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Optimal Control of a Threatened Wildebeest-lion Prey-predator System Incorporating a Constant Prey Refuge in the Serengeti Ecosystem

Thadei Damas Sagamiko[1, *], Nyimvua Shaban[1], Cuthbert Leonard Nahonyo[2], Oluwole Daniel Makinde[3]

[1]Department of Mathematics, University of Dar Es Salaam, Dar Es Salaam, Tanzania
[2]Department of Zoology and Wildlife Conservation, University of Dar Es Salaam, Dar Es Salaam, Tanzania
[3]Faculty of Military Science, Stellenbosch University, Stellenbosch, South Africa

Email address:

tsagamiko@gmail.com (T. D. Sagamiko), shabanmbare@gmail.com (N. Shaban), nahonyo@udsm.ac.tz (C. L. Nahonyo), makinded@gmail.com (O. D. Makinde)

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Abstract: In this paper a two species prey-predator model is developed in which prey is wildebeest and predator is lion and both are threatened by poaching, drought and diseases.The system is found in the Serengeti ecosystem.The model is constructed based on Holling type II functional response incorporating a constant prey refuge. We apply optimal control theory to investigate optimal strategies for controlling the threats in the system where anti-poaching patrols are used for controlling poaching, construction of dams for mitigating drought and vaccination for diseases control. The possible impact of using combinations of three controls either one at a time or two at a time on the threatened system plus a refuge factor is examined. All control strategies have shown significant increase in prey and predator populations . However, the best result is achieved by controlling all threats together. The effect of variation of prey refuge m to the control of threats is studied and results indicate that increase of m causes more prey individuals to be saved and reduces the number of predator individuals saved. This behaviour agrees with theoretical results obtained in co-existence equilibrium point.

Keywords: Optimal Control, Prey-predator System, Prey Refuge, Threat, Gregariousness

1. Introduction

Population dynamics is the dominant branch of mathematical biology that deals with forces affecting changes in population densities or affecting the form of population growth. It is clear that predator population depends on their prey species for survival and lower the survival and fecundity rate of prey species. Therefore, predator population is affected by changes in prey population in a complex predator-prey relationship (Chakraborty and Das [4]).

Prey species have evolved survival strategies on reducing their predation risk through employing techniques like gregariousness, fight, camouflage and fighting back among others. One other strategy is the use of “prey refuge” where prey species are protected from predation as can be described in Rosenzweig-MacArthur Fig. 1. Refuge use by prey decreases predation rate. Typically prey respond to predators presence by increasing their use of refuges, and greater predation risk often results in a stronger shift into refuges (Sih [21]; Sih *et al* [22]).

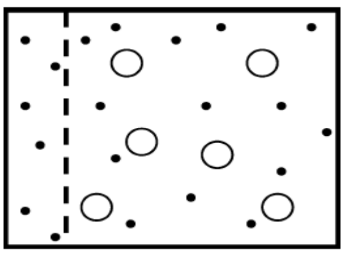

Fig. 1. Conceptual diagram of the Rosenzweig-MacArthur model with prey refuge. Small black circles represent the prey, large circles represent predators. The vertical dashed line represents the boundary of a refuge, where prey are invulnerable to predators (Gonzalez-Olivares and Ramos-Jiliberto [8]).

The study of the consequences of hiding behaviour of prey on the dynamics of predator-prey interactions can be recognized as a major issue in applied mathematics and theoretical ecology (Ma *et al* [15]). In the Serengeti ecosystem the migratory wildebeest uses refuge when they are in the western part of Serengeti where there is human habitation hence reducing their risk of predation since most predators tend to avoid inhabited areas. On the other hand wildebeest have adapted to utilizing human inhabited areas during migration. Wildebeest also practice refuge due to their gregariousness behaviour. By being gregariuos animals tend to reduce predation risk by first lowering the chance of being detected by a predator but also lowering the probability of being preyed upon out of many potential prey individuals. In some cases gregarious animals may join hands to fight back a predator and so increasing their chance of survival (Riipi *et al* [18]; Sillen-Tallberg and Leimar [23]).Actually most species typically preyed by Serengeti lions tend to be gregariuos (Chakraborty and Das [4]).

Wildebeest and lions in the Serengeti ecosytem have been faced with several threats among them are poaching, severe drought and diseases outbreaks (Borner [3]; Roelke *et al* [19]). For example, poaching has become a threat to many migratory populations, particularly as human populations around protected areas increase (Bolger *et al* [2]; Haris *et al* [9]). It has been reported that local consumption of bushmeat from the Serengeti National park and surrounding areas is responsible for approximately 70,000-129,000 wildebeest deaths per year as indicated in GEAS [7] and any further increase in the amount of poaching could lead to decline in the wildebeest population in the Serengeti-Mara ecosystem (Hopcraft et al [11]). However, lion killing not only to Serengeti ecosystem but for the entire East Africa is mainly due to Maasai retaliation as lions prey their livestock triggering retaliatory killings (Ikanda and Packer [12]). This happens because many parts of Maasai land have been preserved as wildelife protected areas (e.g. Serengeti, Tarangire and Amboseli) and Game reserves (e.g. Mkomazi and Loliondo) but none of these protected areas are fenced and lions are reported to frequently kill Maasai cattle in adjacent rangelands (Kolowski and Holekam [14]).

A number of studies such as that of (Chakraborty and Das[4]; Gonzalez-Olivares and Ramos-Jiliberto [8]; Ma et al [15]), have dealt with the role of refuge in prey-predator system but none of them have considered the aspect of Optimal control when the system is threatened. Therefore this paper aims at investigating the application of optimal control theory to a threatened wildebeest-lion prey-predator system found in the Serengeti ecosystem by focusing on effect of prey refuge with Holling type II functional response.

This paper is organized as follows: Section 2, we develop a model, and theoretical results such as boundedness of the solution, equilibrium analysis and dynamic behaviour of the system are carried out. In Section 3, analysis of optimal control where condition of existence is described. However, scenarios for different control strategies are considered and the results are discussed in Section 4 together with the study on the effect of variation of prey refuge m to the optimal control strategies has been taken into consideration. Lastly, Section 5 presents conclusions about the proposed optimal control strategies and the effect of varying prey refuge to the selected optimal control strategies. The outcome tend to display a significant increase in the number of prey individuals as prey refuge increases and decreasing number of predator individuals.

2. The Model with Threats

Consider two populations of different species: x, a prey population, and y, a predator population. The prey species is wildebeest and predator species is lion. We assume that both species are threatened by poaching, drought and disease which are considered to affect their survival. Prey species are assumed to grow logistically to the carrying capacity. The two species are poached and affected with drought at different rates and the rest of the threat affect the prey and predators at the same rate. We also assume that there is a refuge habitat where prey species are protected from predation and nonrefuge habitat in which prey species are exposed to predation. Thus according to Hollying type II functional response (Holling [10]) the two populations are modelled as follows:

$$\begin{aligned}\frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right)-\frac{w(1-m)xy}{1+a(1-m)x}-p_1x-q_1x-dx\\ \frac{dy}{dt} &= -a_2y+\frac{wb(1-m)xy}{1+a(1-m)x}-p_2y-q_2y-dy\end{aligned} \tag{2.1}$$

where x is the prey density at time t, y is the predator density at time t, r is the intrinsic prey growth rate, k is the prey carrying capacity, w is the maximum per capita predation rate, b is the predator biomass to the prey (conversion rate), p_1 and p_2 are poaching rates for prey and predator species respectively, q_1 and q_2 are death rates of prey and predator respectively due to drought , d is death rate of both species due to disease and m is the proportion of prey population not exposed to predation, that it protects mx of the prey and leaves $(1-m)x$ of the prey available to the predator. Note that $m\in[0,1)$ is constant.

2.1. Boundness of the System

The solutions of the system (2.1) represent the densities of the interacting populations and they have their own ecological meaning, that is to say they must be positive and bounded.

Lemma 2.1 *All the solutions of the system (2.1) which start in* R_+^2 *are uniformly bounded.*

Proof 2.1. To prove the theorem, we define a function

$$Z(t)=x(t)+\frac{1}{b}y(t)\ .$$

Therefore, time derivative yields

$$\frac{dz}{dt}=\frac{dx}{dt}+\frac{1}{b}\frac{dy}{dt} \tag{2.2}$$

By substituting the equations of the model (2.1) into (2.2) we get

$$\frac{dz}{dt}=\left[rx\left(1-\frac{x}{k}\right)-\frac{w(1-m)xy}{1+a(1-m)x}-\left(p_1+q_1+d\right)x\right]+\left[\left(\frac{1}{b}\right)\frac{wb(1-m)xy}{1+a(1-m)x}-\frac{1}{b}\left(a_2+p_2+q_2+d\right)y\right]$$

$$\Rightarrow\frac{dz}{dt}=rx\left(1-\frac{x}{k}\right)-\left(p_1+q_1+d\right)x-\frac{1}{b}\left(a_2+p_2+q_2+d\right)y\,;$$

All terms with interspecific competition are cancelled

By letting $E_1=p_1+q_1+d$, and $e_2=a_2+p_2+q_2+d$ we have

$$\frac{dz}{dt}=rx\left(1-\frac{x}{k}\right)-E_1x-\frac{w}{w_1}(e_2)y$$

$$\frac{dz}{dt}\le(r-E_1)x-\frac{rx^2}{k}-\frac{w}{w_1}e_2y$$

Now for each v chosen arbitrary, we have

$$\frac{dz}{dt}+vz\le(r-E_1+v)x-\frac{rx^2}{k}-\frac{w}{w_1}(e_2-v)y$$

But $\max\left[rx\left(1-\frac{x}{k}\right)\right]$ is $\frac{k}{4r}$, This implies that

$$\frac{dz}{dt}+vz\le\frac{k}{4r}\left(r-E_1+v\right)^2-\frac{r}{k}\left(x^2-\left(r-E_1+v\right)\frac{k}{r}x+k^2\left(\frac{r-E_1+v}{4r^2}\right)^2\right)-\frac{w}{w_1}\left(e_2-v\right)y$$

Using the technique of completing the squares we get,

$$\frac{dz}{dt}+vz\le\frac{k}{4r}\left(r-E_1+v\right)^2-\frac{r}{k}\left(x-\left(r-E_1+v\right)\frac{k}{2r}\right)^2-\frac{w}{w_1}\left(e_2-v\right)y$$

Choosing $v<e_2$ and applying the results by (Agnihotri [1]), we have $\frac{dz}{dt}+vz\le L_1$, where $L_1=\frac{k}{4r}\left(r-E_1+v\right)^2$ and solving the resulting differential inequality with integrating factor $I=e^{vt}$, we have

$$z(t)=\frac{L_1}{v}+ce^{-vt} \tag{2.3}$$

where c is a constant of integration.

At $t=0$, $z(x(0),y(0))=z$ and substituting into (2.3) we obtain $z(x(0),y(0))=\frac{L_1}{v}+c$, which implies $c=z(x(0),y(0))-\frac{L_1}{v}$.

Therefore (2.3) becomes;

$$z(t)=\frac{L_1}{v}+z\left(x(0),y(0)-\frac{L_1}{v}\right)e^{-vt}$$
$$=\frac{L_1}{v}\left(1-e^{-vt}\right)+z\left(x(0),y(0)\right)e^{-vt}$$

So,

$$0\le z\left(x(t),y(t)\right)\le\frac{L_1}{v}\left(1-e^{-vt}\right)+z\left(x(0),y(0)\right)e^{-vt} \tag{2.4}$$

As $t\Rightarrow\infty$, it gives

$$0\le z\left(x(t),y(t)\right)\le\frac{L_1}{v}(=K)$$

z is bounded and from positivity of x and y,

$$0\le x\le K\,;\ 0\le y\le K$$

2.2. Equilibrium Analysis

In this part we establish conditions for the existence of the equilibrium points of the system (2.1). By equating (2.1) to zero we find that the system has four possible nonnegative equilibria namely $\alpha_0(0,0)$, $\alpha_1(x^*,0)$, $\alpha_2(0,y^*)$ and the co-existence equilibrium $\alpha_3(x^*,y^*)$. The existence of $\alpha_0(0,0)$ is trivial, Therefore, we show the existence of other equilibria as follows;

- Existence of $\alpha_1(x^*,0)$ with $x^*>0$

Let $y=0$, then equation (2.1) gives:

$$x\left[r\left(1-\frac{x}{k}\right)-\left(p_1+q_1+d\right)\right]=0 \tag{2.5}$$

from which we have $x^*=\frac{k\left[r-\left(p_1+q_1+d\right)\right]}{r}$. Thus

$$\alpha_1(x^*,0)=\alpha_1\left(\frac{k\left[r-\left(p_1+q_1+d\right)\right]}{r},0\right).$$

Therefore from the fact that $x^*>0$, the equilibrium α_1 exists if

$$r > \left(p_1 + q_1 + d\right) \tag{2.6}$$

Thus in the absence of predator y, the total prey death rate due to threats must be less than its intrinsic growth rate for the equilibrium $\alpha_1\left(x^*, 0\right)$ to exist.

- Existence of $\alpha_2\left(0, y^*\right)$ with $y^* > 0$

Let $x = 0$, then equation (2.1) gives:

$-y\left(a_2 + p_2 + q_2 + d\right) = 0$ whcich yields $y^* = 0$ the result which agrees with the model assumption that the predator's only source of food is the prey. Therefore in the absence of prey, the predator goes to extinction

- Co-existence equilibrium point $\alpha_3\left(x^*, y^*\right)$

The co-existence equilibrium point is;

$$x^* = \frac{a_2 + E_2}{\left[wb - a\left(a_2 + E_2\right)\right](1-m)}$$

$$y^* = \frac{rb}{k}\left\{\frac{k\left[wb - \left(E_2 + a_2\right)a\right](1-m) - \left(a_2 + E_2\right)}{\left[\left(wb - a\left(a_2 + E_2\right)\right)(1-m)\right]^2}\right\} - \frac{E_1}{\left[wb - \left(a_2 + E_2\right)a\right](1-m)}$$

where $E_1 = p_1 + q_1 + d$ and $E_2 = p_2 + q_2 + d$

From the expression for $\left(x^*, y^*\right)$, it is clear that a nontrivial (interior) equilibrium point exists for system (2.1) only if the total threat rates E_1 and E_2 satisfy

$$\left[wb - a\left(a_2 + E_2\right)\right] > 0, \tag{2.7}$$

$$\left(a_2 + E_2\right)[rb + ak(1-m)(rb - E_1)] - wb(1-m)k(rb - E_1) < 0 \tag{2.8}$$

Proof for (2.8)

From

$$y^* = \frac{rb}{k}\left\{\frac{k\left[wb - \left(E_2 + a_2\right)a\right](1-m) - \left(a_2 + E_2\right)}{\left[\left(wb - a\left(a_2 + E_2\right)\right)(1-m)\right]^2}\right\}, - \frac{E_1}{\left[wb - \left(a_2 + E_2\right)a\right](1-m)}$$

Let $C = \left[wb - a\left(a_2 + E_2\right)\right](1-m)$; $Q = a_2 + E_2$ and $M = 1 - m$

Therefore $C = \left(wb - aQ\right)M$

From nontrivial $y^* > 0$, we have

$$y^* = \left\{\frac{rb}{k}\left(\frac{kC - Q}{C^2}\right) - \frac{E_1}{C}\right\} > 0$$

and multiplying both sides by $-C^2k$, we obtain

$$CkE_1 - rb\left(kC - Q\right) < 0$$

$$\Rightarrow CkE_1 - rbkC + rbQ < 0$$

$$\Rightarrow kC\left(E_1 - rb\right) + rbQ < 0$$

$$\Rightarrow rbQ - \left(rb - E_1\right)kC < 0$$

and expanding using $C = \left(wb - aQ\right)M$, we get

$$rbQ - kM\left(wb - aQ\right)\left(rb - E_1\right) < 0$$

$$\Rightarrow rbQ + \left(aQkM - wbkM\right)\left(rb - E_1\right) < 0$$

$$\Rightarrow rbQ + aQkM\left(rb - E_1\right) - wbkM\left(rb - E_1\right) < 0$$

$$\Rightarrow Q\left(rb + akM\left(rb - E_1\right)\right) - wbkM\left(rb - E_1\right) < 0$$

But $Q = a_2 + E_2, M = 1 - m$ hence

$$\left(a_2 + E_2\right)[rb + ak(1-m)(rb - E_1)] - wb(1-m)k(rb - E_1) < 0$$

We see that an increase in threat effect to the predator will increase x^*, which is natural as E_2 increases leads to decrease in the predator population and hence enhancing the survival rate of the prey. We also observe that as m increases x^* does the same. More observations are that y^* decreases when E_1 increases and this happens because of the tendency of increasing E_1 reduces prey population hence loss of food for the predator.

2.3. Dynamic Behaviour

In this part we study the stability properties of the equilibrium points α_0, α_1 and α_3. The stability analysis of the equilibrium is studied by computing the variational matrices for each equilibrium point. However, the local stability is established through Jacobian matrix of the systems and finding the eigenvalues evaluated at each equilibrium point. For stability of the equilibrium points, the real parts of the eigenvalues of the Jacobian matrix must be negative. For linearized systems, the Jacobian matrix is given by

$$J\left(\alpha_i\right) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_n} \end{pmatrix},$$

For the system (2.1), its corresponding Jacobian matrix is

$$J(\alpha_i) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

Where

$$A_{11} = r\left(1-\frac{x}{k}\right) - \frac{rx}{k} - \frac{w(1-m)}{1+a(1-m)x} + \frac{w(1-m)^2 axy}{(1+a(1-m)x)^2} - E_1$$

$$A_{12} = -\frac{w(1-m)x}{1+a(1-m)x}$$

$$A_{21} = \frac{wb(1-m)y}{1+a(1-m)x} - \frac{wb(1-m)^2 axy}{(1+a(1-m)x)^2}$$

$$A_{22} = -a_2 + \frac{wb(1-m)x}{1+a(1-m)x} - E_2$$

Hence the Jacobian of the system about the equilibrium point $\alpha_0(0,0)$ is given by

$$J_0 = \begin{pmatrix} r-E_1 & 0 \\ 0 & -(a_2+E_2) \end{pmatrix}$$

We find that the eigenvalues for the eigenvalues for the stead state $(0,0)$ are $r-E_1$ and $-(a_2+E_2)$. Therefore, $\alpha_0(0,0)$ is a sadle point.

For the predator-free equilibrium point $\alpha_1\left(-\frac{k(E_1-r)}{r}, 0\right)$, the Jacobian matrix is

$$J_1 = \begin{pmatrix} r\left(1+\frac{E_1-r}{r}\right)-r & \dfrac{w(1-m)k(E_1-r)}{r\left(1-\frac{a(1-m)k(E_1-r)}{r}\right)} \\ 0 & -a_2 - \dfrac{wb(1-m)k(E_1-r)}{r\left(1-\frac{a(1-m)k(E_1-r)}{r}\right)} - E_2 \end{pmatrix}$$

$$= \begin{pmatrix} -(r-E_1) & \dfrac{-w(1-m)k(r-E_1)}{r+a(1-m)k(r-E_1)} \\ 0 & -(a_2+E_2)+\dfrac{wb(1-m)k(r-E_1)}{r+a(1-m)k(r-E_1)} \end{pmatrix} \quad (2.9)$$

The eigenvalues of the matrix are $-(r-E_1)$ and $-(a_2+E_2)+\frac{wb(1-m)k(r-E_1)}{r+a(1-m)k(r-E_1)}$

Hence α_1 is locally asymptotically stable if and only if

$$-(a_2+E_2)+\frac{wb(1-m)k(r-E_1)}{r+a(1-m)k(r-E_1)} < 0$$

That is

$$E_2 > \left\{\frac{wb(1-m)k(r-E_1)}{r+a(1-m)k(r-E_1)} - a_2\right\}$$

For Co-existence equilibrium point $\alpha_3(x^*, y^*)$,the Jacobian matrix is as follows:

$$J_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where

$$a_{11} = r\left(1+\frac{G}{Bk}\right) + \frac{rG}{Bk} - \frac{b(1-m)Q}{B^2k\left(1-\frac{a(1-m)G}{B}\right)}$$

$$-\frac{b(1-m)^2 GQa}{B^3k\left(1-\frac{a(1-m)G}{B}\right)^2} - q_1E_1$$

$$a_{12} = \frac{b(1-m)G}{B\left(1-\frac{a(1-m)G}{B}\right)}$$

$$a_{21} = \frac{cb(1-m)Q}{B^2k\left(1-\frac{a(1-m)G}{B}\right)} + \frac{cb(1-m)^2 GQa}{B^3k\left(1-\frac{a(1-m)G}{B}\right)^2}$$

$$a_{22} = -a_2 - \frac{cb(1-m)G}{B\left(1-\frac{a(1-m)G}{B}\right)} - q_2E_2$$

Morever Q, B and G are defined as;

$$Q = c\begin{pmatrix} rk(1-m)\left[cb - a_2a - q_2E_2a\right] + q_1E_1(1-m)k \\ *\left[a_2a + q_2E_2a - cb\right] - r\left(a_2+q_2E_2\right) \end{pmatrix}$$

$$B = (1-m)\left[a(a_2+q_2E_2) - cb\right]$$

$$G = a_2 + q_2E_2$$

The local stability α_3 is stated in the following proposition

Proposition 2.3. Suppose $\Delta = a_{11}a_{22} - a_{12}a_{21}$ and $tr = a_{11} + a_{22}$ where Δ and tr stands for determinant and trace respectively, then α_3 is local asymptotically stable if $\Delta > 0$ and $tr < 0$.

3. Analysis of Optimal Control

We introduce into model (2.1), time dependent control efforts on anti-poaching patrols ($u_1(t)$), construction of dams ($u_2(t)$), and vaccination ($u_3(t)$) as controls to curtail the threats to the prey-predator system. The prey-predator model (2.1) thus becomes:

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right)-\frac{w(1-m)xy}{1+a(1-m)x}-\left(1-u_1(t)\right)p_1x-\left(1-u_2(t)\right)q_1x-\left(1-u_3(t)\right)dx \\ \frac{dy}{dt} &= -a_2y+\frac{wb(1-m)xy}{1+a(1-m)x}-\left(1-u_1(t)\right)p_2y-\left(1-u_2(t)\right)q_2y-\left(1-u_3(t)\right)dy \end{aligned} \tag{3.1}$$

We assume the objective is to maximize the species population size at the final time of control while minimizing the cost. These costs are due to control strategies applied such as cost of antipoaching, conservation cost and diseases control cost. We assume the total population $(x+y)$ is to be maximized to the final time, with different relative weights applied to prey and predator populations. For simplicity we take $u_1(t)$ as u_1, $u_2(t)$ as u_2 and $u_3(t)$ as u_3. Thus the objective functional is:

$$J = \max\left[\begin{array}{l}\left(B_1x(T)+B_2x(T)\right) \\ -\int_0^T\left(A_1\frac{u_1^2}{2}+A_2\frac{u_2^2}{2}+A_3\frac{u_3^2}{2}\right)dt\end{array}\right] \tag{3.2}$$

where A_1, A_2, A_3, B_1, B_2 are positive weights. The term $A_1\frac{u_1^2}{2}$ is the cost of control efforts on antipoaching strategy, $A_2\frac{u_2^2}{2}$ is the cost of control efforts on reducing the effect of drought to species using conservation strategies like construction of dams, $A_3\frac{u_3^2}{2}$ is the cost of disease control strategy.

3.1. Existence of an Optimal Control Tripple

Here we prove that at least one optimal control exists that satisfies the optimal control formulation, Equations (3.1)-(3.2).

Theorem 3.1.1. There exist $\vec{u}^* = \left(u^*_1, u^*_2, u^*_3\right) \in U$ which maximizes the objective functional $J(u_1, u_2, u_3)$

Proof. The proof given in Fleming and Rishel (Ref.5) is valid here as such:

- The set of controls and corresponding state variables is non-empty,
- The control set U is closed and convex,
- The integrand of the objective functional (3.2) is concave on the control set U
- The model (3.1) is linear in control variables and is bounded by a linear system in the state and control variables

Therefore the conditions for existence of an optimal control are satisfied.

3.2. Characterization Process

Pontryagin's maximum principle (Pontragin et al [17]) which provides necessary condition for an optimal control problem, converts equations (3.1) and (3.2) into a problem of maximizing point-wise a Hamiltonian H with respect to u_1, u_2 and u_3.

$$\begin{aligned} H = &-\left(A_1\frac{u_1^2}{2}+A_2\frac{u_2^2}{2}+A_3\frac{u_3^2}{2}\right) \\ &+L_1\left\{\begin{array}{l} rx\left(1-\frac{x}{k}\right)-\frac{w(1-m)xy}{1+a(1-m)x}-\left(1-u_1(t)\right)p_1x \\ -\left(1-u_2(t)\right)q_1x-\left(1-u_3(t)\right)dx \end{array}\right\} \\ &+L_2\left\{\begin{array}{l} -a_2y+\frac{wb(1-m)xy}{1+a(1-m)x}-\left(1-u_1(t)\right)p_2y \\ -\left(1-u_2(t)\right)q_2y-\left(1-u_3(t)\right)dy \end{array}\right\} \end{aligned} \tag{3.3}$$

where L_1 and L_2 are the adjoint variables or co-state variables. Applying Pontryagin's Maximum Principle [19] and the existence results for the optimal control from [18], the following preposition is obtained.

Proposition 3.1. For the optimal control tripple u_1^*, u_2^* and u_3^* that maximizes $J(u_1, u_2, u_3)$ over U, then there exists adjoint variables L_1 and L_2 satisfying

$$\begin{aligned} \frac{dL_1}{dt} = -\frac{\partial H}{\partial x} = -\Bigg\{ & L_1\left[r\left(1-\frac{2x}{k}\right)-\frac{wy(1-m)}{\left(1+ax(1-m)\right)^2}\right] \\ & -\left[\left(1-u_1\right)p_1+\left(1-u_2\right)q_1+\left(1-u_3\right)d\right] \\ & +L_2\left(\frac{wby(1-m)}{\left(1+ax(1-m)\right)^2}\right)\Bigg\} \end{aligned}$$

$$\frac{dL_2}{dt} = -\frac{\partial H}{\partial y} = -\left\{ L_1\left[\frac{w(1-m)x}{(1+ax(1-m))}\right] + L_2\left[\begin{array}{l} -a_2 + \dfrac{wb(1-m)x}{1+ax(1-m)} \\ -\left[(1-u_1)p_2 + (1-u_2)q_2 + (1-u_3)d\right] \end{array}\right]\right\} \tag{3.4}$$

and with transversality condition as $L_1(T) = B_1$ and $L_2(T) = B_2$.

By optimality condition, we have $\frac{\partial H}{\partial u} = 0$ at u^*, that is

$$\frac{\partial H}{\partial u_1} = 0 \text{ at } u_1^*, \quad \frac{\partial H}{\partial u_2} = 0 \text{ at } u_2^* \text{ and } \frac{\partial H}{\partial u_3} = 0 \text{ at } u_3^*$$

But

$$\frac{\partial H}{\partial u_1} = -A_1u_1 + L_1p_1x + L_2p_2y = 0 \text{ at } u_1^*,$$

Hence,

$$u_1^* = \frac{L_1p_1x + L_2p_2y}{A_1}. \tag{3.5}$$

$$\frac{\partial H}{\partial u_2} = -A_2u_2 + L_1q_1x + L_2q_2y = 0 \text{ at } u_2^*$$

Hence

$$u_2^* = \frac{L_1q_1x + L_2q_2y}{A_2} \tag{3.6}$$

$$\frac{\partial H}{\partial u_3} = -A_3u_3 + L_1dx + L_2dy = 0 \text{ at } u_3^*,$$

Hence

$$u_3^* = \frac{L_1dx + L_2dy}{A_3}. \tag{3.7}$$

The following characterization holds on the interior of the control set U

$$u_1^* = \min\left\{1, \max\left(0, \frac{L_1p_1x + L_2p_2y}{A_1}\right)\right\},$$

$$u_2^* = \min\left\{1, \max\left(0, \frac{L_1q_1x + L_2q_2y}{A_2}\right)\right\},$$

$$u_3^* = \min\left\{1, \max\left(0, \frac{L_1dx + L_2dy}{A_3}\right)\right\} \tag{3.8}$$

Note that the state system (3.1) has initial time conditions and the co-state system (3.4) has final time conditions.

4. Numerical Simulation and Discussions

In this section, various forms of optimal control strategies that can be applied to control a threatened prey-predator system are studied. State system, costate system and optimal characterization in Equations (3.1), (3.4) and (3.8) respectively, are solved numerically in an algorithmic form as explained in Algorithm 4.1

Algorithm 4.1 : Optimal control

1. Divide the total time interval into N equal subintervals and set the state at different times as $\vec{x} = (x_1, x_2, ..., x_{N+1})$ and the costate variables as $\vec{L} = (L_1, L_2, ..., L_{N+1})$.
2. Assume control takes zero over the time intervals i.e. $\vec{u} = (0, 0, ..., 0)$ for starting iteration.
3. Using the initial condition $x(0) = x_0$ solve the state according to the ODE with the values of $\vec{u}$ forwardly.
4. Using the transversality condition $L_{N+1} = L(T)$, $(T = \text{final time})$ and the values for $\vec{u}$ as well as previously evaluated values for $\vec{x}$, solve $\vec{L}$ in time from costate differential equation in backward process
5. Update the control entering the new $\vec{x}$ and $\vec{L}$ through the rule $u^* = \min(u_{max}, \max(u_{sig}, u_{min}))$, where

$$u^* = \begin{cases} u_{min} & \text{if } \dfrac{\partial H}{\partial u} < 0 \\ u_{sig} & \text{if } \dfrac{\partial H}{\partial u} = 0 \\ u_{max} & \text{if } \dfrac{\partial H}{\partial u} > 0 \end{cases}$$

6. If the solutions of the variables (excluding the control variable) are convergent i.e. if the values of the variables in this iteration and the last iteration are negligibly close, then the last iteration is the complete solution. Otherwise, return to step 3.

From different combinations of the controls, seven strategies are studied numerically with $m = 0.4$ chosen arbitrarily from the interval $m \in [0,1)$ .These strategies are as indicated in Table 4.1.

Table 4.1. *Optimal control strategies*

Strategy	Description
Strategy A	Application of anti-poaching patrols for controlling poaching
Strategy B	Construction of dams for mitigating drought effects
Strategy C	The use of vaccines for controlling diseases
Strategy D	Combination of application of anti-poaching patrols and construction of dams
Strategy E	Combination of application of anti-poaching patrols and the use of vaccines
Strategy F	Combination of construction of dams and the use of vaccines
Strategy G	Combination of application of anti-poaching patrols, construction of dams and the use of vaccines

The system parameters are $w = 0.674$ (Fryxell et al [6]), $b = 0.16$ (Schaller [20]), $r = 1$ (Mduma [16]), $a_2 = 0.01$ (Schaller [20]), $a = 1$ (assumed), $k = 500$, $p_1 = 0.01$ (Fryxell et al [16]), $p_2 = 0.008$ (Schaller [20]), $q_1 = 0.15$ (Sinclair et al [24]), $q_2 = 0.01$ (Sinclair et al [24]), $d = 0.082$ (Mduma [16]), together with the initial states $x(0) = 40$ and $y(0) = 20$. Assume the weights for prey and predator at final time are being kept fixed as $B_1 = 1000$ and $B_2 = 1500$. However the weights of controls u_1, u_2 and u_3 are respectively $A_1 = 1200$, $A_2 = 4000$ and $A_3 = 9000$.

Next, we investigate the effect of the following optimal control strategies on the threatened prey-predator population.

Strategy A: Application of anti-poaching patrols for controlling poaching

With this strategy, only the control on anti-poaching patrols u_1 is used to optimize the objective function J while control on construction of dams u_2 and the control on use of vaccine u_3 are set to zero. The results in Fig. 2a show a significant difference in the prey and predator populations with optimal strategy compared to prey and predator populations without control. This shows that eliminating poaching in the system would lead to direct increase among prey and predator populations. However the increase of population due to control is higher in prey species than predator species and this is because of prey refuge which protects them from predation hence no reduction due to it. The control profile is shown in Fig. 2b, here we see that the optimal anti-poaching patrol control u_1 increases gradually till the time $t = 7$ years where it reaches upper bound and continues to final time. We observe that as the effort of control increase there is also increase in number of individuals saved.

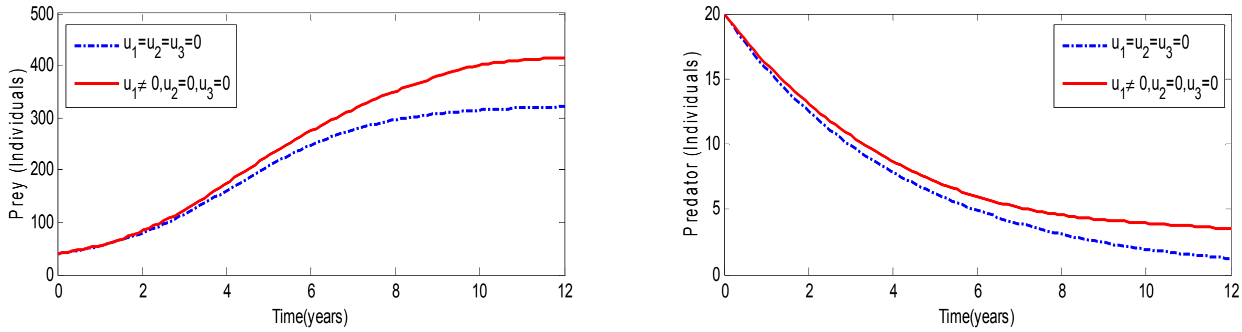

Fig. 2a. *Simulations of a threatened prey-predator model showing the effect of optimal application of anti-poaching patrols.*

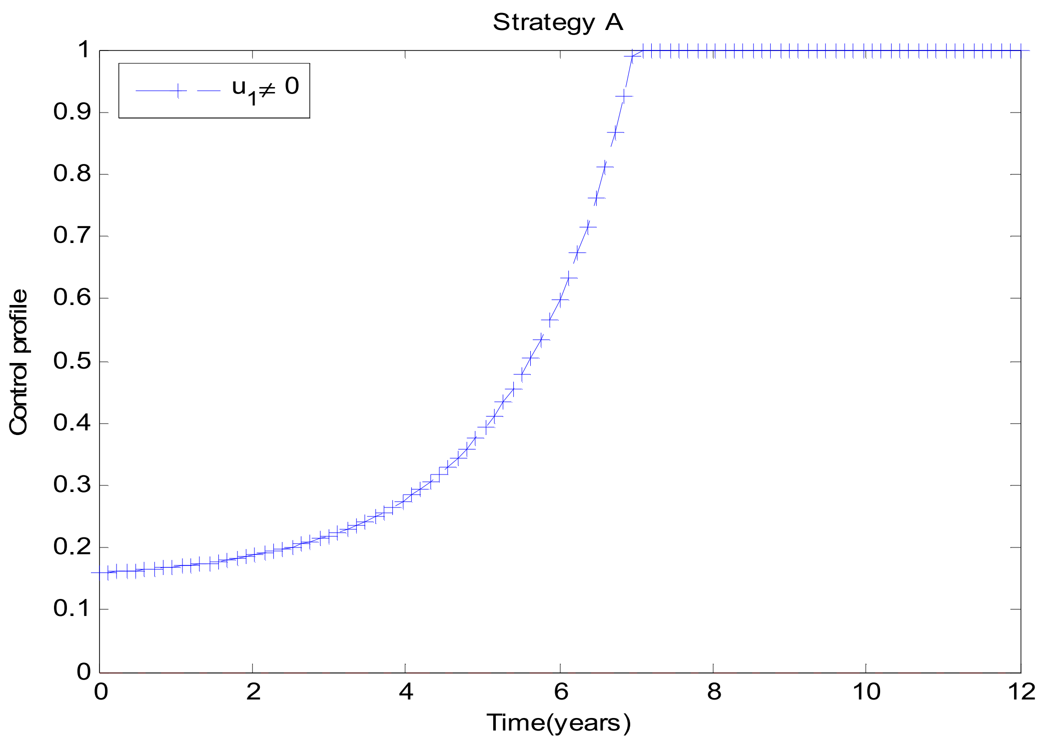

Fig. 2b. *Control profile for* u_1.

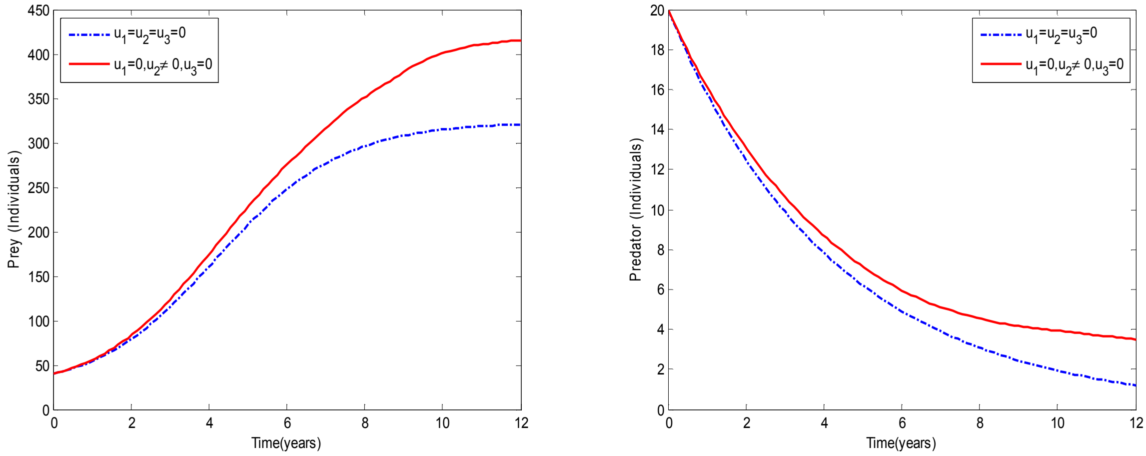

Fig. 3a. *Simulations of a threatened prey-predator model showing the effect of optimal construction of dams.*

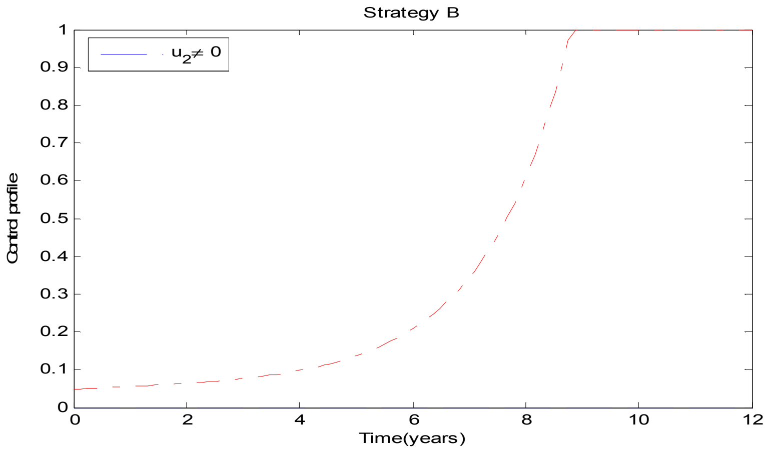

Fig. 3b. *Control profile for* u_2.

Strategy B: Construction of dams for mitigating drought effects.

With this strategy, only the control on the construction of dams u_2 is used to optimize the objective function J while control on anti-poaching patrol u_1 and the control on use of vaccine u_3 are set to zero. The results in Fig. 3a show a significant difference in the prey and predator populations with optimal strategy compared to prey and predator populations without control. This suggests that eliminating drought in the system would lead to direct increase among prey and predator populations. However the increase of population due to control is higher in prey species than predator species and this is because of prey refuge which protects them from predation hence no reduction due to it. The control profile is shown in Fig. 3b, here we see that the optimal anti-poaching patrol control u_1 increases gradually till the time $t = 9$ years where it reaches upper bound and continues to final time. We observe that as the effort of control increases there is also increase in number of individuals saved.

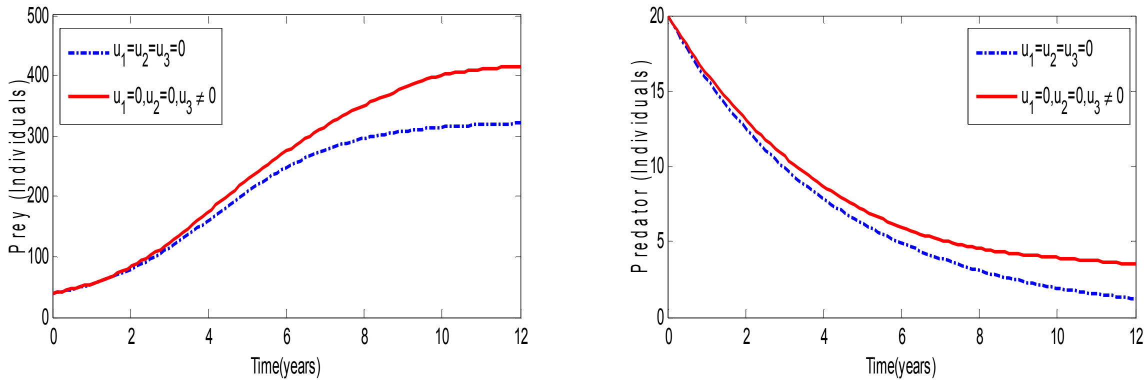

Fig. 4a. *Simulations of a threatened prey-predator model showing the effect of optimal use of vaccine.*

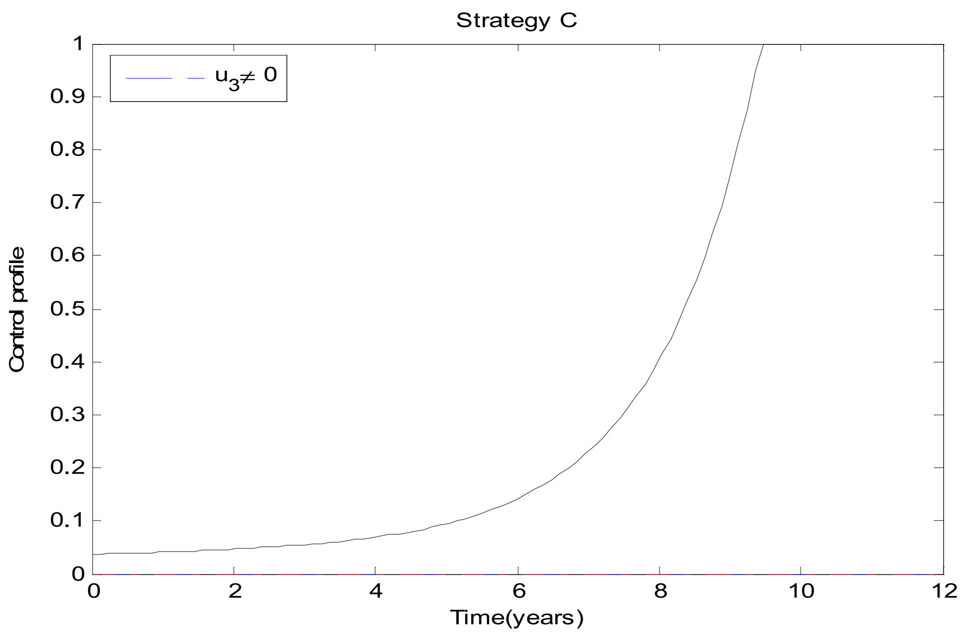

Fig. 4b. *Control profile for* u_3 .

Strategy C: The use of vaccines for control of diseases

With this strategy, only the control on the use of vaccine u_3 is used to optimize the objective function J while control on anti-poaching patrol u_1 and the control on construction of dams u_2 are set to zero. The results in Fig. 4a show a significant difference in the prey and predator populations with optimal strategy compared to prey and predator populations without control. This shows that eliminating diseases in the system would lead to direct increase among prey and predator populations. However the increase of the populations due to control is higher in prey species than predator species and this is because of prey refuge which protects them from predation hence no reduction due to it. The control profile is shown in Fig. 4b, here we see that the optimal anti-poaching patrol control u_1 increases gradually till the time $t = 10$ years where it reaches upper bound and continues to final time. We observe that as the effort of control increases there is also increase in number of individuals saved. Among single control strategies employed in this system, this one has shown highest impact especially to prey population.

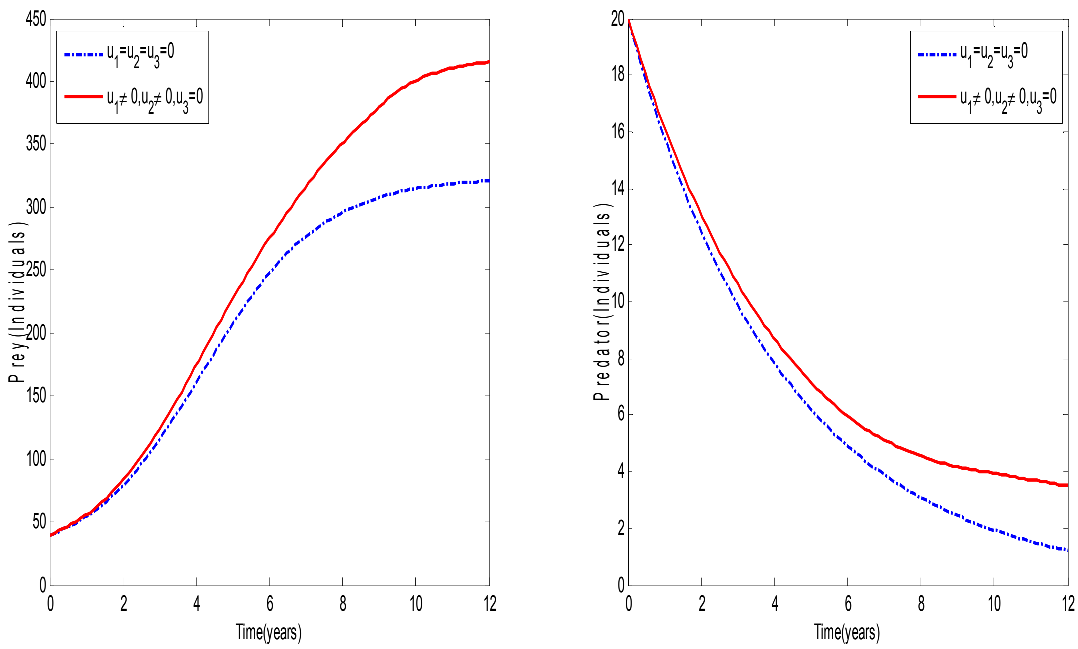

Fig. 5a. *Simulations of a threatened prey-predator model showing the effect of optimal application of anti-poaching patrols and construction of dams.*

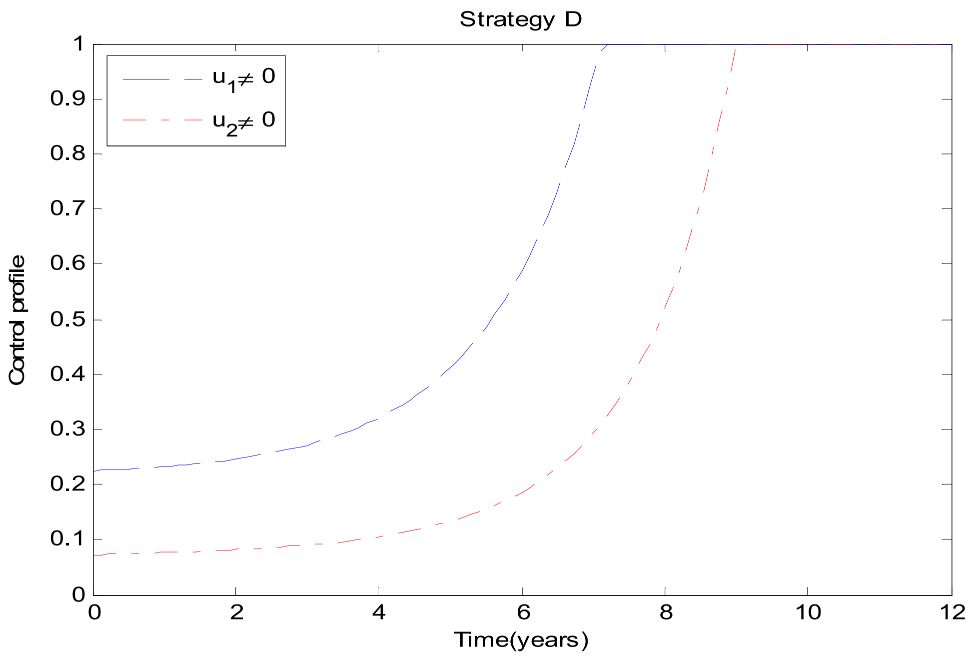

Fig. 5b. *Control profile for* u_1 *and* u_2.

Strategy D: Combination of application of anti-poaching patrols and construction of dams

Here, the control on anti-poaching patrol u_1 and construction of dams u_2 are used to optimize the objective function while control on the use of vaccines u_3 is set to zero. The results in Fig. 5a show a significant difference in the prey and predator populations with optimal strategy compared to prey and predator populations without control. This shows that eliminating poaching and drought in the system would lead to direct increase among prey and predator populations. The control profile is shown in Fig. 5b, here we see that the optimal anti-poaching patrol control u_1 increases gradually till the time $t = 7$ years where it reaches upper bound and continues to final time while optimal construction of dams u_2 control increases till time $t = 9$ years and reaches the upper bound to final time of control.We observe that as the effort of control increase there is also increase in number of indivuals saved.

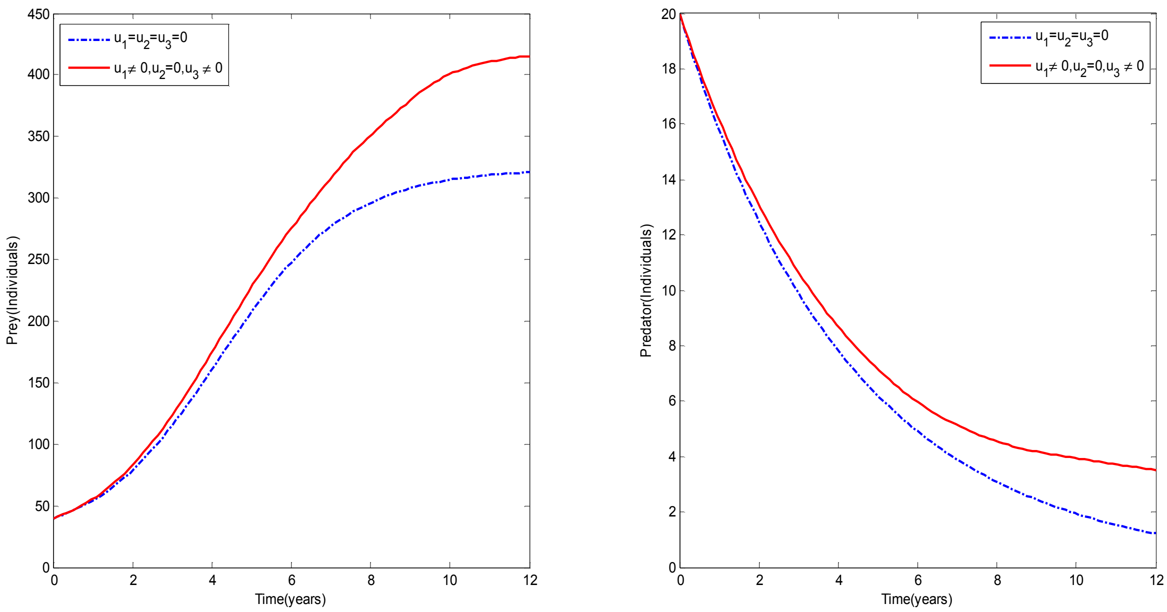

Fig. 6a. *Simulations of a threatened prey-predator model showing the effect of optimal application of anti-poaching patrols and vaccine.*

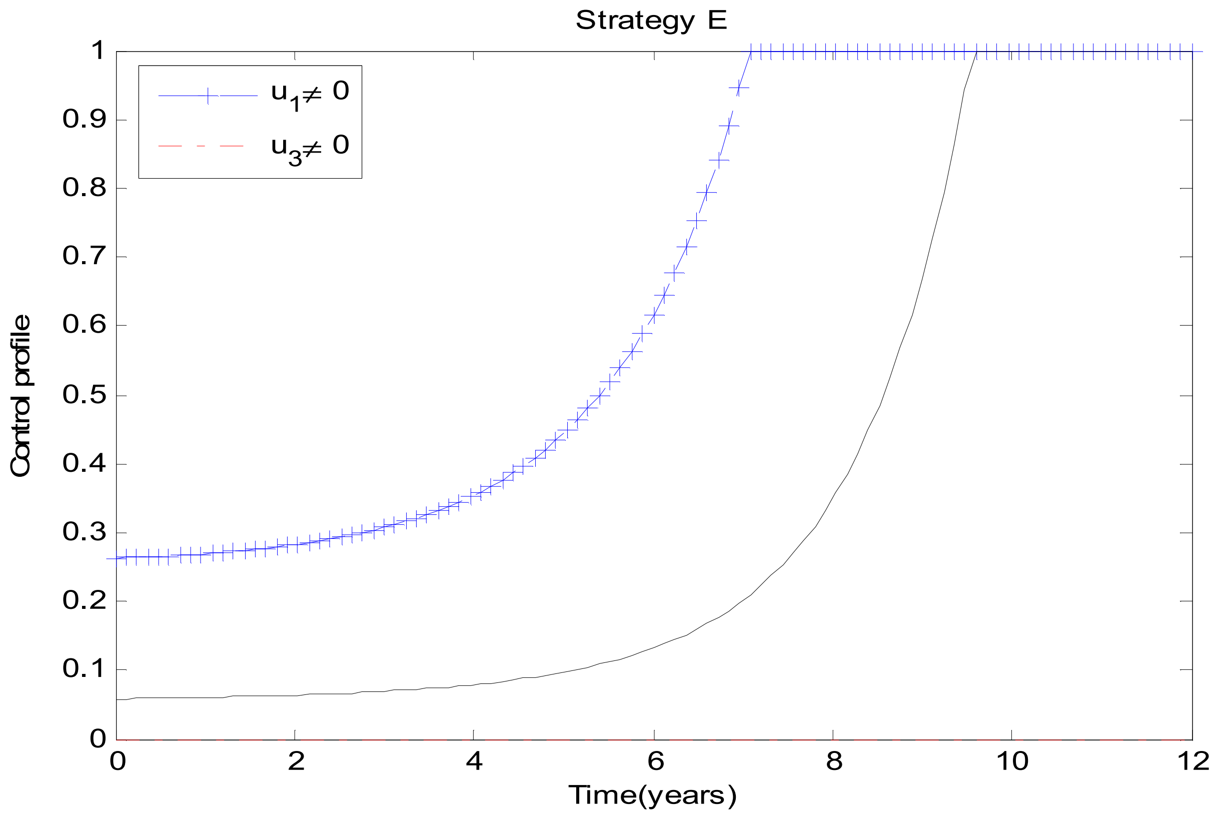

Fig. 6b. *Control profile for* u_1 *and* u_3.

Strategy E: Combination of application of anti-poaching patrols and the use of vaccines

Here, the control on anti-poaching patrol u_1 and the use of vaccines u_3 are used to optimize the objective function J while control on construction of dams u_2 is set to zero. The results in Fig. 6a show a significant difference in the prey and predator populations with optimal strategy compared to prey and predator populations without control. This shows that eliminating poaching and diseases in the system would lead to direct increase among prey and predator populations. The control profile is shown in Fig. 6b, here we see that the optimal anti-poaching patrol control u_1 increases gradually till the time $t = 7$ years where it reaches upper bound and continues to final time while optimal use of vaccine u_3 control increases till time $t = 10$ years and reaches the upper bound to final time of control.We observe that as the effort of control increases there is also increase in the number of individuals saved. However the saving of individuals is more to prey than predators due to presence of refuge.

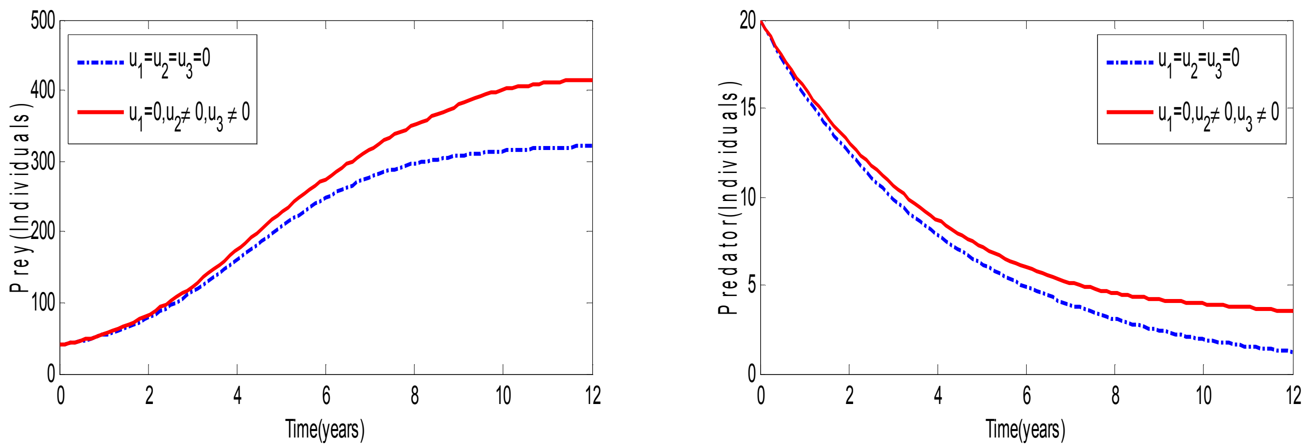

Fig. 7a. *Simulations of a threatened prey-predator model showing the effect of optimal use of dams and vaccines.*

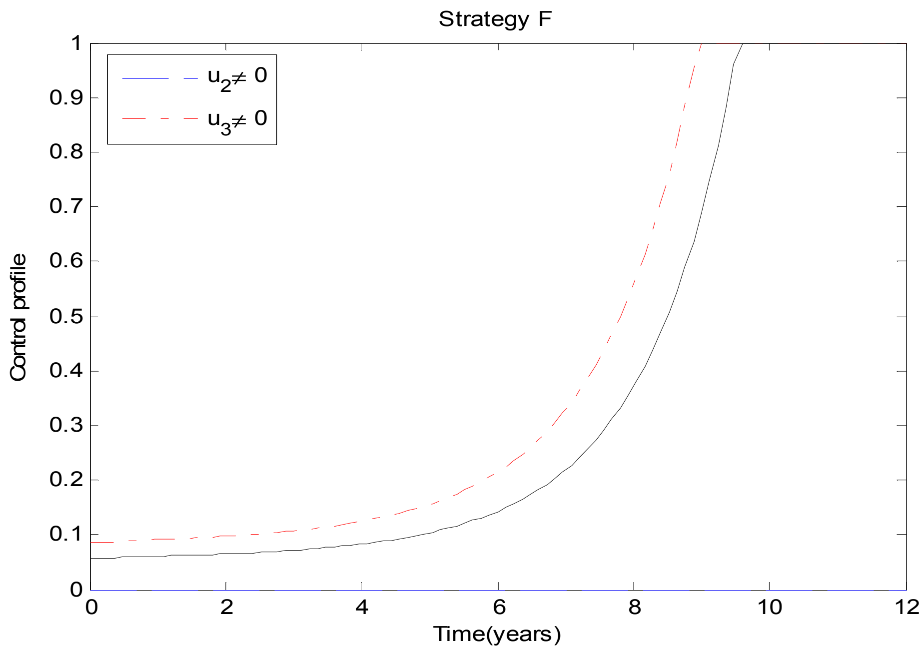

Fig. 7b. *Control profile for* u_2 *and* u_3.

Strategy F: Combination of construction of dams and the use of vaccines

Here, the control on construction of dams u_2 and the use of vaccines u_3 are used to optimize the objective function J while control on anti-poaching patrol u_1 is set to zero. The results in Fig. 7a show a significant difference in the prey and predator populations with optimal strategy compared to prey and predator populations without control. This shows that eliminating drought and disease in the system would lead to direct increase among prey and predator populations. The control profile is shown in Fig. 7b, here we see that the optimal construction of dams u_2 increases gradually till the time $t = 9$ years where it reaches upper bound and continues to final time while optimal use of vaccines u_3 control increases till time $t = 10$ years and reaches the upper bound to final time of control.We observe that as the effort of control increases there is also increase in number of individuals saved. However the saving of individuals is more to prey than predator due to presence of refuge.

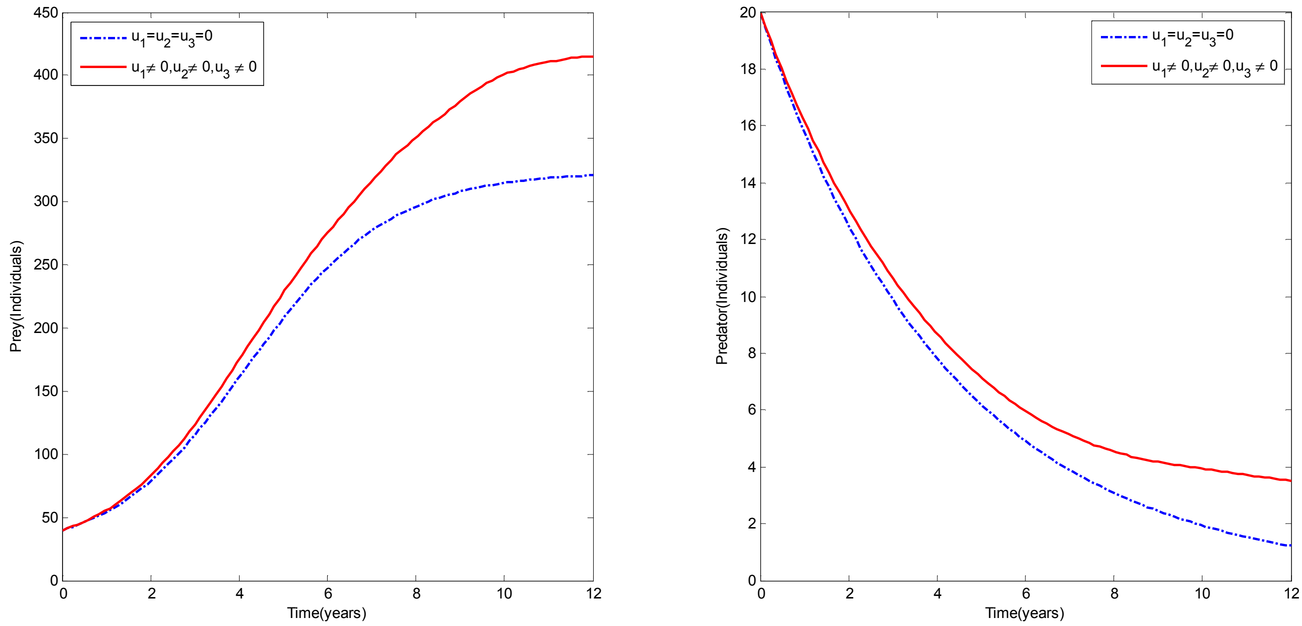

Fig. 8a. *Simulations of a threatened prey-predator model showing the effect of Optimal application of anti-poaching patrols,construction of dams and the use of vaccine.*

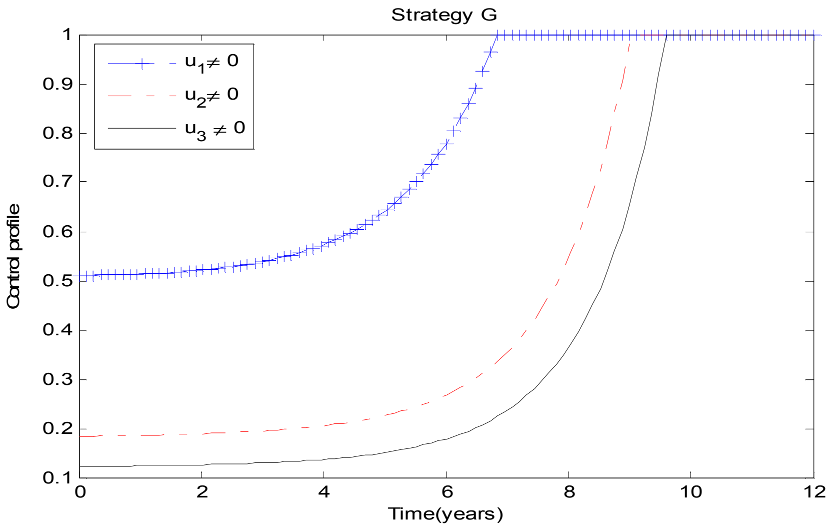

Fig. 8b. *Control profile for* u_1, u_2 *and* u_3.

Strategy G: Combination of application of anti-poaching patrols, construction of dams and the use of vaccines.

Here, all three controls (u_1 , u_2 , and u_3) are used to optimize the objective function J . For this strategy in Fig. 8a we observe that the control strategies resulted in an increase in the number of prey and predator individuals as against in the uncontrolled case. This is the strategy which has shown highest impact not only to prey species but to predators too. This suggests that eliminating poaching, drought and diseases in the system would lead to direct increase among prey and predator populations. The control profile is shown in Fig. 8b, here we see that the optimal anti-poaching patrol control u_1 increases gradually till the time $t = 7$ years where it reaches upper bound and continues to final time while optimal construction of dams u_2 and optimal use of vaccine u_3 control increases respectively till time $t = 9$ years and $t = 10$ years where they reach upper bounds to the final time of control.We observe that as the effort of control increases there is also increase in number of individuals saved. This is the strategy which shows the best result among all others.

4.1. Effect of Variation of Refuge m to Optimal Control Strategies

In this section we investigate numerically effect of varying m to control strategies. Two control strategies are considered here, first is Application of anti-poaching patrol and second is Combination of application of anti-poaching patrol and construction of dams. Since m is bounded such that $m \in [0,1)$ then we chose 0, 0.5,0.8 and 0.95 for simulations

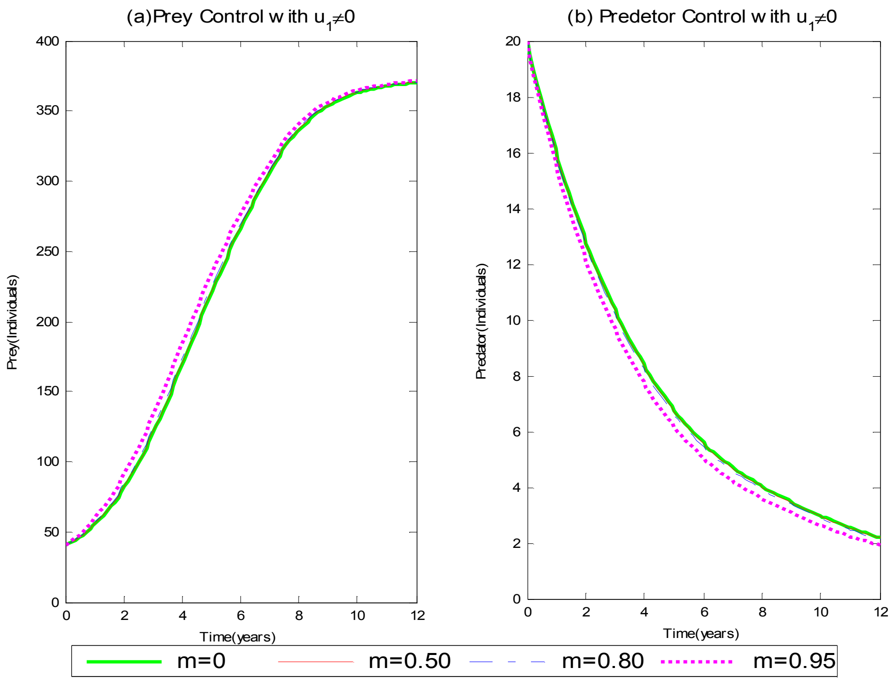

Fig. 9a. *Efffect of variation of prey refuge to the optimal application of anti-poaching patrol.*

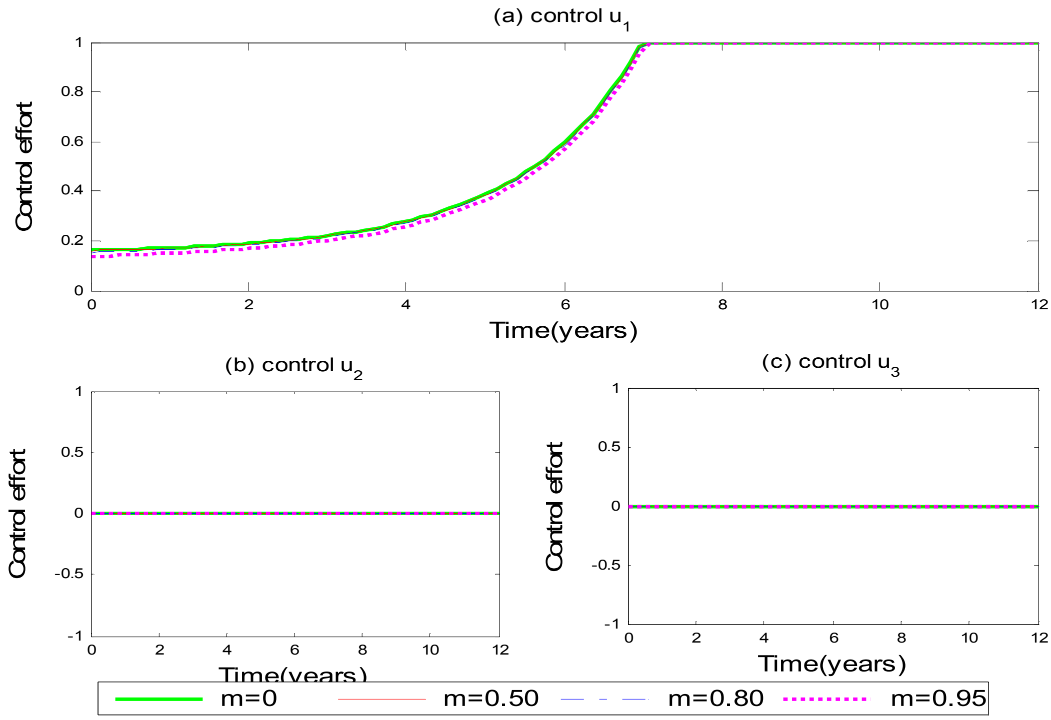

Fig. 9b. *Control profile for Fig.9a.*

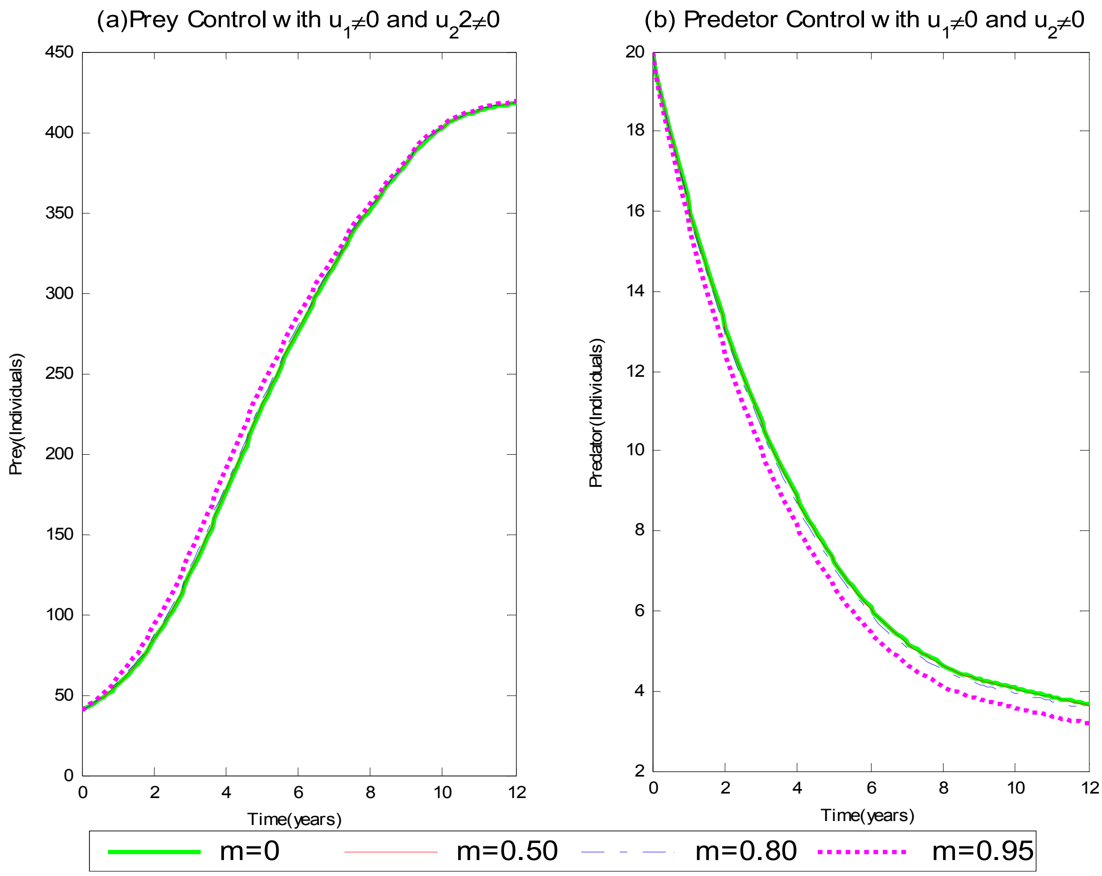

Fig. 10a. *Efffect of variation of prey refuge to the optimal application of the combination of anti-poaching patrol and construction of dams.*

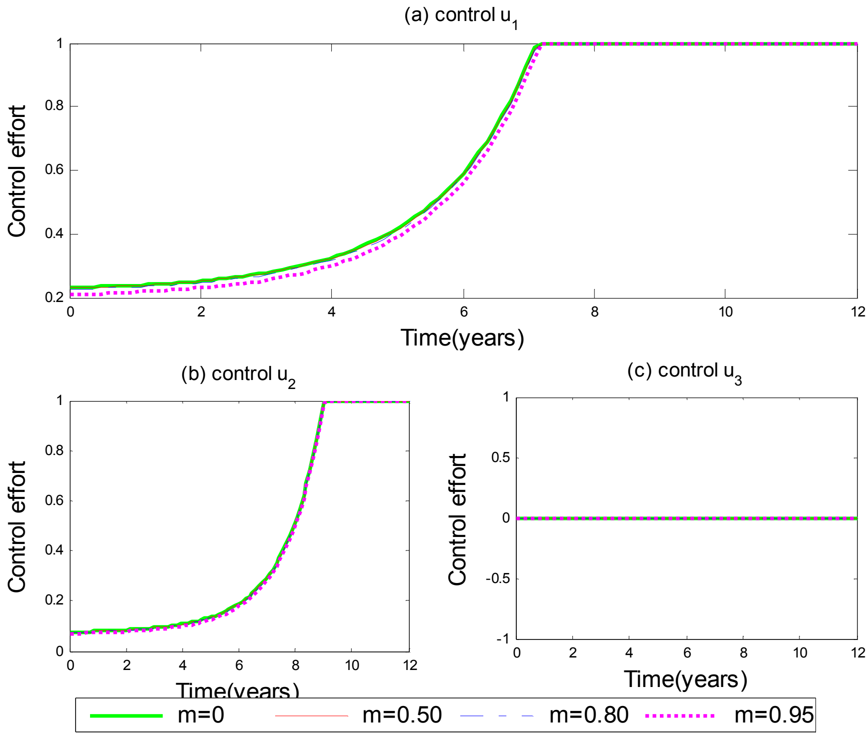

Fig. 10b. *Control profile for Fig.10a.*

4.2. Discussion of Subsetion 4.1

Fig.9a and Fig.10a show the effect varying prey refuge m to optimal control strategies while Fig.9b and Fig.10b their corresponding control profiles . In both figures, that is Fig.9a and Fig.10a we observe that as the value of m increases then prey indivuals increase too. In contrast the increase of m decreases predator individuals, for example the highest value of m taken here which is 0.95 its line is at the upper bound of prey control and lower bound of predator control in each optimal control strategies studied. This behaviour agrees with theoretical results obtained in co-existence equilibrium point as described in section 2.2(iii).

5. Conclusion

In this paper, we presented a threatened prey-predator model using a deterministic system of differential equations. The threats are poaching, drought and diseases. Controls are introduced to the system which are anti-poaching patrols, construction of dams and use of vaccines for controlling poaching, drought and disease respectively. In investigating the effect of optimal control, we use one control at a time, the combination of two controls at a time while setting other(s) to zero to compare the effects of the control strategies on the eradication of threats to prey-predator system. Additionally, the case of all controls was also taken into consideration. Our numerical results suggests that the use of all three controls, anti-poaching patrols, construction of dams and use of vaccines has highest impact on the control of the system threats. The effect of varying prey refuge m to the two selected optimal control strategies is studied and the results show that the increase in m enhances survival of more prey individuals at the same time decreases predator individuals and this due to loss of food.

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