

Dieter Grass · Jonathan P. Caulkins
Gustav Feichtinger · Gernot Tragler
Doris A. Behrens

Optimal Control of Nonlinear Processes

With Applications in Drugs, Corruption,
and Terror

 Springer

Contents

Preface	VII
---------------	-----

Acknowledgments	XI
-----------------------	----

Part I Background

1 Introduction	3
1.1 Taking Rocket Science Beyond the Frontiers of Space	3
1.2 Why Drugs, Corruption, and Terror?	5
1.3 Questions Optimal Control Can Answer	7
2 Continuous-Time Dynamical Systems	9
2.1 Nonlinear Dynamical Modeling	9
2.2 One-Dimensional Systems	10
2.3 A One-Dimensional Corruption Model	14
2.4 Dynamical Systems as ODEs	17
2.4.1 Concepts and Definitions	19
2.4.2 Invariant Sets and Stability	21
2.4.3 Structural Stability	25
2.4.4 Linearization and the Variational Equation	26
2.5 Stability Analysis of a One-Dimensional Terror Model	27
2.6 ODEs in Higher Dimensions	30
2.6.1 Autonomous Linear ODEs	31
2.6.2 Autonomous Nonlinear ODEs	42
2.7 Stability Behavior in a Descriptive Model of Drug Demand ...	51
2.8 Introduction to Bifurcation Theory	55
2.8.1 Terminology and Key Ideas of Bifurcation Theory	56
2.8.2 Normal Forms and the Center Manifold: The Tools of Bifurcation Theory	57
2.8.3 Local Bifurcations in One Dimension	63

2.9	Bifurcation Analysis of a One-Dimensional Drug Model	68
2.10	The Poincaré–Andronov–Hopf Bifurcation	71
2.11	Higher-Dimensional Bifurcation Analysis of a Drug Model	74
2.12	Advanced Topics	78
2.12.1	Stability of Limit Cycles	78
2.12.2	Boundary Value Problems	85
	Exercises	89
	Notes and Further Reading	96

Part II Applied Optimal Control

3	Tour d’Horizon: Optimal Control	101
3.1	Historical Remarks	101
3.2	A Standard Optimal Control Problem	104
3.3	The Maximum Principle of Optimal Control Theory	108
3.3.1	Pontryagin’s Maximum Principle	108
3.3.2	Some General Results	113
3.3.3	The Maximum Principle for Variable Terminal Time	115
3.3.4	Economic Interpretation of the Maximum Principle	117
3.3.5	Sufficiency Conditions	119
3.3.6	Existence of an Optimal Solution	122
3.3.7	How to Solve an Optimal Control Problem: A Simple Consumption vs. Investment Model	124
3.4	The Principle of Optimality	127
3.4.1	The Hamilton–Jacobi–Bellman Equation	127
3.4.2	A Proof of the Maximum Principle	130
3.5	Singular Optimal Control	131
3.5.1	The Most Rapid Approach Path (MRAP)	134
3.5.2	An Example From Drug Control that Excludes Singular Arcs	137
3.5.3	An Example From Terror Control with an MRAP Solution	139
3.6	The Maximum Principle With Inequality Constraints	142
3.6.1	Mixed Path Constraints	144
3.6.2	General Path Constraints	147
3.6.3	Sufficiency Conditions	154
3.7	Infinite Time Horizon	155
3.7.1	Definitions of Optimality for Infinite Horizon Problems	155
3.7.2	Maximum Principle for Infinite Time Horizon Problems	156
3.7.3	Sufficiency Conditions	159
3.8	Discounted Autonomous Infinite Horizon Models	159
3.8.1	The Michel Theorem	160
3.8.2	The Ramsey Model for an Infinite Time Horizon	165

3.8.3	Structural Results on One-State Discounted, Autonomous Systems	167
3.9	An Optimal Control Model of a Drug Epidemic	168
3.9.1	Model Formulation	168
3.9.2	Stability Analysis	170
3.9.3	Phase Portrait Analysis	176
	Exercises	177
	Notes and Further Reading	183
4	The Path to Deeper Insight: From Lagrange to Pontryagin	189
4.1	Introductory Remarks on Optimization	189
4.1.1	Notational Remarks	190
4.1.2	Motivation and Insights	190
4.1.3	A Simple Maximization Problem	192
4.1.4	Finite-Dimensional Approximation of an Infinite-Dimensional Problem	195
4.2	Static Maximization	197
4.2.1	Basic Theorems and Definitions	198
4.2.2	Theory and Geometric Interpretation of Lagrange and Karush–Kuhn–Tucker	202
4.2.3	The Envelope Theorem and the Lagrange Multiplier ...	208
4.2.4	The Discrete-Time Maximum Principle as a Static Maximization Problem	210
4.3	The Calculus of Variations	214
4.3.1	A Simple Variational Example	214
4.3.2	The First Variation	216
4.3.3	Deriving the Euler Equation and Weierstrass–Erdmann Conditions	218
4.4	Proving the Continuous-Time Maximum Principle	223
4.4.1	The Continuous-Time Maximum Principle Revisited ...	223
4.4.2	Necessary Conditions at Junction Points	227
	Exercises	231
	Notes and Further Reading	234
5	Multiple Equilibria, Points of Indifference, and Thresholds	237
5.1	Occurrence of Multiple Equilibria	238
5.2	The Optimal Vector Field	239
5.2.1	Finite vs. Infinite Time Horizon Models	239
5.2.2	Discounted Autonomous Models for an Infinite Time Horizon	243
5.3	A Typical Example	244
5.3.1	Existence and Stability of the Equilibria	245

5.3.2	Determining the Optimal Vector Field and the Optimal Costate Rule	247
5.4	Defining Indifference and DNSS Points	252
5.4.1	Multiplicity and Separability	253
5.4.2	Definitions	254
5.4.3	Conclusions from the Definitions	256
5.5	Revisiting the Typical Example	260
5.6	Eradication vs. Accommodation in an Optimal Control Model of a Drug Epidemic	266
	Exercises	269
	Notes and Further Reading	272

Part III Advanced Topics

6	Higher-Dimensional Models	279
6.1	Controlling Drug Consumption	280
6.1.1	Model of Controlled Drug Demand	280
6.1.2	Deriving the Canonical System	283
6.1.3	The Endemic Level of Drug Demand	286
6.1.4	Optimal Dynamic Policy away from the Endemic State	287
6.1.5	Optimal Policies for Different Phases of a Drug Epidemic	292
6.2	Corruption in Governments Subject to Popularity Constraints	296
6.2.1	The Modeled Incentive for Being Corrupt	297
6.2.2	Optimality Conditions	299
6.2.3	Insights About the Incentive to Be Corrupt	300
6.2.4	Is Periodic Behavior Caused by Rational Optimization?	302
6.3	Is It Important to Manage Public Opinion While Fighting Terrorism?	308
6.3.1	What One Should Know when Fighting Terrorism	309
6.3.2	Derivation of the Canonical System	310
6.3.3	Numerical Calculations	311
6.3.4	Optimal Strategy for a Small Terror Organization	314
	Exercises	316
	Notes and Further Reading	323
7	Numerical Methods for Discounted Systems of Infinite Horizon	327
7.1	General Remarks	327
7.1.1	Problem Formulation and Assumptions	328
7.1.2	Notation	329

7.1.3	Numerical Methods for Solving Optimal Control Problems	330
7.1.4	Boundary Value Problems from Optimal Control	330
7.2	Numerical Continuation	332
7.2.1	Continuation Algorithms	333
7.2.2	Continuing the Solution of a BVP	338
7.3	The Canonical System Without Active Constraints	342
7.4	Calculating Long-Run Optimal Solutions	343
7.4.1	Equilibria	344
7.4.2	Limit Cycles	346
7.5	Continuing the Optimal Solution: Calculating the Stable Manifold	349
7.5.1	Stable Manifold of an Equilibrium	350
7.5.2	Stable Manifold of Limit Cycles	354
7.6	Optimal Control Problems with Active Constraints	359
7.6.1	The Form of the Canonical System for Mixed Path Constraints	360
7.6.2	The Form of the Canonical System for Pure State Constraints	360
7.6.3	Solutions Exhibiting Junction Points	362
7.7	Retrieving DNSS Sets	366
7.7.1	Locating a DNSS Point	366
7.7.2	Continuing a DNSS Point	368
7.8	Retrieving Heteroclinic Connections	368
7.8.1	Locating a Heteroclinic Connection	368
7.8.2	Continuing a Heteroclinic Connection in Parameter Space	369
7.9	Numerical Example from Drug Control	370
7.9.1	Stating the Necessary Conditions	370
7.9.2	Equilibria of the Canonical System	372
7.9.3	Numerical Analysis	372
7.9.4	Optimal Vector Field for $\nu = 4,000$	373
7.9.5	Optimal Vector Field for $\nu = 12,000$	377
	Exercises	380
	Notes and Further Reading	382
8	Extensions of the Maximum Principle	385
8.1	Multi-Stage Optimal Control Problems	386
8.1.1	Necessary Optimality Conditions for Two-Stage Control Problems	386
8.1.2	Two-Stage Models of Drug Control	387
8.1.3	Counter-Terror Measures in a Multi-Stage Scenario	388
8.2	Differential Games	391
8.2.1	Terminology	392
8.2.2	Nash Equilibria	394

8.2.3	Tractable Game Structures	397
8.2.4	A Corrupt Politician vs. the Tabloid Press	397
8.2.5	Leader–Follower Games	404
8.2.6	A Post September 11th Game on Terrorism	407
8.3	Age-Structured Models	417
8.3.1	A Maximum Principle for Distributed Parameter Systems	419
8.3.2	Age-Structured Drug Initiation	420
8.4	Further Optimal Control Issues	422
8.4.1	Delayed Systems	422
8.4.2	Stochastic Optimal Control	424
8.4.3	Impulse Control and Jumps in the State Variables	425
8.4.4	Nonsmooth Systems	426
	Exercises	426
	Notes and Further Reading	436

Part IV Appendices

A	Mathematical Background	443
A.1	General Notation and Functions	443
A.2	Finite-Dimensional Vector Spaces	447
A.2.1	Vector Spaces, Linear Dependence, and Basis	447
A.2.2	Linear Transformations and Matrices	450
A.2.3	Inverse Matrices and Linear Equations	453
A.2.4	Determinants	455
A.2.5	Linear Form and Dual Space	457
A.2.6	Eigenvalues and Eigenvectors	459
A.2.7	Euclidean Vector Space \mathbb{R}^n	461
A.3	Topology and Calculus	463
A.3.1	Open Set, Neighborhood, and Convergence	463
A.3.2	Continuity and Differentiability	464
A.3.3	Maximization of Real-Valued Functions in \mathbb{R}^n	471
A.3.4	Convex Analysis	473
A.3.5	Taylor Theorem and Implicit Function Theorem	475
A.3.6	Integration Theory	477
A.3.7	Distributions	481
B	Derivations and Proofs of Technical Results	483
B.1	Separation Theorems, Farkas Lemma and Supergradient	483
B.2	Proof of the Michel Theorem	486
B.2.1	Augmented and Truncated Problem	487
B.2.2	Optimal Solution of Problem (B.8)	487
B.2.3	Necessary Conditions for Problem (B.8)	488
B.2.4	Limit of Solutions for Increasing Time Sequence	489

B.3	Proof of the Transversality Condition in Proposition 3.74	491
B.4	The Infinite Horizon Transversality Condition Revisited	492
B.5	Monotonicity of the Solution Path	494
B.6	Admissible and Quasi-Admissible Directions	496
B.7	Proof of the Envelope Theorem	498
B.8	The Dimension of the Stable Manifold	499
B.9	Asymptotic Boundary Condition	502
	B.9.1 Equilibrium	502
	B.9.2 Limit Cycle	503
References		505
Glossary		531
Index		535
Author Index		545