Optimal Control of Switched Systems via Nonlinear Optimization Based on Direct Differentiations of Value Functions*

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Abstract

This paper presents an approach for solving optimal control problems of switched systems. In general, in such problems one needs to find both optimal continuous inputs and optimal switching sequences, since the system dynamics vary before and after every switching instant. After formulating the optimal control problem, we propose a two stage optimization methodology for it. Since many practical problems only concern Stage 1 optimization where the number of switchings and the sequence of active subsystems are given, we then concentrate on Stage 1 optimization problems and propose a method for solving them, which is via nonlinear optimization and based on direct differentiations of value functions. Moreover, the method is modified and applied to general switched linear quadratic (GSLQ) problems. Implementation difficulties of the method can be successfully addressed for GSLQ problems. Examples are shown to illustrate the results in the paper.

1 Introduction

A switched system is a particular kind of hybrid system that consists of several subsystems and a switching law specifying the active subsystem at each time instant. Examples of switched systems can be found in chemical processes, automotive systems, and electrical circuit systems, etc.

Recently, optimal control problems of hybrid and switched systems have been attracting researchers from various fields in science and engineering, due to the problems' significance in theory and application. The available results in the literature on such problems can be classified into two categories, i.e., theoretical and practical. [4, 8, 20, 26, 27, 29, 34] are some primarily theoretical results. These results extended the classical maximum principle or the dynamic programming approach to such problems. Among them, the earliest result is [29] which proves a maximum principle for hybrid systems with autonomous switchings only. Another early result is a proof of the existence of optimal control for a system with two subsystems by Seidman in [26]. More complicated versions of maximum principle under various additional assumptions are proved by Sussmann in [27] and by Piccoli in [20]. In [8, 34], Capuzzo Dolcetta and Yong study systems with switchings using the dynamic programming approach to derive the Hamilton-Jacobi-Bellman (HJB) equations

*The research in this paper is supported by the Army Research Office (DAAG 55-98-1-0199) and National Science Foundation (NSF ECS-9912458).

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and prove the existence and uniqueness of viscosity solutions. Branicky in [4] formulates optimal control problems for hybrid systems modeled by his unified model approach; he also proposes some theoretically algorithmic approaches related to some inequalities of the value functions. However, because there are no efficient constructive methodology suggested in these papers for obtaining optimal solutions, there is a significant gap between theoretical results and their applications to real-world examples. As to the second category of practical results, researchers take advantage of the availability of high speed computers and efficient nonlinear optimization techniques to develop approaches for solving such problems (see e.g., [4, 5, 6, 12, 13, 15, 16, 25, 24, 28]). The problem formulations and the methodologies are very diverse in this category. For example, in [4], general formulations and algorithms for optimal control of hybrid systems are proposed. In [16], a novel algorithm using constrained differential dynamic programming is proposed for a class of discrete-time hybrid-state systems. Johansson and Rantzer in [13, 23], by using an inequality of Bellman type, propose upper and lower bounds of optimal cost for quadratic control of piecewise linear systems; however no explicit method for deriving optimal control is given. Riedinger and his coworkers in [25, 24] have tried to apply the hybrid maximum principle to time optimal and linear quadratic control of systems with linear subsystems. More recently, some heuristically oriented methods have been reported. For example, Lincoln in [15] develops an algorithm which prunes the search trees in discrete-time LQR control of switched linear systems; Branicky in [5, 6] proposes fast marching algorithms which are related to the behavioral programming in computer science.

It is worth noting that because there are many different models and optimal control objectives for hybrid systems, the above papers often differ greatly in their problem formulations and approaches. Switched systems, on the other hand, tend to be described by similar models, and similar optimal control problem formulations have appeared in the literature. (e.g, [12, 13, 16, 25, 28, 32]). For an optimal control problem of a switched system, one needs to find both an optimal continuous input and an optimal switching sequence since the system dynamics vary before and after every switching instant. Due to the involvement of switching sequences, such a problem is in general difficult to solve. Interested readers may refer to [32] for an overview of the problem and its difficulties. Most of the methods in the literature that we are aware of are based on some discretization of continuous-time space and/or discretization of state space into grids and use search methods for the resultant discrete model to find optimal/suboptimal solutions. But the discretization approaches may lead to computational combinatoric explosion and the solutions obtained may not be accurate enough (see [30]). In view of this, in this paper, we explore an approach that is not based on discretization of the time space.

This paper presents an approach for solving optimal control problems of switched systems. Unlike the many literature results, the characteristics of our approach is twofold. First, our approach is not based on the discretization of the time space. Second, we emphasize on accurate optimization of switching instants. Optimal control problems for switched systems are first carefully formulated. We then propose a two stage optimization methodology. Since the two stage optimization methodology is still difficult to implement, we then concentrate on Stage 1 optimization where the number of switchings and the order of active subsystems are given. Focusing on Stage 1 problems is appropriate because in many practical situations, we only need to study problems with a fixed number of switchings and a fixed order of active subsystems (e.g., the speeding up of an automobile power train only requires switchings from gear 1 to 2 to 3 to 4) and in such cases the solution to

Stage 1 is indeed optimal for the problem. On the other hand, Stage 1 optimization itself is already challenging enough and solving it is a first step toward solving the general problem which does not possess a good solution up to now. A Stage 1 problem can further be decomposed into Stage 1(a), which is a conventional optimal control problem that finds the optimal cost given the order of active subsystems and the switching instants, and Stage 1(b), which is a nonlinear optimization problem that finds the optimal switching instants. Stage 1(b) poses difficulties because it is hard to obtain the values of the derivatives of the Stage 1(a) optimal cost with respect to the switching instants. To address these difficulties, we then propose a method that approximates such derivatives by direct differentiations of value functions (Theorem 5.1). Moreover, the method is modified and applied to general switched linear quadratic (GSLQ) problems. Implementation difficulties of the method can be successfully addressed for GSLQ problems.

The structure of the paper is as follows. In Section 2, we formulate the optimal control problem we will study in this paper. In Section 3, we show that such a problem can be posed as a two stage optimization problem under some additional assumptions. From Section 4 on, we concentrate on Stage 1 optimization problems. In Section 4, we discuss Stage 1(a) and 1(b) and propose a conceptual algorithm. In Section 5, we derive in detail how to obtain the approximations of the derivatives which are required by Stage 1(b) by direct differentiations of value functions. The method is modified in Section 6 and applied to GSLQ problems. Examples are given in Section 7 to illustrate the effectiveness of the method. Section 8 concludes the paper.

2 Problem Formulation

2.1 Switched Systems

Switched systems

A switched system is a particular kind of hybrid system that consists of several subsystems and a switching logic among them. The feature that distinguishes a switched system from a general hybrid system is that its continuous state does not exhibit jumps at the switching instants. The switched systems we shall consider in this paper are defined as follows.

Definition 2.1 (Switched System) *A switched system is a tuple $\mathcal{S} = (\mathcal{D}, \mathcal{F}, \mathcal{L})$ where*

- $\mathcal{D} = (I, E)$ *is a directed graph indicating the discrete structure of the system. The node set $I = \{1, 2, \cdots, M\}$ is the set of indices for subsystems. The directed edge set E is a subset of $I \times I - \{(i,i) | i \in I\}$ which contains all valid events. If an event $e = (i_1, i_2)$ takes place, the system switches from subsystem i_1 to i_2.*

- $\mathcal{F} = \{f_i : X_i \times U_i \times \mathbb{R} \to \mathbb{R}^n | X_i \subseteq \mathbb{R}^n, U_i \subseteq \mathbb{R}^m,\ i \in I\}$ *with f_i describing the vector field for the i-th subsystem $\dot{x} = f_i(x, u, t)$.*

- $\mathcal{L} = \{\Lambda_e | \Lambda_e \subseteq \mathbb{R}^n,\ e \in E\}$ *provides us with a logic constraint that relates the continuous state and mode switchings. Note for any $e \in E$, $\Lambda_e \neq \emptyset$. Only when $x \in \Lambda_e$, $e = (i_1, i_2)$, a switching from i_1 to i_2 is possible.* □

In view of Definition 2.1, a switched system is a collection of subsystems related by a switching logic described by $\mathcal{D}$ and $\mathcal{L}$. Note that one distinct feature of a switched system is that it has no discontinuities of the state x at the switching instants. If a particular switching law has been specified (the law may be specified by state space partitions or by time involvements), then the switched system can be described as

$$\dot{x}(t) = f_{i(t)}(x(t), u(t), t) \tag{2.1}$$

$$i(t) = \varphi(x(t), i(t^-), t), \tag{2.2}$$

where $\varphi : \mathbb{R}^n \times I \times \mathbb{R} \to I$ determines the active subsystem at instant t. Note that (2.1)-(2.2) are used as the definition of switched systems in some of the literature (e.g., [12]). Here we adopt Definition 2.1 rather than (2.1)-(2.2) because in design problems, in general, φ is not defined a priori and it is a designer's task to find a switching law.

Switching sequences

For a switched system $\mathcal{S}$, the control input of the system consists of both a continuous input $u(t), t \in [t_0, t_f]$ and a switching sequence. We define a switching sequence as follows.

Definition 2.2 (Switching Sequence) *For a switched system $\mathcal{S}$, a switching sequence σ in $[t_0, t_f]$ is defined as*

$$\sigma = ((t_0, i_0), (t_1, e_1), (t_2, e_2), \cdots, (t_K, e_K)), \tag{2.3}$$

with $0 \le K < \infty$, $t_0 \le t_1 \le t_2 \le \cdots \le t_K \le t_f$, and $i_0 \in I$, $e_k = (i_{k-1}, i_k) \in E$ for $k = 1, 2, \cdots, K$. We define $\Sigma_{[t_0,t_f]} \triangleq \{\sigma$'s in $[t_0, t_f]\}$. □

A switching sequence σ as defined above indicates that, if $t_k < t_{k+1}$, then subsystem i_k is active in $[t_k, t_{k+1})$ ($[t_K, t_f]$ if $k = K$); if $t_k = t_{k+1}$, then i_k is switched through at instant t_k ('switched through' means that the system switches from subsystem i_{k-1} to i_k and then to i_{k+1} all at instant t_k). For a switched system to be well-behaved, we generally exclude the undesirable *Zeno* phenomenon, i.e., infinitely many switchings in finite amount of time. Hence in Definition 2.2, we only allow nonZeno sequences which switch at most a finite number of times in $[t_0, t_f]$, though different sequences may have different numbers of switchings. We specify $\sigma \in \Sigma_{[t_0,t_f]}$ as a discrete input to a switched system. The overall control input to the system is therefore a pair (σ, u).

Example 2.1 (An Automotive Control System) A manual transmission car with four gears is a good example of a switched system. If we denote the lateral position as x_1 and the velocity x_2, the system dynamics at gear i can be described by

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \alpha_i(x_2)u,$$

where the nonlinear gear efficiency function $\alpha_i(x_2)$ is depicted in figure 1.

For this system, $I = \{1, 2, 3, 4\}$, all $X_i = \mathbb{R}^n$ and all $U_i = [0, u_{max}]$, where u_{max} is given. If the gear can only be shifted one gear up or down, we have $E = \{(1, 2), (2, 1), (2, 3), (3, 2), (3, 4), (4, 3)\}$; moreover, $\Lambda_{(1,2)} = \Lambda_{(2,1)} = \{x | x_2 \in [10, 20]\}$, $\Lambda_{(2,3)} = \Lambda_{(3,2)} = \{x | x_2 \in [20, 40]\}$, $\Lambda_{(3,4)} = \Lambda_{(4,3)} = \{x | x_2 \in [40, 60]\}$. The control input of this system consists of the continuous input u (the throttle position) and the external switching sequence (gear shifting). □

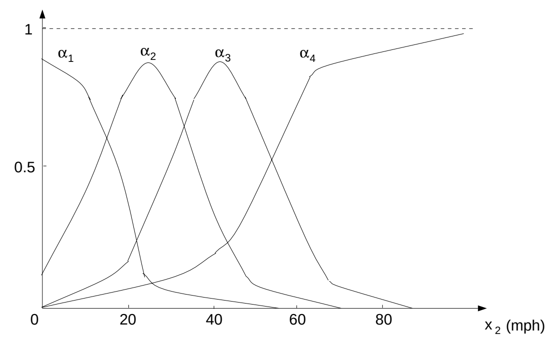

Figure 1: The nonlinear gear efficiency functions α_i.

2.2 An Optimal Control Problem

Note that in the sequel in this paper, we assume that $X_i = \mathbb{R}^n$, $U_i = \mathbb{R}^m$ and $\Lambda_e = \mathbb{R}^n$ for any $i \in I$, $e \in E$ and report results under these assumptions. We assume these because on one hand optimal control problems under these assumptions are already challenging and interesting enough and well deserve our attention; on the other hand problems under more general constraints are still under extensive researches. We also define $\mathcal{U}_{[t_0,t_f]} \triangleq \{u | u \in C_p[t_0, t_f], u(t) \in \mathbb{R}^m\}$; in other words, $\mathcal{U}_{[t_0,t_f]}$ is the set of all piecewise continuous functions for $t \in [t_0, t_f]$ that take values in $\mathbb{R}^m$.

Problem 2.1 *Consider a switched system $\mathcal{S} = (\mathcal{D}, \mathcal{F}, \mathcal{L})$. Given a fixed time interval $[t_0, t_f]$, find a continuous input $u \in \mathcal{U}_{[t_0,t_f]}$ and a switching sequence $\sigma \in \Sigma_{[t_0,t_f]}$ such that the corresponding continuous state trajectory x departs from a given initial state $x(t_0) = x_0$ and meets an $(n - l_f)$-dimensional smooth manifold $S_f = \{x | \phi_f(x) = 0, \phi_f : \mathbb{R}^n \to \mathbb{R}^{l_f}\}$ at t_f and the cost functional*

$$J = \psi(x(t_f)) + \int_{t_0}^{t_f} L(x(t), u(t), t)\ dt \tag{2.4}$$

is minimized. □

Problem 2.1 is a basic optimal control problem with fixed end-time where the final state is on a smooth manifold. As in the usual practice of formulating optimal control problems (see [1]), in the sequel, we assume that f, L are continuous and have continuous partial derivatives with respect to x and t; ϕ_f is assumed to be continuously differentiable; ψ has twice continuous derivatives. Besides these assumptions, in the following, whenever necessary, we will further assume that they possess enough smoothness properties we need in our derivations.

The way we formulate Problem 2.1 with a fixed end-time is mainly for the convenience of subsequent studies in this paper. Note that for a problem with free end-time t_f, we can introduce an additional state variable and transcribe it to a problem with fixed end-time (for more details, see [30]).

Analytical tools such as the maximum principle and the Hamilton-Jacobi-Bellman (HJB) equation for hybrid and switched systems have been derived in the literature (see [20, 27, 29, 32, 34]). However, it is difficult to directly use these tools to find optimal controls even for linear switched systems. For details and comments on the difficulties of using them to obtain optimal control solutions, see [30].

3 Two Stage Optimization

In general, we need to find an optimal control solution (σ^*, u^*) for Problem 2.1 such that

$$J(\sigma^*, u^*) = \min_{\sigma \in \Sigma_{[t_0,t_f]},\ u \in \mathcal{U}_{[t_0,t_f]}} J(\sigma, u). \tag{3.1}$$

Notice that for any fixed switching sequence σ, Problem 2.1 reduces to a conventional optimal control problem for which we only need to find an optimal continuous input u that minimizes $J_\sigma(u) = J(\sigma, u)$. This idea naturally leads us toward considering Problem 2.1 as a two stage optimization problem. Under some additional assumptions, we can prove the following lemma that provides a way to do so.

Lemma 3.1 *For Problem 2.1, if*

(a). an optimal solution (σ^*, u^*) exists and

(b). for any given switching sequence σ, there exists a corresponding $u^* = u^*_\sigma$ such that $J_\sigma(u) = J(\sigma, u)$ is minimized,

then the following equation holds

$$\min_{\sigma \in \Sigma_{[t_0,t_f]},\ u \in \mathcal{U}_{[t_0,t_f]}} J(\sigma, u) = \min_{\sigma \in \Sigma_{[t_0,t_f]}} \min_{u \in \mathcal{U}_{[t_0,t_f]}} J(\sigma, u). \tag{3.2}$$

Proof: First we claim that

$$\min_{\sigma \in \Sigma_{[t_0,t_f]},\ u \in \mathcal{U}_{[t_0,t_f]}} J(\sigma, u) \le \inf_{\sigma \in \Sigma_{[t_0,t_f]}} \min_{u \in \mathcal{U}_{[t_0,t_f]}} J(\sigma, u). \tag{3.3}$$

This is because for any fixed σ, there exists a u^*_σ such that $J(\sigma, u^*_\sigma) = \min_{u \in \mathcal{U}_{[t_0,t_f]}} J(\sigma, u)$. But for every pair (σ, u^*_σ), we must have $J(\sigma^*, u^*) \le J(\sigma, u^*_\sigma)$, therefore from (3.3) we must have

$$J(\sigma^*, u^*) \le \inf_{\sigma \in \Sigma_{[t_0,t_f]}} J(\sigma, u^*_\sigma) = \inf_{\sigma \in \Sigma_{[t_0,t_f]}} \min_{u \in \mathcal{U}_{[t_0,t_f]}} J(\sigma, u). \tag{3.4}$$

While we also have the inequality

$$\inf_{\sigma \in \Sigma_{[t_0,t_f]}} \min_{u \in \mathcal{U}_{[t_0,t_f]}} J(\sigma, u) \le \min_{u \in \mathcal{U}_{[t_0,t_f]}} J(\sigma^*, u) = J(\sigma^*, u^*_{\sigma^*}). \tag{3.5}$$

In (3.5) we can choose $u^*_{\sigma^*} = u^*$, since for any other u, we must have $J(\sigma^*, u^*) \le J(\sigma^*, u)$ due to the optimality of (σ^*, u^*). Hence combining (3.4) and (3.5) we have

$$J(\sigma^*, u^*) \le \inf_{\sigma \in \Sigma_{[t_0,t_f]}} \min_{u \in \mathcal{U}_{[t_0,t_f]}} J(\sigma, u) \le J(\sigma^*, u^*_{\sigma^*}) = J(\sigma^*, u^*). \tag{3.6}$$

Hence all inequalities in (3.6) must be equalities and the $\inf_{\sigma \in \Sigma_{[t_0,t_f]}}$ can be replaced by $\min_{\sigma \in \Sigma_{[t_0,t_f]}}$ so we obtain

$$J(\sigma^*, u^*) = \min_{\sigma \in \Sigma_{[t_0,t_f]},\ u \in \mathcal{U}_{[t_0,t_f]}} J(\sigma, u) = \min_{\sigma \in \Sigma_{[t_0,t_f]}} \min_{u \in \mathcal{U}_{[t_0,t_f]}} J(\sigma, u). \tag{3.7}$$

□

The right hand side of (3.2) needs twice the minimization process. This supports the validity of the following two stage optimization methodology.

A Two Stage Optimization Methodology

Stage 1. Fixing σ, solve the inner minimization problem.

Stage 2. Regarding the optimal cost for each σ as a function

$$J_1 = J_1(\sigma) = \min_{u \in \mathcal{U}_{[t_0,t_f]}} J(\sigma, u), \tag{3.8}$$

minimize J_1 with respect to $\sigma \in \Sigma_{[t_0,t_f]}$. □

In more detail, we can implement the above methodology by the following algorithm.

Algorithm 3.1 (A Two Stage Algorithm)

Stage 1. (a). Fix the total number of switchings to be K and the sequence of active subsystems and let the minimum value of J with respect to u be a function of the K switching instants, i.e., $J_1 = J_1(t_1, t_2, \cdots, t_K)$ for $K \geq 0$ ($t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_K \leq t_f$). Find J_1.

(b). Minimize J_1 with respect to $t_1, t_2, \cdots, t_K$.

Stage 2. (a). Vary the order of active subsystems to find an optimal solution under K switchings.

(b). Vary the number of switchings K to find an optimal solution for Problem 2.1. □

The above algorithm needs further implementations. In practice, many problems only require the solutions of optimal continuous inputs and optimal switching instants for Stage 1 optimization where a fixed number of switchings and a fixed sequence of active subsystems are given. In general, explicit expressions of J_1 are difficult to obtain or quite complicated even for very simple problems. Therefore it is necessary to devise optimization methods that do not require the explicit expression of J_1 as a function of t_k's. In the next section, we shall discuss Stage 1 optimization in detail.

4 More on Stage 1 Optimization

Now we concentrate on Stage 1 optimization. On the one hand, Stage 1 optimization has already presented enough challenge to us. On the other hand, since many real world problems are in fact stage 1 optimization problems, Stage 1 does deserve our attention. For example, the speeding-up of a power train only requires switchings from gear 1 to 2 to 3 to 4. As can be seen from Algorithm 3.1 in Section 3, Stage 1 can be further decomposed into two sub-steps (a) and (b). Stage 1(a) is in essence a conventional optimal control problem which seeks the minimum value of J with respect to u under a given switching sequence $\sigma = ((t_0, i_0), (t_1, e_1), \cdots, (t_K, e_K))$. We denote the corresponding optimal cost as a function $J_1(\hat{t})$, where $\hat{t} \triangleq (t_1, t_2, \cdots, t_K)^T$. Stage 1(b) is in essence a constrained nonlinear optimization problem

$$\begin{array}{c} \min_{\hat{t}} J_1(\hat{t}) \\ \text{subject to } \hat{t} \in T \end{array} \tag{4.1}$$

where $T \triangleq \{\hat{t} = (t_1, t_2, \cdots, t_K)^T | t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_K \leq t_f\}$.

In order to solve a Stage 1 problem, one needs to resort to not only optimal control methods, but also nonlinear optimization techniques. Except for very few classes of problems (e.g. minimum energy problems in [30]), analytical expressions of $J_1(\hat{t})$ are almost impossible to obtain. This is evident from the fact that very few classes of conventional optimal control problems possess analytical solutions. The unavailability of analytical expressions of $J_1(\hat{t})$ henceforth makes Stage 1(b) optimization difficult to carry out. However even without the expressions of $J_1(\hat{t})$, if we can find the values of the derivatives $\frac{\partial J_1}{\partial \hat{t}}$ and $\frac{\partial^2 J_1}{\partial \hat{t}^2}$, we can still solve Stage 1(b) by employing some nonlinear optimization algorithms. Let us elaborate more on Stage 1(a) and 1(b) in the followings.

Stage 1(a)

For Stage 1(a) where a switching sequence $\sigma = ((t_0, i_0), (t_1, e_1), \cdots, (t_K, e_K))$ is given, finding $J_1(\hat{t})$ for the corresponding $\hat{t} = (t_1, \cdots, t_K)^T$ is a conventional optimal control problem. Note that although different subsystems are active in different time intervals, the problem is conventional since these intervals are fixed. In Stage 1(a), we need to find an optimal continuous input u and the corresponding minimum J. In order to find solutions for Stage 1(a) problems, computational methods must be adopted in most cases. Most of the available numerical methods are for unconstrained conventional optimal control problems with fixed end-time can be used. See [17, 21] for surveys of computational methods. Moreover, if all subsystems are linear and the cost functional is quadratic in control and state, then the optimal control and optimal cost can be found by solving a Riccati equation (see [14] for more details).

Stage 1(b)

In Stage 1(b), we need to solve the constrained nonlinear optimization problem (4.1) with simple constraints. Computational methods for the solution of such problems are abundant in the nonlinear optimization literature. For example, feasible direction methods and penalty function methods are two commonly used classes of methods. These methods use the information of first-order derivative $\frac{\partial J_1}{\partial \hat{t}}$ and even second-order derivative $\frac{\partial^2 J_1}{\partial \hat{t}^2}$. In the computation of the examples in this paper, we use the gradient projection method (using $\frac{\partial J_1}{\partial \hat{t}}$) and the constrained Newton's method (using $\frac{\partial J_1}{\partial \hat{t}}$ and $\frac{\partial^2 J_1}{\partial \hat{t}^2}$) and their variations (see Section 2.3 in Bertsekas [3] for details). For more discussions on various methods for solving constrained nonlinear optimization problems, please also see [2, 19].

A Conceptual Algorithm

The following conceptual algorithm provides a framework for the optimization methodologies in the subsequent chapters.

Algorithm 4.1 (A Conceptual Algorithm for Stage 1 Optimization)

(1). Set the iteration index $j = 0$. Choose an initial $\hat{t}^j$.

(2). By solving an optimal control problem (Stage 1(a)), find $J_1(\hat{t}^j)$.

(3). Find $\frac{\partial J_1}{\partial \hat{t}}(\hat{t}^j)$ and $\frac{\partial^2 J_1}{\partial \hat{t}^2}(\hat{t}^j)$.

(4). Use some feasible direction method to update $\hat{t}^j$ to be $\hat{t}^{j+1} = \hat{t}^j + \alpha^j d\hat{t}^j$ (here the stepsize α^j is chosen using the Armijo's rule [3]). Set the iteration index $j = j + 1$.

(5). Repeat Steps (2), (3), (4) and (5), until a prespecified termination condition is satisfied. □

It should be pointed out that the key elements of the above algorithm are

(a). An optimal control algorithm for Step (2).

(b). The derivations of $\frac{\partial J_1}{\partial \hat{t}}$ and $\frac{\partial^2 J_1}{\partial \hat{t}^2}$ for Step (3).

(c). A nonlinear optimization algorithm for Step (4).

In the above discussions, we have already addressed elements (a) and (c). (b) poses an obstacle for the usage of Algorithm 4.1 because $\frac{\partial J_1}{\partial \hat{t}}$ and $\frac{\partial^2 J_1}{\partial \hat{t}^2}$ are not readily available. It is the task of the subsequent sections to address (b) and devise a method for the approximations of the values of $\frac{\partial J_1}{\partial \hat{t}}$ and $\frac{\partial^2 J_1}{\partial \hat{t}^2}$. Lastly, it should be pointed out that in Step (4), when we are searching for α^j, optimal control algorithm for Stage 1(a) will also be used in order to obtain the value of J_1 at the intermediate trial $\hat{t}$'s.

5 Optimization for Stage 1 Problem Based on Direct Differentiations

In the present section, we propose a method to approximate the values of $\frac{\partial J_1}{\partial \hat{t}}$ and $\frac{\partial^2 J_1}{\partial \hat{t}^2}$ which can be used in Stage 1(b) optimizations. The method is based on direct differentiations of the value function. The approach is motivated by the approaches in [9, 10, 11]. Note that in this and next sections, we assume $S_f = \mathbb{R}^n$. See [31, 33] for early versions of the method.

For simplicity of notations, let us assume that we are given a Stage 1 problem where the number of switchings is K and the order of active subsystems is $1, 2, \cdots, K, K+1$. We need to find an optimal switching instant vector $\hat{t} = (t_1, \cdots, t_K)^T$ and an optimal control input u.

Assume that we have a nominal $\hat{t} = (t_1, \cdots, t_K)^T$ and a nominal control input u. If they are both fixed, then the cost J will be a function of $(x(t_0), t_0)$. However, if u is fixed and $\hat{t}$ can be varied in a small neighborhood of the nominal value, then the cost J will be a function of $(x(t_0), t_0, t_1, \cdots, t_K)$. Now let us assume that along with the small variations of $\hat{t}$, u varies correspondingly in the following manner. If $\hat{t}$ varies to $\hat{t} + d\hat{t}$, u varies correspondingly to

$$\hat{u}(t) = \begin{cases} u(t_k-) + (t - t_k)\dot{u}^{k-}, & \text{if } t \in [t_k, t_k + dt_k) \text{ for } dt_k \geq 0 \\ u(t_k+) + (t - t_k)\dot{u}^{k+}, & \text{if } t \in [t_k + dt_k, t_k] \text{ for } dt_k < 0, \\ u(t), & \text{else,} \end{cases} \tag{5.1}$$

where $\dot{u}^{k-} \triangleq \frac{du(t_k-)}{dt}$ and $\dot{u}^{k+} \triangleq \frac{du(t_k+)}{dt}$. We say that u assumes *open-loop variations* in this case. By open-loop variations, we mean that $u(t)$ only has variations in the interval between t_k and $t_k + dt_k$ as shown in figure 2. The reason why we call such variations "open-loop" will be clear in Section 6.2 where we define the so-called closed-loop variations. With the introduction of open-loop variations in u, if we allow $\hat{t}$ to vary in a small neighborhood of the nominal value, the cost J can still be regarded as a function of $(x(t_0), t_0, t_1, \cdots, t_K)$, since u in this case varies corresponding to the variations of $\hat{t}$. We denote such a cost as a value function

$$V^0(x(t_0), t_0, t_1, \cdots, t_K) = \psi(x(t_f)) + \int_{t_0}^{t_1} L(x, u, t)\ dt + \cdots + \int_{t_K}^{t_f} L(x, u, t)\ dt \tag{5.2}$$

where the superscript 0 is to indicate that the starting time for evaluation is t_0. Similarly, we can define the value function at the k-th switching instant as

$$V^k(x(t_k), t_k, t_{k+1}, \cdots, t_K) = \psi(x(t_f)) + \int_{t_k}^{t_{k+1}} L(x,u,t)\ dt + \cdots + \int_{t_K}^{t_f} L(x,u,t)\ dt. \tag{5.3}$$

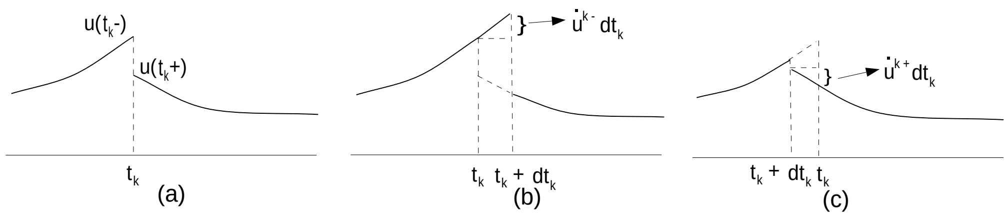

Figure 2: The solid curves are $u(t)$. (a). The nominal input $u(t)$. (b). The open-loop variations of $u(t)$ induced by $dt_k \geq 0$. (c). The open-loop variations of $u(t)$ induced by $dt_k < 0$.

The idea of the method is to approximate $\frac{\partial J_1}{\partial \hat{t}}$ and $\frac{\partial^2 J_1}{\partial \hat{t}^2}$ by $\frac{\partial V^0}{\partial \hat{t}}$ and $\frac{\partial^2 V^0}{\partial \hat{t}^2}$, respectively. Here we assume that for any given nominal $\hat{t}$, we choose a nominal u which is an optimal solution to the corresponding Stage 1(a) problem. From our experience with numerical examples, a suboptimal solution would also suffice to be the nominal u. In the followings, in order to make our presentation clear, we denote $\frac{\partial V}{\partial x}$ for a function V as a row vector V_x, $\frac{\partial^2 V}{\partial x^2}$ as an $n \times n$ matrix V_{xx} and so on.

5.1 Single Switching

Let us first consider the case of a single switching. Assume that we are given a nominal t_1, a nominal u (u may be optimal or suboptimal) and the corresponding nominal state trajectory x. We denote $\hat{u}(t)$ and $\hat{x}(t)$ to be the input and state trajectory after variation dt_1 has taken place. We write a function with a superscript $1-$ (resp. $1+$) whenever it is evaluated at t_1- and the nominal values $x(t_1)$, $u(t_1-)$ (resp. t_1+ and the nominal values $x(t_1)$, $u(t_1+)$). Examples of this notational convention are $f^{1-} = f_1(x(t_1), u(t_1-), t_1-)$, $f^{1+} = f_2(x(t_1), u(t_1+), t_1+)$, $L^{1-} = L(x(t_1), u(t_1-), t_1-)$, $L^{1+} = L(x(t_1), u(t_1+), t_1+)$, $V^{1+} = V^1(x(t_1), t_1+)$ (be careful to distinguish V^{1+} from V^1), etc.

It is not difficult to see that

$$V^0(x_0, t_0, t_1) = V^1(x(t_1), t_1) + \int_{t_0}^{t_1} L(x,u,t)\ dt. \tag{5.4}$$

For a small variation dt_1 of t_1, we have

$$\begin{aligned} & V^0(x_0, t_0, t_1 + dt_1) \\ = \ & V^1(\hat{x}(t_1 + dt_1), t_1 + dt_1) + \int_{t_0}^{t_1 + dt_1} L(\hat{x}, \hat{u}, t)\ dt. \end{aligned} \tag{5.5}$$

The first term in (5.5) can be expanded into second order as

$$\begin{aligned} & V^1(\hat{x}(t_1 + dt_1), t_1 + dt_1) \\ = \ & V^{1+} + V_x^{1+} dx(t_1) + V_{t_1}^{1+} dt_1 + \frac{1}{2}(dx(t_1))^T V_{xx}^{1+} dx(t_1) + \frac{1}{2} V_{t_1 t_1}^{1+} dt_1^2 \\ & + dt_1 V_{t_1 x}^{1+} dx(t_1) + (\text{higher order terms}) \end{aligned} \tag{5.6}$$

where

$$
\begin{aligned}
dx(t_1) &\triangleq \hat{x}(t_1+dt_1) - x(t_1) \\
&= f^{1-}dt_1 + \frac{1}{2}(f_t^{1-} + f_x^{1-}f^{1-} + f_u^{1-}\dot{u}^{1-})dt_1^2 + o(dt_1^2).
\end{aligned} \tag{5.7}
$$

The second order expansion of the second term in (5.5) is derived as follows by distinguishing the case of $dt_1 \geq 0$ and the case of $dt_1 < 0$. If $dt_1 \geq 0$, we have

$$
\begin{aligned}
&\int_{t_0}^{t_1+dt_1} L(\hat{x},\hat{u},t)\ dt = \int_{t_0}^{t_1} L(x,u,t)\ dt + \int_{t_1}^{t_1+dt_1} L(\hat{x},\hat{u},t)\ dt \\
= &\int_{t_0}^{t_1} L(x,u,t)\ dt + L^{1-}dt_1 + \frac{1}{2}dt_1 L_x^{1-}dx(t_1) + \frac{1}{2}dt_1 L_u^{1-}du(t_1) + \frac{1}{2}L_t^{1-}dt_1^2 \\
&+(\text{higher order terms}).
\end{aligned} \tag{5.8}
$$

If $dt_1 < 0$, we have

$$
\begin{aligned}
&\int_{t_0}^{t_1+dt_1} L(\hat{x},\hat{u},t)\ dt = \int_{t_0}^{t_1} L(x,u,t)\ dt + \int_{t_1}^{t_1+dt_1} L(x,u,t)\ dt \\
= &\int_{t_0}^{t_1} L(x,u,t)\ dt + L^{1-}dt_1 + \frac{1}{2}dt_1 L_x^{1-}dx(t_1) + \frac{1}{2}dt_1 L_u^{1-}du(t_1) + \frac{1}{2}L_t^{1-}dt_1^2 \\
&+(\text{higher order terms})
\end{aligned} \tag{5.9}
$$

which has the same expression as (5.8) for $dt_1 \geq 0$ although the derivation is slightly different. Note that in (5.8) and (5.9),

$$
du(t_1) \triangleq \hat{u}((t_1+dt_1)-) - u(t_1-) = \begin{cases} \dot{u}^{1-}dt_1, & \text{for } dt_1 \geq 0, \\ \dot{u}^{1-}dt_1 + o(dt_1), & \text{for } dt_1 < 0. \end{cases} \tag{5.10}
$$

Now substituting (5.7) and (5.10) into the expansions of the terms $V^1(\hat{x}(t_1+dt_1), t_1+dt_1)$, $\int_{t_0}^{t_1+dt_1} L(\hat{x},\hat{u},t)\ dt$ and summing the two terms up, we obtain

$$
\begin{aligned}
&V^0(x_0,t_0,t_1) \\
= \ &V^{1+} + \int_{t_0}^{t_1} L(x,u,t)\ dt + V_x^{1+}dx(t_1) + V_{t_1}^{1+}dt_1 + L^{1-}dt_1 \\
&+\frac{1}{2}(dx(t_1))^T V_{xx}^{1+}dx(t_1) + \frac{1}{2}V_{t_1t_1}^{1+}dt_1^2 + dt_1 V_{t_1x}^{1+}dx(t_1) + \frac{1}{2}dt_1 L_x^{1-}dx(t_1) \\
&+\frac{1}{2}dt_1 L_u^{1-}du(t_1) + \frac{1}{2}L_t^{1-}dt_1^2 + o(dt_1^2) \qquad (5.11) \\
= \ &V^0(x_0,t_0,t_1) + (V_x^{1+}f^{1-} + V_{t_1}^{1+} + L^{1-})dt_1 + \frac{1}{2}[V_x^{1+}(f_t^{1-} + f_x^{1-}f^{1-} + f_u^{1-}\dot{u}^{1-}) \\
&+(f^{1-})^T V_{xx}^{1+}f^{1-} + V_{t_1t_1}^{1+} + 2V_{t_1x}^{1+}f^{1-} + L_x^{1-}f^{1-} + L_u^{1-}\dot{u}^{1-} + L_t^{1-}]dt_1^2 + o(dt_1^2) \\
\triangleq \ &V^0(x_0,t_0,t_1) + V_{t_1}^0 dt_1 + \frac{1}{2}V_{t_1t_1}^0 dt_1^2 + o(dt_1^2) \qquad (5.12)
\end{aligned}
$$

for all dt_1 (no matter $dt_1 \geq 0$ or $dt_1 < 0$ we get the same expression).

Now let us consider V^{1+} (i.e., $V^1(x(t_1),t_1)$) is the value function for fixed $u(t)$, we have the dynamic programming equation for the value function

$$
V_{t_1}^{1+} = -V_x^{1+}f^{1+} - L^{1+}. \tag{5.13}
$$

Note that (5.13) can be derived similarly to the HJB equation. However, the difference between it and the HJB equation is that (5.13) holds for any continuous input.

By differentiating (5.13), we obtain

$$V_{t_1x}^{1+} = -(f^{1+})^T V_{xx}^{1+} - V_x^{1+} f_x^{1+} - L_x^{1+} \tag{5.14}$$

$$\begin{aligned} V_{t_1t_1}^{1+} &= -V_{t_1x}^{1+} f^{1+} - V_x^{1+} f_t^{1+} - L_t^{1+} - (V_x^{1+} f_u^{1+} + L_u^{1+})\dot{u}^{1+} \\ &= (f^{1+})^T V_{xx}^{1+} f^{1+} + (V_x^{1+} f_x^{1+} + L_x^{1+}) f^{1+} - V_x^{1+} f_t^{1+} \\ &\quad -L_t^{1+} - (V_x^{1+} f_u^{1+} + L_u^{1+})\dot{u}^{1+}. \end{aligned} \tag{5.15}$$

By substituting (5.13), (5.14) and (5.15) into (5.12), we can write $V_{t_1}^0$ and $V_{t_1t_1}^0$ in the following form

$$V_{t_1}^0 = L^{1-} - L^{1+} + V_x^{1+}(f^{1-} - f^{1+}), \tag{5.16}$$

$$\begin{aligned} V_{t_1t_1}^0 &= (f^{1-} - f^{1+})^T V_{xx}^{1+} (f^{1-} - f^{1+}) - (V_x^{1+} f_x^{1+} + L_x^{1+})(f^{1-} - f^{1+}) \\ &\quad +(V_x^{1+}(f_x^{1-} - f_x^{1+}) + L_x^{1-} - L_x^{1+}) f^{1-} + V_x^{1+}(f_t^{1-} - f_t^{1+}) + L_t^{1-} - L_t^{1+} \\ &\quad +(V_x^{1+} f_u^{1-} + L_u^{1-})\dot{u}^{1-} - (V_x^{1+} f_u^{1+} + L_u^{1+})\dot{u}^{1+}. \end{aligned} \tag{5.17}$$

5.2 Two or More Switchings

In order to construct a second-order optimization algorithm, for switched systems with two or more switchings, we need more information to derive the derivatives of V^0 with respect to the t_k's. Let us first consider the case of two switchings. Assume that a system switches from subsystem 1 to 2 at t_1 and from subsystem 2 to 3 at t_2 ($t_0 \le t_1 \le t_2 \le t_f$). The value function then is

$$V^0(x_0, t_0, t_1, t_2) = V^1(x(t_1), t_1) + \int_{t_0}^{t_1} L(x, u, t)\, dt \tag{5.18}$$

$$= V^2(x(t_2), t_2) + \int_{t_0}^{t_2} L(x, u, t)\, dt. \tag{5.19}$$

Using (5.18), by holding t_2 fixed, $V_{t_1}^0$, $V_{t_1t_1}^0$ can be derived similarly to Section 5.1. In the same manner, $V_{t_2}^0$, $V_{t_2t_2}^0$ can also be derived using (5.19). However, we need additional information to derive $V_{t_1t_2}^0$. Arguments from the calculus of variations will be used in the followings to derive $V_{t_1t_2}^0$. Now let us define the important notion of incremental change which will be used in our following derivations.

Definition 5.1 (Incremental Change) *Given any variations dt_1 and dt_2, we define $\delta x(t), \min\{t_1, t_1 + dt_1\} \le t \le \max\{t_2, t_2 + dt_2\}$ to be* the incremental change *of the state due to dt_1 and dt_2. In detail, it is defined as follows (see figure 3).*
Case 1: *$dt_1 \ge 0, dt_2 \ge 0$ (see figure 3(a)). In this case, $\delta x(t)$ is defined to be*

$$\delta x(t) = \begin{cases} \hat{x}(t) - x(t), & t \in [t_1 + dt_1, t_2] \\ y_1(t) - x(t), & t \in [t_1, t_1 + dt_1] \\ \hat{x}(t) - z_1(t), & t \in [t_2, t_2 + dt_2] \end{cases} \tag{5.20}$$

where $y_1(t)$ is the solution of

$$\begin{cases} \dot{y}_1(t) = f_2(y_1(t), u(t), t), & t \in [t_1, t_1 + dt_1] \\ y_1(t_1 + dt_1) = \hat{x}(t_1 + dt_1) \end{cases} \tag{5.21}$$

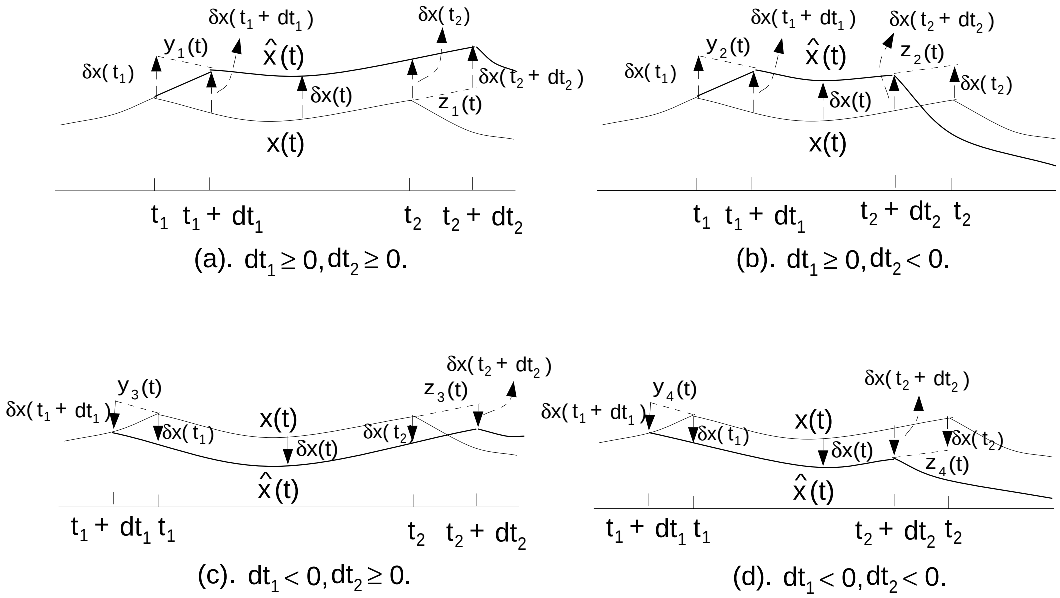

Figure 3: The incremental change $\delta x(t)$ for (a). $dt_1 \geq 0$, $dt_2 \geq 0$; (b). $dt_1 \geq 0$, $dt_2 < 0$; (c). $dt_1 < 0$, $dt_2 \geq 0$; (d). $dt_1 < 0$, $dt_2 < 0$.

and $z_1(t)$ is the solution of

$$\begin{cases} \dot{z_1}(t) = f_2(z_1(t), \hat{u}(t), t), \ t \in [t_2, t_2 + dt_2] \\ z_1(t_2) = x(t_2). \end{cases} \tag{5.22}$$

Case 2: $dt_1 \geq 0, dt_2 < 0$ *(see figure 3(b).) In this case, $\delta x(t)$ is defined to be*

$$\delta x(t) = \begin{cases} \hat{x}(t) - x(t), \ t \in [t_1 + dt_1, t_2 + dt_2] \\ y_2(t) - x(t), \ t \in [t_1, t_1 + dt_1] \\ z_2(t) - x(t), \ t \in [t_2 + dt_2, t_2] \end{cases} \tag{5.23}$$

where $y_2(t)$ is the solution of

$$\begin{cases} \dot{y_2}(t) = f_2(y_2(t), u(t), t), \ t \in [t_1, t_1 + dt_1] \\ y_2(t_1 + dt_1) = \hat{x}(t_1 + dt_1) \end{cases} \tag{5.24}$$

and $z_2(t)$ is the solution of

$$\begin{cases} \dot{z_2}(t) = f_2(z_2(t), u(t), t), \ t \in [t_2 + dt_2, t_2] \\ z_2(t_2 + dt_2) = \hat{x}(t_2 + dt_2). \end{cases} \tag{5.25}$$

Case 3: $dt_1 < 0, dt_2 \geq 0$ *(see figure 3(c).) In this case, $\delta x(t)$ is defined to be*

$$\delta x(t) = \begin{cases} \hat{x}(t) - x(t), \ t \in [t_1, t_2] \\ \hat{x}(t) - y_3(t), \ t \in [t_1 + dt_1, t_1] \\ \hat{x}(t) - z_3(t), \ t \in [t_2, t_2 + dt_2] \end{cases} \tag{5.26}$$

where $y_3(t)$ is the solution of

$$\begin{cases} \dot{y_3}(t) = f_2(y_3(t), \hat{u}(t), t), \ t \in [t_1 + dt_1, t_1] \\ y_3(t_1) = x(t_1) \end{cases} \tag{5.27}$$

and $z_3(t)$ is the solution of

$$\begin{cases} \dot{z_3}(t) = f_2(z_3(t), \hat{u}(t), t), \ t \in [t_2, t_2 + dt_2] \\ z_3(t_2) = x(t_2). \end{cases} \tag{5.28}$$

Case 4: *$dt_1 < 0, dt_2 < 0$ (see figure 3(d).) In this case, $\delta x(t)$ is defined to be*

$$\delta x(t) = \begin{cases} \hat{x}(t) - x(t), \ t \in [t_1, t_2 + dt_2] \\ \hat{x}(t) - y_4(t), \ t \in [t_1 + dt_1, t_1] \\ z_4(t) - x(t), \ t \in [t_2 + dt_2, t_2] \end{cases} \tag{5.29}$$

where $y_4(t)$ is the solution of

$$\begin{cases} \dot{y_4}(t) = f_2(y_4(t), \hat{u}(t), t), \ t \in [t_1 + dt_1, t_1] \\ y_4(t_1) = x(t_1) \end{cases} \tag{5.30}$$

and $z_4(t)$ is the solution of

$$\begin{cases} \dot{z_4}(t) = f_2(z_4(t), u(t), t), \ t \in [t_2 + dt_2, t_2] \\ z_4(t_2 + dt_2) = \hat{x}(t_2 + dt_2). \end{cases} \tag{5.31}$$

□

Remark 5.1 In plain words, $\delta x(t)$ defines the difference between $\hat{x}(t)$ and $x(t)$ in the time interval where subsystem 2 is active. Moreover, by extending the trajectories $\hat{x}$ and x under subsystem 2 dynamics to the time interval $\min\{t_1, t_1 + dt_1\} \leq t \leq \max\{t_2, t_2 + dt_2\}$ where at least one of $\hat{x}(t)$ and $x(t)$ evolves along subsystem 2, $\delta x(t)$ even defines the difference for this time interval. □

In the followings, let us derive the expressions of $\delta x(t_2)$ and $dx(t_2)$. The following important Lemma will be used frequently in the proofs of the lemmas in this section (for details see Appendix).

Lemma 5.1 ([22]) *Let $g(t, u)$ be a real continuous function of the pair of variables $t \in (a, b)$, $u \in U$ and let $u(t)$, $a < t < b$, be a piecewise continuous function with values in U. If θ is a point in (a, b), we have*

$$\int_{\theta + p\epsilon}^{\theta + q\epsilon} g(t, u(t)) \ dt = \epsilon(q - p) g(\theta, u(\theta)) + o(\epsilon). \tag{5.32}$$

Here p and q are arbitrary real numbers, ϵ is a sufficiently small positive number, and $o(\epsilon)$ is an infinitesimal of higher order than ϵ, i.e., $\lim_{\epsilon \to 0} \frac{o(\epsilon)}{\epsilon} = 0$. □

Remark 5.2 In [22], Lemma 5.1 is said to hold for any measurable function u. Here for our purpose in this paper, we restrict u to be piecewise continuous functions. □

Lemma 5.2 *The expressions of $\delta x(t_2)$ and $\delta x(t_2+dt_2)$ are as follows*

$$\begin{aligned} \delta x(t_2) &= A(t_2,t_1)(f^{1-}-f^{1+})dt_1 + o(dt_1), \qquad (5.33) \\ \delta x(t_2+dt_2) &= A(t_2,t_1)(f^{1-}-f^{1+})dt_1 + f_x^{2-}A(t_2,t_1)(f^{1-}-f^{1+})dt_1dt_2 \\ &\quad +(\textit{other terms in } dt_1^2, dt_2^2 \textit{ and higher order terms}), \qquad (5.34) \end{aligned}$$

where $A(t_2,t_1)$ is the state transition matrix for the variational time-varying equation

$$\dot{y}(t) = \frac{\partial f(x(t),u(t),t)}{\partial x} y(t) \qquad (5.35)$$

for $y(t), t \in [t_1,t_2]$; in (5.35), f is the corresponding active subsystem vector field (here it is f_2) in $[t_1,t_2]$ and u, x are the current nominal input and state.

Proof: See Appendix. □

In fact, from the proof of Lemma 5.2 (see Appendix), we can observe that $\delta x(t) = A(t,t_1)\delta x(t_1)$ for any $t \in [\min\{t_1,t_1+dt_1\}, \max\{t_2,t_2+dt_2\}]$. The following important principle can be obtained directly from this observation. We refer to it as *the forward decoupling principle* in the subsequent discussions. It reveals some intrinsic relationship among different switching instants.

The Forward Decoupling Principle: If u assumes open-loop variations, then

(a). The value of the incremental change $\delta x(t_1)$ at t_1 will not be depending on dt_2.

(b). The value of the incremental change $\delta x(t_2)$ at t_2 will be depending on dt_2. □

The forward decoupling principle tells us that a variation of an earlier switching instant will affect the value of the incremental change at a later switching instant, but not vice versa.

Lemma 5.3 *The expression of $dx(t_2)$ is*

$$\begin{aligned} dx(t_2) &= A(t_1,t_1)(f^{1-}-f^{1+})dt_1 + f_x^{2-}A(t_2,t_1)(f^{1-}-f^{1+})dt_1dt_2 + f^{2-}dt_2 \\ &\quad +(\textit{other terms in } dt_1^2, dt_2^2 \textit{ and higher order terms}). \qquad (5.36) \end{aligned}$$

Proof: The proof follows directly from the fact that

$$\begin{aligned} dx(t_2) &= \delta x(t_2+dt_2) + f_2(x(t_2),u(t_2-),t_2)dt_2 \\ &\quad +(\text{other terms in } dt_1^2, dt_2^2 \text{ and higher order terms}). \qquad (5.37) \end{aligned}$$

for all four cases of the signs of dt_1, dt_2. □

Remark 5.3 It is very important to point out that in the expression of $dx(t_2)$, we deliberately express the term $f_x^{2-}A(t_2,t_1)(f^{1-}-f^{1+})dt_1dt_2$ explicitly because it will contribute the the coefficient of dt_1dt_2 as can be seen from the discussions below. □

Now that we have the expressions for $\delta x(t_2)$, $\delta x(t_2+dt_2)$ and $dx(t_2)$, we are ready to derive the coefficient for $dt_1 dt_2$ in the expansion of

$$V^0(x_0, t_0, t_1+dt_1, t_2+dt_2) = V^2(\hat{x}(t_2+dt_2), t_2+dt_2) + \int_{t_0}^{t_2+dt_2} L(\hat{x}(t), \hat{u}(t), t)\ dt. \qquad (5.38)$$

There are two terms in (5.38). Let us look at their Taylor expansions one by one in order to find each term's contribution to the coefficient of $dt_1 dt_2$.

Similar to the single switching case, the Taylor expansions of the first term is

$$\begin{aligned} & V^2(\hat{x}(t_2+dt_2), t_2+dt_2) \\ = \ & V^{2+} + V_x^{2+} dx(t_2) + V_{t_2}^{2+} dt_2 + \frac{1}{2}(dx(t_2))^T V_{xx}^{2+} dx(t_2) + \\ & \frac{1}{2} V_{t_2 t_2}^{2+} dt_2^2 + dt_2 V_{t_2 x}^{2+} dx(t_2) + o(dt_2^2). \end{aligned} \qquad (5.39)$$

In (5.39), all those terms that will possibly contribute to the coefficient of $dt_1 dt_2$ are those terms containing $dx(t_2)$. They are

$$V_x^{2+} dx(t_2), \frac{1}{2}(dx(t_2))^T V_{xx}^{2+} dx(t_2), dt_2 V_{t_2 x}^{2+} dx(t_2). \qquad (5.40)$$

Substituting the expression of $dx(t_2)$ into (5.40) and summing them, we obtain the contribution of the first term to the coefficient of $dt_1 dt_2$ as

$$[V_x^{2+} f_x^{2-} + (f^{2-})^T V_{xx}^{2+} + V_{t_2 x}^{2+}] A(t_2, t_1)(f^{1-} - f^{1+}). \qquad (5.41)$$

For the second term, we have the following Lemma.

Lemma 5.4 *The contribution of* $\int_{t_0}^{t_2+dt_2} L(\hat{x}, \hat{u}, t)\ dt$ *to the coefficient of* $dt_1 dt_2$ *is*

$$L_x^{2-} A(t_2, t_1)(f^{1-} - f^{1+}). \qquad (5.42)$$

Proof: See Appendix. □

Remark 5.4 The above results still holds even when $t_1 = t_2$ (we can consider $t_2 > t_1$ first and then let $t_2 \to t_1$ to prove this). □

Combining (5.41) and (5.42) and the expression of $V_{t_2 x}^{2+}$ which can be similarly derived as $V_{t_1 x}^{1+}$ (see 5.14), we conclude that the coefficient of $dt_1 dt_2$ (i.e., V_{t1t2}^0 in the expansion of $V^0(x_0, t_0, t_1 + dt_1, t_2 + dt_2)$ is

$$\begin{aligned} V_{t_1 t_2}^0 &= [V_x^{2+} f_x^{2-} + (f^{2-})^T V_{xx}^{2+} + V_{t_2 x}^{2+} + L_x^{2-}] A(t_2, t_1)(f^{1-} - f^{1+}) \\ &= [V_x^{2+}(f_x^{2-} - f_x^{2+}) + (f^{2-} - f^{2+})^T V_{xx}^{2+} + L_x^{2-} - L_x^{2+}] A(t_2, t_1)(f^{1-} - f^{1+}). \end{aligned} \qquad (5.43)$$

The above result can also be similarly extended to the case of K switchings to relate $\delta x(t_l)$ and dt_k $(k < l)$. The expression for $V_{t_k t_l}^0$ can similarly be obtained. We summarize and extend the results obtained in this section into the following theorem.

Theorem 5.1 *For a switched system with K switchings,*

$$\begin{aligned}
& V^0(x_0, t_0, t_1 + dt_1, t_2 + dt_2, \cdots, t_K + dt_K) \\
= \; & V^0(x_0, t_0, t_1, t_2, \cdots, t_K) + \sum_{k=1}^{K} V^0_{t_k} dt_k + \frac{1}{2}\sum_{k=1}^{K} V^0_{t_k t_k} dt_k^2 + \sum_{1 \le k < l \le K} V^0_{t_k t_l} dt_k dt_l \\
& + (\textit{higher order terms}) \qquad (5.44)
\end{aligned}$$

where

$$\begin{aligned}
V^0_{t_k} &= L^{k-} - L^{k+} + V_x^{k+}(f^{k-} - f^{k+}), \qquad (5.45) \\
V^0_{t_k t_k} &= (f^{k-} - f^{k+})^T V_{xx}^{k+}(f^{k-} - f^{k+}) - (V_x^{k+} f_x^{k+} + L_x^{k+})(f^{k-} - f^{k+}) \\
&\quad + (V_x^{k+}(f_x^{k-} - f_x^{k+}) + L_x^{k-} - L_x^{k+}) f^{k-} + V_x^{k+}(f_t^{k-} - f_t^{k+}) \\
&\quad + L_t^{k-} - L_t^{k+} + (V_x^{k+} f_u^{k-} + L_u^{k-})\dot{u}^{k-} - (V_x^{k+} f_u^{k+} + L_u^{k+})\dot{u}^{k+}, \qquad (5.46) \\
V^0_{t_k t_l} &= [V_x^{l+}(f_x^{l-} - f_x^{l+}) + (f^{l-} - f^{l+})^T V_{xx}^{l+} \\
&\quad + L_x^{l-} - L_x^{l+}] A(t_l, t_k)(f^{k-} - f^{k+}). \qquad (5.47)
\end{aligned}$$

□

5.3 The Implementation of the Algorithm

Once the values of $\frac{\partial V^0}{\partial \hat{t}}$ and $\frac{\partial^2 V^0}{\partial \hat{t}^2}$ are obtained as approximations to $\frac{\partial J_1}{\partial \hat{t}}$ and $\frac{\partial^2 J_1}{\partial \hat{t}^2}$, the following Algorithm which is a modified version of the conceptual Algorithm 4.1 can be used for Stage 1 optimization.

Algorithm 5.1 (An Algorithm for Stage 1 Optimization)

(1). Set the iteration index $j = 0$. Choose an initial $\hat{t}^j$.

(2). By solving an optimal control problem for the current $\hat{t}^j$ (Stage 1(a)), find the corresponding optimal or suboptimal control input u^j.

(3). For the current $\hat{t}^j$ and its corresponding u^j, supposing that u^j assumes open-loop variations, find $\frac{\partial V^0}{\partial \hat{t}}(\hat{t}^j)$ and $\frac{\partial^2 V^0}{\partial \hat{t}^2}(\hat{t}^j)$ as approximations to $\frac{\partial J_1}{\partial \hat{t}}$ and $\frac{\partial^2 J_1}{\partial \hat{t}^2}$.

(4). Use some feasible direction method to update $\hat{t}^j \le 0$ to be $\hat{t}^{j+1} = \hat{t}^j + \alpha^j d\hat{t}^j$. Set the iteration index $j = j + 1$.

(5). Repeat Steps (2), (3), (4) and (5), until a prespecified termination condition is satisfied. □

It should be pointed out that in order to compute $\frac{\partial V^0}{\partial \hat{t}}(\hat{t}^j)$ and $\frac{\partial^2 V^0}{\partial \hat{t}^2}(\hat{t}^j)$ using the formulae (5.45)-(5.47), we need to know the values of V_x^{k+}, V_{xx}^{k+}, $\dot{u}^{k-}$, $\dot{u}^{k+}$ and $A(t_l, t_k)$. However, given nominal $\hat{t}$, u and x, these values are not readily available. In general, numerical methods need to be used to approximate their values. These added numerical computations usually demands extra computational effort. The numerical method we use is described in the followings.

Suppose that the current switching instant vector is $\hat{t} = (t_1, \cdots, t_K)^T$ and the sequence of active subsystems are subsystems $1, 2, \cdots, K, K+1$. We can use discrete-time optimal control algorithms (see e.g., [11]) for the discretized version of the continuous problem to find the Stage 1(a) solution. Our computation of V_x^{k+}, V_{xx}^{k+}, $\dot{u}^{k-}$, $\dot{u}^{k+}$ and $A(t_l, t_k)$ will be based on the solution of the discretized optimal control problem.

In order not to lose the state, control information at the switching instant, instead of discretizing the whole range $[t_0, t_f]$ using the same discretization level h (in this case, a switching instant may fail to be grid point), we discretize each interval $[t_{k-1}, t_k]$ into N intervals of equal length h_k. In other words, for each $[t_{k-1}, t_k]$ we specify a different discretization level h_k (h_{K+1} for $[t_K, t_f]$), hence the total number of discretized intervals are $(K+1)N$. In this way, we can write our discretized system as

$$x(i+1) = \hat{f}(x(i), u(i), i) \tag{5.48}$$

$$= \begin{cases} x(i) + h_1 f_1(x(i), u(i), ih_1), \ 0 \le i < N, \\ x(i) + h_2 f_2(x(i), u(i), Nh_1 + (i-N)h_2), \ N \le i < 2N, \\ \vdots \\ x(i) + h_{K+1} f_{K+1}(x(i), u(i), N(h_1 + h_2 + \cdots + h_K) \\ \qquad +(i - KN)h_{K+1}), \ KN \le i < (K+1)N. \end{cases} \tag{5.49}$$

The corresponding discrete-time value function is

$$V(x(i), u(i), \cdots, u((K+1)N), i) = \hat{\psi}(x((K+1)N)) + \Sigma_{j=i}^{(K+1)N-1} \hat{L}(x(j), u(j), j) \tag{5.50}$$

where

$$\hat{\psi}(x((K+1)N)) = \psi(x((K+1)N)), \tag{5.51}$$

$$\hat{L}(x(i), u(i), i) = \begin{cases} h_1 L(x(i), u(i), ih_1), \ 0 \le i < N, \\ h_2 L(x(i), u(i), Nh_1 + (i-N)h_2), \ N \le i < 2N, \\ \vdots \\ h_{K+1} L(x(i), u(i), N(h_1 + h_2 + \cdots + h_K) \\ \qquad +(i - KN)h_{K+1}), \ KN \le i < (K+1)N. \end{cases} \tag{5.52}$$

In the followings, we denote $V(x(i), u(i), \cdots, u((K+1)N), i)$ simply as $V(i)$. It can be shown that the value function for the discretized problem satisfies the following backward difference equations (They are actually the equations for the derivatives of the value function for discrete-time optimal control problems stated in [11]).

$$V_x((K+1)N) = \hat{\psi}_x((K+1)N), \tag{5.53}$$

$$V_x(i) = V_x(i+1)\hat{f}_x(i) + \hat{L}_x(i). \tag{5.54}$$

We can then approximate V_x^{k+} by

$$V_x^{k+} \cong V_x(kN). \tag{5.55}$$

Therefore, V_x^{k+} can be derived from solution of (5.53) and (5.54).

Differentiating (5.53) and (5.54). with respect to x, we obtain

$$V_{xx}((K+1)N) = \hat{\psi}_{xx}(x((K+1)N)), \tag{5.56}$$

$$V_{xx}(i) = \hat{f}_x^T(i) V_{xx}(i+1) \hat{f}_x(i) + V_x(i+1)\hat{f}_{xx}(i) + \hat{L}_{xx}(i). \tag{5.57}$$

In (5.57), $\hat{f}_{xx}(i)$ is an $n \times n \times n$ array whose (j_1, j_2, j_3) element is $\frac{\partial^2 \hat{f}_{j_1}}{\partial x_{j_2} \partial x_{j_3}}$ and the notation $V_x(i+1)\hat{f}_{xx}(i)$ denotes an $n \times n$ matrix which has its (j_2, j_3)-th element as

$$\Sigma_{j_1=1}^{n} V_{x_{j_1}}(i+1) \frac{\partial^2 \hat{f}_{j_1}}{\partial x_{j_2} \partial x_{j_3}}. \tag{5.58}$$

By solving (5.56) and (5.57) backwards in time, we can obtain $V_{xx}(i)$. Then we can approximate V_{xx}^{k+} by

$$V_{xx}^{k+} \cong V_{xx}(kN). \tag{5.59}$$

Furthermore, we approximate $\dot{u}^{k-}$, $\dot{u}^{k+}$ by

$$\dot{u}^{k-} = \frac{\hat{f}_k(kN-1) - \hat{f}_k(kN-2)}{h_k}, \tag{5.60}$$

$$\dot{u}^{k+} = \frac{\hat{f}_{k+1}(kN+2) - \hat{f}_{k+1}(kN+1)}{h_{k+1}}. \tag{5.61}$$

Finally, we derive the approximation for $A(t_l, t_k)$. Note that $A(t_l, t_k)$ is the state transition matrix for

$$\dot{y}(t) = \frac{\partial f(x,u,t)}{\partial x} y(t). \tag{5.62}$$

For $k < l$, the discretized version of (5.62) for $t \in [t_k, t_l]$ is

$$y(i+1) = \begin{cases} y(i) + h_{k+1}\frac{\partial f_{k+1}}{\partial x}(x(i), u(i), N(h_1 + \cdots + h_k) + (i - kN)h_{k+1})y(i), \\ \quad \text{for } kN \le i < (k+1)N, \\ \vdots \\ y(i) + h_l \frac{\partial f_l}{\partial x}(x(i), u(i), N(h_1 + \cdots + h_{l-1}) \\ \quad +(i-(l-1)N)h_l)y(i), \\ \quad \text{for } (l-1)N \le i < lN. \end{cases} \tag{5.63}$$

Find the solution $y^{(1)}(i), \cdots, y^{(n)}(i)$ corresponding to initial conditions

$$y^{(1)}(kN) = e_1, \ \cdots, \ y^{(n)}(kN) = e_n \tag{5.64}$$

respectively, where e_j is the unit column vector with all 0's except that the j-th element is 1, $j = 1, 2, \cdots, n$. From linear systems theory, we observe that $A(t_l, t_k)$ can be approximated by the square matrix whose j-th column is $y^{(j)}(lN)$, i.e.

$$A(t_l, t_k) \cong [y^{(1)}(lN), \cdots, y^{(n)}(lN)]. \tag{5.65}$$

Having described the above numerical method, we note that for the special class of optimal control problems which are called generalized switched linear quadratic (GSLQ) problems, the method in Sections 5.1 and 5.2 can be modified so that these values can be easily obtained. We will elaborate on this in the next section.

6 General Switched Linear Quadratic Problems

In this section, we modify the approach in Section 5 and apply it to Stage 1 problem of the following general switched linear quadratic (GSLQ) Problem 6.1. Using the modified approach for this class of problems, the implementation difficulties mentioned at the end of Section 5.3 can be successfully addressed.

Problem 6.1 (GSLQ Problem) *Consider a switched system $\mathcal{S} = (\mathcal{D}, \mathcal{F}, \mathcal{L})$ with linear subsystems $\dot{x} = A_i x + B_i u, i \in I$. Given a fixed time interval $[t_0, t_f]$, find a continuous input $u \in \mathcal{U}_{[t_0,t_f]}$ and a switching sequence $\sigma \in \Sigma_{[t_0,t_f]}$ such that the cost functional in general quadratic form*

$$\begin{aligned} J &= \frac{1}{2}x(t_f)^T Q_f x(t_f) + M_f x(t_f) + W_f + \int_{t_0}^{t_f} (\frac{1}{2}x^T Q x \\ &\quad + x^T V u + \frac{1}{2}u^T R u + Mx + Nu + W)\ dt \end{aligned} \tag{6.1}$$

is minimized. Here t_0, t_f and $x(t_0) = x_0$ are given; $Q_f, M_f, W_f, Q, V, R, M, N, W$ are matrices of appropriate dimensions with $Q_f \geq 0$, $Q \geq 0$ and $R > 0$. □

6.1 Solution for a Single Linear System

Note that for the general quadratic control of a single linear system $\dot{x} = Ax + Bu$, we can use the dynamic programming approach to obtain the following results (the method is similar to the method for solving conventional linear quadratic regulator problem reported in, e.g., [7]).

The optimal value function is

$$V^*(x,t) = \frac{1}{2}x^T P(t) x + S(t) x + T(t) \tag{6.2}$$

where $P(t) = P^T(t)$ and

$$-\dot{P}(t) = Q + P(t)A + A^T P(t) - (P(t)B + V)R^{-1}(B^T P(t) + V^T), \tag{6.3}$$

$$-\dot{S}(t) = M + S(t)A - (N + S(t)B)R^{-1}(B^T P(t) + V^T), \tag{6.4}$$

$$-\dot{T}(t) = W - \frac{1}{2}(N + S(t)B)R^{-1}(B^T S^T(t) + N^T). \tag{6.5}$$

The optimal control is in the feedback form

$$u(x(t), t) = -K(t)x(t) - E(t) \tag{6.6}$$

where

$$K(t) = R^{-1}(B^T P(t) + V^T), \tag{6.7}$$

$$E(t) = R^{-1}(B^T S^T(t) + N^T). \tag{6.8}$$

Remark 6.1 For a GSLQ Stage 1(a) problem, if we assume that subsystem (A_k, B_k) is active in $t \in [t_{k-1}, t_k)$, the above results also hold except that (6.3)-(6.8) should be modified by substituting A and B with A_k and B_k in the time interval $[t_{k-1}, t_k)$ (A_{K+1} and B_{K+1} in $[t_K, t_f]$). □

6.2 Modified Method for GSLQ Problems

Now we modify the method developed in Section 5 so that it will suit better for Stage 1 optimization for GSLQ problems and V_x^{k+}, V_{xx}^{k+}, $\dot{u}^{k-}$ and $\dot{u}^{k+}$ can be obtained more easily without much extra computational effort. Assume we are given nominal switching instants and the corresponding nominal optimal continuous input u in feedback form (6.6). Unlike Section 5, here we choose the nominal $K(\cdot)$ and $E(\cdot)$ rather than $u(\cdot)$ to assume open-loop variations. This can give us the flexibility of letting u vary as a function of x since here u depends on x (see (6.6)). Consequently we have (compare with (5.14), (5.15))

$$V_{t_k x}^{k+} = -(f^{k+})^T V_{xx}^{k+} - V_x^{k+} f_x^{k+} - L_x^{k+} - (V_x^{k+} f_u^{k+} + L_u^{k+}) u_x^{k+}, \tag{6.9}$$

$$\begin{aligned} V_{t_k t_k}^{k+} &= -V_{t_k x}^{k+} f^{k+} - V_x^{k+} f_t^{k+} - L_t^{k+} - (V_x^{k+} f_u^{k+} + L_u^{k+}) u_t^{k+} \\ &= (f^{k+})^T V_{xx}^{k+} f^{k+} + (V_x^{k+} f_x^{k+} + L_x^{k+}) f^{k+} - V_x^{k+} f_t^{k+} - L_t^{k+} \\ &\quad + (V_x^{k+} f_u^{k+} + L_u^{k+})(u_x^{k+} f^{k+} - u_t^{k+}). \end{aligned} \tag{6.10}$$

Note that the expressions $V_{t_k x}^{k+}$ and $V_{t_k t_k}^{k+}$ are different from those in Section 5.1. Similar to the derivation in Section 5.1, it can be shown that $V_{t_k}^0$ is of the same form as in Theorem 5.1 and

$$\begin{aligned} V_{t_k t_k}^0 = &(f^{k-} - f^{k+})^T V_{xx}^{k+} (f^{k-} - f^{k+}) - (V_x^{k+} f_x^{k+} + L_x^{k+})(f^{k-} - f^{k+}) \\ &+ (V_x^{k+}(f_x^{k-} - f_x^{k+}) + L_x^{k-} - L_x^{k+}) f^{k-} + V_x^{k+}(f_t^{k-} - f_t^{k+}) + L_t^{k-} - L_t^{k+} \\ &+ (V_x^{k+} f_u^{k-} + L_u^{k-}) \dot{u}^{k-} - (V_x^{k+} f_u^{k+} + L_u^{k+}) \dot{u}^{k+} \\ &- 2(V_x^{k+} f_u^{k+} + L_u^{k+}) u_x^{k+} (f^{k-} - f^{k+}), \end{aligned} \tag{6.11}$$

where

$$\dot{u}^{k-} = u_x^{k-} f^{k-} + u_t^{k-}, \tag{6.12}$$

$$\dot{u}^{k+} = u_x^{k+} f^{k+} + u_t^{k+}. \tag{6.13}$$

$V_{t_k t_l}^0$ can also be derived similarly to the derivation in Section 5.2 except for the difference described below. Here we can substitute $u(x(t), t) = -K(t)x(t) - E(t)$ into the system state equation and the cost functional. Note that K and E assume open-loop variations and u is a function of x; hence a variation δx will cause a variation $\delta u = -K(t)\delta x$. Similar to the derivations in Section 5.2, we can prove that the expression of $\delta x(t_l + dt_l)$ is

$$\begin{aligned} \delta x(t_l + dt_l) = &A(t_l, t_k)(f^{k-} - f^{k+}) dt_k + (f_x^{l-} + f_u^{l-} u_x^{l-}) A(t_l, t_k)(f^{k-} - f^{k+}) dt_k dt_l \\ &+ (\text{other terms in } dt_k^2, dt_l^2 \text{ and higher order terms}), \end{aligned} \tag{6.14}$$

where $A(t_l, t_k)$ is the state transition matrix for

$$\delta \dot{x} = \frac{\partial f(x,u,t)}{\partial x} \delta x + \frac{\partial f(x,u,t)}{\partial u} \delta u = (A_{i(t)} - B_{i(t)} K(t)) \delta x. \tag{6.15}$$

Once we have the expression of $\delta x(t_l + dt_l)$, we can similarly obtain

$$\begin{aligned} dx(t_l) = &A(t_l, t_k)(f^{k-} - f^{k+}) dt_k + (f_x^{l-} + f_u^{l-} u_x^{l-}) A(t_l, t_k)(f^{k-} - f^{k+}) dt_k dt_l + f^{l-} dt_l \\ &+ (\text{other terms in } dt_k^2, dt_l^2 \text{ and higher order terms}). \end{aligned} \tag{6.16}$$

Moreover, similarly to the derivation in Section 5.2, we can derive the coefficient for $dt_k dt_l$ in the expansion of $V^0(x_0, t_0, t_1 + dt_1, \cdots, t_k + dt_k, \cdots, t_l + dt_l, \cdots, t_K + dt_K)$ as

$$\begin{aligned} V^0_{t_k t_l} &= [V_x^{l+}(f_x^{l+} - f_x^{l-}) + (f^{l-} - f^{l+})^T V_{xx}^{l+} + (L_x^{l-} - L_x^{l+}) \\ &\quad + (L_u^{l-} u_x^{l-} - L_u^{l+} u_x^{l+})] A(t_l, t_k)(f^{k-} - f^{k+}). \end{aligned} \tag{6.17}$$

It can now be seen from the expressions of $V^0_{t_k}$, $V^0_{t_k t_k}$ and $V^0_{t_k t_l}$ that all terms necessary for the evaluation of them are readily available. In this case,

$$V_x^{k+} = x^T(t_k) P^{k+} + S^{k+}, \tag{6.18}$$
$$V_{xx}^{k+} = P^{k+}, \tag{6.19}$$
$$\dot{u}^{k-} = -\dot{K}^{k-} x(t_k) - K^{k-} f^{k-} - \dot{E}^{k-}, \tag{6.20}$$
$$\dot{u}^{k+} = -\dot{K}^{k+} x(t_k) - K^{k+} f^{k+} - \dot{E}^{k+}, \tag{6.21}$$
$$u_x^{k-} = -K^{k-}, \tag{6.22}$$
$$u_x^{k+} = -K^{k+}, \tag{6.23}$$
$$u_x^{l-} = -K^{l-}, \tag{6.24}$$
$$u_x^{l+} = -K^{l+}, \tag{6.25}$$

where x, P, S are continuous at t_k; $\dot{K}^{k-}$,$\dot{K}^{k+}$,$\dot{E}^{k-}$, $\dot{E}^{k+}$ are functions of P, S obtainable by substituting the expressions of $\dot{P}$ and $\dot{S}$ into the differentiation of (6.7) and (6.8). The advantage of applying the approach to GSLQ problems is that here V_x^{k+}, V_{xx}^{k+}, $\dot{u}^{k-}$ and $\dot{u}^{k+}$ can be obtained easily without resorting to extra computational methods. $A(t_l, t_k)$ is the state transition matrix from t_k to t_l for the time varying linear system

$$\dot{y}(t) = (A_{i(t)} - B_{i(t)} K(t)) y(t) \tag{6.26}$$

which can be calculated by numerical integrations as described at the end of Section 5.3.

Now that we have the expressions for $V^0_{t_k}$, $V^0_{t_k t_k}$ and $V^0_{t_k t_l}$, we can use Algorithm 5.1 except that Step (3) should be revised as "suppose K and E assumes open-loop variations".

Remark 6.2 It should be pointed out that only closed-loop variations for u can give us the convenience of computing V_x^{k+}, V_{xx}^{k+}, $\dot{u}^{k-}$ and $\dot{u}^{k+}$. If open-loop variations for u is adopted, the relationship $V(x, t) = \frac{1}{2} x^T P(t) x + S(t) x + T(t)$ is no longer true; hence (6.18)-(6.25) cannot be obtained. In such a case, extra computational effort must be spent to find approximations for the required values. □

7 Some Examples

In this section, we illustrate the effectiveness of the approach developed in this section using several examples. The first two examples use the method derived in Section 5 and the numerical implementation described in Section 5.3.

Example 7.1 Consider a switched system consisting of

$$\text{subsystem 1: } \dot{x} = x + 2xu, \tag{7.1}$$
$$\text{subsystem 2: } \dot{x} = -x - 3xu. \tag{7.2}$$

Assume that $t_0 = 0$, $t_f = 2$ and the system switches once at $t = t_1$ $(0 \le t_1 \le 2)$ from subsystem 1 to 2. We want to find an optimal switching instant t_1 and an optimal input u such that the cost functional $J = \frac{1}{2}(x(2) - 1)^2 + \frac{1}{2}\int_0^2 u^2(t)\ dt$ is minimized. Here $x(0) = 1$.

For this problem, we choose an initial nominal $t_1 = 1.2$. By using Algorithm 5.1 with the gradient projection method, after 40 iterations we find that the optimal switching instant is $t_1 = 1.0013$ and the corresponding optimal cost is 1.3393×10^{-10}. The corresponding continuous control and state trajectory are shown in Figure 4 (a) and (b). Note that the theoretical optimal solutions for this problem are $t_1^{opt} = 1$, $u^{opt} \equiv 0$ and $J^{opt} = 0$, so the result we obtain is quite accurate. Figure 5 shows the optimal cost for different t_1's. □

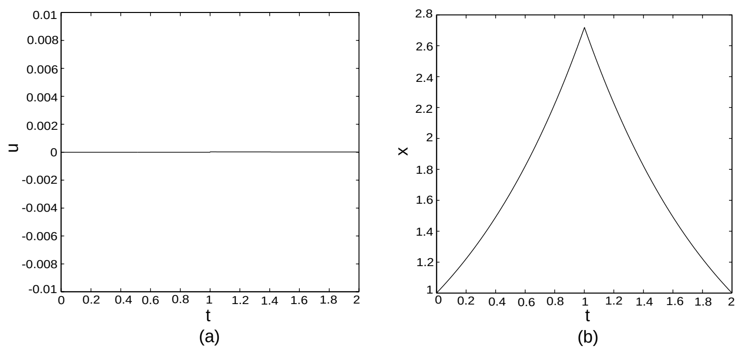

Figure 4: Example 7.1: (a) The control input. (b) The state trajectory $x(t)$.

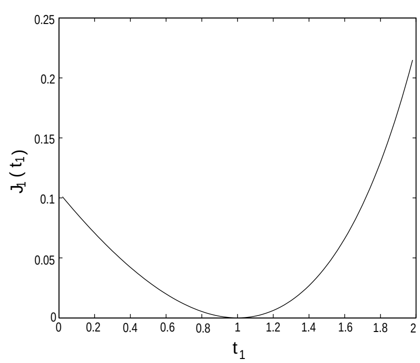

Figure 5: The optimal cost for Example 7.1 for different t_1's.

Example 7.2 Consider a switched system consisting of

$$\text{subsystem 1:} \begin{cases} \dot{x}_1 = -x_1 + 2x_1 u \\ \dot{x}_2 = x_2 + x_2 u \end{cases} \tag{7.3}$$

$$\text{subsystem 2:} \begin{cases} \dot{x}_1 = x_1 - 3x_1 u \\ \dot{x}_2 = 2x_2 - 2x_2 u \end{cases} \tag{7.4}$$

$$\text{subsystem 3:} \begin{cases} \dot{x}_1 = 2x_1 + x_1 u \\ \dot{x}_2 = -x_2 + 3x_2 u \end{cases} \tag{7.5}$$

Assume that $t_0 = 0$, $t_f = 3$ and the system switches at $t = t_1$ from subsystem 1 to 2 and at $t = t_2$ from subsystem 2 to 3 ($0 \le t_1 \le t_2 \le 3$). We want to find optimal switching instants t_1, t_2 and an optimal input u such that the cost functional $J = \frac{1}{2}(x_1(3) - e^2)^2 + \frac{1}{2}(x_2(3) - e^2)^2 + \frac{1}{2}\int_0^3 u^2(t)\ dt$ is minimized. Here $x_1(0) = 1$ and $x_2(0) = 1$.

For this problem, we choose initial nominal $t_1 = 1.1$, $t_2 = 2.1$. By using Algorithm 5.1 with the constrained Newton's method, after 20 iterations we find that the optimal switching instants are $t_1 = 0.9914$, $t_2 = 2.0140$ and the corresponding optimal cost is 2.6919×10^{-4}. The corresponding continuous control and state trajectory are shown in Figure 6 (a) and (b). Note that the theoretical optimal solutions for this problem are $t_1^{opt} = 1$, $t_2^{opt} = 2$, $u^{opt} \equiv 0$ and $J^{opt} = 0$. □

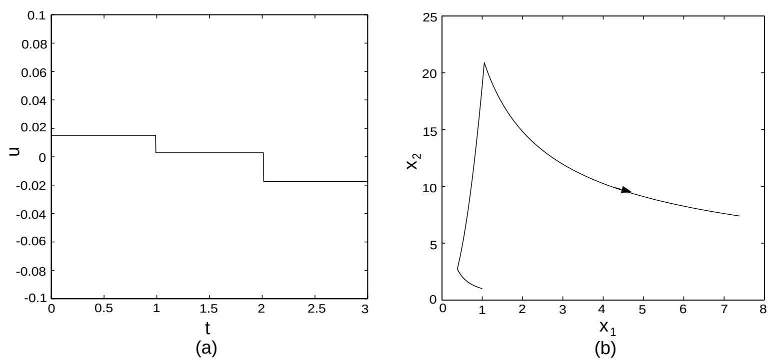

Figure 6: Example 7.2: (a) The control input. (b) The state trajectory.

In the following three examples, we applied the modified method for GSLQ problems.

Example 7.3 Consider a switched system consisting of

$$\text{subsystem 1: } \dot{x} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \tag{7.6}$$

$$\text{subsystem 2: } \dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u. \tag{7.7}$$

Assume that $t_0 = 0$, $t_f = 2$ and the system switches once at $t = t_1$ ($0 \le t_1 \le 2$) from subsystem 1 to 2. We want to find an optimal switching instant t_1 and an optimal input u such that the cost functional $J = \frac{1}{2}\int_0^2 u^2(t)\ dt$ is minimized. Here $x(0) = [1,\ 1]^T$ and $x(2)$ is required to be close to $[e,\ e]^T$.

For this GSLQ problem, we adjoin a penalty term $\frac{1}{2}[(x_1(2) - e)^2 + (x_2(2) - e)^2]$ to J and then consider the expanded cost functional J_{exp}. We use the modified method for GSLQ problems to obtain approximations to $\frac{dJ_1}{dt_1}$. From an initial nominal $t_1 = 1.5$, by using Algorithm 5.1 with the gradient projection method, after 6 iterations we find that the optimal switching instant is $t_1 = 0.9998$ and the corresponding optimal cost is 1.3458×10^{-5}. The corresponding continuous control and state trajectory are shown in figure 7 (a) and (b). Note that the theoretical optimal solutions for this problem are $t_1^{opt} = 1$, $u^{opt} \equiv 0$ and $J_{exp}^{opt} = 0$. Figure 8 shows the optimal cost for different t_1's. □

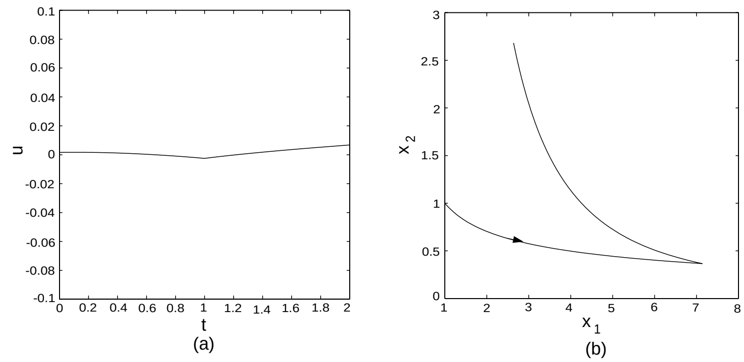

Figure 7: Example 7.3: (a) The control input. (b) The state trajectory.

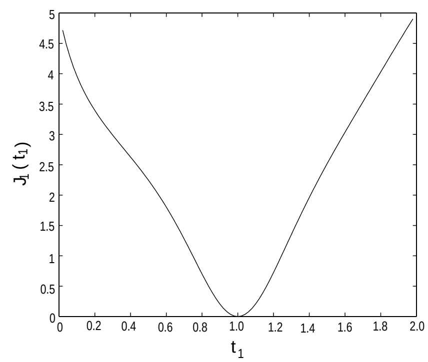

Figure 8: The optimal cost for Example 7.3 for different t_1's.

Example 7.4 Consider a switched system consisting of

$$\text{subsystem 1: } \dot{x} = \begin{bmatrix} 0.6 & 1.2 \\ -0.8 & 3.4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \tag{7.8}$$

$$\text{subsystem 2: } \dot{x} = \begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ -1 \end{bmatrix} u. \tag{7.9}$$

Assume that $t_0 = 0$, $t_f = 2$ and the system switches once at $t = t_1$ $(0 \leq t_1 \leq 2)$ from subsystem 1 to 2. We want to find an optimal switching instant t_1 and an optimal input u such that the cost functional

$$J = \frac{1}{2}(x_1(2) - 4)^2 + \frac{1}{2}(x_2(2) - 2)^2 + \frac{1}{2}\int_0^2 (x_2(t) - 2)^2 + u^2(t)\ dt \tag{7.10}$$

is minimized. Here $x(0) = [0,\ 2]^T$.

For this GSLQ problem, we can use the modified method for GSLQ problems to obtain approximations to $\frac{dJ_1}{dt_1}$. From an initial nominal $t_1 = 1.0$, by using Algorithm 5.1 with the gradient projection method, after 13 iterations we find that the optimal switching instant is $t_1 = 0.1897$ and the corresponding optimal cost is 9.7667. The corresponding continuous control and state trajectory are shown in figure 9 (a) and (b). Figure 10 shows the optimal cost for different t_1's. □

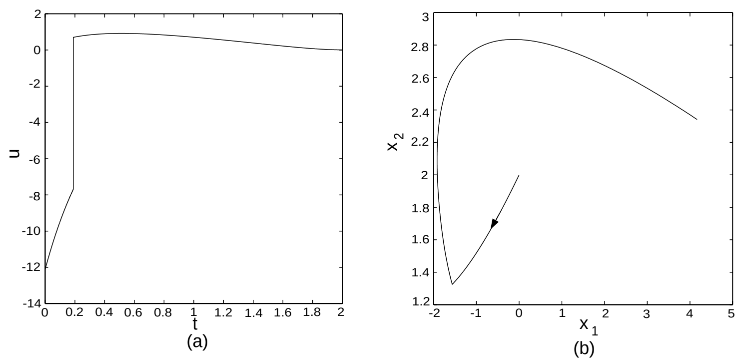

Figure 9: Example 7.4: (a) The control input. (b) The state trajectory.

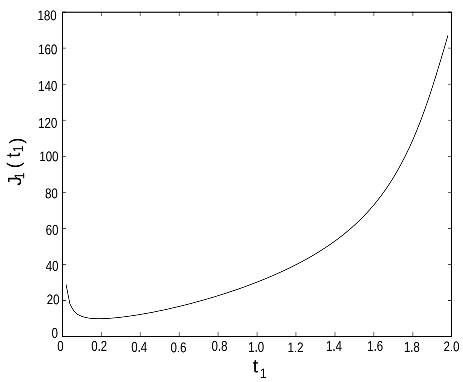

Figure 10: The optimal cost for Example 7.4 for different t_1's.

Example 7.5 Consider a switched system consisting of

$$\text{subsystem 1: } \dot{x} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \tag{7.11}$$

$$\text{subsystem 2: } \dot{x} = \begin{bmatrix} 0.5 & 5.3 \\ -5.3 & 0.5 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u, \tag{7.12}$$

$$\text{subsystem 3: } \dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1.5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u. \tag{7.13}$$

Assume that $t_0 = 0$, $t_f = 3$ and the system switches at $t = t_1$ from subsystem 1 to 2 and at $t = t_2$ from subsystem 2 to 3 ($0 \leq t_1 \leq t_2 \leq 3$). We want to find optimal switching instants t_1, t_2 and an optimal input u such that the cost functional $J = \frac{1}{2}\int_0^3 u^2(t)\ dt$ is minimized. Here $x(0) = [4,\ 4]^T$ and $x(3)$ is required to be close to $[-4.1437,\ 9.3569]^T$.

For this problem, we adjoin a penalty term $[(x_1(3) + 4.1437)^2 + (x_2(3) - 9.3569)^2]$ to J and then consider the expanded cost functional J_{exp}. We can use the modified method for GSLQ problems to obtain approximations to $\frac{\partial J_1}{\partial \hat{t}}$ and $\frac{\partial^2 J_1}{\partial \hat{t}^2}$. From initial nominal values $t_1 = 0.8$, $t_2 = 1.8$, by using Algorithm 5.1 with the constrained Newton's method, after 43 iterations we find that the optimal switching instant is $t_1 = 1.0002$, $t_2 = 2.0008$ and the corresponding optimal cost is 6.3146×10^{-5}. The corresponding continuous control and state trajectory are shown in figure 11 (a) and (b). Note that the theoretical optimal solutions for this problem are $t_1^{opt} = 1$, $t_2^{opt} = 2$, $u^{opt} \equiv 0$ and $J_{exp}^{opt} = 0$, so the result we obtained is quite accurate. Figure 12 shows the optimal cost for different $t_1 < t_2$. $\square$

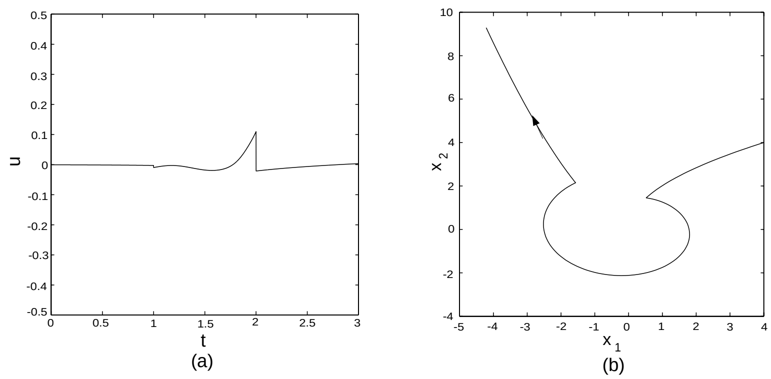

Figure 11: Example 7.5: (a) The control input. (b) The state trajectory.

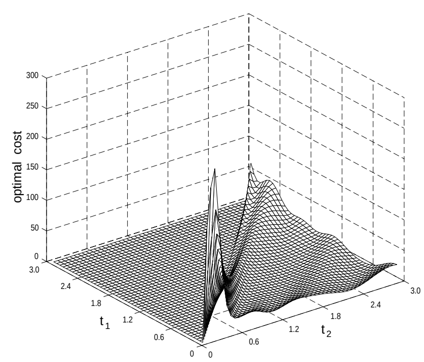

Figure 12: The optimal cost for Example 7.5 for different (t_1, t_2)'s $(0 \le t_1 \le t_2 \le 3)$.

It can be observed from figure 12 that the function $J_1(t_1, t_2)$ has several ripples. Hence it is not convex even for this simple GSLQ problem; that is why such problems pose significant difficulties.

8 Conclusion

In this paper, we formulated an optimal control problem of switched systems and proposed a two stage optimization methodology for it. Then we focused on Stage 1 optimization problems which can further be decomposed into Stage 1(a) and Stage 1(b). We proposed a method to obtain approximations of the values of the derivatives that are necessary for Stage 1(b) optimizations. The method is based on direct differentiations of value functions (Theorem 5.1). In particular, a modified version of the method was proposed for general switched linear quadratic (GSLQ) problems which can successfully address some of the implementation difficulties of the method. Note that earlier results of this paper have appeared in [32, 31, 33] and a more complete version can be found in [30]. There are many further research topics that are worth exploring since this paper is only a first attempt to solve optimal control problems for switched systems. They include the development of numerical methods for deriving accurate values of $\frac{\partial J_1}{\partial \hat{t}}$, $\frac{\partial^2 J_1}{\partial \hat{t}^2}$; development of methods for Stage 1 optimization for systems with state and control constraints; explorations of the properties of the function $J_1(\hat{t})$; etc. We will report progresses in these topics in our future papers.

Appendix: Some Proofs

Proof of Lemma 5.2: Although the results in the Lemma hold for all cases in the definition of $\delta x(t)$, we need to discuss each case in order to show the validity of them.
Case 1: $dt_1 \geq 0$, $dt_2 < 0$.

$$\delta x(t_1 + dt_1) = \int_{t_1}^{t_1+dt_1} f_1(\hat{x}(t), \hat{u}(t), t)\ dt - \int_{t_1}^{t_1+dt_1} f_2(x(t), u(t), t)\ dt. \tag{A.1}$$

Using Lemma 5.1 (here we can let $\epsilon = dt_1$, $q = 1$ and $p = 0$) and noting that $\hat{x}(t_1) = x(t_1)$, $\hat{u}(t_1) = u(t_1-)$, we have

$$\begin{aligned}\int_{t_1}^{t_1+dt_1} f_1(\hat{x}(t), \hat{u}(t), t)\ dt &= f_1(\hat{x}(t_1), \hat{u}(t_1), t_1)dt_1 + o(dt_1)\\ &= f_1(x(t_1), u(t_1-), t_1)dt_1 + o(dt_1)\\ &= f^{1-}dt_1 + o(dt_1). \end{aligned}\tag{A.2}$$

Using Lemma 5.1, we have

$$\begin{aligned}\int_{t_1}^{t_1+dt_1} f_2(x(t), u(t), t)\ dt &= f_2(x(t_1), u(t_1+), t_1)dt_1 + o(dt_1)\\ &= f^{1+}dt_1 + o(dt_1)\end{aligned}\tag{A.3}$$

Hence $\delta x(t_1 + dt_1) = (f^{1-} - f^{1+})dt_1 + o(dt_1)$ and we conclude from the property of the variational equation that

$$\begin{aligned}\delta x(t_2) &= A(t_2, t_1 + dt_1)\delta x(t_1 + dt_1) + o(dt1)\\ &= [A(t_2, t_1) + A_{t_1}dt_1 + o(dt_1)][(f^{1-} - f^{1+})dt_1 + o(dt_1)] + o(dt_1)\\ &= A(t_2, t_1)(f^{1-} - f^{1+})dt_1 + o(dt_1), \qquad\qquad (A.4)\\ \delta x(t_2 + dt_2) &= [\hat{x}(t_2) + \int_{t_2}^{t_2+dt_2} f_2(\hat{x}(t), \hat{u}(t), t)\ dt] - [z_1(t_2) + \int_{t_2}^{t_2+dt_2} f_2(z_1(t), \hat{u}(t), t)\ dt]\\ &= \delta x(t_2) + \int_{t_2}^{t_2+dt_2} [f_2(\hat{x}(t), \hat{u}(t), t) - f_2(z_1(t), \hat{u}(t), t)]\ dt\\ &= \delta x(t_2) + [f_2(\hat{x}(t_2), u(t_2-), t_2) - f_2(x(t_2), u(t_2-), t_2)]dt_2 + o(dt_2)\\ &= \delta x(t_2) + f_x^{2-}\delta x(t_2)dt_2 + o(dt_2)\\ &= A(t_2, t_1)(f^{1-} - f^{1+})dt_1 + f_x^{2-}A(t_2, t_1)(f^{1-} - f^{1+})dt_1 dt_2\\ &\quad +(\text{other terms in } dt_1^2, dt_2^2 \text{ and higher order terms}). \qquad\qquad (A.5)\end{aligned}$$

Case 2: $dt_1 \geq 0$, $dt_2 < 0$.

The arguments for proving (A.4) in Case 1 can be applied in this case to show its validity. In this case,

$$\begin{aligned}
\delta x(t_2+dt_2) &= z_2(t_2+dt_2) - x(t_2+dt_2) \\
&= [z_2(t_2) + \int_{t_2}^{t_2+dt_2} f_2(z_2(t),u(t),t)\ dt] - [x(t_2) + \int_{t_2}^{t_2+dt_2} f_2(x(t),u(t),t)\ dt] \\
&= \delta x(t_2) + \int_{t_2}^{t_2+dt_2} [f_2(z_2(t),u(t),t) - f_2(x(t),u(t),t)]\ dt \\
&= \delta x(t_2) + [f_2(z_2(t_2),u(t_2-),t_2) - f_2(x(t_2),u(t_2-),t_2)]dt_2 + o(dt_2) \\
&= \delta x(t_2) + f_x^{2-}\delta x(t_2)dt_2 + o(dt_2) \\
&= A(t_2,t_1)(f^{1-} - f^{1+})dt_1 + f_x^{2-}A(t_2,t_1)(f^{1-} - f^{1+})dt_1 dt_2 \\
&\quad +(\text{other terms in } dt_1^2, dt_2^2 \text{ and higher order terms}). \qquad (A.6)
\end{aligned}$$

Case 3: $dt_1 < 0$, $dt_2 \geq 0$.

In this case, we have

$$\begin{aligned}
\delta x(t_1) &= \int_{t_1+dt_1}^{t_1} f_2(\hat{x}(t),\hat{u}(t),t)dt - \int_{t_1+dt_1}^{t_1} f_1(x(t),u(t),t)dt \\
&= f_2(x(t_1+dt_1), u(t_1+) + \dot{u}^{1+}dt_1, t_1+dt_1)(-dt_1) \\
&\quad -f_1(x(t_1+dt_1), u(t_1+dt_1), t_1+dt_1)(-dt_1) \\
&\quad +o(dt_1) \\
&= f_1(x(t_1),u(t_1-),t_1)dt_1 - f_2(x(t_1),u(t_1+),t_1)dt_1 + o(dt_1) \\
&= (f^{1-} - f^{1+})dt_1 + o(dt_1). \qquad (A.7)
\end{aligned}$$

In reaching the second to the last equations in (A.7), we use the relationship

$$x(t_1+dt_1) = x(t_1) + \dot{x}(t_1-)dt_1 + o(dt_1), \qquad (A.8)$$

$$u(t_1+dt_1) = u(t_1-) + \dot{u}(t_1-)dt_1 + o(dt_1), \qquad (A.9)$$

and the Taylor expressions of f_2 and f_1. Therefore, we have

$$\begin{aligned}
\delta x(t_2) &= A(t_2,t_1)\delta x(t_1) + o(dt_1) \\
&= A(t_2,t_1)(f^{1-} - f^{1+})dt_1 + o(dt_1) \qquad (A.10) \\
\delta x(t_2+dt_2) &= [\hat{x}(t_2) + \int_{t_2}^{t_2+dt_2} f_2(\hat{x}(t),\hat{u}(t),t)\ dt] \\
&\quad -[z_3(t_2) + \int_{t_2}^{t_2+dt_2} f_2(z_3(t),\hat{u}(t),t)\ dt] \\
&= \delta x(t_2) + \int_{t_2}^{t_2+dt_2} [f_2(\hat{x}(t),\hat{u}(t),t) - f_2(z_3(t),\hat{u}(t),t)]\ dt \\
&= \delta x(t_2) + [f_2(\hat{x}(t_2),u(t_2-),t_2) - f_2(x(t_2),u(t_2-),t_2)]dt_2 + o(dt_2) \\
&= \delta x(t_2) + f_x^{2-}\delta x(t_2)dt_2 + o(dt_2) \\
&= A(t_2,t_1)(f^{1-} - f^{1+})dt_1 + f_x^{2-}A(t_2,t_1)(f^{1-} - f^{1+})dt_1 dt_2 \\
&\quad +(\text{other terms in } dt_1^2, dt_2^2 \text{ and higher order terms}). \qquad (A.11)
\end{aligned}$$

Case 4: $dt_1 < 0$, $dt_2 < 0$.

The arguments for proving (A.10) in Case 3 can be applied in this case to show its validity. In this case, we have

$$\begin{aligned}
\delta x(t_2+dt_2) &= [z_4(t_2) + \int_{t_2}^{t_2+dt_2} f_2(z_4(t), u(t), t)\ dt] \\
&\quad -[x(t_2) + \int_{t_2}^{t_2+dt_2} f_2(x(t), u(t), t)\ dt] \\
&= \delta x(t_2) + \int_{t_2}^{t_2+dt_2} [f_2(z_4(t), u(t), t) - f_2(x(t), u(t), t)]\ dt \\
&= \delta x(t_2) + [f_2(z_4(t_2), u(t_2-), t_2) - f_2(x(t_2), u(t_2-), t_2)]dt_2 + o(dt_2) \\
&= \delta x(t_2) + f_x^{2-}\delta x(t_2)dt_2 + o(dt_2) \\
&= A(t_2,t_1)(f^{1-} - f^{1+})dt_1 + f_x^{2-}A(t_2,t_1)(f^{1-} - f^{1+})dt_1 dt_2 \\
&\quad +(\text{other terms in } dt_1^2, dt_2^2 \text{ and higher order terms}). \qquad \text{(A.12)}
\end{aligned}$$

□

Proof of Lemma 5.4: We first note that

$$\int_t^{t_2+dt_2} L(\hat{x}, \hat{u}, t)dt = \int_{t_0}^{\max\{t_1, t_1+dt_1\}} L(\hat{x}, \hat{u}, t)dt + \int_{\max\{t_1, t_1+dt_1\}}^{t_2+dt_2} L(x+\delta x, \hat{u}, t)dt. \qquad \text{(A.13)}$$

In the light of the forward decoupling principle, the first term in (A.13) will not depend on dt_2; therefore, it will not contribute to the coefficient of $dt_1 dt_2$.

For the second term, we discuss as follows.

Case 1. $dt_2 > 0$.

In this case, we have

$$\int_{\max\{t_1, t_1+dt_1\}}^{t_2+dt_2} L(\hat{x}, \hat{u}, t)\ dt = \int_{\max\{t_1, t_1+dt_1\}}^{t_2} L(x+\delta x, \hat{u}, t)\ dt + \int_{t_2}^{t_2+dt_2} L(\hat{x}, \hat{u}, t)\ dt. \qquad \text{(A.14)}$$

The first term in (A.14) will not be contributing due to the reason that

$$\begin{aligned}
\delta x(t) &= A(t,t_1)(f^{1-} - f^{1+})dt_1 + o(dt_1), \qquad \text{(A.15)} \\
\hat{u}(t) &= u(t), \qquad \text{(A.16)}
\end{aligned}$$

for $t \in [t_0, \max\{t_1, t_1+dt_1\})$ and therefore they are not depending on dt_2.

The second term is shown to be

$$\begin{aligned}
&\int_{t_2}^{t_2+dt_2} L(\hat{x}(t_2), \hat{u}(t), t)\ dt = L(\hat{x}(t_2), u(t_2-), t_2)dt_2 + o(dt_2) \\
&= L(x(t_2), u(t_2-), t_2)dt_2 + L_x^{2-}\delta x(t_2)dt_2 \\
&+(\text{other terms in } dt_2, dt_2^2 \text{ and terms higher than order 2}). \qquad \text{(A.17)}
\end{aligned}$$

By substituting the expression of $\delta x(t_2)$ into (A.17), we obtain the coefficient of $dt_1 dt_2$ contributed by this term as

$$L_x^{2-}A(t_2,t_1)(f^{1-} - f^{1+}). \qquad \text{(A.18)}$$

Case 2. $dt_2 < 0$.

In this case, since $x(t) + \delta x(t) = \hat{x}(t)$ and $\hat{u}(t) = u(t)$, for $t \in [\max\{t_1, t_1 + dt_1\}, t_2 + dt_2]$, we have

$$\int_{\max\{t_1,t_1+dt_1\}}^{t_2+dt_2} L(\hat{x}, \hat{u}, t)\ dt = \int_{\max\{t_1,t_1+dt_1\}}^{t_2+dt_2} L(x + \delta x, u, t)\ dt$$
$$= \int_{\max\{t_1,t_1+dt_1\}}^{t_2} L(x + \delta x, u, t)\ dt + \int_{t_2}^{t_2+dt_2} L(x + \delta x, u, t)\ dt. \tag{A.19}$$

Similar to Case 1, the first term in (A.19) will not be contributing. The second term is shown to be

$$\int_{t_2}^{t_2+dt_2} L(x + \delta x, u, t)\ dt = L(x(t_2) + \delta x(t_2), u(t_2-), t_2)dt_2 + o(dt_2)$$
$$= L(x(t_2), u(t_2-), t_2)dt_2 + L_x^{2-}\delta x(t_2)dt_2$$
$$+(\text{other terms in } dt_2, dt_2^2 \text{ and terms higher than order 2}). \tag{A.20}$$

Therefore, by substituting the expression of $\delta x(t_2)$ into (A.20), we obtain the same coefficient (A.18). □

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