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Czechoslovak Mathematical Journal, Vol. 59 (2009), No. 2, 317–342

Persistent URL: http://dml.cz/dmlcz/140483

Czechoslovak Mathematical Journal, 59 (134) (2009), 317–342

OPTIMAL CONTROL OF LINEAR STOCHASTIC EVOLUTION EQUATIONS IN HILBERT SPACES AND UNIFORM OBSERVABILITY

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(Received May 9, 2007)

Abstract. In this paper we study the existence of the optimal (minimizing) control for a tracking problem, as well as a quadratic cost problem subject to linear stochastic evolution equations with unbounded coefficients in the drift. The backward differential Riccati equation (BDRE) associated with these problems (see [2], for finite dimensional stochastic equations or [21], for infinite dimensional equations with bounded coefficients) is in general different from the conventional BDRE (see [10], [18]). Under stabilizability and uniform observability conditions and assuming that the control weight-costs are uniformly positive, we establish that BDRE has a unique, uniformly positive, bounded on $\mathbb{R}_+$ and stabilizing solution. Using this result we find the optimal control and the optimal cost. It is known [18] that uniform observability does not imply detectability and consequently our results are different from those obtained under detectability conditions (see [10]).

Keywords: Riccati equation, stochastic uniform observability, stabilizability, quadratic control, tracking problem

MSC 2010: 93E20, 49K45

1. Introduction

There are many recent papers (see [15], [16], [17], [5], [21])) where the linear stochastic quadratic control problems are solved under stabilizability and uniform observability conditions. This approach is an alternative to the one adopted in [12], [10], where similar problems are treated under stabilizability and detectability conditions. It is known (see [18] for example) that uniform observability does not imply detectability as we expected by analogy with the deterministic case. Hence the results obtained by using the stochastic uniform observability property are different from those where the detectability property is assumed. The purpose of this paper is to solve a general tracking problem, as well as a linear quadratic control problem for stochastically uniformly observable systems with an unbounded coefficient in the drift. Unlike the papers quoted above the control problems considered in this paper are more general (the control appears both in the drift and in the stochastic part of the equation) and the coefficients are unbounded. Moreover, the backward differential Riccati equation (BDRE) arising in our control problems is not the well-known conventional Riccati equation considered in [18], [12] or [10]. We have proved that under stabilizability and the stochastic uniform observability conditions this Riccati equation has a unique, uniformly positive, bounded and stabilizing solution. The result is similar to those obtained in [16], respectively in [18], for finite dimensional systems, respectively for stochatisc systems in Hilbert spaces, and the conventional Riccati equation. We note that it is based on Theorem 6, which state that there exists a unique solution for the considered BDRE with final condition in the class of strongly continuous families of nonnegative linear operators. These results play the key role in solving the optimal control problems addressed in this paper (see Theorems 21, 22). The deterministic characterization of the stochastic uniform observability property given in [22] allow us to provide examples of stochastic uniformly observable systems. One of this example (see Example 23) help us in illustrating our theory.

2. Preliminaries and statement of the problem

Let H, U, V be separable real Hilbert spaces. $L(H, V)$ is the Banach space of all bounded linear operators from H into V (if $H = V$ we put $L(H, V) = L(H)$). We write $\langle \cdot, \cdot \rangle$ for the inner product and $\| \cdot \|$ for norms of elements and operators. We say that $A \in L(H)$ is nonnegative if A is self-adjoint and $\langle Ax, x \rangle \geqslant 0$, $x \in H$. Let $L^+(H)$ be the subset of $L(H)$ of all nonnegative operators. We denote by I the identity operator on H. For any $S \in L(H)$ we put $\|S\|_1 = Tr|S| \leqslant \infty$ and denote by $C_1(H)$ the set $\{S \in L(H) / \|S\|_1 < \infty\}$ (the trace class of operators). It is known that $C_1(H)$ is a Banach space with the usual operations and the norm $\| \cdot \|_1$. By $\mathscr{N}$ we will denote the subspace of $C_1(H)$ containing all self adjoint operators from $C_1(H)$.

Let $J \subset \mathbb{R}_+ = [0, \infty)$ be an interval and E a Banach space. We denote by $C(J, E)$ the space of all mappings $G\colon J \to E$ that are continuous.

We also use the notation $C_s(J, L(H))$ and $C_b(J, L(H))$ for the space of all strongly continuous mappings $G\colon J \to L(H)$ and for the subspace of $C_s(J, L(H))$ which consist of all mappings G such that $\sup\limits_{t \in J} \|G(t)\| < \infty$, respectively. We recall that

$C_s([0,T], L(H))$, $T \in \mathbb{R}_+$ is a Banach space endowed with the usual operations and the norm $|P|_T = \sup\limits_{s\in[0,T]} \|P(s)\|$.

If $Z \in C_b(\mathbb{R}_+, L(H,V))$, we set $\widetilde{Z} = \sup\limits_{0\leqslant r<\infty} \|Z(r)\| < \infty$. An element $G \in C_s(\mathbb{R}_+, L(H))$ is called uniformly positive iff there exists $\gamma > 0$ such that $G(t) \geqslant \gamma I$ for all $t \in \mathbb{R}_+$. For a fixed $T > 0$ we will denote $\Delta(T) = \{(t,s),\ 0 \leqslant s \leqslant t \leqslant T\}$. If $T = \infty$ we set $\Delta = \Delta(\infty)$.

We need the following definitions:

Definition 1 (see Definition 5.3 in [8]). A family $\{V(t,s)\}_{(t,s)\in\Delta(T)} \subset L(H)$ is an evolution operator (system) iff

1. $V \in C_s(\Delta(T), L(H))$ and
2. $V(s,s) = I, V(t,s)V(s,r) = V(t,r)$ for all $0 \leqslant r \leqslant s \leqslant t \leqslant T$ (the semigroup property).

Definition 2 (see Definition 4.2 in [6]). A strong evolution operator is an evolution operator $U(t,s)$ for which there exists a closed linear and densely defined operator $A(t)$, $t \geqslant 0$, with domain $D(A(t))$ such that

1. $U(t,s)\colon\ D(A(s)) \to D(A(t))$ for $t > s$,
2. $\partial U(t,s)x/\partial t = A(t)U(t,s)x$ for every $x \in D(A(s))$ and $t > s \geqslant 0$.

We will say that the family $\{A(t)\}_{t\geqslant 0}$ is the generator of $U(t,s)$ or that $A(t)$ generates the evolution operator $U(t,s)$.

Throughout the paper we assume the following hypotheses if we do not specify otherwise:

(P1) $U(t,s)$ is a strong evolution operator generated by the family $A(t)$, $t \in [0,\infty)$; there exists a sequence $\{A_n\}_{n\in\mathbb{N}} \subset C_s([0,\infty), L(H))$ such that for every $n \in \mathbb{N}$, the family $\{A_n(t)\}_{t\geqslant 0}$ generates the strong evolution operator $U_n(t,s)$ and

$$U_n(t,s)x \underset{n\to\infty}{\to} U(t,s)x,\ x \in H \tag{1}$$

uniformly on bounded subsets of Δ.

(P2) $B, H_i \in C_b(\mathbb{R}_+, L(U,H))$, $B^*, H_i^* \in C_b(\mathbb{R}_+, L(H,U))$, $C \in C_b(\mathbb{R}_+, L(H,V))$, $C^* \in C_b(\mathbb{R}_+, L(V,H))$, $G_i, G_i^* \in C_b(\mathbb{R}_+, L(H))$, $i = 1,\dots,m$, $m \in \mathbb{N}^*$, $K \in C_b(\mathbb{R}_+, L(U))$ and there exists $\delta > 0$ such that for all $t \in \mathbb{R}_+$,

$$K(t) \geqslant \delta I. \tag{2}$$

If $A^*(t)$ denotes the adjoint operator of $A(t)$ and the families $A^*(t), A_n^*(t)$, $n \in \mathbb{N}$, $t \in [0,\infty)$ satisfy the hypothesis (P1), we will say that P1(A^*) holds. In what follows we will assume that P1(A^*) holds. Although P1(A^*) is not explicitly used in this paper, we base our results on Lemma 3 in [13] or Theorem 10 in [23], which require this hypothesis.

We note that P1(A^*) ensures that $U_n^*(t,s)$ converges strongly for $n \to \infty$ to $U^*(t,s)$, uniformly on any bounded subset of Δ. Indeed, it is known that $A_n^*(t_0-t)$, $t \in [0,\ t_0]$, $n \in \mathbb{N}$, are the generators of the evolution operators $U_n^*(t_0-s,\ t_0-t)$, $0 \leqslant s \leqslant t \leqslant t_0$, $n \in \mathbb{N}$ (see Chapter 9 in [3]). Letting $n \to \infty$, it follows that $U_n^*(t_0-s,\ t_0-t)$ is strongly convergent to $U^*(t_0-s,\ t_0-t)$ uniformly on $\Delta(t_0)$. Hence $\lim\limits_{n\to\infty} U_n^*(t,s)x = U^*(t,s)x$ uniformly on $\Delta(t_0)$.

Now, let us recall the following perturbation result:

Proposition 3 [22]. *Let $D \in C_s([0,T], L(H))$ and let $U,\ U_n \in C_s(\Delta(T), L(H))$ be evolution operators such that $\lim\limits_{n\to\infty} U_n(t,s)x = U(t,s)x$, $x \in H$ uniformly for $(t,s) \in \Delta(T)$. Then there exist unique solutions $U_D,\ U_{D,n} \in C_s(\Delta(T), L(H))$ of the following integral equations*

$$U_D(t,s)x = U(t,s)x + \int_s^t U(t,r)D(r)U_D(r,s)x\,\mathrm{d}r, \tag{3}$$

$$U_{D,n}(t,s)x = U_n(t,s)x + \int_s^t U_n(t,r)D(r)U_{D,n}(r,s)x\,\mathrm{d}r. \tag{4}$$

Moreover, U_D and $U_{D,n}$ are evolution operators and $\lim\limits_{n\to\infty} U_{D,n}(t,s)x = U_D(t,s)x$, $x \in H$, uniformly for $(t,s) \in \Delta(T)$.

The evolution operators U_D and $U_{D,n}$ are called the perturbed evolution operators corresponding to the perturbation D or, simply, the perturbations of U and U_n by D.

Let $(\Omega, \mathscr{F}, \mathscr{F}_t, t \in [0,\infty), \mathscr{P})$ be a stochastic basis and let $w_i, i = 1,\ldots,m$, $m \in \mathbb{N}^*$ be independent real Wiener processes relative to $\mathscr{F}_t$.

We will denote by $\{A : B;\ G_i : H_i\}$ the stochastic equation with control

$$\mathrm{d}x(t) = A(t)x(t)\,\mathrm{d}t + B(t)u(t)\,\mathrm{d}t + \sum_{i=1}^m (G_i(t)x(t) + H_i(t)u(t))\,\mathrm{d}w_i(t), \tag{5}$$

$$x(s) = x \in H. \tag{6}$$

Here the set of admissible controls is $U_{ad} = \{u \in L^2(\mathbb{R}_+ \times \Omega, U)$, u is $\mathscr{F}_t$-adapted such that $\sup\limits_{t\geqslant s} E\|x(t)\|^2 < \infty$, where $x(t)$ is the solution of (5)–(6)$\}$.

If $B = 0$ and $H_i = 0, i \in \{1,\ldots,m\}$ then we will denote (5) by $\{A; G_i\}$.

We recall [10] that $\{A; G_i\}$ with the initial condition $x(s) = x \in H$ has a unique mild solution $x = x(\cdot, s, x) \in C([s,T]; L^2(\Omega; H))$ that is adapted to $\mathscr{F}_t$; namely the solution of

$$x(t) = U(t,s)x + \sum_{i=1}^m \int_s^t U(t,r)G_i(r)x(r)\,\mathrm{d}w_i(r). \tag{7}$$

Tracking problem. Given a signal $r \in C_b(\mathbb{R}_+, H)$ we want to minimize the cost

$$(8) \qquad J(s,u) = \overline{\lim_{t\to\infty}} \frac{1}{t-s} E \int_s^t \|C(\sigma)(x(\sigma) - r(\sigma))\|^2 + \langle K(\sigma)u(\sigma), u(\sigma)\rangle \, \mathrm{d}\sigma$$

in the class of admissible controls

$$\mathscr{U}_{ad} = \left\{ u \in U_{ad}, \overline{\lim_{t\to\infty}} \frac{1}{t-s} E \int_s^t \|u(\sigma)\|^2 \, \mathrm{d}\sigma < \infty \right\}$$

subject to the equation (5)–(6).

Quadratic control problem. Let us consider the set $\mathscr{U}_{ad}^q = \{u \in U_{ad}$ such that $\lim\limits_{t\to\infty} E\|x(t)\|^2 \to 0$, where $x(t)$ is the solution of (5)–(6)$\}$. We look for an optimal control $u \in \mathscr{U}_{ad}^q$ which minimizes the quadratic cost

$$(9) \qquad I_{s,x}(u) = E \int_s^\infty \|C(t)x(t)\|^2 + < K(t)u(t), u(t) > \, \mathrm{d}t.$$

3. Riccati equation of stochastic control

In this section we consider the backward Riccati differential equation (BDRE) (10) associated with the optimal control problems (5)–(6), (8) or (5)–(6), (9) and we investigate the existence of a global bounded solution. As in the classical situation [10], we prove that, under stabilizability conditions, a global, bounded on $\mathbb{R}_+$ solution of the BDRE is the strong limit of solutions of the BDREs with final conditions. The problem of existence of the solutions for the conventional BDREs with final conditions in infinite dimensions was solved in [9] and [13] by using the principle of contractions. Since in our case the last term of (10) is more complicated we cannot use the same way of proof as in the classical case. Thus, in the next subsection we will use the algorithm proposed in [2] for finite dimensional spaces and the results in [9] to prove the existence of these solutions. We also must handle many problems arising from the unboundedness of the coefficients of the equations. We note that in [2] only the solvability of the BDRE (10) with a final condition is studied (the control weight cost being not necessarily uniformly positive), because the control problems are considered over finite intervals. Unlike this case, in our situation the control problems are considered on infinite intervals and the existence of a nonnegative, bounded on $\mathbb{R}_+$ solution of the BDRE is required. In the second subsection we prove that, under uniform observability and stabilizability conditions, the Riccati equation (10) has a unique, bounded on $\mathbb{R}_+$ and stabilizing solution. Therefore we extend the results obtained in [18] and [15], [16] for infinite respectively finite, dimensions and conventional BDRE to the case of nonconventional BDRE.

3.1. Existence of solutions for the backward Riccati equations associated with the stochastic control. We consider the linear operator $\mathscr{B}$: $C_s(\mathbb{R}_+, L(H)) \to C_s(\mathbb{R}_+, L(H,U))$,

$$\mathscr{B}(P)(s) = B^*(s)P(s) + \sum_{i=1}^{m} H_i^*(s)P(s)G_i(s)$$

and the function $\mathscr{K}\colon\ C_s(\mathbb{R}_+, L^+(H)) \to C_s(\mathbb{R}_+, L^+(U))$,

$$\mathscr{K}(P)(s) = K(s) + \sum_{i=1}^{m} H_i^*(s)P(s)H_i(s).$$

By (2) it follows that $\mathscr{K}(P)$ is uniformly positive for any $P \in L^+(H)$.

Consequently, $\mathscr{K}(P)(s)$ is invertible for any $s \in \mathbb{R}_+$ and $P \in L^+(H)$. We will denote by $[\mathscr{K}(P)]^{-1}$ the element of $C_s(\mathbb{R}_+, L^+(U))$ defined by $s \to [\mathscr{K}(P)(s)]^{-1}$.

Let us introduce the differential Riccati equation

$$P' + A^*P + PA + \sum_{i=1}^{m} G_i^* P G_i + C^*C - [\mathscr{B}(P)]^*[\mathscr{K}(P)]^{-1}\mathscr{B}(P) = 0. \tag{10}$$

We say that $P \in C_s(J, L^+(H))$ is a mild solution on an interval $J \subseteq [0,\infty)$ of the BDRE (10), if it satisfies the following integral equation

$$\begin{aligned} P(s)x = U^*(t,s)P(t)U(t,s)x + \int_s^t U^*(r,s)\Big\{ \sum_{i=1}^{m} G_i^*(r)P(r)G_i(r) \\ + C^*(r)C(r) - [\mathscr{B}(P)(r)]^*[\mathscr{K}(P)(r)]^{-1}\mathscr{B}(P)(r)\Big\} U(r,s)x\,\mathrm{d}r \end{aligned} \tag{11}$$

for all $s \leqslant t, s,\ t \in J$. Moreover, if P is a mild solution on $\mathbb{R}_+$ of (10) and $\sup_{s\in\mathbb{R}_+} \|P(s)\| < \infty$, then P is said to be a bounded on $\mathbb{R}_+$ (global) solution. We also will denote by $P(T,s;R)$ the mild solution on an interval $J = [0,T]$, $T > 0$ of the Riccati equation (10), with the final condition $P(T) = R \in L^+(H)$.

If P is a mild solution on an interval $J \subseteq [0,\infty)$ of the BDRE (10) then we define the function $S\colon\ J \to L(H,U)$,

$$S(s) = [\mathscr{K}(P)(s)]^{-1}\mathscr{B}(P)(s), \tag{12}$$

which clearly is strongly continuous on J. Let $Q \in C_s([0,T], L^+(H))$ and $R \in L^+(H)$. We consider the Lyapunov equations

$$X' + A^*X + XA + \sum_{i=1}^{m} G_i^* X G_i + Q = 0, \ X(T) = R, \tag{13}$$

$$X_n' + A_n^* X_n + X_n A_n + \sum_{i=1}^{m} G_i^* X_n G_i + Q = 0, \ X_n(T) = R, \ n \in \mathbb{N} \tag{14}$$

and the integral equations

$$\begin{aligned} X(s)x &= U^*(t,s)RU(t,s)x \\ &\quad + \int_s^t U^*(r,s)\Big[\sum_{i=1}^{m} G_i^*(r)X(r)G_i(r) + Q(r)\Big]U(r,s)x\,\mathrm{d}r, \end{aligned} \tag{15}$$

$$\begin{aligned} X_n(s)x &= U_n^*(t,s)RU_n(t,s)x \\ &\quad + \int_s^t U_n^*(r,s)\Big[\sum_{i=1}^{m} G_i^*(r)X_n(r)G_i(r) + Q(r)\Big]U_n(r,s)x\,\mathrm{d}r \end{aligned} \tag{16}$$

As in the case of the BDRE (10), we define the mild solution $X = X(T,\cdot,R) \in C_s([0,T], L^+(H))$ on $[0,T]$ of (13) via (15). Let $\mathscr{E}$ be the Banach subspace of $L(H)$ formed by all self-adjoint operators. By a strong (classical) solution of (14) we mean a mapping $X_n \in C_s([0,T]; L^+(H))$ such that $t \to X_n(t)x\colon [0,T) \to H$ is differentiable in $[0,T)$ for all $x \in H$, $\mathrm{d}X_n/\mathrm{d}t \in C_s([0,T);\mathscr{E})$ and X_n fulfils the equation (14) (see [1]).

By Lemma 3 in [13] it follows that there exists a unique mild solution $X(T,\cdot,R)$ of (13) on $[0,T]$ given by (15) and unique classical solution $X_n(T,\cdot,R)$ of (14) on $[0,T]$ given by (16) and for each $x \in H$,

$$X_n(T,s,0)x \underset{n\to\infty}{\to} X(T,s,0)x \tag{17}$$

uniformly on $[0,T]$.

Arguing as in the proof of Proposition 4.64 [7] and using Dini's theorem we can prove the following lemma.

Lemma 4. *If $(Q_n)_{n\in\mathbb{N}^*}$ is an increasing sequence in $C_s([0,T], L^+(H))$ such that $Q_n(t) \leqslant I$, for all $t \in [0,T]$ (I is the identity operator on H), then there exists $Q \in C_s([0,T], L^+(H))$ such that $Q_n(t)x \underset{n\to\infty}{\to} Q(t)x$, $x \in H$ uniformly for $t \in [0,T]$.*

The next result follows directly from the proof of Lemma 4.2 in [9].

Lemma 5. *If $(A_n)_{n\in\mathbb{N}^*}, (B_n)_{n\in\mathbb{N}^*}$ are two sequences in $C_s([0,T],L^+(H))$ such that for every $x\in H$, $A_n(t)x \underset{n\to\infty}{\to} A(t)x$, $B_n(t)x \underset{n\to\infty}{\to} B(t)x$ uniformly with respect to $t\in[0,T]$, then $A_n(t)B_n(t)x \underset{n\to\infty}{\to} A(t)B(t)x$ uniformly with respect to $t\in[0,T]$ (obviously $A,B,AB\in C_s([0,T],L(H))$).*

Theorem 6. *Let $R\in L^+(H)$. The Riccati equation* (10) *with the final condition $P(T)=R$ has a unique mild solution $P(T,s;R)\in C_s([0,T],L^+(H))$. It is given by:*

$$(18)\qquad P(s)x = U^*(T,s)RU(T,s)x + \int_s^T U^*(r,s)\Big\{\sum_{i=1}^m G_i^*(r)P(r)G_i(r) + C^*(r)C(r) - [\mathscr{B}(P)(r)]^*[\mathscr{K}(P)(r)]^{-1}\mathscr{B}(P)(r)\Big\}U(r,s)x\,\mathrm{d}r.$$

Moreover, $P(T,s;R)$ is monotone in the sense that $P(T,s;R_1)\leqslant P(T,s;R)$ for any $0\leqslant R_1\leqslant R$.

Proof. *Existence.* Let $P_k\in C_s([0,T],L^+(H))$ be fixed and let us denote $S_k=[\mathscr{K}(P_k)]^{-1}\mathscr{B}(P_k)$, $\widehat{A}_k=A-BS_k$, $\widehat{G}_{k,i}=G_i-H_iS_k$, $i=1,\dots,m$.

Then the Lyapunov equation

$$(19)\quad P'_{k+1}+A^*P_{k+1}+P_{k+1}A+\sum_{i=1}^m G_i^*P_{k+1}G_i + C^*C - S_k^*(r)\mathscr{B}(P_{k+1})(r) - [\mathscr{B}(P_{k+1})(r)]^*S_k(r) + S_k^*(r)[\mathscr{K}(P_{k+1})(r)]S_k(r)=0,$$

$$(20)\quad P_{k+1}(T)=R$$

can be equivalently rewritten as

$$(21)\qquad P'_{k+1}+\widehat{A}_k^*P_{k+1}+P_{k+1}\widehat{A}_k+\sum_{i=1}^m \widehat{G}_{k,i}^*P_{k+1}\widehat{G}_{k,i}+C^*C+S_k^*KS_k=0,$$

$$(22)\qquad P_{k+1}(T)=R.$$

Since $-BS_k\in C_s([0,T],L(H))$, it follows by Proposition 3 that there exists $U_{-BS_k}(\cdot,\cdot)$, the perturbed evolution operator of the evolution operator $U(\cdot,\cdot)$ corresponding to the perturbation $-BS_k$.

Then Lemma 3 in [13] implies that (21)–(22) (and consequently (19)–(20)) has a unique mild solution $P_{k+1}\in C_s([0,T],L^+(H))$ which satisfies the integral equation

$$(23)\ P_{k+1}(s)x = U^*_{-BS_k}(T,s)RU_{-BS_k}(T,s)x + \int_s^T U^*_{-BS_k}(r,s)\Big[\sum_{i=1}^m \widehat{G}_{k,i}^*(r)P_{k+1}(r)\widehat{G}_{k,i}(r)+C^*C+S_k^*KS_k\Big]U_{-BS_k}(r,s)x\,\mathrm{d}r.$$

It is not difficult to see that (23) is equivalent to

$$
\begin{aligned}
(24)\quad P_{k+1}(s)x = U^*(T,s)RU(T,s)x + \int_s^T U^*(r,s)\Big\{ &\sum_{i=1}^m G_i^*(r)P_{k+1}(r)G_i(r) \\
&+ C^*(r)C(r) - S_k^*(r)\mathscr{B}(P_{k+1})(r) \\
&- [\mathscr{B}(P_{k+1})(r)]^* S_k(r) + S_k^*(r)[\mathscr{K}(P_{k+1})(r)]S_k(r)\Big\} U(r,s)x\,\mathrm{d}r.
\end{aligned}
$$

Now we consider the iterative scheme proposed in [2] to construct the mild solution of (10).

Let $k = 0$ and $P_0 = I$. Reasoning as above we deduce that there exists a mild solution $P_1 \in C_s([0,T],\, L^+(H))$ of (19)–(20) that satisfies (24). Using induction it follows that for any $k = 2,3,\ldots$ the equation (19)–(20) has a unique mild solution P_k, which belongs to $C_s([0,T], L^+(H))$ and satisfies (24).

It is easy to see that $\Lambda_{k+1} = P_k - P_{k+1}$ verifies the equation

$$
\begin{aligned}
(25)\qquad &\Lambda'_{k+1} + \widehat{A}_k^*\Lambda_{k+1} + \Lambda_{k+1}\widehat{A}_k + \sum_{i=1}^m \widehat{G}_{k,i}^*\Lambda_{k+1}\widehat{G}_{k,i} \\
&\qquad + (S_k - S_{k-1})^*\mathscr{K}(P_k)(S_k - S_{k-1}) = 0, \\
&\Lambda_{k+1}(T) = 0.
\end{aligned}
$$

Using again Lemma 3 in [13] we obtain that (25) has a unique mild solution $\Lambda_{k+1} \in C_s([0,T], L^+(H))$ for all $k \in \mathbb{N}^*$. Consequently, the sequence $\{P_k(\cdot)\}$ is monotone decreasing. Setting $M_{1,T} = \sup\limits_{r\in[0,T]} \|P_1(r)\|$ we have $M_{1,T} < \infty$ by the uniform boundedness principle.

We apply Lemma 4 to the nonnegative and monotone increasing sequence $\{P_1(\cdot) - P_k(\cdot)\}$, $P_1(r) - P_k(r) \leqslant M_{1,T}I, r \in [0,T]$, $k \in \mathbb{N}^*$ and we deduce that there exists $P \in C_s([0,T], L^+(H))$ such that for all $x \in H$, $P_k(r)x \underset{k\to\infty}{\to} P(r)x$ uniformly for $r \in [0,T]$. By direct computation we obtain

$$
\begin{aligned}
&\|[\mathscr{K}(P_{k+1})(r)]^{-1} - [\mathscr{K}(P)(r)]^{-1}x\| \\
&\qquad\qquad \leqslant \frac{1}{\delta}\sum_{i=1}^m \|[\mathscr{K}(P_{k+1})(r) - \mathscr{K}(P)(r)][\mathscr{K}(P)(r)]^{-1}x\|.
\end{aligned}
$$

Now, it is clear that $[\mathscr{K}(P_{k+1})(r)]^{-1} - [\mathscr{K}(P)(r)]^{-1}x \underset{k\to\infty}{\to} 0$ uniformly for $r \in [0,T]$.

Similarly it follows that $\mathscr{B}(P_{k+1})(r)x \underset{k\to\infty}{\to} \mathscr{B}(P)(r)x$ and $S_k(r)x \underset{k\to\infty}{\to} S(r)x$ uniformly for $r \in [0,T]$. (Here S is defined by (12).)

Letting $k \to \infty$ in (24), we deduce by the dominated convergence theorem of Lebesgue that P satisfies the integral equation (18). Hence, there exists a solution

$P \in C_s([0,T], L^+(H))$ of the Riccati equation (10) with the final condition $P(T) = R \in L^+(H)$. Using Gronwall's inequality we see that for all $0 \leqslant s \leqslant T$,

$$\|P(s)\| \leqslant L_T = M_T^2\biggl(\|R\| + \int_s^T \|C(r)\|^2 \,\mathrm{d}r\biggr)\mathrm{e}^{M_T^2 \sum\limits_{i=1}^m \int_s^T \|G_i(r)\|^2 \,\mathrm{d}r},$$

where $M_T = \sup\limits_{0\leqslant s\leqslant t<T} \|U(t,s)\|$.

Uniqueness. Assume that Q is another solution of (10) satisfying $Q(T) = R \in L^+(H)$. Then Q also satisfies (18) and setting $S(Q) = [\mathscr{K}(Q)]^{-1}\mathscr{B}(Q)$ we get

$$\begin{aligned}\langle [P(s) - Q(s)]x, x\rangle = \int_s^{T_0} &\sum_{i=1}^m \{\langle [G_i^*(r)(P(r) - Q(r))G_i(r) \\ &+ \mathscr{B}^*(Q)(r)[\mathscr{K}(P)(r)]^{-1}H_i^*(P(r) - Q(r))H_iS(Q)(r)]U(r,s)x, U(r,s)x\rangle\} \\ &- \langle [\mathscr{B}(P-Q)(r)]^*[\mathscr{K}(P)(r)]^{-1}[\mathscr{B}(P+Q)(r)]U(r,s)x, U(r,s)x\rangle \,\mathrm{d}r.\end{aligned}$$

Since P and Q are bounded on $[0,T]$ ($\|P(s)\|, \|Q(s)\| \leqslant L_T$ for all $0 \leqslant s \leqslant T$) we apply again Gronwall's inequality to deduce that $\|P(s) - Q(s)\| = 0$. Thus the mild solution of the Riccati equation (10) with the final condition $P(T) = R \in L^+(H)$ is unique.

Monotonicity. Let us assume that $P(s) = P(T,s;R)$ and $P_1(s) = P(T,s;R_1)$, $R, R_1 \in L^+(H)$ are two solutions of the Riccati equation (10) and $R - R_1 \geqslant 0$. If we denote $\Lambda = P - P_1$, $S_1 = [\mathscr{K}(P_1)]^{-1}\mathscr{B}(P_1)$, $\widehat{A}_1 = A - BS_1$, $\widehat{G}_{1,i} = G_i - H_iS_1$, $i \in \{1,\ldots,m\}$, then Λ is the solution of the Lyapunov equation

$$\begin{aligned}&\Lambda' + \widehat{A}_1^*\Lambda + \Lambda\widehat{A}_1 + \sum_{i=1}^m \widehat{G}_{1,i}^*\Lambda\widehat{G}_{1,i} + (S_1 - S)^*\mathscr{K}(P)(S_1 - S) = 0, \\ &\Lambda(T) = R - R_1.\end{aligned} \tag{26}$$

From Lemma 3 in [13] it follows that (26) has a unique mild solution satisfying $\Lambda(s) \geqslant 0$, $s \in [0,T]$. Hence $P(s) - P_1(s) \geqslant 0$ and we obtain the conclusion. □

Let us introduce the following hypothesis:

(P3) *There exists $p > 0$ such that $Z(t) = Z(t+p)$ for all $t \geqslant 0$*, where $Z = A, B, C, K, H_i, G_i$, $i = 1,\ldots,m$.

It is easy to verify (see [24], [12], [3]) that if (P1) and (P3) hold then the evolution operator generated by the family $A(t)$, $t \geqslant 0$ is p-periodic, that is

$$U(t+p, s+p) = U(t,s) \quad \text{for all } t \geqslant s \geqslant 0. \tag{27}$$

Proposition 7. *Assume that* (P3) *holds. Then the unique mild solution* $P(T,s;R)$ *of the Riccati equation* (10) *is p-periodic that is* $P(T+p,s+p;R) = P(T,s;R)$.

Proof. Using the above theorem and (P3) it is easy to see that both $P(T+p, s+p;R)$ and $P(T,s;R)$ satisfy the integral equation (18). The conclusion follows by Gronwall's inequality. $\square$

Proposition 8. *Let $P \in C_s([0,T], L^+(H))$ be the unique mild solution of the Riccati equation* (10) *with the final condition $P(T) = R \in L^+(H)$ and let $P_n \in C([0,T], L^+(H))$ (denoted $P_n(T,s;R)$) be the strong solution of the Lyapunov equation*

$$
\begin{aligned}
(28)\qquad & P_n'(s) + \widehat{\mathscr{A}_n^*}(s)P_n(s) + P_n(s)\widehat{\mathscr{A}_n}(s) + \sum_{i=1}^m \widehat{\mathscr{G}_i^*}(s)P_n(s)\widehat{\mathscr{G}_i}(s) \\
&\quad + C^*(s)C(s) + S^*(s)K(s)S(s) = 0, \\
& P_n(T) = R,
\end{aligned}
$$

where $\widehat{\mathscr{A}_n} = A_n - BS$, $\widehat{\mathscr{G}_i} = G_i - H_iS$, $i = 1,\dots,m$ and S is defined by (12). *Then $P_n(t)x \underset{n\to\infty}{\to} P(t)x$ uniformly for $t \in [0,T]$.*

Proof. Since $-BS \in C_s([0,T], L(H))$ we apply Proposition 3 and Lemma 3 in [13] to deduce that (28) has a unique solution $P_n \in C_s([0,T], L^+(H))$ which converges, as $n \to \infty$, strongly and uniformly on $[0,T]$ to the unique mild solution $\widetilde{P} \in C_s([0,T], L^+(H))$ of the Lyapunov equation

$$
\begin{aligned}
(29)\qquad & \widetilde{P}'(s) + A^*(s)\widetilde{P}(s) + \widetilde{P}(s)A(s) + \sum_{i=1}^m G_i^*(s)\widetilde{P}(s)G_i(s) - S^*(s)\mathscr{B}(\widetilde{P})(s) \\
&\quad - \mathscr{B}^*(\widetilde{P})(s)S(s) + C^*(s)C(s) + S^*(s)\mathscr{K}(\widetilde{P})(s)S(s) = 0, \\
& \widetilde{P}(T) = R.
\end{aligned}
$$

We note that P, the mild solution of (10) with the final condition $P(T) = R \in L^+(H)$, is also a solution of (29). Hence $\widetilde{P} = P$ and the conclusion follows. $\square$

3.2. Bounded solutions of Riccati equation of stochastic control under uniform observability and stabilizability conditions. In this subsection we will prove that under uniform observability and stabilizability conditions the Riccati equation (10) has a unique bounded on $\mathbb{R}_+$, uniformly positive and stabilizing solution.

Let $x(t,s;x)$ be the mild solution of $\{A;G_i\}$ with the initial condition $x(s) = x \in H$ and let C be a family of operators satisfying (P2). We consider the system $\{A, G_i; C\}$ formed by the stochastic equation $\{A; G_i\}$ and the observation relation $z(t) = C(t)x(t,s,x)$.

Definition 9 (see [15], [18]). We say that the system $\{A, C; G_i\}$ *is stochastically uniformly observable if there exist* $\tau > 0$ *and* $\gamma > 0$ such that

$$E\int_s^{s+\tau} \|C(t)x(t,s;x)\|^2 \,\mathrm{d}t \geqslant \gamma\|x\|^2$$

for all $s \in \mathbb{R}_+$ and $x \in H$.

We recall the following characterization of the stochastic uniform observability property of the system $\{A, C; G_i\}$.

Theorem 10 [22], [23]. *The following statements are equivalent:*

1) $\{A, C; G_i\}$ *is stochastically uniformly observable;*
2) *there exist* $\sigma > 0$, $\gamma > 0$ *such that* $X(T-\sigma) \geqslant \gamma I$ *for all* $T \geqslant \sigma$, *where* $X \in C_s([0,T],\, L^+(H))$ *is the unique mild solution of the problem*

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} + A^*(t)X(t) + X(t)A(t) + \sum_{i=1}^m G_i^*(t)X(t)G_i(t) + C^*(t)C(t) = 0, \tag{30}$$

$$X(T) = 0 \in L(H). \tag{31}$$

We note that, following [23], the above result can be proved if (P1), P1(A^*) hold, $G_i \in C_s(\mathbb{R}_+, L(H))$, $i = 1, \dots, m$ and $C \in C_s(\mathbb{R}_+, L(H, V))$. That is, we do not need the assumptions $C^* \in C_s(\mathbb{R}_+, L(V, H))$ and $C^*C \in C_s(\mathbb{R}_+, L(H))$ or the boundedness of the families of operators.

Definition 11. We say that $\{A; G_i\}$ is uniformly exponentially stable if there exist the constants $M \geqslant 1$, $\omega > 0$ such that $E\|x(t,s;x)\|^2 \leqslant M\mathrm{e}^{-\omega(t-s)}\|x\|^2$ for all $t \geqslant s \geqslant 0$ and $x \in H$.

Definition 12. We say that $\{A : B; G_i : H_i\}$ is stabilizable if there exists $F \in C_b([0,\infty), L(H,U))$ such that $\{A + BF; G_i + H_iF\}$ is uniformly exponentially stable.

Assume that (10) has a bounded solution $P(s)$ and let $S(s)$ be defined by (12). It is not difficult to see that $S, S^* \in C_b([0,\infty), L(H,U))$.

Definition 13. A bounded solution of (10) is called stabilizing for $\{A : B; G_i : H_i\}$ if $\{A - BS; G_i - H_iS\}$ is uniformly exponentially stable.

Proposition 14 (see [10]). *The Riccati equation* (10) *has at most one bounded solution, which is stabilizing for* $\{A : B; G_i : H_i\}$.

Proof. Let P_1 and P_2 be two stabilizing and bounded solutions of (10) and let $\mathscr{S}_k(s) = [\mathscr{K}(P_k)(s)]^{-1}\mathscr{B}(P_k)(s), k = 1, 2$. Let A_n be the families of operators introduced by hypothesis (P1) and let $x_{n,k}(t)$ be the unique strong solution of the stochastic equation $\{A_n - B\mathscr{S}_k; G_i - H_i\mathscr{S}_k\}$ with the initial condition $x_{n,k}(s) = x$, $k = 1, 2$. It is known (see [18]) that $x_{n,k}(t) \underset{n\to\infty}{\to} x_k(t)$ uniformly for $t \in [0, T]$, where $x_k(t)$ is the unique mild solution of the stochastic equation $\{A - B\mathscr{S}_k; G_i - H_i\mathscr{S}_k\}$ with the initial condition $x_k(s) = x$, $k = 1, 2$. Using the notation introduced in Proposition 8, we denote by $P_{n,1}$ and $P_{n,2}$, the unique strong solutions of the approximating Lyapunov equations

$$\begin{aligned}
&P'_{n,k}(s) + \widehat{\mathscr{A}^*_{n,k}}(s)P_{n,k}(s) + P_{n,k}(s)\widehat{\mathscr{A}_{n,k}}(s) + \sum_{i=1}^{m}\widehat{\mathscr{G}}^*_{i,k}(s)P_{n,k}(s)\widehat{\mathscr{G}}_{i,k}(s)\\
&\qquad + C^*(s)C(s) + S_k^*(s)K(s)S_k(s) = 0,\\
&P_{n,k}(t) = P_k(t), k = 1, 2.
\end{aligned}$$

Then $\Lambda_n = P_{n,2} - P_{n,1}$ is a solution of

$$\begin{aligned}
(32)\qquad &\Lambda'_n + \widehat{\mathscr{A}^*_{n,2}}\Lambda_n + \Lambda_n\widehat{\mathscr{A}_{n,1}} + \sum_{i=1}^{m}\widehat{\mathscr{G}}^*_{i,2}\Lambda_n\widehat{\mathscr{G}}_{i,1}\\
&\quad + (S_1^* - S_2^*)\mathscr{B}(P_{n,1} - P_1) + \mathscr{B}^*(P_{n,2} - P_2)(S_1 - S_2)\\
&\quad + S_2^*(\mathscr{K}(P_{n,2}) - \mathscr{K}(P_2))S_2 - S_1^*(\mathscr{K}(P_{n,1}) - \mathscr{K}(P_1))S_1 = 0,\\
&\Lambda_n(t) = P_2(t) - P_1(t).
\end{aligned}$$

Now, we apply Ito's formula for the stochastic process $(x_{n,1}(t), x_{n,2}(t))$ and the function $F_n(t, x_1, x_2) = \langle\Lambda_n(t)x_1, x_2\rangle$ ($F_n\colon \mathbb{R}_+ \times H \times H \to \mathbb{R}$). Using (32) and taking expectations, we obtain

$$\begin{aligned}
(33)\ \ &E\langle\Lambda_n(t)x_{n,1}(t), x_{n,2}(t)\rangle = \langle\Lambda_n(s)x, x\rangle\\
&\quad - E\int_s^t \langle[(S_1^* - S_2^*)\mathscr{B}(P_{n,1} - P_1) + \mathscr{B}^*(P_{n,2} - P_2)(S_1 - S_2).\\
&\quad + S_2^*(\mathscr{K}(P_{n,2}) - \mathscr{K}(P_2))S_2 - S_1^*(\mathscr{K}(P_{n,1}) - \mathscr{K}(P_1))S_1]x_{n,1}(r), x_{n,2}(r)\rangle\,\mathrm{d}r
\end{aligned}$$

for all $t \geqslant s$. Let $\Lambda(t) = P_2(t) - P_1(t)$. Letting $n \to \infty$ in (33) and using Proposition 8 and the dominated convergence theorem of Lebesgue we get

$$E\langle\Lambda(t)x_1(t), x_2(t)\rangle = \langle\Lambda(s)x, x\rangle.$$

Since P_1 and P_2 are bounded and stabilizing it follows that there exist $M, a > 0$, such that $E\|x_k(t)\|^2 \leqslant M\mathrm{e}^{-a(t-s)}\|x\|^2, k = 1, 2$. Letting $\widetilde{M} = M \sup_{t \geqslant 0} \|\Lambda(t)\| < \infty$, we see that $\langle \Lambda(s)x, x\rangle \leqslant \widetilde{M}\mathrm{e}^{-a(t-s)}\|x\|^2, 0 \leqslant s \leqslant t$. As $t \to \infty$ we obtain $\|\Lambda(s)\| = 0$, for all $s \in [0, \infty)$. The conclusion follows. □

Reasoning as in [10] (see Theorem 3.1) and stochasticizing the proof we obtain the following result.

Proposition 15. *If $\{A : B; G_i : H_i\}$ is stabilizable then there exists a (global) nonnegative and bounded on $\mathbb{R}_+$ solution of the Riccati equation* (10).

Proof. Assume that there exists $F \in C_b([0, \infty), L(H, U))$ such that $\{A + BF; G_i + H_iF\}$ is uniformly exponentially stable.Let $x_n(t)$ be the unique mild solution of the equation $\{A_n + BF; G_i + H_iF\}$ with the initial condition $x_n(s) = x$, $x \in H$. Analogously, $x(t)$ is the unique mild solution of the equation $\{A + BF; G_i + H_iF\}$ satisfying $x(s) = x$.

Set $T \in \mathbb{R}_+$. It is known [18] that $x_n(t) \underset{n\to\infty}{\to} x(t)$ uniformly for $t \in [0, T]$.

Let $P(s) = P(T, s; 0)$ and $P_n(s) = P_n(T, s; 0)$ be the mild, respectively strong, solution of (10), respectively of the approximating Lyapunov equation (28). (They exist by Theorem 6 and Proposition 8.) Let $S_n = [\mathscr{K}(P_n)(s)]^{-1}\mathscr{B}(P_n)(s)$. Applying Ito's formula for the function $G_n \colon [0, T] \times H \to \mathbb{R}_+$, $G_n(t, x) = \langle P_n(t)x, x\rangle$ and the stochastic process $x_n(t)$ and taking expectations we obtain

$$
\begin{aligned}
(34) \qquad \langle P_n(s)x, x\rangle = E \int_s^T & \|C(r)x_n(r)\|^2 + \langle K(r)F(r)x_n(r), F(r)x_n(r)\rangle \,\mathrm{d}r \\
& - E \int_s^T \langle \mathscr{B}(P_n)(r)x_n(r), 2[S(r) - S_n)](r)x_n(r)\rangle \\
& + \langle \mathscr{K}(P_n)(r)S_n(r)x_n(r), S_n(r)x_n(r)\rangle \\
& - \langle \mathscr{K}(P_n)(r)S(r)x_n(r), S(r)x_n(r)\rangle \,\mathrm{d}r \\
& - E \int_s^T \|[\mathscr{K}(P_n)(r)]^{1/2}[F(r) + S_n(r)]x_n(r)\| \,\mathrm{d}r.
\end{aligned}
$$

By Proposition 8, it follows that for any $x \in H$, $S_n(t)x \underset{n\to\infty}{\to} S(t)x$, $\mathscr{B}(P_n)(t)x \underset{n\to\infty}{\to} \mathscr{K}(P)(t)S(t)x$ uniformly on $t \in [0, T]$.

Letting $n \to \infty$ in (34) we see that the second integral in the right hand side converges to 0. Consequently,

$$
\langle P(s)x, x\rangle \leqslant \int_s^T E\|C(r)x(r)\|^2 + E\langle K(r)F(r)x(r), F(r)x(r)\rangle \,\mathrm{d}r.
$$

Since $\{A+BF; G_i+H_iF\}$ *is* uniformly exponentially stable we deduce that there exist $\beta, a > 0$ such that $E\|x(r)\|^2 \leqslant \beta \mathrm{e}^{-a(r-s)}\|x\|^2$ for any $r \geqslant s$, $x \in H$ and $\langle P(s)x, x\rangle \leqslant [\widetilde{C}^2 + \widetilde{K}\widetilde{F}^2]\beta \int_s^T \mathrm{e}^{-a(r-s)}\,\mathrm{d}r\|x\|^2$. (Recall that $\widetilde{Z} = \sup\limits_{0\leqslant r<\infty} \|Z(r)\| < \infty$, $Z = C, K, F$.) Now it is clear that there is $M \in \mathbb{R}_+$ such that for any $T \in \mathbb{R}_+$ and $s \in [0, T]$, $P(T, s; 0) \leqslant MI$.

Using Theorem 6 it follows that $P(T_1, s; 0) \leqslant P(T_2, s; 0) \leqslant MI$ for all $s \leqslant T_1 \leqslant T_2$. Thus we apply Lemma 4 and deduce that there exists $P_\infty \in C_b(R_+, L(H))$ such that for any $x \in H$,

$$\lim_{T\to\infty} P(T, s; 0)x = P_\infty(s)x. \tag{35}$$

Moreover, for any $s \leqslant t \leqslant T$ we have

$$\begin{aligned} P(T, s; 0)x = U^*(t, s)P(T, t; 0)U(t, s)x + \int_s^t U^*(r, s)\Big[\sum_{i=1}^m G_i^*(r)P(r)G_i(r) \\ + C^*(r)C(r) - [\mathscr{B}(P)(r)]^*[\mathscr{K}(P)(r)]^{-1}\mathscr{B}(P)(r)\Big]U(r, s)x\,\mathrm{d}r. \end{aligned}$$

As $T \to \infty$ in the above equality, it follows that $P_\infty(s)$ is a bounded mild solution of (10). The proof is complete. □

Now, we assume that the system $\{A : B; G_i : H_i\}$ is in the time invariant case, that is the following hypothesis holds:

(P4) $Z(t) = Z$ for all $t \in [0, \infty)$ where $Z = A, B, C, K, G_i, H_i$, $i = 1, \ldots, m$.

Let us introduce the algebraic Riccati equation

$$A^*P + PA + \sum_{i=1}^m G_i^*PG_i + C^*C - [\mathscr{B}(P)]^*[\mathscr{K}(P)]^{-1}\mathscr{B}(P) = 0. \tag{36}$$

Corollary 16. *Assume* (P4). *If* $\{A : B; G_i : H_i\}$ *is stabilizable then there exists a nonnegative solution of the algebraic Riccati equation* (36).

P r o o f. If (P4) holds then (P3) holds for all $p > 0$. It follows by Proposition 7 that $P(T + p, s + p; 0) = P(T, s; 0)$ for any $p > 0$. We deduce from (35) that $P_\infty(s + p) = P_\infty(s)$ for any $p > 0, s \geqslant 0$. Hence $P_\infty(t) = P_\infty(0) = P, t \geqslant 0$ and $P_\infty(t)$ satisfies (36) as we will prove below. First, we note that the evolution operator $U(t, s)$ generated by A is now a C_0-semigroup denoted by $S(t)$ and $U(t, s) = S(t-s)$. Therefore it is not difficult to see that P satisfies the integral equation

$$\begin{aligned} \langle Px, y\rangle &= \langle S^*(t)PS(t)x, y\rangle \\ &+ \int_0^t \Big\langle S^*(r)\Big[\sum_{i=1}^m G_i^*PG_i + C^*C - [\mathscr{B}(P)]^*[\mathscr{K}(P)]^{-1}\mathscr{B}(P)\Big]S(r)x, y\Big\rangle\,\mathrm{d}r \end{aligned}$$

for all $t > 0$. Let $x, y \in D(A)$. Differentiating the above relation with respect to t we get

$$0 = \langle A^* S^*(t) P S(t) x, y\rangle + \langle S^*(t) P S(t) A x, y\rangle$$
$$+ \Big\langle S^*(t) \Big[\sum_{i=1}^{m} G_i^* P G_i + C^* C - [\mathscr{B}(P)]^* [\mathscr{K}(P)]^{-1} \mathscr{B}(P) \Big] S(t) x, y \Big\rangle.$$

Rewriting the last equality we obtain

$$0 = \langle S^*(t) P S(t) x, Ay\rangle + \langle Ax, S^*(t) P S(t) y\rangle$$
$$+ \Big\langle S^*(t) \Big[\sum_{i=1}^{m} G_i^* P G_i + C^* C - [\mathscr{B}(P)]^* [\mathscr{K}(P)]^{-1} \mathscr{B}(P) \Big] S(t) x, y \Big\rangle.$$

Now it is clear that letting $t \to 0$ we have

$$0 = \langle Px, Ay\rangle + \langle Ax, Py\rangle + \Big\langle \Big[\sum_{i=1}^{m} G_i^* P G_i + C^* C - [\mathscr{B}(P)]^* [\mathscr{K}(P)]^{-1} \mathscr{B}(P) \Big] x, y \Big\rangle.$$

Thus P is a nonnegative solution of (36). The proof is complete. $\square$

We make the following assumption:

(P5) $U(t,s)$ generated by $A(t)$ has an exponential growth, that is there exist $M_1 \geqslant 1$ and $\omega_1 > 0$ such that $\|U(t,s)\| \leqslant M_1 \mathrm{e}^{\omega_1 (t-s)}$ for all $t \geqslant s \geqslant 0$

The following result is the analogue of Theorem 1 in [18], (see also [15] for the finite dimensional case) and the proof is very similar, but we recall the main steps of the proof for the readers' convenience.

Theorem 17. *Assume that $\{A, G_i; C\}$ is stochastically uniformly observable and* (P5) *holds. If $P(t)$ is a nonnegative and bounded solution of* (10) *then*

a) *there exists $\delta_0 > 0$ such that $P(t) \geqslant \delta_0 I$ for all $t \in \mathbb{R}_+$ (P is uniformly positive on $\mathbb{R}_+$);*

b) *P is a stabilizing solution for $\{A : B; G_i : H_i\}$).*

P r o o f. Let $P(t)$ be a nonnegative bounded solution of the Riccati equation (10) and let S be the family of operators defined by (12). In this proof the constants γ, τ are those introduced in Definition 9. We will denote by $z(t) = z(t,s;x)$ the mild solution of $\{A - BS; G_i - H_i S\}$ with the initial condition $z(s) = x$. Arguing as in the proof of the above proposition and using Ito's lemma we have

$$\langle P(s)x, x\rangle = E\langle P(T) z(T), z(T)\rangle + E \int_s^T \|C(r) z(r)\|^2 + \langle K(r) S(r) z(r), S(r) z(r)\rangle \,\mathrm{d}r \tag{37}$$

for all $T \geqslant s$. Now it is easy to see that if we prove that the linear, bounded and nonnegative operator $Q(s)$ that satisfies the relation
(38)
$$\langle Q(s)x, x\rangle = E\int_s^{s+\tau} [\|C(r)z(r,s;x)\|^2 + \langle S^*(r)K(r)S(r)z(r,s;x), z(r,s;x)\rangle]\,\mathrm{d}r,$$
is uniformly positive, then P is uniformly positive on $\mathbb{R}_+$ and the statement a) follows. Since $K(t) \geqslant \delta I$ for all $t \in \mathbb{R}_+$ we have

$$\langle Q(s)x, x\rangle \geqslant \delta E\int_s^{s+\tau} \|S(t)z(t,s;x)\|^2\,\mathrm{d}t \tag{39}$$

Let $x(t) = x(t,s;x)$ be the mild solution of $\{A; G_i\}$ with the initial condition $x(s) = x$. It is clear that $z(t,s;x) - x(t,s;x)$ is the solution of a nonlinear equation (see [11]) with the initial condition $z(s) - x(s) = 0$, whose unique mild solution satisfies the integral equation

$$\begin{aligned} z(t) - x(t) &= \int_s^t U(t,r)[-B(r)S(r)z(r)]\,\mathrm{d}r \\ &\quad + \sum_{i=1}^m \int_s^t U(t,r)\{G_i(t)[z(r) - x(r)] - H_i(r)S(r)z(r)\}\,\mathrm{d}w_i(r). \end{aligned}$$

Taking the mean square and using (P2) and (39) we deduce that there exists $M > 0$ such that for all $t \in [s, s+\tau]$

$$E\|z(t,s;x) - x(t,s;x)\|^2 \leqslant M\left[\langle Q(s)x, x\rangle + \int_s^t E\|z(r,s;x) - x(r,s;x)\|^2\,\mathrm{d}r\right].$$

Applying Gronwall's inequality we see that for all $t \in [s, s+\tau]$

$$E\|z(t,s;x) - x(t,s;x)\|^2 \leqslant M\mathrm{e}^{M\tau}\langle Q(s)x, x\rangle. \tag{40}$$

On the other hand, we get

$$\begin{aligned} E\int_s^{s+\tau} \|C(r)z(r,s;x)\|^2\,\mathrm{d}r \geqslant \tfrac{1}{2}E\int_s^{s+\tau} &\|C(r)x(r,s;x)\|^2\,\mathrm{d}r \\ &- \widetilde{C}\int_s^{s+\tau} E\|z(r,s;x) - x(r,s;x)\|^2\,\mathrm{d}r, \end{aligned} \tag{41}$$

where $\widetilde{C} = \sup_{t\geqslant 0} \|C(t)\|^2$. Using the uniform observability condition and (40) it follows that for all $s \in \mathbb{R}_+$

$$\langle Q(s)x, x\rangle \geqslant \tfrac{1}{2}\gamma\|x\|^2 - \tau\widetilde{C}M\mathrm{e}^{M\tau}\langle Q(s)x, x\rangle.$$

Thus $Q(s)$ is uniformly positive and the proof of a) is complete.

b) Since P is bounded on $\mathbb{R}_+$, we can find $\delta_1 > 0$ such that $\|P(t)\| < \delta_1, t \in \mathbb{R}_+$. Taking $T = s + \tau$ we deduce by (37) and the above inequality that

$$\langle P(s)x, x\rangle - E\langle P(s+\tau)z(s+\tau), z(s+\tau)\rangle \geqslant \frac{\delta_0}{\delta_1}\langle P(s)x, x\rangle,$$

where $\delta_0 = \gamma/[2 + 2\tau\widetilde{C}M\mathrm{e}^{M\tau}]$. Obviously $q = 1 - \delta_0/\delta_1 \in (0, 1)$ and

$$E\langle P(s+\tau)z(s+\tau), z(s+\tau)\rangle \leqslant q\langle P(s)x, x\rangle, s \in \mathbb{R}_+. \tag{42}$$

By Theorem 10 in [23] it follows that there exists an evolution operator $\mathscr{V}(t, s)$ on $\mathscr{N}$ associated with the equation $\{A - BS; G_i - H_iS\}$ given by

$$\begin{aligned} \mathscr{V}(t, s)X &= U_{-BS}(t, s)XU^*_{-BS}(t, s) \\ &+ \int_s^t U_{-BS}(t, r)\Big[\sum_{i=1}^m (G_i - H_iS)(r)\mathscr{V}(r, s)(X)(G_i - H_iS)^*(r)\Big]U^*_{-BS}(t, r)\,\mathrm{d}r \end{aligned} \tag{43}$$

such that

$$E[z(t, s; x) \otimes z(t, s; x)] = \mathscr{V}(t, s)(x \otimes x).$$

By $E(\xi \otimes \xi)$, where $\xi \in L^2(\Omega, H)$, we mean the linear and bounded operator acting on H defined as

$$E(\xi \otimes \xi)(x) = E(\langle x, \xi\rangle\xi).$$

Evidently, $E(\xi \otimes \xi) \in \mathscr{N}$. Further, (42) can be equivalently rewritten to

$$Tr[P(s+\tau)\mathscr{V}(s+\tau, s)(x \otimes x)] \leqslant qTr[P(s)(x \otimes x)],\ x \in H.$$

By a standard argument (see [19]) we deduce that

$$Tr[P(s+\tau)\mathscr{V}(s+\tau, s)(N)] \leqslant qTr[P(s)(N)]$$

for all $N \in \mathscr{N}$. Setting $N = \mathscr{V}(s, p)(x \otimes x), p \leqslant s$ we get

$$Tr[P(s+\tau)\mathscr{V}(s+\tau, p)(x \otimes x)] \leqslant qTr[P(s)\mathscr{V}(s, p)(x \otimes x)]. \tag{44}$$

Let $t > p$. There exist $c \in \mathbb{N}, r \in [0, \tau)$ such that $t = c\tau + r + p$. Using the induction method and (44) we obtain

$$Tr[P(t)\mathscr{V}(t, p)(x \otimes x)] \leqslant q^cTr[P(p+r)\mathscr{V}(p+r, p)(x \otimes x)].$$

Since $U(t,s)$ has an exponential growth, we use Gronwall's inequality and (43) to deduce that there exists $M_\tau > 0$ such that $\|\mathscr{V}(p+r,p)\| \leqslant M_\tau$ for all $p \in \mathbb{R}_+$ and $r \in [0,\tau)$. Taking $a = q^{1/\tau}$ and $\beta = q^{-r/\tau} M_\tau \delta_1$ we have

$$E\langle P(t)z(t,p;x), z(t,p;x)\rangle = Tr[P(t)\mathscr{V}(t,p)(x\otimes x)] \leqslant \beta a^{t-p}\|x\otimes x\|_1.$$

Further,

$$\delta_0 E\|z(t,p;x)\|^2 \leqslant \beta a^{t-p}\|x\|^2$$

for all $t \geqslant p \geqslant 0$ and $x \in H$ by a). Therefore $\{A - BS; G_i - H_i S\}$ is uniformly exponentially stable and the proof is complete. □

The next result is a consequence of the above theorem and of Proposition 15 (see also [16] for the finite dimensional case and the conventional Riccati equation).

Theorem 18. *Assume $\{A : B; G_i : H_i\}$ is stabilizable, $\{A, G_i; C\}$ is stochastically uniformly observable and* (P5) *holds. Then the Riccati equation* (10) *has a unique uniformly positive and bounded on $\mathbb{R}_+$ solution $P(t)$, which is stabilizing for $\{A : B; G_i : H_i\}$.*

4. Optimal control

The following lemma and remark can be proved only under the hypotheses (P1) and (P2). P1(A^*) is not necessary here.

Lemma 19. *Assume $L \in C_b([0,\infty), L(H))$ and $h \in C_b(\mathbb{R}_+, H)$. Then the equation*

$$g_n'(t) = -(A_n^* + L^*)g_n(t) - h(t), g_n(T) = x_0 \in H, t \leqslant T, \tag{45}$$

where the weak differentiability is considered, has a unique solution. The functions $(t,x) \to \langle g_n'(t), x\rangle$, $n \in \mathbb{N}$ are continuous on $[0,\infty)\times H$. Moreover, if $U_{L,n}$ (respectively U_L) are the perturbations of U (respectively of U_n) by L, then for all $y \in H$ and $t \in [0,T]$ we have

$$\langle g_n(t), y\rangle \underset{n\to\infty}{\longrightarrow} \langle U_L^*(T,t)x_0, y\rangle + \left\langle \int_t^T U_L^*(\sigma,t)h(\sigma)\,\mathrm{d}\sigma, y\right\rangle. \tag{46}$$

Here $L^* : [0,\infty) \to L(H)$, $L^*(t) = (L(t))^*$.

Proof. Since for all $x \in H$, $\partial U_{L,n}(t,s)x/\partial s = -U_{L,n}(t,s)(A_n(s) + L(s))x$ [20], it is not difficult to see that $g_n(t) = U_{L,n}^*(T,t)x_0 + \int_t^T U_{L,n}^*(\sigma,t)h(\sigma)\,\mathrm{d}\sigma$ is the unique solution of the equation (45). Using Lemma 3 in [18] it follows that for each $y \in H$, $\lim\limits_{n\to\infty} U_{L,n}(t,s)y = U_L(t,s)y$ uniformly with respect to $t \in [s,T]$, $0 \leqslant s \leqslant T$ and (46) holds. □

Remark 20. Assume that $\{A : B; G_i : H_i\}$ is stabilizable with the stabilizing sequence $F \in C_b([0,\infty), L(H,U))$ and let $h \in C_b(\mathbb{R}_+, H)$. Since $\{A+BF, G_i+H_iF\}$ is uniformly exponentially stable it follows that there exist constants $M \geqslant 1, \omega > 0$such that $\|U_{BF}(\sigma,t)\| \leqslant M\mathrm{e}^{-\omega(\sigma-t)}, \sigma \geqslant t \geqslant 0$. Hence, the integral

$$g(t) = \int_t^\infty U_{BF}^*(\sigma,t)h(\sigma)\,\mathrm{d}\sigma \tag{47}$$

is convergent in H and $g(\cdot)$ is bounded on $\mathbb{R}_+$. Moreover,

$$\left\langle \int_t^T U_{BF}^*(\sigma,t)h(\sigma)\,\mathrm{d}\sigma, y \right\rangle \underset{T\to\infty}{\to} \left\langle \int_t^\infty U_{BF}^*(\sigma,t)h(\sigma)\,\mathrm{d}\sigma, y \right\rangle.$$

If we consider the solution of (45) with the initial condition $g_n(T) = g(T)$ it is not difficult to see that $\langle g_n(t), y\rangle \underset{n\to\infty}{\to} \langle g(t), y\rangle$ for all $y \in H$.

4.1. Tracking problem.

The following result is similar to the one obtained in [17] for finite dimensional spaces and the conventional Riccati equation or in [21] for Hilbert spaces and bounded coefficients of the conventional Riccati equation.

Theorem 21. *Assume the hypotheses of Theorem* 18 *are satisfied. Let P be the unique, bounded on $\mathbb{R}_+$ and stabilizing solution of the Riccati equation* (10) *and let $g(t)$ be given by* (47), *where $F(t) = -[\mathscr{K}(P)(t)]^{-1}\mathscr{B}(P)(t)$ and $h(t) = -C^*(t)C(t)r(t)$. Then the optimal cost for the problem* (5)–(6), (8) *is*

$$\begin{aligned} J(s) &= \inf_{u\in\mathscr{U}_{ad}} J(s,u) \\ &= \varlimsup_{t\to\infty} \frac{1}{t-s}\left[\int_s^t \|C(\sigma)r(\sigma)\|^2\,\mathrm{d}\sigma - \int_s^t \|[\mathscr{K}(P)(\sigma)]^{-1/2}B^*(\sigma)g(\sigma)\|^2\,\mathrm{d}\sigma\right], \end{aligned}$$

and it is obtained for the optimal control

$$u(t) = -[\mathscr{K}(P)(t)]^{-1}[\mathscr{B}(P)(t)x(t) + B^*(t)g(t)],$$

where $x(t)$ is the solution of (5)–(6).

Proof. Let P be the unique, bounded on $\mathbb{R}_+$ and stabilizing solution of the Riccati equation (10) and let $S(s)$ be the family of operators introduced by (12). We recall (see Theorem 18) that $F(t) = -S(t)$ is a stabilizing sequence for the stochastic equation $\{A : B; G_i : H_i\}$. Let $P_n(t_1, s; P(t_1))$ be the solution of the approximating Lyapunov equation (28) satisfying $P_n(t_1) = P(t_1)$ and let $g_n(s)$ be the solution of (45) with the final condition $g_n(t_1) = g(t_1)$, where $L(t) = -B(t)S(t)$. We consider the function $G_n(t,x) = \langle P_n(t)x, x\rangle + 2\langle g_n(t), x\rangle$, which is continuous together its partial derivatives $G_{n,t}$, $G_{n,x}$, $G_{n,xx}$ on $[0,\infty)\times H$. Let $u \in \mathscr{U}_{ad}$ and let $x_n(t)$ be its response. Using Ito's formula for $G_n(t,x)$ and the strong solution of $\{A_n : B; G_i : H_i\}$ we get

$$\begin{aligned}
&E\langle P_n(t)x_n(t), x_n(t)\rangle + 2E\langle g_n(t), x_n(t)\rangle - \langle P_n(s)x, x\rangle - 2\langle g_n(s), x\rangle \\
&\quad = -E\int_s^t \|C(\sigma)[x_n(\sigma) - r(\sigma)]\|^2 + \langle K(\sigma)u(\sigma), u(\sigma)\rangle \,\mathrm{d}\sigma + \int_s^t \|C(\sigma)r(\sigma)\|^2 \,\mathrm{d}\sigma \\
&\qquad + E\int_s^t [2\langle \mathscr{B}(P_n)(\sigma)x_n(\sigma) + B^*(\sigma)g_n(\sigma), u(\sigma) + S(\sigma)x_n(\sigma)\rangle \\
&\qquad - \langle \mathscr{K}(P_n)(\sigma)S(\sigma)x_n(\sigma), S(\sigma)x_n(\sigma)\rangle + \langle \mathscr{K}(P_n)(\sigma)u(\sigma), u(\sigma)\rangle] \,\mathrm{d}\sigma.
\end{aligned}$$

Letting $n \to \infty$ and using the dominated convergence theorem of Lebesgue we obtain

$$\begin{aligned}
&E\langle P(t)x(t), x(t)\rangle + 2E\langle g(t), x(t)\rangle - \langle P(s)x, x\rangle - 2\langle g(s), x\rangle \\
&\quad = -E\int_s^t \|C(\sigma)[x(\sigma) - r(\sigma)]\|^2 + \langle K(\sigma)u(\sigma), u(\sigma)\rangle \,\mathrm{d}\sigma \\
&\qquad + E\int_s^t \|\mathscr{K}(P)(\sigma)^{1/2}[u(\sigma) + [\mathscr{K}(P)(\sigma)]^{-1}[\mathscr{B}(P)(\sigma)x(\sigma) + B^*(\sigma)g(\sigma)]]\|^2 \\
&\qquad + \int_s^t \|C(\sigma)r(\sigma)\|^2 \,\mathrm{d}\sigma - \int_s^t \|[\mathscr{K}(P)(\sigma)]^{-1/2}B^*(\sigma)g(\sigma)\|^2 \,\mathrm{d}\sigma.
\end{aligned}$$

Since $P(t)$ and $g(t)$ are bounded on $\mathbb{R}_+$, we multiply the last relation by $1/(t-s)$ and taking the limit as $t \to \infty$ and then the infimum we get the conclusion. □

4.2. Quadratic control.

Theorem 22. *Assume that the hypotheses of Theorem* 18 *are fulfilled and consider the control problem* (5)–(6), (9). *The optimal control is given by the feedback law*

$$\tilde{u}(t) = -[\mathscr{K}(P)(t)]^{-1}\mathscr{B}(P)(t)x(t), \tag{48}$$

where P is the unique bounded on $\mathbb{R}_+$ and uniformly positive solution of (10) *($x(t)$ is the corresponding solution of* (5)–(6)*) and the optimal cost is*

$$I_{s,x}(\tilde{u}) = \langle P(s)x, x\rangle. \tag{49}$$

P r o o f. Let P be the unique, bounded on $\mathbb{R}_+$ and stabilizing solution of the Riccati equation (10). Let $u \in \mathscr{U}_{ad}^q$ and let $x(t)$ be its response ($x(t)$ is the solution of the stochastic equation $\{A : B; G_i : H_i\}$ with the initial condition $x(s) = x$). Arguing as in the proof of Theorem 21 we deduce that

$$E\langle P(t)x(t), x(t)\rangle - \langle P(s)x, x\rangle = -E\int_s^t \|C(\sigma)x(\sigma)\|^2 + \langle K(\sigma)u(\sigma), u(\sigma)\rangle \, \mathrm{d}\sigma$$
$$+ E\int_s^t \|\mathscr{K}(P)(\sigma)^{1/2}[u(\sigma) + [\mathscr{K}(P)(\sigma)]^{-1}\mathscr{B}(P)(\sigma)x(\sigma)]\|^2.$$

It is clear that the control $\tilde{u}(t)$ given by (48) belongs to $\mathscr{U}_{ad}^q$ and $I_s(\tilde{u}) = \langle P(s)x, x\rangle$. Let $u \in \mathscr{U}_{ad}^q$. Since $P(t)$ is bounded on $\mathbb{R}_+$ and stabilizing, we take the limit (for $t \to \infty$) in the above relation and see that $\langle P(s)x, x\rangle \leqslant I_s(u)$. The conclusion follows. □

Example 23. Let $\Lambda\colon D(\Lambda) = H_0^1(0,1) \cap H^2(0,1) \to L^2(0,1), \Lambda y = -\partial^2 y/\partial\xi^2$. It is known that the linear operator Λ is the infinitesimal generator of an analytic semigroup on the Hilbert space $L^2(0,1)$ [8]. Recall that the eigenvalues of Λ are $n^2\pi^2$, $n \in \mathbb{N}^*$ and the corresponding eigenvectors are $f_n = \sqrt{2}\varphi_n$, where $\varphi_n = \sin n\pi\xi$. Let $H = D(\Lambda^{1/2}) \oplus L^2(0,1)$ be Hilbert space [3] endowed with the inner product

$$\langle x, u\rangle_H = \langle \Lambda^{1/2}x_1, \Lambda^{1/2}u_1\rangle_{L^2(0,1)} + \langle x_2, u_2\rangle_{L^2(0,1)},$$
$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in H.$$

On the Hilbert space H we consider the stochastic controlled system

(50)
$$\begin{cases} \mathrm{d}y_t(\xi,t) = \dfrac{\partial^2 y(\xi,t)}{\partial\xi^2}\,\mathrm{d}t + u_2(\xi,t)\,\mathrm{d}t + y_t(\xi,t)\,\mathrm{d}w(t), \\ y(0,t) = 0, y(1,t) = 0, t \geqslant 0, \\ y(\xi,0) = y_0(\xi), y_t(\xi,0) = y_1(\xi), \xi \in [0,1] \end{cases}$$

with the observation relation

(51)
$$z(t) = (0, y_t(\xi,t))^T.$$

Our problems are

i) to find the optimal control which minimizes the cost functional

(52)
$$I_{0,(y_0,y_1)}(u) = E\int_0^\infty \|y_t(\xi,t)\|_{L^2(0,1)}^2 + \|u(t)\|_H^2 \,\mathrm{d}t;$$

ii) to solve the tracking problem defined by the given signal $r(t) = \mathrm{e}^{-(1/2)t}\varrho_o$, $\varrho_0 = (0, \sqrt{2}\sin(\pi\xi))^T \in H$, $t \geqslant 0$, (50)–(51) and

$$J(0,u) = \overline{\lim_{t\to\infty}} \frac{1}{t} E \int_0^t \|y_t(\xi,\sigma) - \mathrm{e}^{-(1/2)\sigma}\sqrt{2}\sin(\pi\xi)\|^2_{L^2(0,1)} + \|u(\sigma)\|^2_H \,\mathrm{d}\sigma.$$

Now, let us introduce the linear operator

$$A\colon D(\Lambda) \oplus D(\Lambda^{1/2}) \to H, A(x) = \begin{pmatrix} 0 & I \\ -\Lambda & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

It is known (see [11], [3]) that A generates the contraction semigroup

$$S(t)(x) = \begin{pmatrix} \sum\limits_{n=1}^{\infty} 2\Big[\langle x_1, \varphi_n\rangle_{L^2(0,1)} \cos n\pi t + \dfrac{1}{n\pi}\langle x_2, \varphi_n\rangle_{L^2(0,1)} \sin n\pi t\Big]\varphi_n \\ \sum\limits_{n=1}^{\infty} 2[-n\pi\langle x_1, \varphi_n\rangle_{L^2(0,1)} \sin n\pi t + \langle x_2, \varphi_n\rangle_{L^2(0,1)} \cos n\pi t]\varphi_n \end{pmatrix}.$$

Moreover, $D(A^*) = D(A)$ and $A^* = -A$. It is easy to see that the system (50)–(51) can be equivalently rewritten as

$$\begin{aligned} \mathrm{d}x(t) &= Ax(t)\,\mathrm{d}t + Bu(t)\,\mathrm{d}t + Gx(t)\,\mathrm{d}w(t),\ x(0) = x_0, \\ z(t) &= C(t)x(t), \end{aligned}$$

where $x = (x_1, x_2)^T$, $x_0(\xi) = (y_0(\xi),\ y_1(\xi))$, $u = (u_1, u_2)$ $B = G = C = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$. Comparing it with (5), (8) and (9) we also see that $m = 1$, $H(t) = H_1(t) = 0$ and $K(t) = I$, $t \geqslant 0$, $U = V = H$. Obviously the system is in the time invariant case and (P2) holds. Also the hypotheses (P1) and P1(A^*) fulfilled since A is the infinitesimal generator of a C_0-semigroup.

We need to prove that $\{A, G; C\}$ is stochastically uniformly observable and $\{A : B, G : H\}$ is stabilizable. Arguing as in Example 23 in [22] (see also [23]) and using Theorem 10 it follows that $\{A, G; C\}$ is stochastically uniformly observable. The algebraic Riccati equation associated with the discussed control problems is

$$A^*P + PA + G^*PG + C^*C - PBB^*P = 0. \tag{53}$$

Since $A^* = -A$ it is clear that $P = (\frac{1}{2}\sqrt{5} + \frac{1}{2})I$ is a nonnegative solution of (53). Obviously P is a nonnegative and bounded solution of (10). The hypotheses of Theorem 17 are verified and we conclude that P is stabilizing for the system $\{A : B, G_i : 0\}$.

i) By Theorem 22 it follows that the optimal cost of the quadratic control problem (i) is $I_{0,x_0}(u) = (\frac{1}{2}\sqrt{5} + \frac{1}{2})\|x_0\|_H^2$ and the optimal control is

$$\tilde{u}(\xi, t) = (0, -(\sqrt{5}/2 + 1/2)\tilde{y}_t(\xi, t))^T,$$

where $\tilde{y}(\xi, t)$ is the solution of the equation

$$\begin{aligned} \mathrm{d}\tilde{y}_t(\xi, t) &= \frac{\partial^2 \tilde{y}(\xi, t)}{\partial \xi^2}\,\mathrm{d}t - (\sqrt{5}/2 + 1/2)\tilde{y}_t(\xi, t)\,\mathrm{d}t + \tilde{y}_t(\xi, t)\,\mathrm{d}w(t), \\ \tilde{y}(\xi, 0) &= y_0(\xi), \quad \tilde{y}_t(\xi, 0) = y_1(\xi). \end{aligned}$$

ii) Let $P = \big(\sqrt{5}/2 + 1/2\big)I$ be the unique nonnegative solution of (53). Since the stochastic system $\{A - BB^*P, G_i\}$ is uniformly exponentially stable, we deduce easily by Theorem 9 in [23] that the deterministic system $\{A - BB^*P\}$ is uniformly exponentially stable. Obviously $A - BB^*P = A - \big(\sqrt{5}/2 + 1/2\big)B$ generates a C_0-semigroup $T(t)$. Then $T^*(t)$ is also a C_0-semigroup, generated by $A^* - \big(\sqrt{5}/2 + 1/2\big)B = -A - \big(\sqrt{5}/2 + 1/2\big)B$. Let us denote $a = \big(\sqrt{5}/2 + 1/2\big)$ and $\Lambda_a = \sqrt{\Lambda - (a/2)^2 I}$. After an easy computation we obtain the formula

$$T^*(t) = \mathrm{e}^{-(a/2)t} \begin{pmatrix} \cos(\Lambda_a t) + \frac{a/2}{\Lambda_a}\sin(\Lambda_a t) & -\frac{1}{\Lambda_a}\sin(\Lambda_a t) \\ (\Lambda_a + \frac{(a/2)^2}{\Lambda_a})\sin(\Lambda_a t) & \cos(\Lambda_a t) - \frac{a/2}{\Lambda_a}\sin(\Lambda_a t) \end{pmatrix}.$$

Here $\cos(\Lambda_a t)(y) = \sum_{n=1}^{\infty} \cos(\sqrt{n^2\pi^2 - a^2/4}t)\langle y, f_n\rangle_{L^2(0,1)} f_n$. The remainders of the operators are defined similarly. In view of (47) we get

$$g(t) = \int_t^\infty \mathrm{e}^{-(1/2)\sigma} T^*(\sigma - t)(\varrho_0)\,\mathrm{d}\sigma = \mathrm{e}^{-(1/2)t} \int_0^\infty \mathrm{e}^{-(1/2)u} T^*(u)(\varrho_0)\,\mathrm{d}u.$$

Since $\varrho_0 = (0, f_1)$, we have

$$g(t) = (g_1(t), g_2(t))^T = \mathrm{e}^{-(1/2)t}\Big(-\frac{4}{2 + \sqrt{5} + 4\pi^2}, \frac{2}{2 + \sqrt{5} + 4\pi^2}\Big)^T \sqrt{2}\sin \pi\xi.$$

Applying Theorem 21 it follows that the optimal control for the tracking problem is

$$\tilde{u}(t) = \big(0, -\big[\big(\sqrt{5}/2 + 1/2\big)y_t(t) + g_2(t)\big]\big)^T,$$

where $y(t)$ is the mild solution of the corresponding control system $\{A : B, G_i : 0\}$. Also

$$\begin{aligned} J(0, \tilde{u}) &= \varlimsup_{t\to\infty} \frac{1}{t}\Big[\int_0^t \mathrm{e}^{-\sigma}\|\sqrt{2}\sin(\pi\xi)\|^2_{L^2(0,1)}\,\mathrm{d}\sigma - \int_s^t \|g_2(\sigma)\|^2_{L^2(0,1)}\,\mathrm{d}\sigma\Big] \\ &= \varlimsup_{t\to\infty} \frac{1}{t}\int_0^t \mathrm{e}^{-\sigma}\Big(1 - \frac{2}{2 + \sqrt{5} + 4\pi^2}\Big)\,\mathrm{d}\sigma = 0. \end{aligned}$$

We note that if we consider the tracking problem ii) for $r(t) = \varrho_0$, then evidently the optimal cost $J(0, \tilde{u}) = (\sqrt{5} + 4\pi^2)/(2 + \sqrt{5} + 4\pi^2)$ is obtained for

$$\tilde{u}(t) = \left(0, -\left[\left(\frac{\sqrt{5}}{2} + \frac{1}{2}\right) y_t(t) + \frac{2}{2 + \sqrt{5} + 4\pi^2} f_1\right]\right)^T.$$

5. Concluding Remarks

In this paper we solved a linear quadratic, as well as a tracking problem, under uniform observability and stabilizability conditions. In view of Theorem 10 we see that the stochastic uniform observability property can be easily verified as compared with the detectability condition (see [10]). However, there are some situations where the stochastic uniform observability property cannot be achieved. This is the case when the family $A(t)$, $t \geqslant 0$ generates a compact evolution operator and the Hilbert space H is of infinite dimension (we refer to [23] for details). Obviously, in this case we cannot use the results in present paper. However, if stabilizability and detectability conditions are fulfilled, then we can solve the optimal control problems following the way indicated by G. Da Prato and A. Ichikawa in [10], [12]. Therefore, on the one side, our results make the ones in [10], [12] more complete, since they cover the case of stochastically uniformly observable systems which are not detectable and, on the other hand, they are an alternative to the approach of G. Da Prato and A. Ichikawa. (We refer here to the cases where both the detectability and stochastic uniform observability properties hold and the latter is easier to be verified.)

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