} \cdot S_+ v^{(n)})]. \end{aligned}$$

Proof. By Theorem 2.11 we can find a solution $(c^{(n)}, w^{(n)}, v^{(n)}) \in \mathcal{S}^w_{\varphi^{(n)}} u^{(n)}$ for every $n \in \mathbb{N}$ satisfying

$$\begin{aligned} &\operatorname{Tr} v^{(n)} = u^{(n)}, \ \ c_0^{(n)} = c_a, \ \ w_0^{(n)} = 0, \ \ v_0^{(n)} = v_a, \\ &(-\Delta)^{-1} D^+ c^{(n)} - \Delta S_+ c^{(n)} + S_+ a^{(n)} - I S_- c^{(n)} + (-\Delta)^{-1} (\nabla S_- c^{(n)} \cdot S_+ v^{(n)}) \\ &\quad = (-\Delta)^{-1} f^c, \\ &S_+ w^{(n)} = (-\Delta)^{-1} \left[f^c - (D^+ c^{(n)} + \nabla S_- c^{(n)} \cdot S_+ v^{(n)}) \right], \\ &D^+ v^{(n)} - \Delta S_+ v^{(n)} + (S_- v^{(n)} \cdot \nabla) S_+ v^{(n)} - S_+ w^{(n)} \nabla S_- c^{(n)} = f^v \end{aligned}$$

for $a^{(n)} \in A^{(n)} c^{(n)}$. Proposition 2.12 shows that the sequence $(c^{(n)}, w^{(n)}, v^{(n)})$ is bounded in the space $(W_1 \times W_1 \times V_1)^{M+1}$. Hence, we can pass to a subsequence $(c^{(m)}, w^{(m)}, v^{(m)}, u^{(m)})$ that converges weakly to some $(c^{(\infty)}, w^{(\infty)}, v^{(\infty)}, u^{(\infty)})$ in $(W_1 \times W_1 \times V_1 \times U_1)^{M+1}$. Exploiting the weak continuity of linear, bounded operators and the total continuity of $(\hat{c}, \hat{v}) \mapsto \nabla \hat{c} \cdot \hat{v}$, $(\hat{c}, \hat{v}) \mapsto \hat{w} \nabla \hat{c}$ and $(\hat{v}_1, \hat{v}) \mapsto (\hat{v}_1 \cdot \nabla) \hat{v}$ we conclude that $S_+ a^{(m)}$ converges weakly to some $S_+ a^{(\infty)}$ in W_{-1}^M and

$$\begin{aligned} &\operatorname{Tr} v^{(\infty)} = u^{(\infty)}, \ \ c_0^{(\infty)} = c_a, \ \ w_0^{(\infty)} = 0, \ \ v_0^{(\infty)} = v_a, \\ &(-\Delta)^{-1} D^+ c^{(\infty)} - \Delta S_+ c^{(\infty)} + S_+ a^{(\infty)} - I S_- c^{(\infty)} + (-\Delta)^{-1} (\nabla S_- c^{(\infty)} \cdot S_+ v^{(\infty)}) \\ &\quad = (-\Delta)^{-1} f^c, \\ &S_+ w^{(\infty)} = (-\Delta)^{-1} \left[f^c - (D^+ c^{(\infty)} + \nabla S_- c^{(\infty)} \cdot S_+ v^{(\infty)}) \right], \\ &D^+ v^{(\infty)} - \Delta S_+ v^{(\infty)} + (S_- v^{(\infty)} \cdot \nabla) S_+ v^{(\infty)} - S_+ w^{(\infty)} \nabla S_- c^{(\infty)} = f^v. \end{aligned}$$

In order to finish the proof, it remains to show the boundedness of $(c^{(n)})$ and $(a^{(n)})$ in $H^2(\Omega)$, respectively, W_0, in the case of (H$_3$), and moreover, the strong convergence properties and that $(c_i^{(\infty)}, a_i^{(\infty)}) \in A^{(\infty)}$ for $i = 1, \ldots, M$. This will be done by induction over i together with $-\Delta c_i^{(m)} \in W_0$ for the case of (H$_3$). We obviously have $c_0^{(n)} = c_a \in V_1 \cap H^2(\Omega)$ for all $n \in \mathbb{N}^*$. This concludes the basis step of the induction. Let us suppose that $c_i^{(m)} \to c_i^{(\infty)}$ in W_1 for some $i = 0, \ldots, M-1$ as well as $w_i^{(m)} \rightharpoonup w_i^{(\infty)}$ in W_1 and $v_i^{(m)} \rightharpoonup v_i^{(\infty)}$ in V_1. Using the equations above we obtain

$$\begin{aligned}
&-\Delta(c_{i+1}^{(m)} - c_{i+1}^{(\infty)}) + a_{i+1}^{(m)} - a_{i+1}^{(\infty)} \\
&\quad = -\Big[(-\Delta)^{-1}(D^+c_i^{(m)} - D^+c_i^{(\infty)}) - I(c_i^{(m)} - c_i^{(\infty)}) \\
&\qquad + (-\Delta)^{-1}(\nabla c_i^{(m)} \cdot v_{i+1}^{(m)} - \nabla c_i^{(\infty)} \cdot v_{i+1}^{(\infty)})\Big]
\end{aligned}$$

and therefore by the compactness of $I\ (-\Delta)^{-1}I, I : W_1 \to W_{-1}$, and $(\hat{v}_1, \hat{v}) \mapsto (\hat{v}_1 \cdot \nabla)\hat{v} : V_1 \times V_1 \to \widetilde{V}_{-1}$ that

$$\lim_{m\to\infty} \Big[\Big\langle -\Delta(c_{i+1}^{(m)} - c_{i+1}^{(\infty)}), c_{i+1}^{(m)} - c_{i+1}^{(\infty)} \Big\rangle_{W_1} + \Big\langle a_{i+1}^{(m)} - a_{i+1}^{(\infty)}, c_{i+1}^{(m)} - c_{i+1}^{(\infty)} \Big\rangle_{W_1} \Big] = 0.$$

Hence, we find

$$\begin{aligned}
&\overline{\lim_{m\to\infty}} \Big\langle a_{i+1}^{(m)} - a_{i+1}^{(\infty)}, c_{i+1}^{(m)} - c_{i+1}^{(\infty)} \Big\rangle_{W_1} \\
&\quad = -\underline{\lim}_{m\to\infty} \Big\langle -\Delta(c_{i+1}^{(m)} - c_{i+1}^{(\infty)}), c_{i+1}^{(m)} - c_{i+1}^{(\infty)} \Big\rangle_{W_1} \\
&\quad = -\underline{\lim}_{m\to\infty} \|c_{i+1}^{(m)} - c_{i+1}^{(\infty)}\|_{W_1}^2 \le 0.
\end{aligned} \tag{3.1}$$

If (H$_2$) is satisfied, it follows directly that $(c_{i+1}^{(\infty)}, a_{i+1}^{(\infty)}) \in A^{(\infty)}$ as well as the convergence $\langle a_{i+1}^{(m)} - a_{i+1}^{(\infty)}, c_{i+1}^{(m)} - c_{i+1}^{(\infty)} \rangle_{W_1} \to 0$ and therefore $c_{i+1}^{(m)} \to c_{i+1}^{(\infty)}$ in W_1. In the case of (H$_3$), notice that from

$$\begin{aligned}
-\Delta c_{i+1}^{(n)} + a_{i+1}^{(n)} &= (-\Delta)^{-1}(f_i^c - \nabla c_i^{(n)} \cdot v_{i+1}^{(n)} - D^+c_i^{(n)}) + Ic_i^{(n)} \\
&= w_{i+1}^{(n)} + Ic_i^{(n)} := g^{(n)}
\end{aligned} \tag{3.2}$$

and $g^{(n)} \in W_0$ as well as $A^{(n)}(W_1) \subset W_0$ it follows that $-\Delta c_{i+1}^{(n)} \in W_0$. Moreover, since $(g^{(n)})$ is bounded in W_0, the assumption (H$_3$) yields

$$\begin{aligned}
\|a_{i+1}^{(n)}\|_{W_0}^2 = \Big(g^{(n)} + \tfrac{1}{2}\Delta c_{i+1}^{(n)} \Big| a_{i+1}^{(n)}\Big)_{W_0} &\le \Big(g^{(n)} \Big| a_{i+1}^{(n)}\Big)_{W_0} \\
&\le \|g^{(n)}\|_{W_0} \|a_{i+1}^{(n)}\|_{W_0}.
\end{aligned}$$

Consequently, $(a_{i+1}^{(n)})$ is bounded in W_0 and therefore $(c_{i+1}^{(n)})$ in $H^2(\Omega)$ by (3.2). Hence, we can assume without loss of generality that $(a_{i+1}^{(m)})$ converges weakly in W_0 and strongly in W_{-1} to $a_{i+1}^{(\infty)}$. By (H$_3$) it follows that $(c_{i+1}^{(\infty)}, a_{i+1}^{(\infty)}) \in A^{(\infty)}$ and by (3.1) that $c_{i+1}^{(m)} \to c_{i+1}^{(\infty)}$ in W_1. This completes the proof. ∎

Remark 3.3. It is well known that for a maximal monotone operator $A \subset W_1 \times W_{-1}$ its Yosida approximations $A^{(n)} := A_{\lambda_n}$ with parameter $\lambda_n := \frac{1}{n}$ as well as the sequence $A^{(n)} := A$ itself satisfy the condition (H$_2$). This condition is used in order to show the existence of minimizers for a fixed potential φ, whereas (H$_3$) will be the appropriate condition for the approximation procedure which we apply below.

The existence result is stated next.

THEOREM 3.2. *For every proper, convex, and lower-semicontinuous functional $\varphi : W_1 \to \overline{\mathbb{R}}$ fulfilling* (H_1) *and* $\varphi(c_a) < \infty$, *problem* (P_φ) *admits a minimizer.*

Proof. Let $(c^{(n)}, u^{(n)}) \in U_1^{M+1} \times W_1^{M+1}$ be an infimizing sequence for problem (P_φ) and $(w^{(n)}, v^{(n)}) \in W_1^{M+1} \times V_1^{M+1}$ such that $(c^{(n)}, w^{(n)}, v^{(n)}) \in \mathcal{S}_\varphi^w u^{(n)}$. The coercivity of J in the second component and the boundedness imply that $(u^{(n)})$ is bounded in U_1^{M+1}. We choose $\varphi^{(n)} := \varphi$ in Proposition 3.1 and Remark 3.3 to conclude the existence of a weakly convergent subsequence $(c^{(n)}, w^{(n)}, v^{(n)})$ in $(W_1 \times W_1 \times V_1)^{M+1}$ with limit within $\mathcal{S}_\varphi^w u^{(\infty)}$ and $(u^{(n)})$ in U_1^{M+1} with limit $u^{(\infty)}$. The weak lower semicontinuity of J implies that this limit is indeed a minimizer of (P_φ). This finishes the proof. ∎

For the proof of the subsequent theorem we need the following auxiliary result.

LEMMA 3.3. *Let $\varphi : W_1 \to \overline{\mathbb{R}}$ be a proper, convex, and lower-semicontinuous functional with a single-valued subdifferential $A := \partial\varphi$, which is defined on all of W_1. Moreover, suppose that $A : W_1 \to W_{-1}$ is continuously Fréchet differentiable in $w_0 \in W_1$ with derivative $Q := DA(w_0) \in \mathcal{L}(W_1; W_{-1})$. Then $\varphi_0(w) := \frac{1}{2}\langle Qw, w\rangle_{W_1}$ is a proper, convex, and lower-semicontinuous functional on W_1 and its subdifferential $A_0 := \partial\varphi_0$ is single-valued, defined on all W_1, and continuously Fréchet differentiable with $DA_0(w) = Q$ for all $w \in W_1$.*

Proof. Since Aw is a singelton and continuous for every $w \in W_1$, φ is Gâteaux differentiable (cf. Showalter [38]). Consequently, $Q \in \mathcal{L}(W_1; W_{-1})$ is symmetric and positive. Hence, the assertion follows. ∎

Next we study the adjoint system pertinent to (P_φ). This system is relevant for deriving first order optimality conditions of approximate versions of (P_φ) with a smooth potential φ.

THEOREM 3.4. *Assume that (c^o, u^o) is a minimizer of (P_φ), $(c^o, w^o, v^o) \in \mathcal{S}_\varphi u^o$, and that $A = \partial\varphi \subset W_1 \times W_{-1}$ is single-valued, defined on W_1, and maps W_3 continuously Fréchet differentiable into W_1. Moreover, for $c \in W_3$, $DA(c)$ can be continuously extended to an operator $\overline{DA(c)} \in \mathcal{L}(W_1; W_1)$. Then there exists $(p, q) \in \widetilde{Y}_1^M$ such that*

$$
\begin{aligned}
0 &= -D^+p + (-\Delta + DA(S_+S_-c^o))^*(-\Delta S_-p - \nabla S_-S_-c^o \cdot S_-q)\\
&\quad -I^*(-\Delta S_+p - \nabla S_+S_-c^o \cdot S_+q) - \operatorname{div}(S_+pS_+S_+v^o)\\
&\quad + \operatorname{div}(S_+S_+w^oS_+q),\\
J_{W_1}(c_M^o - c_e) &= \frac{1}{\tau}p_{M-1} + (-\Delta + DA(c_M^o))^*(-\Delta p_{M-1} - \nabla c_{M-1}^o \cdot q_{M-1}),\\
(J_{U_1}P_{U_1}\ Tr)^*S_+u^o &= -I^*D^+q - \Delta^*S_-q + b_1(S_+S_+v^o, S_+q) + b_2(S_-S_-v^o, S_-q)\\
&\quad + S_-p\nabla S_-S_-c^o,\\
(J_{U_1}P_{U_1}\ Tr)^*u_M^o &= \Big(\frac{1}{\tau}I^* - \Delta^*\Big)q_{M-1} + b_2(v_{M-1}^o, q_{M-1}) + p_{M-1}\nabla c_{M-1}^o,
\end{aligned}
$$

where the last two equations are understood in the sense of $(Z^*)^{M-1}$ and Z^* for $Z := \{v \in V_1 : P_{U_1^\perp}\ Tr\, v = 0\}$ with P_{U_1} and $P_{U_1^\perp}$ denoting the orthogonal projections of S_1 onto U_1 and its orthogonal complement $U_1^\perp$ and w is given according to (2.4).

Proof. We split the proof into three steps. Our goal is to apply the theory developed by Zowe and Kurcyusz [44]. For this purpose, in the first step of the proof we define relevant quantities. Then we apply [44] in the second step. And finally, in the third step, we rearrange terms in order to derive the asserted adjoint system.

1. In the proof we utilize the sets and spaces

$$
\begin{aligned}
X_1 &:= W_3^{M+1} \times V_1^{M+1} \times U_1^{M+1},\\
X_2 &:= W_{-1}^{M} \times \widetilde{V}_{-1}^{M} \times U_{-1}^{M},\\
C_{(c_b,v_b)} &:= \{(c,v,u) \in X_1 \,:\, (c_0,v_0) = (c_b,v_b),\ \operatorname{Tr} v_b = u_0,\ P_{U_1^\perp} \operatorname{Tr} v = 0\}
\end{aligned}
$$

and the mappings $Q : X_1 \to X_2$ and $g : X_1 \to \mathbb{R}$ defined by

$$
\begin{aligned}
Q(c,v,u) \,:=& \Big(D^+c - \Delta((-\Delta + A)S_+c - IS_-c) + \nabla S_-c \cdot S_+v - f^c\,,\\
& D^+v - \Delta S_+v + (S_-v\cdot\nabla)S_+v - ((-\Delta + A)S_+c - IS_-c)\nabla S_-c - f^v\,,\\
& S_+J_{U_1}(P_{U_1}\operatorname{Tr} v - u)\Big),\\
g(c,v,u) \,:=& J(c,u),
\end{aligned}
$$

where J_{U_1} denotes the duality mapping from U_1 to U_{-1}. With DA being the Fréchet derivative of $A : W_3 \to W_1$, their derivatives at $(c,v,u) \in X_1$ in direction $(c^\delta, v^\delta, u^\delta) \in X_1$ read as follows:

$$
\begin{aligned}
&DQ(c,v,u;c^\delta,v^\delta,u^\delta)\\
&= \Big(D^+c^\delta - \Delta((-\Delta + DA(S_+c))S_+c^\delta - IS_-c^\delta) + \nabla S_-c^\delta\cdot S_+v + \nabla S_-c\cdot S_+v^\delta,\\
&\quad D^+v^\delta - \Delta S_+v^\delta + (S_-v^\delta\cdot\nabla)S_+v + (S_-v\cdot\nabla)S_+v^\delta\\
&\qquad -((-\Delta + A)S_+c - IS_-c)\nabla S_-c^\delta - ((-\Delta + DA(S_+c))S_+c^\delta - IS_-c^\delta)\nabla S_-c,\\
&\quad S_+J_{U_1}(P_{U_1}\operatorname{Tr} v^\delta - u^\delta)\Big),
\end{aligned}
$$

$$
Dg(c,v,u;c^\delta,v^\delta,u^\delta) \,=\, \gamma\left(c_M - c_e \middle| c_M^\delta\right)_{W_0} + \left(u\middle|u^\delta\right)_{U_1^{M+1}}.
$$

2. The triple (c^o, v^o, u^o) is a minimizer of g on the set $Z := C_{(c_a,v_a)} \cap Q^{-1}(0)$; cf. Remark 2.3. For the application of the existence result of Lagrange multipliers by Zowe and Kurcyusz [44], it has to be shown that (c^o, v^o, u^o) is regular in the sense of [44]. For this purpose we fix an arbitrary $(\hat f^c, \hat f^v, \hat f^u) \in X_2$ and show the existence of a $(c^\delta, v^\delta, u^\delta) \in C_{(0,0)}$ such that $DQ(c^o, v^o, u^o; c^\delta, v^\delta, u^\delta) = (\hat f^c, \hat f^v, \hat f^u)$. This is equivalent to

$$
c_0^\delta = 0, \qquad v_0^\delta = 0, \qquad u_0^\delta = 0, \qquad S_+J_{U_1}(P_{U_1}(\operatorname{Tr} v^\delta - u^\delta)) = \hat f^u, \tag{3.3}
$$

$$
D^+c^\delta - \Delta((-\Delta + DA(S_+c))S_+c^\delta - IS_-c^\delta) + \nabla S_-c^o\cdot S_+v^\delta = g^c(c^\delta, v^\delta, u^\delta), \tag{3.4}
$$

$$
D^+v^\delta - \Delta S_+v^\delta + (S_-v^o\cdot\nabla)S_+v^\delta - ((-\Delta + DA(S_+c))S_+c^\delta - IS_-c^\delta)\nabla S_-c^o = g^v(c^\delta, v^\delta, u^\delta), \tag{3.5}
$$

where the right-hand-sides g^c and g^v correspond to

$$
\begin{aligned}
g^c(c^\delta, v^\delta, u^\delta) &:= \hat f^c - \nabla S_-c^\delta \cdot S_+v^o,\\
g^v(c^\delta, v^\delta, u^\delta) &:= \hat f^v - (S_-v^\delta\cdot\nabla)S_+v^o + ((-\Delta + A)S_+c - IS_-c)\nabla S_-c^\delta.
\end{aligned}
$$

As in Theorem 2.11, the existence of a triple $(c^\delta, v^\delta, u^\delta)$ satisfying the above equations will be proved by induction over M'. Moreover, we show that it is possible to satisfy the additional condition $\operatorname{Tr} v^\delta = 0$. In case of $M' = 0$, the conclusion is evident. Now, we assume that the system is satisfied for some M' with $0 \le M' < M$. We apply the assumption on A, the previous lemma, Proposition 2.10, and Remark 2.3 in order to conclude the existence of $(c^2, v^2) \in (W_1 \times V_1)^2$ such that

$$\begin{aligned} &c_0^2 = c_{M'}, \quad v_0^2 = v_{M'}, \quad u_0^2 = 0, \quad u^2 = -J_{U_1}^{-1}(\hat{f}^u_{M'}, \hat{f}^u_{M'+1}),\\ &D^+c^2 - \Delta((-\Delta + DA(S_+c^o))S_+c^2 - IS_-c^2) + \nabla S_-c^o \cdot S_+v^2 = g^c(c^\delta, v^\delta, u^\delta),\\ &D^+v^2 - \Delta S_+v^2 + (S_-v^o \cdot \nabla)S_+v^2 - ((-\Delta + DA(S_+c^o))S_+c^2 - IS_-c^2)\nabla S_-c^o\\ &\quad = g^v(c^\delta, v^\delta, u^\delta). \end{aligned}$$

Consequently, $((c_0, \ldots, c_{M'}, c_1^2), (v_0, \ldots, v_{M'}, v_1^2), (u_0, \ldots, u_{M'}, u_1^2))$ solves the system (3.3)–(3.5) for $M' + 1$.

3. The result of Zowe–Kurcyusz now implies that for some $(p, q, r) \in W_1^M \times \widetilde{V}_1^M \times U_1^M \cong X_2^*$

$$Dg(c^o, v^o, u^o; c^\delta, v^\delta, u^\delta) \;=\; \left\langle DQ(c^o, v^o, u^o; c^\delta, v^\delta, u^\delta), (p, q, r)\right\rangle_{X_2^*}$$

for all $(c^\delta, v^\delta, u^\delta) \in C_{(0,0)}$ holds true.

First notice that for $c_1, c_2 \in W_1,\ v, v_1, v_2 \in V_1$ we have

$$\begin{aligned} \left\langle(-\Delta)^{-1}(\nabla c_1 \cdot v), c_2\right\rangle_{W_1} &= \left\langle c_2, (-\Delta)^{-1}(\nabla c_1 \cdot v)\right\rangle_{W_1} \;=\; \left\langle \nabla c_1 \cdot v, (-\Delta)^{-1}c_2\right\rangle_{W_1}\\ &= \left\langle -\operatorname{div}((-\Delta)^{-1}c_2 v), c_1\right\rangle_{W_1} \;=\; \left\langle(-\Delta)^{-1}c_2\nabla c_1, v\right\rangle_{V_1} \end{aligned}$$

as well as

$$\begin{aligned} &\left\langle(-\Delta)^{-1}(\nabla c_2 \cdot v_2)\nabla c_1, v_1\right\rangle_{V_1}\\ &\quad= \left\langle \nabla c_1 \cdot v_1, (-\Delta)^{-1}(\nabla c_2 \cdot v_2)\right\rangle_{W_1} \;=\; \left\langle \nabla c_2 \cdot v_2, (-\Delta)^{-1}(\nabla c_1 \cdot v_1)\right\rangle_{W_1}\\ &\quad= \left\langle(-\Delta)^{-1}(\nabla c_1 \cdot v_1)\nabla c_2, v_2\right\rangle_{V_1}. \end{aligned}$$

Choosing $(c^\delta, 0, 0) \in C_{(0,0)}$, passing to adjoint operators, and collecting terms involving c_i^δ we obtain

$$\begin{aligned} &\gamma\left(c_M^o - c_e \middle| c_M^\delta\right)_{W_0}\\ &= \left\langle D^+c^\delta - \Delta((-\Delta + DA(S_+c))S_+c^\delta - IS_-c^\delta) + \nabla S_-c^\delta \cdot S_+v^o, p\right\rangle_{W_1^M}\\ &\quad - \left\langle(-\Delta + A)S_+c - IS_-c\nabla S_-c^\delta + ((-\Delta + DA(S_+c))S_+c^\delta - IS_-c^\delta)\nabla S_-c, q\right\rangle_{\widetilde{V}_1^{M+1}}\\ &= \sum_{i=1}^{M-1} \Big\langle \frac{1}{\tau}(p_{i-1} - p_i) + (-\Delta + DA(c_i^o))^*(-\Delta)p_{i-1} + I^*\Delta p_i - div(p_i v_{i+1}^0)\\ &\qquad\qquad + \operatorname{div}((-\Delta + A)c_{i+1} - Ic_iq_i) - (-\Delta + DA(c_i))^*(\nabla c_{i-1} \cdot q_{i-1})\\ &\qquad\qquad + I^*(\nabla c_i \cdot q_i)\,,\ c_i^\delta \Big\rangle_{W_1}\\ &\quad + \left\langle \frac{1}{\tau}p_{M-1} + (-\Delta + DA(c_M))^*[-\Delta p_{M-1} - \nabla c_{M-1} \cdot q_{M-1}], c_M^\delta \right\rangle_{W_1}. \end{aligned}$$

Hence, we can choose c_i^δ arbitrarily for $i > 0$, which implies the assertion on p. Next, we use $(0, 0, u^\delta) \in C_{(0,0)}$ and find $\left(u^o \middle| u^\delta\right)_{U_1^{M+1}} = \left(-S_+u^\delta \middle| r\right)_{U_1^M}$ and hence $r + S_+u^o = 0$. Finally, choosing $(0, v^\delta, 0) \in C_{(0,0)}$ and proceeding as before yields

$$
\begin{aligned}
0 &= \left\langle D^+ v^\delta - \Delta S_+ v^\delta + (S_- v^\delta \cdot \nabla) S_+ v^o + (S_- v^o \cdot \nabla) S_+ v^\delta, q \right\rangle_{\widetilde{V}_1^M} \\
&\quad + \left\langle \nabla S_- c^o \cdot S_+ v^\delta, p \right\rangle_{W_1^M} + \left(S_+ P_{U_1} \operatorname{Tr} v^\delta \middle| r \right)_{U_1^M} \\
&= \sum_{i=1}^{M-1} \Big\langle \frac{1}{\tau} I^*(q_{i-1} - q_i) - \Delta^* q_{i-1} + b_1(v_{i+1}^o, q_i) + b_2(v_{i-1}^o, q_{i-1}) \\
&\qquad\qquad + p_{i-1} \nabla c_{i-1}^o - \operatorname{Tr}^* P_{U_1}^* u_i^o \,,\; v_i^\delta \Big\rangle_{V_1} \\
&\quad + \Big\langle \frac{1}{\tau} I^* q_{M-1} - \Delta^* q_{M-1} + b_2(v_{M-1}^o, q_{M-1}) + p_{M-1} \nabla c_{M-1}^o - \operatorname{Tr}^* P_{U_1}^* u_M^o \,,\; v_M^\delta \Big\rangle_{V_1}.
\end{aligned}
$$

This concludes the proof. ∎

Lemma 3.5. *Let* $(c, v) \in Y_1$, $(f^p, f^q) \in \widetilde{Y}_{-1}$, *and* Ψ *be the set of all solutions* (p, q, A) *to*

$$\frac{1}{\tau} p + (-\Delta + A)^*(-\Delta p - \nabla c \cdot q) = f^p, \tag{3.6}$$

$$\Big(\frac{1}{\tau} I^* - \Delta^*\Big) q + b_2(v, q) + p \nabla c = f^q \tag{3.7}$$

with $(p, q) \in \widetilde{Y}_1$ *and* $A \in \mathcal{L}(W_1; W_{-1})$ *monotone and such that* $A(W_1) \subset W_1$, $\langle -\Delta r, Ar \rangle_{W_1} \geq 0$ *for all* $r \in W_3$. *Then* $\{(p, q) : (p, q, A) \in \Psi\}$ *is bounded in* $\widetilde{Y}_1$.

Proof. Let $(p, q, A) \in \Psi$. Then, there is an $r \in W_1$ with $(-\Delta + A) r = p$. Moreover, $-\Delta r = p - Ar \in W_1$ implies that r belongs to W_3. Testing r with this equation, it follows that

$$\langle p, r \rangle_{W_1} \;\geq\; \langle (-\Delta + A) r, r \rangle_{W_1} \;\geq\; \|r\|_{W_1}^2.$$

Testing (3.6) with r we obtain

$$
\begin{aligned}
\frac{1}{2\tau} \|r\|_{W_1}^2 + \frac{\tau}{2} \|f^p\|_{W_{-1}}^2 &\geq \|f^p\|_{W_{-1}} \|r\|_{W_1} \\
&\geq \frac{1}{\tau} \langle p, r \rangle_{W_1} + \langle -\Delta p - \nabla c \cdot q, (-\Delta + A) r \rangle_{W_1} \\
&\geq \frac{1}{\tau} \|r\|_{W_1}^2 + \|p\|_{W_1}^2 - g(c, p, q).
\end{aligned}
$$

Testing (3.7) with q it follows that

$$\frac{1}{2} \|q\|_{\widetilde{V}_1}^2 + \frac{1}{2} \|f^q\|_{\widetilde{V}_{-1}}^2 \;\geq\; \|f^q\|_{\widetilde{V}_{-1}} \|q\|_{\widetilde{V}_1} \;\geq\; \frac{1}{\tau} \|q\|_{V_0}^2 + \|q\|_{\widetilde{V}_1}^2 + 0 + g(c, p, q),$$

and summing both inequalities

$$\frac{1}{2\tau} \|r\|_{W_1}^2 + \|p\|_{W_1}^2 + \frac{1}{2} \|q\|_{\widetilde{V}_1}^2 + \frac{1}{\tau} \|q\|_{V_0}^2 \;\;\leq\;\; \frac{\tau}{2} \|f^p\|_{W_{-1}}^2 + \frac{1}{2} \|f^q\|_{\widetilde{V}_{-1}}^2$$

gives the result. ∎

Theorem 3.6. *Let* $(\varphi^{(n)})_{n \in \mathbb{N}^*}$ *be a sequence of proper, convex, and lower-semi-continuous functionals from* W_1 *into* $\overline{\mathbb{R}}$ *satisfying* (H_1) *for a common constant* $C = C_{\varphi^{(n)}}$ *for all* $n \geq 1$ *as well as* (H_2) *or* (H_3) *and let* $A^{(n)} = \partial \varphi^{(n)}$ *fulfill the assumptions of Theorem 3.4. Suppose that* $(c^{(n)}, u^{(n)}) \in W_1^{M+1} \times V_1^{M+1} \times U_1^{M+1}$ *are minimizers for*

$(\mathrm{P}_{\varphi^{(n)}})$ for all $n \in \mathbb{N}$, $(c^{(n)}, w^{(n)}, v^{(n)}) \in \mathcal{S}_{\varphi^{(n)}} u^{(n)}$ with $(J(c^{(n)}, u^{(n)}))$ being bounded. Then there exist $(p^{(n)}, q^{(n)}) \in W_1^M \times \widetilde{V}_1^M$ satisfying

$$
\begin{aligned}
0 = &-D^+ p^{(n)} + (-\Delta + DA^{(n)}(S_+ S_- c^{(n)}))^* \big(-\Delta S_- p^{(n)} \\
&- \nabla S_- S_- c^{(n)} \cdot S_- q^{(n)} \big) - I^*(-\Delta S_+ p^{(n)} \\
&- \nabla S_+ S_- c^{(n)} \cdot S_+ q^{(n)}) - \operatorname{div}(S_+ p^{(n)} S_+ S_+ v^{(n)}) \\
&+ \operatorname{div}(S_+ S_+ w^{(n)} S_+ q^{(n)}),
\end{aligned} \tag{3.8}
$$

$$
\begin{aligned}
J_{W_1}(c_M^{(n)} - c_e) = &\frac{1}{\tau} p_{M-1}^{(n)} \\
&+ (-\Delta + DA^{(n)}(c_M^{(n)}))^* (-\Delta p_{M-1}^{(n)} - \nabla c_{M-1}^{(n)} \cdot q_{M-1}^{(n)}),
\end{aligned} \tag{3.9}
$$

$$
\begin{aligned}
(J_{U_1} P_{U_1}\ Tr)^* S_+ u^o = &-I^* D^+ q^{(n)} - \Delta^* S_- q^{(n)} + b_1(S_+ S_+ v^{(n)}, S_+ q^{(n)}) \\
&+ b_2(S_- S_- v^{(n)}, S_- q^{(n)}) + S_- p^{(n)} \nabla S_- S_- c^{(n)},
\end{aligned} \tag{3.10}
$$

$$
(J_{U_1} P_{U_1}\ Tr)^* u_M^o = \Big(\frac{1}{\tau} I^* - \Delta^*\Big) q_{M-1}^{(n)} + b_2(v_{M-1}^{(n)}, q_{M-1}^{(n)}) + p_{M-1}^{(n)} \nabla c_{M-1}^{(n)}, \tag{3.11}
$$

where the last two equations are understood in the sense of $(Z^*)^{M-1}$ and Z^* for $Z := \{v \in V_1 : P_{U_1^\perp}\ Tr\, v = 0\}$. For a subsequence (denoted by index m) it holds that

$$
\begin{aligned}
c^{(m)} &\to c^{(\infty)} && \text{in } W_1^{M+1}, \\
(w^{(m)}, v^{(m)}, u^{(m)}) &\rightharpoonup (w^{(\infty)}, v^{(\infty)}, u^{(\infty)}) && \text{in } W_1^{M+1} \times V_1^{M+1} \times U_1^{M+1}, \\
(p^{(m)}, q^{(m)}) &\rightharpoonup (p^{(\infty)}, q^{(\infty)}) && \text{in } W_1^M \times V_1^M, \\
DA^{(m)}(S_+ c^{(m)})^* (-\Delta p^{(m)} - \nabla S_c^{(m)} \cdot q^{(m)}) &\rightharpoonup \lambda^{(\infty)} && \text{in } W_{-3}^M.
\end{aligned}
$$

Moreover, $(c^{(\infty)}, u^{(\infty)})$ is a minimizer of $(\mathrm{P}_{\varphi^{(\infty)}})$ and we have that

$$
\begin{aligned}
0 = &-D^+ p^{(\infty)} - \Delta^*(-\Delta S_- p^{(\infty)} - \nabla S_- S_- c^{(\infty)} \cdot S_- q^{(\infty)}) \\
&+ \lambda^{(\infty)} - I^*(-\Delta S_+ p^{(\infty)} - \nabla S_+ S_- c^{(\infty)} \cdot S_+ q^{(\infty)}) \\
&- \operatorname{div}(S_+ p^{(\infty)} S_+ S_+ v^{(\infty)}) + \operatorname{div}(S_+ S_+ w^{(\infty)} S_+ q^{(\infty)}),
\end{aligned} \tag{3.12}
$$

$$
J_{W_1}(c_M^{(\infty)} - c_e) = \frac{1}{\tau} p_{M-1}^{(\infty)} - \Delta^*(-\Delta p_{M-1}^{(\infty)} - \nabla c_{M-1}^{(\infty)} \cdot q_{M-1}^{(\infty)}) + \lambda_{M-1}^{(\infty)}, \tag{3.13}
$$

$$
\begin{aligned}
(J_{U_1} P_{U_1}\ Tr)^* S_+ u^o = &-I^* D^+ q^{(\infty)} - \Delta^* S_- q^{(\infty)} + b_1(S_+ S_+ v^{(\infty)}, S_+ q^{(\infty)}) \\
&+ b_2(S_- S_- v^{(\infty)}, S_- q^{(\infty)}) + S_- p^{(\infty)} \nabla S_- S_- c^{(\infty)},
\end{aligned} \tag{3.14}
$$

$$
(J_{U_1} P_{U_1}\ Tr)^* u_M^o = \Big(\frac{1}{\tau} I^* - \Delta^*\Big) q_{M-1}^{(\infty)} + b_2(v_{M-1}^{(\infty)}, q_{M-1}^{(\infty)}) + p_{M-1}^{(\infty)} \nabla c_{M-1}^{(\infty)}, \tag{3.15}
$$

Proof. We split the proof into three steps. We first prove the strong convergence of a subsequence $(c^{(m)})$ and the weak convergence of a subsequence $(w^{(m)}, v^{(m)}, u^{(m)})$ and then we establish the weak convergence of the adjoint state. Finally, we pass to the limit in the first order system.

1. Using Theorem 3.4 we find sequences $(p^{(n)})_{n\geq 1}$ in W_1^M and $(q^{(n)})_{n\geq 1}$ in V_1^M satisfying the desired system. Moreover, the coercivity of J in u and the boundedness of $(J(c^{(n)}, u^{(n)}))$ imply that $(u^{(n)})$ is bounded in U_1^{M+1}. Taking advantage of Proposition 3.1 it follows that we can pass to subsequences $(c^{(m)})$ and $(v^{(m)}, u^{(m)})$ that converge strongly to $c^{(\infty)}$ in W_1^{M+1} and weakly to $(v^{(\infty)}, u^{(\infty)})$ in $V_1^{M+1} \times U_1^{M+1}$, respectively, with $(c^{(\infty)}, v^{(\infty)}, u^{(\infty)})$ being a minimizer of $(\mathrm{P}_{\varphi^{(\infty)}})$.

2. Now we show that $(p^{(n)})$ and $(q^{(n)})$ are bounded in W_1^M, respectively, $\widetilde{V}_1^M$. This is done by induction. First observe that for $n \in \mathbb{N}$, $i \in \{0, \ldots, M-1\}$, $(p_i^{(n)}, q_i^{(n)})$ satisfies

$$\frac{1}{\tau}p_i^{(n)} + (-\Delta + DA(c_{i+1}^{(n)}))^*(-\Delta p_i^{(n)} - \nabla c_i^{(n)} \cdot q_i^{(n)}) = g_i^{(n)},$$
$$\left(\frac{1}{\tau}I^* - \Delta^*\right)q_i^{(n)} + b_2(v_i^{(n)}, q_i^{(n)}) + p_i^{(n)}\nabla c_i^{(n)} = h_i^{(n)}$$

for

$$g_i^{(n)} := \begin{cases} \frac{1}{\tau}p_{i+1}^{(n)} + I^*(-\Delta p_{i+1}^{(n)} - \nabla c_{i+1}^{(n)} \cdot q_{i+1}^{(n)}) \\ \quad + \operatorname{div}(p_{i+1}^{(n)} v_{i+2}^{(n)}) - \operatorname{div}(w_{i+2}^{(n)} q_{i+1}^{(n)}) & \text{if } i < M-1, \\ J_{W_1}(c_M^{(n)} - c_e) & \text{if } i = M-1, \end{cases}$$
$$h_i^{(n)} := \begin{cases} (J_{U_1} P_{U_1} \operatorname{Tr})^* u_{i+1}^{(n)} + \frac{1}{\tau} I^* q_{i+1}^{(n)} + b_1(v_{i+2}^{(n)}, q_{i+1}^{(n)}) & \text{if } i < M-1, \\ (J_{U_1} P_{U_1} \operatorname{Tr})^* u_M^{(n)} & \text{if } i = M-1. \end{cases}$$

Since $(c_M^{(n)}, u_M^{(n)})$ is bounded in $W_1 \times U_1$, so is $(g_{M-1}^{(n)}, h_{M-1}^{(n)})$ in $\widetilde{Y}_{-1}$. Therefore, let us assume that $(g_i^{(n)}, h_i^{(n)})$ is bounded in $\widetilde{Y}_{-1}$ for some $i = 0, \ldots, M-1$ as well as $(p_j^{(n)}, q_j^{(n)})$ in $\widetilde{Y}_1$ for all $j > i$. Lemma 3.5 now implies that also $(p_i^{(n)}, q_i^{(n)})$ is bounded in $\widetilde{Y}_1$ and thus $(g_{i-1}^{(n)}, h_{i-1}^{(n)})$ in $\widetilde{Y}_{-1}$ if $i > 0$.

3. By (3.8), $\lambda^{(n)} := DA^{(n)}(S_+S_-c^{(n)})^*(-\Delta S_-p^{(n)} - \nabla S_-S_-c^{(n)} \cdot S_-q^{(n)})$ remains bounded in W_{-3}^{M-1}. Therefore, we pass to subsequences (denote by index m again) to obtain the desired convergence result for $(p^{(m)})$, $(q^{(m)})$, and $(\lambda^{(m)})$. Using the strong and weak convergences and the properties of the operators involved and passing to the limit in (3.8)–(3.11) as $m \to \infty$ we finally end up with (3.13)–(3.15). ∎

4. Application to the double-obstacle potential. In this section we consider the case where φ is given by the indicator function of a special convex subset of W_1. This corresponds to the Cahn–Hilliard system with double-obstacle potential. Moreover, the $\varphi^{(n)}$ are defined as mollified versions of the Moreau–Yosida approximations in W_0 of $\varphi := \varphi^{(\infty)}$. In this setting, the optimization problem (P_φ) becomes a mathematical program with complementarity constraints since (2.2) indeed becomes a variational inequality. In this context our approach yields a function space version of C-stationarity; see [26] for the latter.

In this section, we use the notation of the previous sections and suppose Assumption 3.1 to hold.

Double-obstacle potential. Let $k_1, k_2 \in \mathbb{R}$ with $k_1 < 0 < k_2$. We define

$$K := [k_1, k_2], \qquad \psi := \imath_K : \mathbb{R} \to \overline{\mathbb{R}}, \qquad \theta := \partial\psi \subset \mathbb{R} \times \mathbb{R},$$
$$K_0 := \{c \in W_0 : c(x) \in K \text{ for a.e. } x \in \Omega\}, \quad K_1 := K_0 \cap W_1.$$

Then $\varphi := \imath_{K_1} : W_1 \to \overline{\mathbb{R}}$ defines a so-called double-obstacle potential.

Moreover, let $\rho \in C^1(\mathbb{R})$ denote a fixed mollifier with $\operatorname{supp}\rho \subset [-1, 1]$, $\int_{\mathbb{R}} \rho = 1$, and $0 \le \rho \le 1$ a.e. on $\mathbb{R}$, and let $\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+$ be a function with $\varepsilon(\alpha) > 0$ and $\frac{\varepsilon(\alpha)}{\alpha} \to 0$ as $\alpha \to 0$. We consider the Yosida approximation θ_α (with parameter $\alpha > 0$) of θ (for the general definition we refer to [6]) and define

$$\rho_\varepsilon(s) := \frac{1}{\varepsilon}\rho\left(\frac{s}{\varepsilon}\right), \qquad \beta_\alpha := \theta_\alpha * \rho_{\varepsilon(\alpha)}, \qquad \widetilde{\theta}_\alpha(s) := \int_0^s \theta_\alpha, \qquad \varphi_\alpha(c) := \int_\Omega \widetilde{\theta}_\alpha \circ c,$$

where $*$ denotes the usual convolution operator. Let $\tau_1 > 0$ be fixed such that the sequence of functionals $(\varphi^{(n)})_{n\in\mathbb{N}^*}$ given by

$$\varphi^{(\infty)} := \varphi, \qquad \alpha_n := \tau_1 n^{-1}, \qquad \varphi^{(n)} := \varphi_{\alpha_n}$$

satifies conditions (H_3) and (H_1) for some common constant C. Finally, $A^{(n)} := \partial\varphi^{(n)}$.

In what follows we collect a few useful properties of φ and its approximations $\varphi^{(n)}$. For proofs and further details we refer to [28].

Remark 4.1.

1. The mapping $\beta_\alpha : \mathbb{R} \to \mathbb{R}$ is a regularization of the Yosida approximation θ_α of θ and $\varphi_\alpha : W_1 \to \mathbb{R}$ a regularization of the Moreau–Yosida approximation of φ in $L^2(\Omega)$.
2. The existence of $\tau_1 > 0$ such that $(\varphi^{(n)})_{n\in\mathbb{N}^*}$ satisfies conditions (H_3) and (H_1) for some common constant C was shown in Proposition 4.3 of [28].
3. The subdifferentials $A^{(n)} = \partial\varphi^{(n)}$ meet the assumption on the operator A given in Theorem 3.6. For a proof we again refer to the arguments provided in [28].
4. For sufficiently small α, β_α vanishes identically in a neighborhood of 0. Note further that we could choose different mollifiers ρ^1 and ρ^2 instead of ρ in the definition of $\beta_\alpha(s)$ for either positive s or negative s, respectively. Thus, conditions (H_1) and (H_3) remain true also in this case.

THEOREM 4.1. *Consider the setting of this section and suppose that $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function with $h(k_1) = h(k_2) = 0$. Then, the optimization problem* ($\mathrm{P}_{\varphi^{(n)}}$) *admits a minimizer $(c^{(n)}, u^{(n)})$. Moreover, we can find $(c^{(n)}, w^{(n)}, v^{(n)}) \in \mathcal{S}^w_\varphi u^{(n)}$ and $(p^{(n)}, q^{(n)})$ according to Theorem* 3.6 *such that for the sequences $(a^{(n)})$ and $(\lambda^{(n)})$ given by $a_0^{(n)} = 0$ and*

$$\begin{aligned}
S_+a^{(n)} &:= (-\Delta)^{-1} f^c \\
&\qquad -[(-\Delta)^{-1}D^+c^{(n)} - \Delta S_+c^{(n)} - IS_-c^{(n)} + (-\Delta)^{-1}(\nabla S_-c^{(n)}\cdot S_+v^{(n)})],\\
\lambda^{(n)} &:= DA^{(n)}(S_+c^{(n)})^* p^{(n)},\\
\xi^{(n)} &:= DA^{(n)}(S_+c^{(n)})^*(-\Delta p^{(n)} - \nabla S_-c^{(n)}\cdot q^{(n)}),
\end{aligned}$$

there exist subsequences (denoted by index m) with

$$\begin{aligned}
c^{(m)} &\to c^{(\infty)} && \text{in } W_1^{M+1},\\
(w^{(m)}, a^{(m)}, v^{(m)}) &\rightharpoonup (w^{(\infty)}, a^{(\infty)}, v^{(\infty)}) && \text{in } W_1^{M+1}\times W_{-1}^{M+1}\times V_1^{M+1},\\
(u^{(m)}, p^{(m)}, q^{(m)}, \xi^{(m)}) &\rightharpoonup (u^{(\infty)}, p^{(\infty)}, q^{(\infty)}, \xi^{(\infty)}) && \text{in } U_1^{M+1}\times W_1^{M}\times V_1^{M}\times W_{-3}^{M},
\end{aligned}$$

such that $(c^{(\infty)}, u^{(\infty)})$ is a minimizer of (P_φ) *and*

$$\begin{aligned}
f^c &= D^+c^{(\infty)} - \Delta S_+w^{(\infty)} + \nabla S_-c^{(\infty)}\cdot S_+v^{(\infty)},\\
S_+w^{(\infty)} &= -\Delta S_+c^{(\infty)} - IS_-c^{(\infty)} + a^{(\infty)},\\
f^v &= D^+v^{(\infty)} - \Delta S_+v^{(\infty)} + (S_+v^{(\infty)}\cdot\nabla)S_+v^{(\infty)}\\
&\quad - S_+w^{(\infty)}\nabla S_-c^{(\infty)},\\
0 &= -D^+p^{(\infty)} - \Delta^*(-\Delta S_-p^{(\infty)} - \nabla S_-S_-c^{(\infty)}\cdot S_-q^{(\infty)}) + \xi^{(\infty)}\\
&\quad - I^*(-\Delta S_+p^{(\infty)} - \nabla S_+S_-c^{(\infty)}\cdot S_+q^{(\infty)})\\
&\quad - \operatorname{div}(S_+p^{(\infty)}S_+S_+v^{(\infty)}) + \operatorname{div}(S_+S_+w^{(\infty)}S_+q^{(\infty)}),
\end{aligned}$$

$$J_{W_1}(c_M^{(\infty)} - c_e) = \frac{1}{\tau} p_{M-1}^{(\infty)} - \Delta^*(-\Delta p_{M-1}^{(\infty)} - \nabla c_{M-1}^{(\infty)} \cdot q_{M-1}^{(\infty)}) + \xi_{M-1}^{(\infty)},$$

$$\begin{aligned}(J_{U_1} P_{U_1}\ Tr)^* S_+ u^{(\infty)} &= -I^* D^+ q^{(\infty)} - \Delta^* S_- q^{(\infty)} + b_1(S_+ S_+ v^{(\infty)}, S_+ q^{(\infty)}) \\ &\quad + b_2(S_- S_- v^{(\infty)}, S_- q^{(\infty)}) + S_- p^{(\infty)} \nabla S_- S_- c^{(\infty)}, \\ (J_{U_1} P_{U_1}\ Tr)^* u_M^{(\infty)} &= \Big(\frac{1}{\tau} I^* - \Delta^*\Big) q_{M-1}^{(\infty)} + b_2(v_{M-1}^{(\infty)}, q_{M-1}^{(\infty)}) + p_{M-1}^{(\infty)} \nabla c_{M-1}^{(\infty)}.\end{aligned}$$

Furthermore, if $(\lambda^{(n)})$ *remains bounded in* $(H^1(\Omega)^*)^M$ *with* $\lambda^{(m)} \rightharpoonup \lambda^{(\infty)}$*, then we have for a subsequence* $(\lambda^{(k)})$ *of* $(\lambda^{(m)})$ *and all* $i = 0, \ldots, M-1$ *that*

$$\begin{gathered}\big(\, S_+ a^{(\infty)} \,|\, h(S_+ c^{(\infty)}) \,\big)_{L^2} = 0, \quad \big(\, \lambda^{(\infty)} \,|\, h(S_+ c^{(\infty)}) \,\big)_{L^2} = 0, \\ \lim \big(\, S_+ a^{(m)} \,|\, p^{(m)} \,\big)_{L^2} = 0, \quad \underline{\lim} \big(\, \lambda^{(m)} \,|\, p^{(m)} \,\big)_{L^2} \geq 0, \\ \lambda_i^{(k)} \to 0 \text{ a.e. on } \{x \in \Omega : k_1 < c_{i+1}(x) < k_2\}.\end{gathered}$$

Proof. 0. Since the double-obstacle potential φ satisfies the conditions (H_3) and (H_1) for a common constant and since all $A^{(n)} = \partial\varphi^{(n)}$ satisfy the assumptions of Theorem 3.6, it remains to show that $(J(c^{(n)}, u^{(n)}))$ is bounded for a sequence of minimizers $(c^{(n)}, u^{(n)})$ as given in Theorem 3.6. But this is easily seen by our energy estimates, the fact that $J(c,u) \leq C + E_{\text{free}}^{\varphi^{(n)}}(c_M) + \frac{1}{2}\|u\|^2_{U_1^{M+1}}$ for a constant C and sufficiently large n and by choosing $u_+ = 0$.

1. We start by showing the complementarity condition $\big(\, S_+ a^{(\infty)} \,|\, h(S_+ c^{(\infty)}) \,\big)_{L^2} = 0$. Since $(S_+ c^{(\infty)}, S_+ a^{(\infty)}) \in A$ by Proposition 3.1 and since A is the superposition operator of $\theta = \partial\imath_K \subset \mathbb{R} \times \mathbb{R}$, we conclude that $(c_i^{(\infty)}, a_i^{(\infty)}) \in \theta$ for almost all $x \in \Omega$ and $i > 0$ and therefore $a_i^{(\infty)}(x) h(c_i^{(\infty)}(x)) = 0$ since one of the factors equals 0. Integration yields the complementarity condition.

2. Next, we prove $\lim \big(\, \lambda^{(m)} \,|\, h(S_+ c^{(m)}) \,\big)_{L^2} = 0$. Denoting the metric projection of $\mathbb{R}$ onto $K = [k_1, k_2]$ by p_K and the metric projection of W_0 onto $\{f \in L^2(\Omega) : f(x) \in K \text{ a.e. on } \Omega\}$ by P (which is the superposition operator of p_K), respectively, and taking advantage of the continuity of the superposition operator of h on W_1 (cf. [36]), it follows that $P(W_1) \subset H^1(\Omega)$ and $\lim Pc^{(m)} = Pc^{(\infty)} = c^{(\infty)}$, $\lim h(Pc^{(m)}) = h(Pc^{(\infty)}) = h(c^{(\infty)}) = \lim h(c^{(m)})$ in $(H^1(\Omega))^{M+1}$. Moreover, it holds that $|\beta'_\alpha(s)| \leq \frac{1}{\alpha}$ for all s and $\beta'_\alpha(s) = 0$ for $k_1 + \varepsilon(\alpha) \leq s \leq k_2 - \varepsilon(\alpha)$; see [28] for details. If L_h is the Lipschitz constant of h, then $|h(s)| \leq L_h \min(|s - k_1|, |s - k_2|)$ for $r \in \mathbb{R}$. Consequently, it follows that

$$\begin{aligned}|\big(\, \lambda^{(m)} \,|\, h(PS_+ c^{(m)}) \,\big)_{L^2}|^2 &= |\big(\, p^{(m)} \,|\, DA^{(m)}(S_+ c^{(m)}) h(PS_+ c^{(m)}) \,\big)_{L^2}|^2 \\ &\leq \|\, p^{(m)} \,\|_{L^2}^2 \sum_{i=1}^{M} \int_\Omega |\beta'_{\alpha_m}(c_i^{(m)}) h(Pc_i^{(m)})|^2 \\ &\leq \left(M|\Omega| \,\|\, p^{(m)} \,\|_{L^2}\, L_h \frac{\varepsilon(\alpha_m)}{\alpha_m} \right)^2 \ \to\ 0\end{aligned}$$

as $m \to \infty$. Moreover, since $(\lambda^{(m)})$ is supposed to be bounded in $(H^1(\Omega)^*)^M$ we have that

$$\begin{aligned}&\lim \big(\, \lambda^{(m)} \,|\, h(S_+ c^{(m)}) \,\big)_{L^2} \\ &\quad = \lim \big(\, \lambda^{(m)} \,|\, h(PS_+ c^{(m)}) \,\big)_{L^2} + \lim \Big\langle \lambda^{(m)}, h(S_+ c^{(m)}) - h(PS_+ c^{(m)}) \Big\rangle_{H^1(\Omega)^M} \\ &\quad = 0.\end{aligned}$$

3. We set $g_m(s) := \beta_{\alpha_m}(s) - \beta'_{\alpha_m}(s)\kappa(s)$ with $s - p_K(s) = \kappa(s)$. Then we obtain

$$\begin{aligned}\left(S_+ a^{(m)} \,|\, p^{(m)} \right)_{L^2} &= \left(p^{(m)} \,|\, \beta_{\alpha_m}(S_+ c^{(m)}) \right)_{L^2} \\ &= \left(p^{(m)} \,|\, g_m(S_+ c^{(m)}) \right)_{L^2} + \left(\lambda^{(m)} \,|\, S_+ c^{(m)} - P S_+ c^{(m)} \right)_{L^2}.\end{aligned}$$

By Lemma 4.2 in [28], for m sufficiently large it holds that $|g_m(s)| = |\beta_{\alpha_m}(s) - \beta'_{\alpha_m}(s)\kappa(s)| \leq C\,\frac{\varepsilon(\alpha_m)}{\alpha_m}$. Hence, the first term on the right-hand side converges to 0. This is also true for the second since $(\lambda^{(m)})$ is bounded in $(H^1(\Omega)^*)^M$ and $(c^{(m)})$ and $(Pc^{(m)})$ both converge to $c^{(\infty)}$ in $H^1(\Omega)^M$.

4. The fact that $\underline{\lim}\left(\lambda^{(m)} \,|\, p^{(m)} \right)_{L^2} \geq 0$ is a consequence of the monotonicity of $DA^{(n)}(c) : W_1 \to W_{-1}$ for every $c \in W_1$. Indeed, given $\overline{c} \in W_1$ we have

$$\left\langle DA^{(n)}(c)\overline{c}, \overline{c} \right\rangle_{W_1} = \lim_{t \to 0} \frac{1}{t^2} \left\langle A^{(n)}(c + t\overline{c}) - A^{(n)}c, (c + t\overline{c}) - c \right\rangle_{W_1} \geq 0$$

by the monotonicity of $A^{(n)}$.

5. Let us fix $i \in \{0, \ldots, M-1\}$ and representatives of the equivalence classes $c^{(\infty)}, (c^{(m)})$. Further, define $Z := \{x \in \Omega : k_1 < c^{(\infty)}_{i+1}(x) < k_2\}$. Since $c^{(m)}_{i+1}$ converges to $c^{(\infty)}_{i+1}$ in W_1, a subsequence converges almost everywhere on Ω. Without loss of generality, we assume that $(c^{(m)}_{i+1})$ itself has this convergence property. Moreover, we know that $\varepsilon(\alpha_m) \to 0$. Hence, for almost all $x \in Z$ there exists $m_0(x)$ such that $k_1 + \varepsilon(\alpha_m) < c^{(m)}_{i+1}(x) < k_2 - \varepsilon(\alpha_m)$ for all $m \geq m_0(x)$.

From the properties of β_α it follows that $\lambda^{(m)}_i(x) = 0$ for almost all $x \in Z$ and $m \geq m_0(x)$. Consequently, $\lambda^{(m)}_i$ converges to 0 almost everywhere on Z. ∎

We remark that compared to weaker forms of stationarity, for instance, those contained in [7] for certain classes of optimal control problems for variational inequalities, C-stationarity represents a sharper stationarity notion avoiding spurious stationarity points. A numerical realization based on an extension of the algorithms in [27] to the CH-NS setting will be the subject of future work.

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