OPTIMAL CONTROL OF SYSTEMS WITH STATE-DEPENDENT TIME DELAY

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CHAPTER I

INTRODUCTION

In past years engineers have utilized various methods of analysis and synthesis for the design of automatic control systems. These methods include one of the crudest methods, "build and try", to more sophisticated methods such as designing in the frequency domain via Bode plots, root locus plots, and Nyquist plots.

Due to increased demands for more accurate and better performing systems, control engineers started to develop more sophisticated methods for the synthesis of control systems. The rapid growth of the computer industry helped the control engineer in his development of better techniques in that the high-speed digital computer became readily available to him. Thus, he is able to utilize the digital computer not only as a problem solving tool for applying advanced analytical techniques, but with the advent of the mini-computer he is able to actually use a computer as an element of his control loop. This factor now and even more in the future makes the more sophisticated control synthesis techniques usable to the practicing engineer.

Thus, since both military and industrial applications are calling for better performing systems, the development of what is generally known as control theory will continue at the rapid pace it has enjoyed the last decade. The development will be not only to find better methods to analyze and synthesize deterministic linear systems which various prior methods were limited, but also, for example, to find methods to design systems with nonlinearities and systems with stochastic parameters and inputs.

This research develops the basis for the synthesis of optimal control for a special class of control systems. The class of systems considered are systems with state dependent time delay.

Mathematical Model

The first step in any control problem is that of obtaining a suitable mathematical model of the system. The mathematical model must be of sufficient accuracy to model the system. Yet, it must be simple enough such that the task of analyzing the system is not impossible. However, failure to correctly model the system may lead to a design that is inadequate for the particular task the system is to accomplish and may cause various undesirable system problems such as instability.

Many dynamical systems may be best modeled by differential-difference equations. That is, equations of the form

$$\dot{x}(t) = f[x(t), x_h, u(t), u_h, t]$$

where:

$$x_h = x(t-h)$$

$$u_h = u(t-h),$$

may be used to best represent many physical systems. The variable h is a non-negative function and may be a constant, a function of time, a function of x(t), or a stochastic process.

The state of a system or of a mathematical process is a minimum set of numbers which along with the knowledge of the future inputs contain sufficient information about the history of the system or process to allow computation of future behavior. In a finite-dimensional system the variable $x(t)$ would define the state of the system. However, for a system modeled by differential-difference equations, the state at a time t is a continuous vector function as shown

$$x(\tau),\ t-h \leq \tau \leq t.$$

Systems modeled by differential-difference equations are also referred to as systems with time delay or systems with transport lag. With this mathematical introduction to systems with time delays, typical applications of such systems will be discussed.

Applications of Time Delay

Applications where time delays may occur include problems of guidance and control of distant space vehicles (63), control of complex processing plants (27), economic systems, biological systems, human operator models, and remote control of lunar surface vehicles (43). Typical examples of systems with time delays are as follows.

The first problem is that of high accuracy, ground-based attitude control of space vehicles. Sabroff (58) presents a sound case for Earth-based attitude control of deep-space satellites assuming that the control problem created by the time delay can be solved. Measurement devices aboard the spacecraft may send attitude error signals to the Earth-based controller via a data link. This controller will calculate a control law to reduce the attitude error to some tolerable level and send the control signal back to the satellite. Pictorially, the problem may be seen as follows.

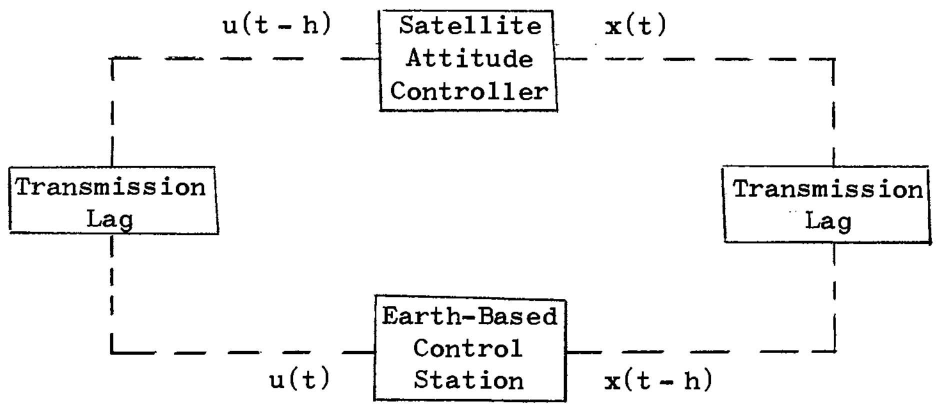

Figure 1. Schematic of Ground-Based Attitude Control System

The signals x(t) denote the error signals sent from the vehicle at time t. However, since the signal must travel over a long distance to reach the Earth-based control station, the control station will receive the signal at some time later than when the measurement occurred. It then must use this delayed signal to calculate the required control signal to send back to the vehicle to correct for attitude errors. But the control problem is further complicated since the control signal must again travel over a long distance to reach the satellite. Thus, there is a time delay or transmission lag associated with the paths the signals must travel over to reach their respective destinations. Also, unless the satellite is in a circular orbit, the time delay will be a function of time, thus further complicating the problem. The time delay may be predicted from an ephemeris of the spacecraft. It is assumed that there is no computational delay in obtaining the correct control law. That is, the ground-based controller may instantaneously compute the control law.

A similar problem of guidance and control of deep-space vehicles was presented in reference (63). In this reference, the author presented a deterministic guidance problem of an Earth-to-Mars mission. Again, the problem of transmission lags occurred and were considered in the guidance philosophy.

High-accuracy, Earth-based, guidance and control of spacecraft on outer-planet missions make the problem of time delays extremely important since the time delays incurred are much larger than the Earth-to-Mars voyage considered in (63). Also, moon roving surface vehicles remotely controlled from the earth make the time delay problem extremely important in this aspect since delays on the order of three seconds occur (43).

Another problem where time delay models are frequently used are in human operator models such as pilot models for aircraft control design purposes. Considerable research is still being accomplished in obtaining a representative model for the human in different control tasks. However, an example of a particular model in Laplace transform notation is

$$H(s) = \frac{Ke^{-hs}\,(1 + \tau_1 s)}{(1 + \tau_2 s)(1 + \tau_3 s)} .$$

The variable h denotes the magnitude of the time delay.

Many complex industrial processing plants have large multiple time delays. Not only are large delays inherent in flow lines, mixing processes, and heating processes, but high-order complex nonlinear systems may be approximated by a linear time delay system. That is, the time constants and the time delays are adjusted until the system response due to this model fits actual measured system response data. An example of a particular model that is used in the chemical processing industry to model complex components in a processing plant is

$$G(s) = \frac{Ke^{-hs}}{(1+\tau_1 s)(1+\tau_2 s)} .$$

An example of a simple control loop that may just be a secondary control loop of a large processing plant that has many more control loops will now be illustrated. This example was taken from reference (27).

> An aqueous stream is diluted continuously in a 500-gal. tank equipped with a 16 in. turbine. A portion of the exit stream is sent to a controller which adjusts the flow of water to the tank. The total flow is about 100 gpm, and there is a 10-sec. delay in the measurement line.

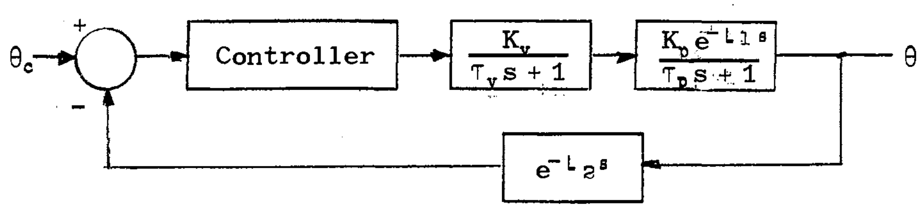

Figure 2. Typical Processing System With Time Delay

> The variable θ_c represents some desired set point. The delays L_1 and L_2 are of the order of 12 sec. and 10 sec., respectively.

Again, this may represent only a portion of a highly complex processing plant that may have many time delays of various magnitudes. This is not uncommon in many industrial processes.

There are three basic ways time delays may occur in systems:

1. A delay in the control variable may occur when the control signal has to travel over a long distance to reach the

system to be controlled. An example is the attitude control of distant space vehicles via Earth-based controllers.

2. A delay in one or more of the state variables may occur in man-in-the-loop systems, process control plants, and economic systems.

3. A delay may occur in measuring one or more of the output variables of a control system. Examples of such delays include measurements of the temperature of the output of a nuclear reactor downstream from the reactor and measurement of the position of a deep-space vehicle via Earth-based radar.

In all physical systems there is inherent time delay; however, in some systems the time delays may be small enough to be ignored. Failure to consider time delays in a system may lead to a system that is unstable, or may lead to a set of controls that is not optimal if optimality is the design criterion.

Since time delays may significantly affect system performance, it becomes extremely important that the control engineer considers time delay in the mathematical model of his system.

There are several forms that the time delay argument may have. First, the time delays in a system may be a constant. This assumption is most often used in the literature. However, this assumption may be erroneous in some systems since the time delay may be time-varying such as in the example of the remote attitude control of deep-space vehicles. If the position of the vehicle can be determined from an ephemeris of the vehicle, the time delay can be found as a function of time. Another example where the constant time delay assumption may be invalid is that there may be a particular known bias that is a function of time causing the time delay to be time-varying. Thus, the second type of time delay is that of a time-varying delay. The third type is that of a state-dependent time delay. This implies that the time delay is a function of the state of the system. Note that the meaning of "state" in this instance is actually the leading terminus of the state function at time t and not the function that actually denotes the state of the system. Examples of systems where the time delay is a function of the state include any control system of a conveyor process where measurement devices may be located several feet from a process. The feedback from the measurement devices may be used to control the conveyor velocity and the process. Since the time delay for the system may be proportional to the velocity, and the velocity is one of the state variables, the time delay may be said to be state-dependent. In a deep-space guidance and control problem with an Earth-based ground control the time delay of signals propagating to and from the spacecraft is a function of the position of the spacecraft measured from an Earth-based coordinate system. If the position vector components are treated as state variables, then the problem is that of a state-dependent time delay.

Optimal Control Theory

One of the more important synthesis techniques for design of con-systems is that of optimal control. A system designed via optimal control can be claimed to be "best" according to some prescribed criterion of performance. The philosophy of optimal control is to choose a set of inputs, or controls, to the system such that it most satisfactorily represents the desired goals of the system. The desired goals of the system must be expressed in the form of a scalar mathematical criterion, called the performance index. Once the mathematical model of the system is obtained and the performance index is chosen, then optimal control theory may be applied to obtain a set of controls such that the performance index is minimized (or maximized). This set of controls, which may, also, be constrained to a particular set, is then called the optimal control for the particular system model, the particular performance index, and the particular constraints. It is always desirable to constrain the control set to be a function of the output of the system. This type of control is called feedback control. The reason feedback control is always desirable is that the system may be subject to disturbances such as uncertainties in the dynamic model or uncertainties in the operating environment. A feedback control can compensate for these errors and still obtain optimality since the control depends only on the state of the system. A control calculated as a function of time may become, at best, suboptimal if these errors become too large.

Once the optimal controls are found they may be implemented into the system. If economic considerations are such that the added complexity of implementing the optimal controls cannot be achieved, then the optimal controls and the value of the optimal performance index may be used as a standard. The designer may use this standard to find controls that are suboptimal in the sense that they do not minimize the mathematical performance index. Yet, he is able to compare the value of the standard to that of his suboptimal controls in order to obtain a tradeoff between system performance and economic considerations. Thus, if the optimal control cannot be implemented, then it is still of benefit to the designer to know what the optimal control is in order to better approximate it, and yet, still obtain a degree of optimality.

There are many archive journal articles on the applications of optimal control theory. Two excellent books on the general topic of optimal control which include applications are by Bryson and Ho (12) and Athans and Falb (3). However, these books do not include systems with time delay.

Research Objectives

The class of systems considered in this research are modeled by nonlinear differential-difference equations where the time delay is a piecewise differentiable scalar function, $h(x(t),t)$, of time and of $x(t)$, where $x(t) \in R^n$ and is determined from the state of the system. The systems to be considered are modeled by nonlinear differential-difference equations

$$\dot{x}(t) = f[x(t), x_h, u(t), u_h, t] \tag{1}$$

with the definitions

$$x_h = x[t-h(x(t),t)]$$

$$u_h = u[t-h(x(t),t)]$$

The objective of this research was to present a basis for the synthesis of an optimal control for systems with state-dependent time delay. Necessary conditions for an optimal control of a general class of dynamical systems involving time delays and constraints are developed. The development and results are shown in Chapter III.

The necessary conditions were utilized to obtain the optimal control for a system that is linear in the explicit variables, but the function for the time delay is not necessarily linear in the state variables. A quadratic performance index is used. A numerical algorithm is proposed to solve this problem by using the resulting necessary conditions. Examples of systems with state-dependent time delay are shown in Chapter V.

Extensions to a gradient algorithm developed by Sebesta (60) were developed and the gradient algorithm is outlined in Chapter IV.

The next chapter gives the results of a literature survey of optimal control of systems with time delay.

CHAPTER II

LITERATURE SURVEY

The literature in control theory for systems with time delay has become quite extensive in the past several years. This chapter gives the results of a literature survey for optimal control of systems with time delay. The survey was divided into four categories: systems with a constant time delay, systems with a time varying time delay, systems with a state-dependent time delay, and numerical algorithms for systems with time delay.

There are several books treating differential-difference equations (9), (26), (51). Also, there is a book (48) that is devoted to the theory of control systems with time delay.

The majority of time delay control systems can be classified as that of systems with a constant time delay. However, in some instances this may be only approximate in that the time delay may vary as time or the time delay may even be a stochastic process.

Systems With Constant Time Delay

Kharatishvili (52) derived necessary conditions for optimality for systems with time delay by extending Pontryagin's maximum principle. In this work, the results were formulated for a system with a time delay in the state variable; however, in (30) Kharatishvili has extended the previous results to systems with a time delay in the state and in the control. The necessary conditions derived in Chapter III reduce to the results obtained by the extension of the maximum principle.

Chyung and Lee (14) give both necessary and sufficient conditions for optimality of linear systems with time delay and a quadratic performance index. The conditions for optimality are obtained by using set theoretic arguments similar to (41). The results are identical to that obtained by extension of the maximum principle, but sufficiency is also proved. A theorem giving necessary and sufficient conditions for controllability of a linear system with a time delay in the state is given without proof. The controllability matrix has the same form as the controllability matrix of a linear system without time delay and differs only in that the transition matrix is that for the delayed system.

Krasovskii (36), (37) states without proof the form of the optimal cost and the form of a feedback controller for a system with a time delay in the state and a quadratic performance index. His motivation for this form of controller is to obtain a control law which stabilizes the system. He does not give the form of certain coefficient matrices in the feedback control law that he states. Alekal (2) proved that this form of the feedback control law is optimal and derived the equations for the coefficient matrices. The equations are coupled ordinary and partial differential equations. Eller et al. (17) independently proved these results and derived the equations for the coefficient matrices. Garrard (23) used a technique similar to that of (32) in order to obtain a suboptimal controller to circumvent the solution of the coupled ordinary and partial differential equations. A suboptimal control law was given and a numerical algorithm derived to obtain the suboptimal coefficient matrices.

Ross and Flugge-Lotz (56), (57) solved a specific case of the above problem. The problem solved was that of an infinite final time quadratic performance index. The results were obtained by a method similar to that used by Alekal et al. The equations for the coefficient gain matrices are a coupled set of algebraic equations, and ordinary and partial differential equations. An approximate solution for the optimal control was also discussed.

Oguztöreli (48), (49), (50) has considered the time optimal problem in detail for systems with a time delay in the state. The optimal control is similar to that of the non-delay problem in that the optimal control is bang-bang. The case of multiple time delays was also treated. Westdal (64) also treats the case of a time optimal system. This paper used a higher order non-delayed system to approximate the time delay system. The maximum principle was then used to obtain an optimal control. An example was shown.

Friedman (19) extended the maximum principle to hereditary processes of the form

$$x(t) = x(t_0) + \int_{t_0}^{t} h^T(t-\tau) f(\tau, u(\tau), x(\tau))\, d\tau .$$

Bates (8) has considered in detail the problem of optimal control of systems described by linear differential-integral equations of the form

$$\dot{x} = \int_{-\infty}^{t} [F(t,\tau)\ x(\tau) + D(t,\tau)\ u(\tau)]\, d\tau .$$

The optimal control problem was studied with the quadratic, the time optimal, and the minimum effort performance index. McClamrock (44), (45) has obtained necessary and sufficient conditions for linear hereditary processes.

Fuller (20) has studied in detail a particular control system with time delay for the integral-square error performance index. In this paper Fuller gives two references to older papers that show the approximation of a high-order nonlinear system by a lower order system with time delay can be quite accurate. This type approximation is used extensively in the chemical processing industry.

Kramer (35) and Jen Wei (29) considered the control problem for a linear system with a constant delay in the state variable. The method used is dynamic programming. Merriam (46) used dynamic programming to solve the problem of a time delay in the control variable.

Halanay (26) rigorously obtained a maximum principle for a general class of delayed systems. The necessary conditions admit as special cases hereditary equations and differential-difference equations. Some results were also obtained for systems with variable time delay.

Khatri (31) used a Laplace transform approach to obtain the optimal control as a function of time for a quadratic performance index. The control is not in a feedback form. As one of the reviewers pointed out, the problem has a solution for a feedback control, (60).

Budelis (13) analytically solved a specific example in which the system contains both an undelayed control variable and a delayed control variable. The results are valid if the final time is less than twice the time delay. This problem occurred in an economic situation.

Banks (7) has obtained a rigorous maximum principle for systems described by functional differential equations.

Day (16) obtained a feedback control results for a linear system with an infinite final time quadratic performance index by discretizing the problem.

Koepche (33) used dynamic programming to solve the problem of discrete-time optimization with delay in the control.

Systems with Variable Time Delay

Various situations where the time delay may not be a constant, but a function of time, result when the system to be controlled is moving relative to the controller or when the medium through which a signal is propagated changes properties as a function of time. Examples of systems with variable time delay include ground based attitude control of spacecraft, guidance of spacecraft, and any process control plant where the medium through which the signal is propagated changes properties.

Sebesta (60), (61) has derived necessary conditions for a nonlinear system with time varying delay in the state variable and in the control variable. A lemma proved in (10) was utilized as the basis for the work. The performance index was of the Bolza type, and terminal constraints were included in the problem. An exact feedback law was obtained for a linear system with a delay in the control, and an approximate feedback law was obtained for systems with a small time delay. Sebesta (63) applied these results to a spacecraft guidance problem. In (62) the results were extended to systems with state and control variable inequality constraints. Budelis (13) has also obtained necessary conditions for the control variable inequality constraint problem. The results are the same as that of Sebesta and not as general; i.e., state variable inequality constraints were not considered.

Banks (6) has obtained a rigorous maximum principle for systems with time varying delays. The work included transversality conditions for variable initial function.

Systems with State-Dependent Time Delay

Examples of systems where the time delay is a function of the state include any control system of conveyor process where measurement devices may be located several feet from a process. The feedback from the measurement devices may be used to control the conveyor velocity and the process (see example in Chapter V). Since the time delay of the system is proportional to the velocity, and velocity is one of the state variables, the time delay may be said to be state-dependent. Also if the measurement device is moved as a function of the state or if there was a digital computer in the loop that purposely was programmed to delay the feedback information an amount dependent on the current value of the state, then the problem would be of this class. In a deep-space guidance problem with an Earth-based ground control the time delay of signals propagating to and from the spacecraft is a function of the position of the spacecraft measured from an Earth-based coordinate system. If the position vector components are treated as state variables, then the problem is that of a state-dependent time delay.

Ragg (54) had attempted to derive necessary conditions for a state-dependent time dealy. However, the necessary conditions published were incorrect as pointed out in (4) and are correct only for a constant time delay.

Schweizer (59) has obtained necessary conditions for a state-dependent time delay problem without constraints. Gabasov (21) has also considered this problem by proving a maximum principle. The work in this thesis was accomplished independently of the two previous papers, and also represents an extension since both state and control variable inequality constraints were included. The necessary conditions obtained in this report agree with the unconstrained problem in the previously cited work. The results also reduce to that of Sebesta (60) for the case of the time varying delay.

Computational Algorithms

Little work has been done to numerically solve the time delay problem. Mueller (47), Kurzweil (38), and Koepke (33) have investigated computational methods for special types of systems with time delay. Mueller obtained an algorithm for a linear system with a time delay in the state. Kurzweil and Koepke studied the time optimal problem for linear time invariant delay systems.

Sebesta (60) developed a gradient algorithm for nonlinear systems with time-varying delays in both the state and control variables.

MacKinnon (42) developed an algorithm for systems with a time delay in the state. Some unpublished reports have extended MacKinnon's results to various other forms of the time delay problem; however, the validity of these algorithms has not been verified.

The next chapter contains the derivation of the necessary conditions for the state-dependent time delay problem. The results are utilized to obtain the optimal control for a system that is linear in the explicit state and control variables, but does not necessarily have a time delay, h, that is linear in the state.

CHAPTER III

OPTIMAL CONTROL OF SYSTEMS WITH TIME DELAY

In this chapter necessary conditions for optimal control of systems containing a time delay that is a function of the state of the system and of time are derived by utilizing calculus of variations. The time delay may be in the state vector and in the control vector. The state vector and the control vector can be constrained by inequality constraints. A transformation to eliminate state variable inequality constraints by increasing the dimensions of state space, developed by Jacobson (28) for an undelayed system, is extended to a system with time delays.

The necessary conditions are utilized to obtain the optimal control for a system that is linear in the explicit variables, but the function for the time delay is not necessarily linear in the state variables. A numerical algorithm is proposed to solve this problem by use of the resulting necessary conditions.

In the next section, the mathematical problem is formulated.

Statement of the Problem

The system to be considered is modeled by a set of non-linear differential-difference equations

$$\dot{x}(t) = f[x(t),\ x_h, u(t), u_h, t] \tag{1}$$

with the definitions

$$x_h = x[t - h(x(t),t)]$$

$$u_h = u[t - h(x(t),t)]$$

and where $x(t)$ is a continuous n-vector denoting the leading terminus of the state function at time t.

$u(t)$ is a m-vector denoting the control variables at time t and is piecewise continuous and has at most a finite number of discontinuities.

$x(t-h)$ is a n-vector denoting the trailing terminus of the state function and is x at time $t - h(x(t),t)$.

$u(t-h)$ is a m-vector denoting the delayed control variables at time $t - h(x(t),t)$.

$h(x(t),t)$ is a scalar piecewise differentiable function which must satisfy the following conditions:

$$h[x(t),t] \geq 0 \tag{2}$$

$$\frac{dh[x(t),t]}{dt} \neq 1. \tag{3}$$

Equation (2) is evident from physical considerations in that the system is nonanticipatory. Equation (3) is required in order for the equations formulating the necessary conditions to be non-singular.

Since a differential-difference equation is infinite-dimensional, the initial condition must be an initial function. The initial condition function

$$x(\tau) = \sigma(\tau), \ \min[t - h(x(t),t)] \leq \tau \leq t_0 \tag{4}$$

is assumed given.

The problem is to find the set of controls, $u(t)$, in order that the state of the system intersects a terminal surface

$$\psi[x(t_f),\ t_f] = 0 \tag{5}$$

where ψ is a q-vector, $q \leq n$, and such that the performance index

$$J = G[x(t_f),t_f] + \int_{t_0}^{t_f} Q[x(t),x_h,u(t),u_h,t]\,dt \tag{6}$$

is minimized.

The control trajectory and the state trajectory are assumed constrained by the following inequality constraints:

$$S[x(t),t] \leq 0\ , \tag{7}$$

where S is a r-vector of state variable inequality constraints, and

$$C[x(t),x_h,u(t),u_h,t] \leq 0 \tag{8}$$

where C is a k-vector of control variable inequality constraints.

Equation (2) may be put into the form of a state variable inequality constraint as in (7) by multiplying both sides of (3) by a minus one. This inequality constraint will be assumed to be an element of (7).

Since the control is contained explicitly in (8), the control $u(t)$ may be found such as to not violate this constraint. However, when the control is not contained explicitly in the constraint as in (7), additional consideration must be given to the determination of the control such that the constraint is not violated. The next section discusses a procedure to eliminate both types of inequality constraints.

Inequality Constraints

Control variable inequality constraints may be reduced to equality constraints by use of a real variable known as Valentine's device or as a slack variable. The slack variable vector Z_c is defined by the following equation:

$$C(x,u,t) + Z_c^2 = 0 \tag{9}$$

where $Z_c^2 = [Z_1^2\ Z_2^2\ \dots\ Z_k^2]_c^T$ and is constrained to be a real variable, and $C(x,u,t)$ is the k-vector of inequality constraints (the subscript, c, denotes the slack variable, Z, associated with the control variable inequality constraints. The equality constraint may now be adjoined to the performance index.

The state variable inequality constraints can be eliminated by a transformation that increases the dimensions of state space. This was developed by Jacobson for an undelayed system. This transformation will now be developed.

The q^{th} order inequality constraint is defined as the lowest time derivative of $S[x(t),t]$ that contains either u or u_h. That is, the lowest order of $\frac{d^q S}{dt^q}$ such that

$$\frac{\partial}{\partial u_h}\left(\frac{d^q S}{dt^q}\right) + \frac{\partial}{\partial u}\left(\frac{d^q S}{dt^q}\right) = 0. \tag{10}$$

Thus, the q^{th}-order inequality constraint may contain either u_h or u. It is assumed that the state variable inequality constraints (7) are first-order inequality constraints in the subsequent analyis.

By using a slack variable, Z_s, the state variable inequality constraint may be written as an equality constraint

$$S(x,t), + \tfrac{1}{2}Z_s^2 = 0 \tag{11}$$

where $Z_s^2 = [Z_1^2 \; Z_2^2 \; \ldots \; Z_r^2]_s^T$. The superscript, T, denotes the transpose, and the subscript, s, denotes the slack variable, Z, associated with the state variable inequality constraints.

Each element S_i of the vector S may now be differentiated.

$$\dot{S}_i + Z_{i_s} \dot{Z}_{i_s} = 0, \; i = 1, 2, \ldots, r. \tag{12}$$

Since S is a first-order inequality constraint, it will contain either u(t) or u_h explicitly.

The vector v may be defined as $v = [Z_1 \; Z_2 \; \ldots \; Z_r]^T$ and state space increased by defining a new state vector

$$X = \begin{bmatrix} x \\ \cdots \\ v \end{bmatrix}.$$

The new state vector, X, is a n + r vector. A vector, w, a psuedo control vector to be determined, is defined as $w = [\dot{Z}_{1_s} \; \dot{Z}_{1_s} \; \ldots \; \dot{Z}_{r_s}]^T$. The additional state equations may be written as

$$\dot{v}(t) = w(t) \tag{13}$$

where the initial conditions for v may be found from

$$S_i[x(t_0), t_0] + v_i^2(t_0) = 0, \; i = 1, 2, \ldots, r.$$

Equation (12) may now be written in terms of the elements of v and w; that is,

$$\dot{S}_i + v_i w_i = 0, \; i = 1, 2, \ldots, r. \tag{14}$$

If one assumes that all r equations in (14) are independent, r elements of u or u_h may be eliminated in terms of the remaining elements of u and u_h, the new state vector, X(t), and the psuedo control vector, w(t). It may be advantageous to eliminate u_h. However, if any elements of u or u_h appear linearly, then that element which appears linearly may be the variable solved for and then utilized to eliminate that variable from (1), (6), and (8).

At this point in the development, the control variable inequality constraint has now been transformed into an equality constraint by use of a slack variable. Also, the state variable inequality constraints have been eliminated by increasing the dimensions of the state space and by eliminating r elements of u or u_h in terms of the remaining control variables, the new state vector, and the pseudo control vector.

In the subsequent analysis it will be assumed that the state variable inequality constraints have been eliminated by the preceding technique. Thus, the state vector x(t) is assumed to be the augmented state vector, and the control vector, u(t), is taken to be the control vector containing the remaining elements of u(t) and the elements of the pseudo control vector w(t).

Development of Necessary Conditions

In this section necessary conditions for an optimal control are derived by use of calculus of variations. The necessary conditions are a basis for synthesis by indirect methods of optimal control problems for systems with time delay. The development allows discontinuities to appear in the variables. Corner conditions for points of discontinuity are derived, and it is shown that the Lagrange multipliers are continuous. This would not be the case if the transformation to eliminate the state variable inequality constraints was not utilized (39). This is important since this eliminates the problem of finding the locations of the points of discontinuity and of discontinuous Lagrange multipliers. Also, the final time is allowed to be free.

The nomenclature used is the same as the preceding sections, except that the state variable inequality constraints are assumed eliminated by use of the transformation.

The problem is to determine necessary conditions in order to determine u(t), for all $t \in [\min(t-h), t_f]$ such that the performance index, (6) is minimized while satisfying all constraints.

For simplicity of nomenclature and for comparison with Pontryagin's maximum principle a scalar Hamiltonian may be defined as

$$H(x, x_h, u, u_h, \lambda, \varphi, t) = Q + \lambda^T f + \varphi^T C \tag{15}$$

where λ is a n-vector of time-varying Lagrange multipliers and φ is a k-vector of time-varying Lagrange multipliers.

The augmented performance index may be written as

$$J = G + \nu^T \psi + \sum_{j=1}^{N} \int_{t_{j-1}^{+}}^{t_j^{-}} (H - \lambda^T \dot{x} + \varphi^T Z^2)\,dt + \int_{\min[t - h(\mathbf{x}(t),t)]}^{t_0^{-}} \varphi^T (C + Z^2)\,dt \tag{16}$$

where the t_j's, j = 1, 2, ..., N, represent points of discontinuity of u, $\dot{x}$, $\dot{h}$, $\dot{x}_h$, and $\dot{u}_h$, and t_N^- represents the final time t_f. The slack variable Z is taken to be the variable Z_C of the last section, and ψ is the terminal surface of Equation (5). Also, ν is a q-vector of constant Lagrange multipliers. The last integral of (16) is due to the requirement that the control variable inequality constraints must also be satisfied for $t \in [\min(t-h), t_0]$.

If existence of an optimal control is assumed, then the necessary conditions for a minimum is that the first variation of the performance index vanish (24); i.e.,

$$\delta J = 0 \tag{17}$$

and the second variation must be non-negative; i.e.,

$$\delta^2 J \geq 0. \tag{18}$$

The first variation of (16) may be written as

$$\delta J = \left[\left(\frac{\partial G^T}{\partial x} + \nu^T \frac{\partial \psi}{\partial x}\right) dx + \left(\frac{\partial G}{\partial t} + \nu^T \frac{\partial \psi}{\partial t}\right) dt + \psi d\nu\right]_{t=t_f} + \delta \sum_{j=1}^{N} \int_{t_{j-1}^+}^{t_j^-} (H - \lambda^T \dot{x} + \varphi^T Z^2)\, dt . \tag{19}$$

The following term of (19) may be separated for simplicity and expanded by use of Leibnitz's rule; i.e.,

$$\delta \int_{t_j^+}^{t_j^-} (H - \lambda^T \dot{x} + \varphi^T Z^2)\, dt = [H - \lambda^T \dot{x} + \varphi^T Z^2]_{t = t_{j-1}^+}^{t = t_j^-} dt + \int_{t_{j-1}^+}^{t_j^-} \delta (H - \lambda^T \dot{x} + \varphi^T Z^2)\, dt . \tag{20}$$

If the following variables are defined as

$$\tau(x,t) = t - h(x,t)$$

$$a(\tau) = x(\tau) = x_h$$

$$b(\tau) = u(\tau) = u_h ,$$

then the last term in (20) may be written as

$$\int_{t_{j-1}^{+}}^{t_j^{-}} (H - \lambda^T \dot{x} + \varphi^T Z^2)\, dt = \int_{t_{j-1}^{+}}^{t_j^{-}} \frac{\partial H^T}{\partial x} \delta x + \frac{\partial H^T}{\partial u} \delta u + \frac{\partial H^T}{\partial a(\tau)} \delta a(\tau)$$

$$+ \frac{\partial H^T}{\partial b(\tau)} \delta b(\tau) + \frac{\partial H^T}{\partial \lambda} \delta\lambda + \frac{\partial H^T}{\partial \varphi} \delta\varphi - \lambda^T \delta\dot{x} - \dot{x}^T \delta\lambda + \delta\varphi^T Z^2 + \varphi^T \delta(Z^2))\, dt. \tag{21}$$

The reason for the change in nomenclature is due to possible confusion as to exactly what is the total variation in x_h. It is assumed that t is held constant under the integral. Thus, a variation in $a(\tau)$ is due to a variation in a holding τ constant and, also, due to a variation in τ. That is,

$$\delta a = \delta x_h + \frac{dx_h}{d\tau} d\tau . \tag{22}$$

Similarly,

$$\delta b = \delta u_h + \frac{du_h}{d\tau} d\tau . \tag{23}$$

A graphical picture of Equation (22) can be seen in Figure 3. Also,

$$d\tau = \frac{d\tau^T}{dx} \delta x = - \frac{\partial h^T}{\partial x} \delta x ;$$

therefore, Equations (22) and (23) may be written as

$$\delta a(\tau) = \delta x_h - \frac{dx_h}{d\tau} \frac{\partial h^T}{\partial x} \delta x$$

$$\delta b(\tau) = \delta u_h - \frac{du_h}{d\tau} \frac{\partial h^T}{\partial x} \delta x . \tag{24}$$

Since the final time is free,

$$dx = \delta x + \dot{x} dt .$$

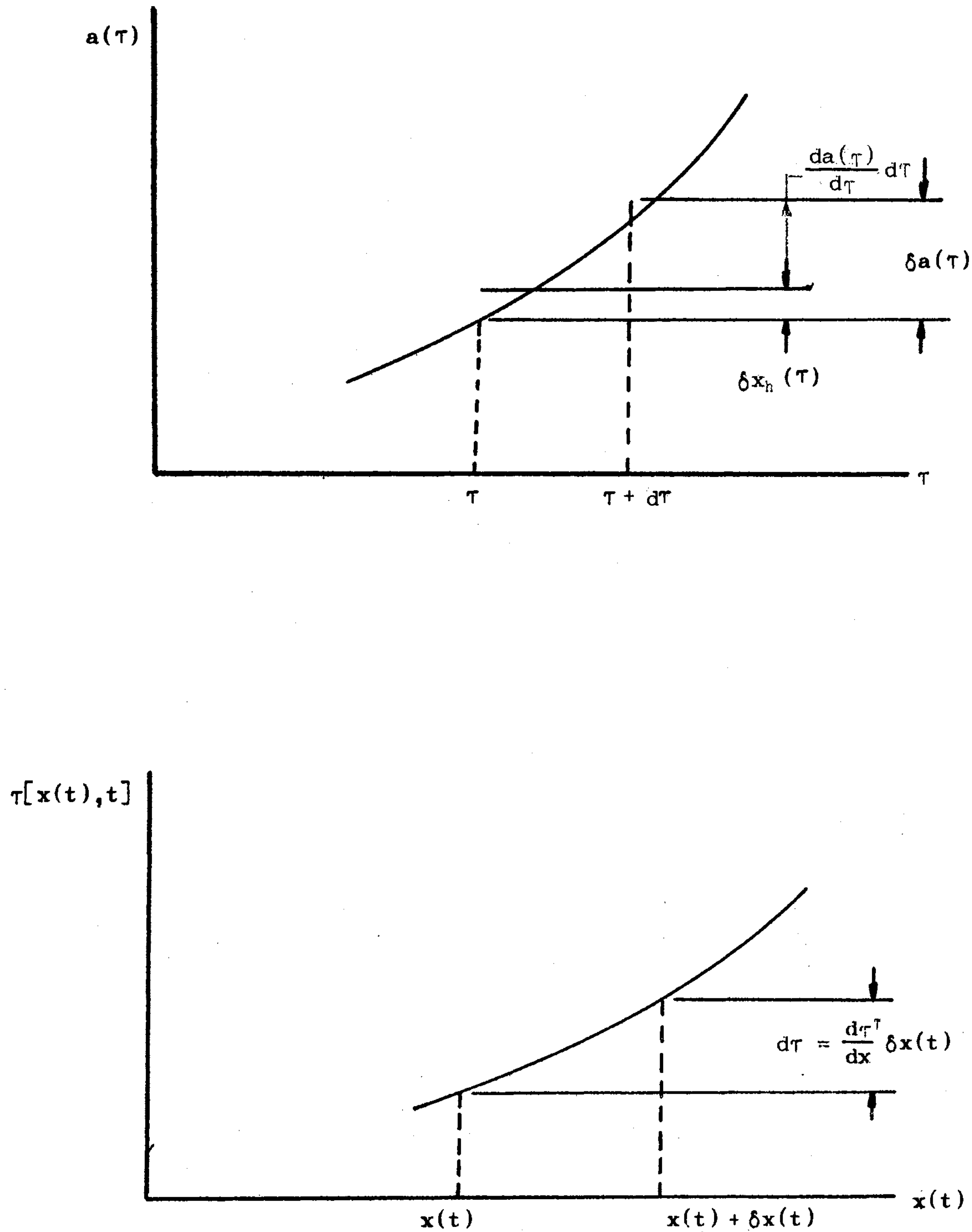

Figure 3. Variations in the Delayed Variables

The term in (21) containing the derivative of the variation of the state may be written as

$$\int_{t_{j-1}^{+}}^{t_j^{-}} \lambda^T \delta\dot{x}\, dt = \lambda^T (dx - \dot{x}\, dt) \Big|_{t = t_{j-1}^{+}}^{t = t_j^{-}} - \int_{t_{j-1}^{+}}^{t_j^{-}} \dot{\lambda}^T \delta x\, dt \,. \tag{25}$$

Also,

$$\delta \sum_{i=1}^{k} Z_i^2 = \sum_{i=1}^{k} 2Z_i \; \delta Z_i \,.$$

Therefore, the first variation may be written as

$$\delta J = \left[\left(\frac{\partial G^T}{\partial x} + \nu^T \frac{\partial \psi}{\partial x} - \lambda^T\right) dx + \left(\frac{\partial G}{\partial t} + \nu^T \frac{\partial \psi}{\partial t} + H + \varphi^T Z^2\right) dt + \psi^T d\nu\right]_{t=t_f}$$

$$+ \sum_{i=1}^{N-1} \left[(H + \varphi^T Z^2)\, dt + \lambda^T dx\right]_{t=t_j^{+}}^{t=t_j^{-}} + \sum_{i=1}^{N} \int_{t_{j-1}^{+}}^{t_j^{-}} \left[\left(\frac{\partial H^T}{\partial x} - \frac{\partial H^T}{\partial x_h} \frac{dx_h}{d\tau} \frac{\partial h^T}{\partial x}\right.\right.$$

$$\left. - \frac{\partial H^T}{\partial u_h} \frac{du_h}{d\tau} \frac{\partial h^T}{\partial x} + \dot{\lambda}^T\right) \delta x + \frac{\partial H^T}{\partial x_h} \delta x_h + \frac{\partial H^T}{\partial u} \delta u + \frac{\partial H^T}{\partial u_h} \delta u_h + \delta x^T\left(\frac{\partial H}{\partial \lambda} - \dot{x}\right)$$

$$\left. + \delta\varphi^T\left(Z^2 + \frac{\partial H}{\partial \varphi}\right) + \sum_{j=1}^{k} 2\varphi_i Z_i \delta Z_i\right] dt \,. \tag{26}$$

The variations x and x_h, and u and u_h are not independent. Consequently, a transformation must be accomplished to eliminate x and u at more than one value of the independent variable. Let $t = s - h(x,s)$ and the inverse of this equation be $s = g(t)$ and utilize this transformation on the delayed parts of the integral (see reference (60)).

$$\int_{t_{j-1}^+}^{t_j^-} \frac{\partial H^T}{\partial x_h}\, \delta x_h\, dt = \int_{t_{j-1}^+ - h(x(t_{j-1}^+),t_{j-1}^+)}^{t_j^- - h(x(t_j^-),t_j^-)} \frac{1}{1-\dot{h}(s)} \frac{\partial H^T}{\partial x_h}(s)\, \delta x\, dt \tag{27}$$

and

$$\int_{t_{j-1}^+}^{t_j^-} \frac{\partial H^T}{\partial u_h}\, \delta u_h\, dt = \int_{t_{j-1}^+ - h(x(t_{j-1}^+),t_{j-1}^+}^{t_j^- - h(x(t_j^-),t_j^-)} \frac{1}{1-\dot{h}(s)} \frac{\partial H^T}{\partial u_h}(s)\, \delta u\, dt \tag{28}$$

where

$$\dot{h}(t) = \frac{\partial h^T}{\partial x}(t)\, f(t) + \frac{\partial h}{\partial t}(t)\,. \tag{29}$$

The use of (26) and (27) in (25) along with the transformation

$$\frac{d}{d\tau} = \frac{1}{1-\dot{h}} \frac{d}{dt}\,, \tag{30}$$

allows the necessary conditions to be written as follows:

Transversality conditions at $t = t_f$ are

$$\left(\lambda^T - \frac{\partial G^T}{\partial x} - \nu^T \frac{\partial \psi}{\partial x}\right) \mathbf{dx} = 0 \tag{31}$$

$$\left(\frac{\partial G}{\partial t} + \nu^T \frac{\partial \psi}{\partial t} + H + \varphi^T Z^2\right) dt = 0 \tag{32}$$

$$\psi^T d\nu = 0\,. \tag{33}$$

Corner conditions at $t = t_j$, $j = 1, 2, \ldots, N-1$ are

$$[\lambda^T(t_j^-) - \lambda^T(t_j^+)]\, \mathbf{dx} = 0 \tag{34}$$

$$[(H + \varphi^T Z^2)_{t=t_j^-} - (H + \varphi^T Z^2)_{t=t_j^+}]\, dt = 0\,. \tag{35}$$

Euler-Lagrange equations are:

For $\forall\ t \in [\min(t - h(x(t),t)), t_0]$

$$\frac{1}{1-\dot{h}(s)}\frac{\partial H^T}{\partial u_h}(s)+\varphi^T\frac{\partial C}{\partial u}=0 \qquad (36)$$

$$C+Z^2=0 \qquad (37)$$

$$Z_i\varphi_i=0,\ i=1,2,\dots,k. \qquad (38)$$

For $\forall t\in[t^+_{j-1},t^-_j]$, $j=1,2,\dots,N-1$, where $t_{N-1}=t_f-h(x(t_f),t_f)$

$$\dot{x}=f(x,u,x_h,u_h,t) \qquad (39)$$

$$Z^2+C=0 \qquad (40)$$

$$Z_i\varphi_i=0,\ i=1,2,\dots,k \qquad (41)$$

$$\dot{\lambda}^T=-\left[\frac{\partial H^T}{\partial x}-\frac{1}{1-\dot{h}}\left(\frac{\partial H^T}{\partial x_h}\frac{dx_h}{dt}\frac{\partial h^T}{\partial x}+\frac{\partial H^T}{\partial u_h}\frac{du_h}{dt}\frac{\partial h^T}{dx}\right)\right]-$$

$$\frac{1}{1-\dot{h}(s)}\frac{\partial H^T}{\partial x_h}(s) \qquad (42)$$

$$\frac{\partial H}{\partial u}+\frac{1}{1-\dot{h}(s)}\frac{\partial H}{\partial u_h}(s)=0. \qquad (43)$$

For $\forall t\in[t^+_{N-1},t_f]$

$$\dot{x}=f(x,u,x_h,u_h,t) \qquad (44)$$

$$Z^2+C=0 \qquad (45)$$

$$Z_i\varphi_i=0,\ i=1,2,\dots,k \qquad (46)$$

$$\dot{\lambda}^T=-\left[\frac{\partial H^T}{\partial x}-\frac{1}{1-\dot{h}}\left(\frac{\partial H^T}{\partial x_h}\frac{dx_h}{dt}\frac{\partial h^T}{\partial x}+\frac{\partial H^T}{\partial u_h}\frac{du_h}{dt}\frac{\partial h^T}{\partial x}\right)\right] \qquad (47)$$

$$\frac{\partial H}{\partial u}=0. \qquad (48)$$

It is assumed that no discontinuities occur in the intervals

$\min[t - h(x(t),t)] \leq t \leq t_0$ and $t_f - h(x(t_f),t_f) \leq t \leq t_f$.

The difference in the necessary conditions developed by Sebesta (60) and the necessary conditions developed in this work is the addition of terms in the equations for the Lagrange multipliers that are dependent on the time derivative of the delayed state and the delayed control. When the time delay is not a function of state, the necessary conditions reduce to the conditions developed by Sebesta.

The Quadratic Criteria Problem With State-Dependent Time Delay

One of the more important problems in optimal control is that of minimizing a performance index that is quadratic in the state and in the control. The problem is that of obtaining a state near zero at the final time with minimum control energy expenditure. The state might represent an error variable to be reduced.

The system to be controlled is

$$\dot{x}(t) = A(t)\ x(t) + B(t)\ x_h + C(t)\ u(t) \tag{49}$$

with the initial functions

$$\left.\begin{aligned} x(\tau) &= \sigma(\tau), \\ u(\tau) &= 0 \end{aligned}\right\} \forall\ \tau \leq t_0. \tag{50}$$

The problem is to find $u(t)$, $\forall\, t \in [t_0, t_f]$ such that the following performance index is minimized.

$$J = \tfrac{1}{2} x^T(t_f)\, S x(t_f) + \tfrac{1}{2}\int_{t_0}^{t_f} [x^T(t)\, Q(t)\, x(t) + u^T(t)\, R(t)\, u(t)]\, dt \tag{51}$$

where

Q is a $n \times n$ positive semi-definite matrix

S is a n X n positive semi-definite matrix

R is a m X m positive definite matrix

and $h = h(x,t)$ is a scalar function, assumed to be positive semi-definite in order to avoid problem-oriented inequality constraints. (The time delay h must be constrained to be greater than or equal to zero if it is not a positive semi-definite function. However, the form of h is problem-oriented.)

The necessary conditions previously developed may be utilized to obtain the required equations,

$$\dot{x} = Ax + B\,x_h + u, \quad t \,\epsilon [t_0, t_f]$$

$$\dot{\lambda} = \begin{cases} -Qx - A^T\lambda + \frac{1}{1-\dot{h}} \frac{\partial h}{\partial x} \frac{dx_h}{dt} B^T\lambda - \\ \frac{1}{1-\dot{h}(s)} B^T(s)\lambda(s), \ \forall\, t\, \epsilon [t_0, t_f - h(x(t_f), t_f] \\ -Qx - A^T\lambda + \frac{1}{1-\dot{h}} \frac{\partial h}{\partial x} \frac{dx_h^T}{dt} B^T\lambda, \ \forall\, t\, \epsilon [t_f - h(x)t_f), \\ t_f), t_f] \\ \quad u = -R^{-1} C^T \lambda, \forall\, t\, \epsilon [t_0, t_f]. \end{cases} \tag{52}$$

Boundary conditions are

$$x(\tau) = \sigma(\tau), \forall\, t\, \epsilon\, [\min(t-h), t_0] \tag{53}$$

$$\lambda(t_f) = Sx(t_f).$$

An algorithm that may be used to find the control is as follows:

1. An initial guess of the optimal control may be taken.
2. With the guessed control, the state equations may be

integrated forward in time. The state trajectory is stored.

3. The equations for the Lagrange multipliers may be integrated backward in time.

4. The control may be modified at each instant of time by the following equation where u_{old} represents the current value of the control:

$$u_{new} = u_{old} - \epsilon \frac{\partial H}{\partial u} .$$

The quantity ϵ is an arbitrary positive number to be chosen, and

$$\frac{\partial H}{\partial u} = Ru + C^T \lambda .$$

The equation

$$t = s - h(x(s), s)$$

may be inverted at each value of t by a Newton-Raphson technique. A variable P may be defined as

$$P = t - s + h(x(s)\ s)$$

and

$$\frac{dP}{ds} = -1 + \frac{dh}{ds} .$$

The iterative formula is

$$s_{i+1} = s_i - \frac{P_i}{\frac{dP_i}{ds}} .$$

An important point to note for this problem is that an optimal feedback controller cannot be deduced from the necessary conditions established here. This fact should be explored further and a suboptimal feedback controller developed.

CHAPTER IV

COMPUTATIONAL ALGORITHM

A gradient procedure for nonlinear systems with state-dependent time delay is outlined in this chapter. The gradient procedure is a direct optimization technique whereby the problem is to be solved by directly iterating to the optimal control. This is in contrast with using the necessary conditions in a numerical algorithm to solve the problem.

The gradient technique has rapid convergence at the start of the optimization process, but it tends to oscillate about the optimal solution. Consequently, convergence properties are poor. However, it may give a control that is close to the optimal.

The gradient procedure outlined is similar in form to Bryson (11) and is an extension of Sebesta's gradient procedure for nonlinear systems with time-varying time delay.

Development of the Algorithm

The system equations as given in Chapter III

$$\dot{x}(t) = f[x(t),\ x_h,\ u(t),\ u_h,\ t] \tag{1}$$

where the initial condition functions

$$x(t) = \sigma(t),\ \forall\, t \in [\min(t-h), t_0] \tag{2}$$

$$u(t) = \alpha(t), \ \forall t \in [\min(t-h), t_0]$$

are assumed given.

The problem is to determine $u(t)$, $\forall t \in [t_0, t_f]$, such that the performance index

$$J = G[x(t_f), t_f] \tag{3}$$

is minimized and such that the following constraints are satisfied:

$$\psi[x(t_f), t_f] = 0 \tag{4}$$

and

$$\Omega[x(t_f), t_f] = 0 \tag{5}$$

where

ψ is a $(q-1)$ vector of terminal constraints

Ω is a scalar stopping condition, monotonic over (t_0, t_f).

A linearized trajectory may be obtained by expanding the state equations in a Taylor series about the nominal trajectory with the following definitions:

$$a = x[t - h(x(t),\ t)]$$

$$b = u[t - h(x(t),\ t)]\ .$$

The linearized equations are

$$\delta\dot{x} = \frac{\partial f}{\partial x}\,\delta x + \frac{\partial f}{\partial a}\,\delta a + \frac{\partial f}{\partial u}\,\delta u + \frac{\partial f}{\partial b}\,\delta b \tag{6}$$

where the coefficient matrices are Jacobian matrices of the proper dimensions and are evaluated along the nominal trajectory.

As was shown earlier in this thesis, the variations δa and δb are

$$\delta a = \delta x_h - \frac{dx_h}{d\tau}\frac{\delta h^T}{\delta x}\delta x$$

$$\delta b = \delta u_h - \frac{du_h}{d\tau}\frac{\partial h^T}{\partial x}\delta x .$$

Therefore, the linearized state equations may be written as

$$\delta \dot{x} = A_0 \delta x + A_1 \delta x_h + B_0 \delta u + B_1 \delta u_h \tag{7}$$

where

$$A_0 = \left[\frac{\partial f}{\partial x} - \frac{1}{1-\dot{h}}\left(\frac{\partial f}{\partial x_h}\frac{dx_h}{dt}\frac{\partial h^T}{\partial x} + \frac{\partial f}{\partial u_h}\frac{du_h}{dt}\frac{\partial h^T}{\partial x}\right)\right]$$

$$A_1 = \frac{\partial f}{\partial x_h}$$

$$B_0 = \frac{\partial f}{\partial u}$$

$$B_1 = \frac{\partial f}{\partial u_h} .$$

The adjoint equations for (7) are

$$\dot{\lambda} = -A_0^T \lambda - \frac{1}{1-\dot{h}(s)} A_1^T (s)\lambda(s) \tag{8}$$

where s is the inverse of

$$t = s - h[x(s), s] .$$

With the preceding modifications implemented, the remaining development of the computational algorithm follows that of Sebesta (60).

Outline of the Algorithm

This section gives the steps to be followed to apply the algorithm.

1. The state equations are integrated by using the initial

function and a nominal control until the stopping condition is satisfied. The nominal trajectory is stored.

2. The final conditions for the sensitivity functions are set

$$\lambda_{G\Omega}(t_f) = \left(\frac{\partial G}{\partial x} - \frac{\dot{G}}{\dot{\Omega}}\frac{\partial \Omega}{\partial x}\right)_{t=t_f}$$

$$\lambda_{G\Omega}(t) = 0, \; \forall\, t \,\varepsilon [t_f, s_f]$$

where s_f is the inverse of $t_f = s_f - h[x(s_f), s_f]$

$$\lambda_{\psi\Omega}(t_f) = \left(\frac{\partial \psi}{\partial x} - \frac{\dot{\psi}}{\dot{\Omega}}\frac{\partial \Omega}{\partial x}\right)_{t=t_f}$$

$$\lambda_{\psi\Omega}(t) = 0, \; \forall\, t \,\varepsilon [t_f, s_f]$$

$$I_{GG}(t_f) = 0$$

$$I_{\psi G}(t_f) = 0$$

$$I_{\psi\psi}(t_f) = 0 .$$

3. The correction of u must be made small in order to stay within the linear region of the linearized state equations and adjoint systems. Consequently, a control effort constraint must be placed on the correction to u.

$$(dR)^2 = \tfrac{1}{2}\int_{t_0}^{t_f} \delta u^T W \delta u dt ,$$

where W(t) is a symmetric weighting matrix to be chosen, and $(dR)^2$ is the control effort constraint.

4. The following differential-difference equations are

integrated backward in time from t_f to t_o to obtain the required sensitivity functions.

$$\dot{\lambda}_{G\Omega} = -A_o^T \lambda_{G\Omega} - 1/(1-\dot{h}(s))\ A_1^T\ (s)\ \lambda_{G\Omega}(s)$$

$$\dot{\lambda}_{\psi\Omega} = -A_o^T \lambda_{\psi\Omega} - 1/(1-\dot{h}(s))\ A_1^T\ (s)\ \lambda_{\psi\Omega}(s)$$

$$\dot{I}_{GG} = \Phi_{G\Omega} W^{-1}\ \Phi^T_{G\Omega}$$

$$\dot{I}_{\psi G} = \Phi_{\psi\Omega} W^{-1} \bar{\Phi}^T_{G\Omega}$$

$$\dot{I}_{\psi\psi} = \Phi_{\psi\Omega} W^{-1} \bar{\Phi}^T_{\psi\Omega}$$

where

$$\Phi_{G\Omega} = \lambda^T_{G\Omega}\ B_o + 1/1(1-\dot{h}(s))\ \lambda^T_{G\Omega}\ (s)\ B_1(s)$$

$$\Phi_{\psi\Omega} = \lambda^T_{\psi\Omega}\ B_o + 1/1(1-\dot{h}(s))\ \lambda^T_{\psi\Omega}\ (s)\ B_1(s)\ .$$

The functions $\Phi_{\psi\Omega}(t)$ and $\Phi_{G\Omega}(t)$ and the resulting integrals $I_{GG}(t_o)$, $I_{\psi G}(t_o)$, and $I_{\psi\psi}(t_o)$ are stored.

5. A reasonable terminal condition change is selected, if required, to bring the next iteration close to the desired end conditions specified by the constraints $\psi = 0$.
6. A reasonable control effort constraint, $(dR)^2$, is specified and the predicted change in the performance index is computed from

$$dG = \pm\sqrt{\left((dR)^2 - d\psi^T\ I^{-1}_{\psi\psi}\ d\psi\right)\left(I_{GG} - I^T_{\psi G}\ I^{-1}_{\psi\psi}\ I_{\psi G}\right)} + I^T_{\psi G}\ \bar{I}^{-1}_{\psi\psi}\ d\psi .$$

If dG and $d\psi$ are sufficiently small, the procedure has converged. If the quantity under the radical sign is negative, then either decrease $d\psi$ or increase $(dR)^2$. The + sign is used to maximize G, and the - sign is used to minimize G.

7. If the procedure has not converged, then obtain a new nominal control by letting $u(t)_{new} = u(t)_{old} + \delta u(t), \ \forall t \in [t_o, t_f]$ where

$$\delta u(t) = \pm \sqrt{\frac{(dR)^2 - d\psi^T I_{\psi\psi}^{-1} d\psi}{I_{GG} - I_{\psi G}^T I_{\psi\psi}^{-1} I_{\psi G}}} \; W^{-1}\left[\Phi_{G\Omega}^T - \Phi_{\psi\Omega}^T I_{\psi\psi}^{-1} I_{\psi G}\right] + W^{-1}\Phi_{G\Omega}^T I_{\psi\psi}^{-1} d\psi .$$

8. The procedure is repeated until convergence is obtained.

CHAPTER V

APPLICATIONS

This chapter contains examples of optimization of systems with time delay. The first example is not an optimal control problem but that of a parameter optimation problem where the parameter to be optimized is the time delay. This was placed in the thesis because of its interesting nature and because of the possible feasibility of using the necessary conditions derived herein for parameter optimization or identification problems of this type. The second example is that of a regulator for a steel rolling mill. The control problem contains both a state-dependent time delay and a state variable inequality constraint. The methods outlined in Chapter III are used to obtain the equations necessary to find the optimal control for the system.

Optimal Time Delays

Since some systems contain an inherent time delay, it may be advantageous to see if this delay (assumed constant and under a designers influence) may be chosen in an optimal manner. Koivuniemi (34) has shown by use of numerical algorithm for linear time delay systems and two examples that the value of the time delay greatly affects the value of the performance index. However, necessary conditions for optimality of the time delay were not given.

Since the necessary conditions for a system with state-dependent time delays have been derived it becomes a simple matter to obtain the required necessary conditions. The time delay is considered to be a constant. The state vector may be augmented by the following equation:

$$\dot{x}_{n+1} = 0 \tag{1}$$

where the constant time delay is

$$h = x_{n+1}. \tag{2}$$

The problem then becomes a parameter optimization problem where the parameter to be optimized is the constant value of the time delay. Necessary conditions follow directly by application of the results in Chapter III.

The above technique, for instance, may be used to obtain the optimal location for a sensor in a feedback control system in which the sensor location determines the delay magnitude, or the technique might be used to identify the time delay parameter in a human operator model.

The following example will illustrate the use of the necessary conditions by using the necessary conditions to find an optimal time constant and an optimal time delay such that the system considered follows an ideal desired response.

The example problem worked is trivial and seems to have a singular characteristic. However, it illustrates that the necessary conditions might be utilized to work a more realistic parameter optimization problem or identification problem where one of the parameters is a time delay. This idea could prove useful in identification of the magnitude of time delays in large processing plants from measured response data if the computational problem could be solved.

The block diagram of the system to be considered is as follows:

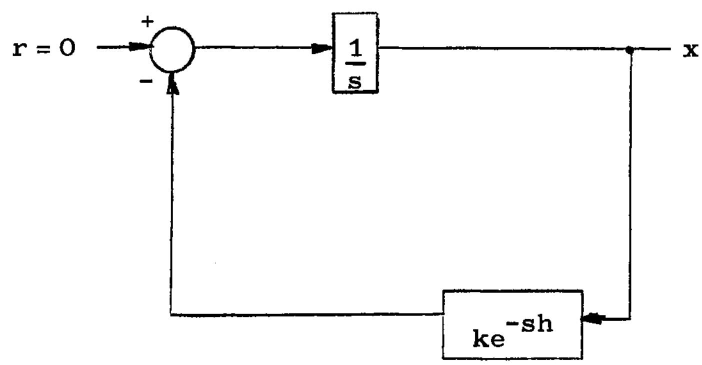

Figure 4. Block Diagram of Example 1

The system equation may be written as

$$\dot{x} = -k\ x(t-h).$$

The system is constrained to be nonanticipatory. Also, hardware implementation of the time delay element will be deferred at this time. However, an approximation to this element will be given and the system hardware implemented.

The problem is to determine the time constant, 1/k and the time delay, h, such that the system follows some desired response over a given period of time.

The performance index

$$J = \int_0^1 (x - x_d)^2 \, dt \qquad (3)$$

where x is the desired response and is shown in Figure 6, and the final time is taken to be one second.

The equations are transformed into a problem with state-dependent time delay by augmenting the state equations as follows:

$$\dot{x}_1 = -x_2 x_1 (t - x_3)$$

$$\dot{x}_2 = 0 \qquad (4)$$

$$\dot{x}_3 = 0$$

where x_2 denotes k and x_3 denotes h.

Since the system is constrained to be nonanticipatory, the admissible region of state space for a solution excludes the region where x_3 is negative. A constraint must be added to the necessary conditions; i.e.,

$$z^2 - x_3 = 0 \qquad (5)$$

where z is a real variable. Unlike the optimal control problem the state may be constrained in this manner. Thus, if the boundary is reached, then z must be zero. When not in the boundary, z must be a real variable. This excludes the possibility that x_3 is negative.

Therefore, the necessary conditions for a minimum of the performance index

$$J = \int_0^1 [x_1 - (1-t)]^2 \, dt \tag{6}$$

may be written as

$$\left.\begin{aligned} \dot{x}_1 &= -x_2 x_1 (t - x_3) \\ \dot{x}_2 &= 0 \\ \dot{x}_3 &= 0 \\ z^2 - x_3 &= 0 \\ z\varphi &= 0 \end{aligned}\right\} \forall\, t \in [0,\ 1] \tag{7}$$

$$\begin{aligned} \dot{\lambda}_1 &= \begin{cases} -2x_1 + 2(1-t) + \lambda_1 (t + x_3)\ x_2 (t + x_3), & \forall\, t \in [0,\ 1 - x_3] \\ -2x_1 + 2(1-t), & \forall\, t \in [1 - x_3,\ 1] \end{cases} \\ \dot{\lambda}_2 &= \lambda_1 x_{1_h}, \ \forall\, t \in [0,1] \\ \dot{\lambda}_3 &= \varphi - \lambda_1 x_2 \dot{x}_{1_h}, \ \forall\, t \in [0,\ 1] \end{aligned} \tag{8}$$

with boundary conditions

$$\begin{aligned} x_1(t) &= 1, \forall\, t \in [-x_3,\ 0] \\ \lambda_1(1) &= 0 \\ \lambda_2(0) &= \lambda_2(1) = 0 \\ \lambda_3(0) &= \lambda_3(1) = 0 \end{aligned} \tag{9}$$

The boundary conditions at the initial time for λ_2 and λ_3 are zero because the initial states for x_2 and x_3 are not given. Thus, the variation $x_2(0)$ and $x_3(0)$ in the necessary conditions are arbitrary; therefore, the corresponding Lagrange multipliers are zero. This development was not shown in the report; however, it is easily obtained.

Before a solution of the necessary conditions is undertaken, the following parameter set may be utilized in Equation (7) and the differential equations solved.

$$x_2 = 1$$
$$x_3 = 1 \tag{10}$$

The solution for the system response using these values of the parameters yields the exact desired response

$$x_1(t) = 1 - t. \tag{11}$$

The value of the performance index by using (10) is zero.

Since the performance index is positive semidefinite, this parameter set gives an absolute minimum of the performance index. However, the parameter set (10) is optimal only for the initial function given in (9).

The solutions to the state equations are

$$\left.\begin{aligned} x_1 &= 1 - t \\ x_2 &= 1 \\ x_3 &= 1. \end{aligned}\right\} \forall\, t \in [0, 1] \tag{12}$$

Since x_3 is equal to one, z must also be equal to one. This would imply that φ is equal to zero. Thus, the solution to the equations for the Lagrange multipliers is

$$\lambda_1 = \lambda_2 = \lambda_3 = 0, \ \forall t \in [0, 1]. \tag{13}$$

Therefore, the chosen parameters satisfy the necessary conditions.

Figure 6 compares the desired response with the actual response for several values of h and k. The actual response for k = 1.0 and h h = 1.0 agrees exactly with the desired response.

The solution has a singular characteristic in that all values of h greater than or equal to one will also satisfy the necessary conditions. Thus, the parameter set is not unique.

Figure 7 is a contour plot of constant performance index for this particular example.

An approximation for the e^{-hs} element is the well known Pade approximation

$$e^{-hs} = \frac{1-hs/2}{1+hs/2} \tag{14}$$

The optimal system may now be approximated by

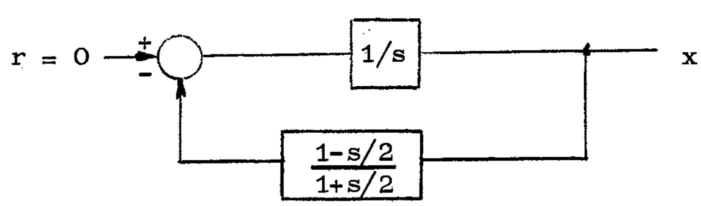

Figure 5. Hardware Implementation of the Optimal Delay

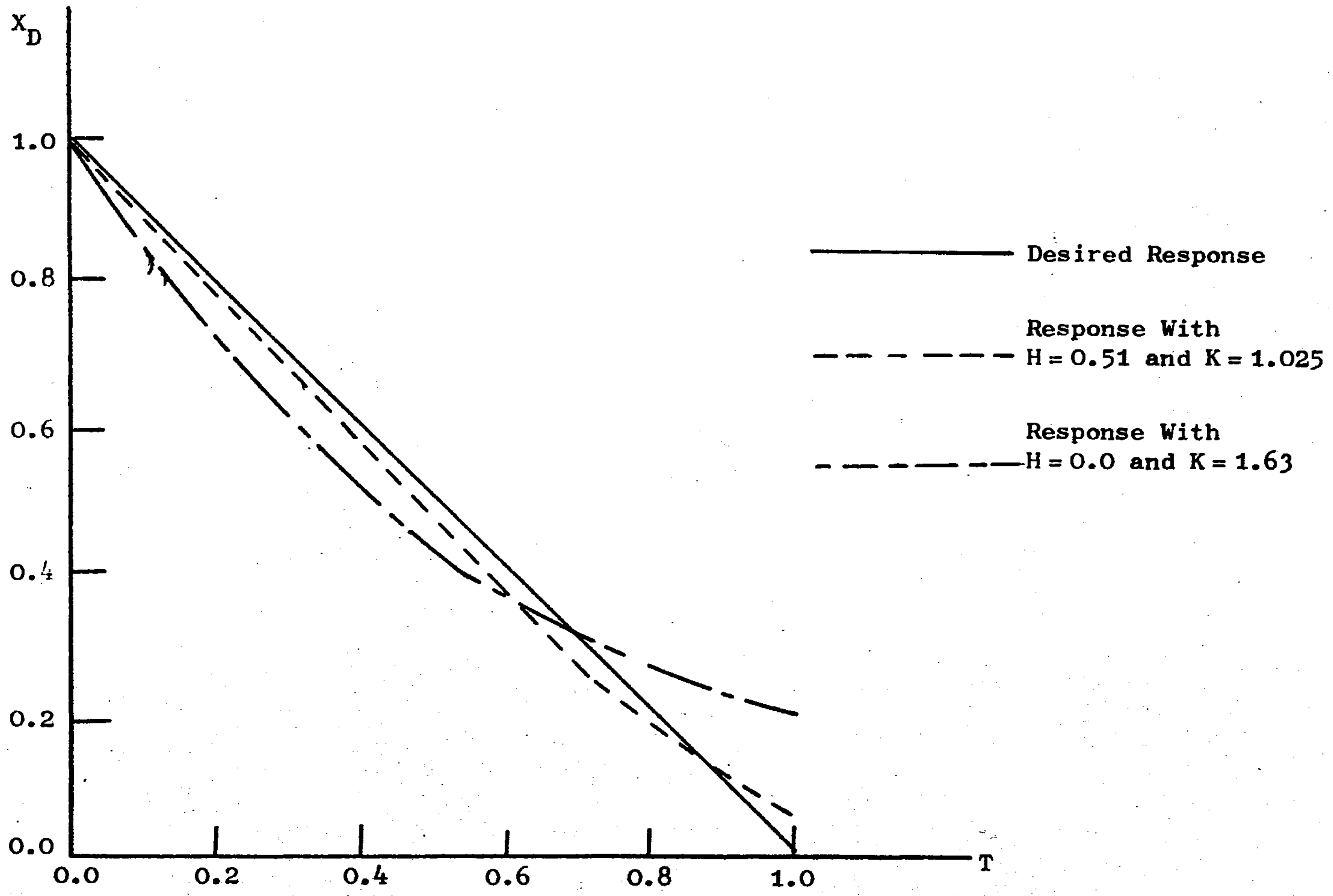

Figure 6. Desired and Actual Responses Versus Time

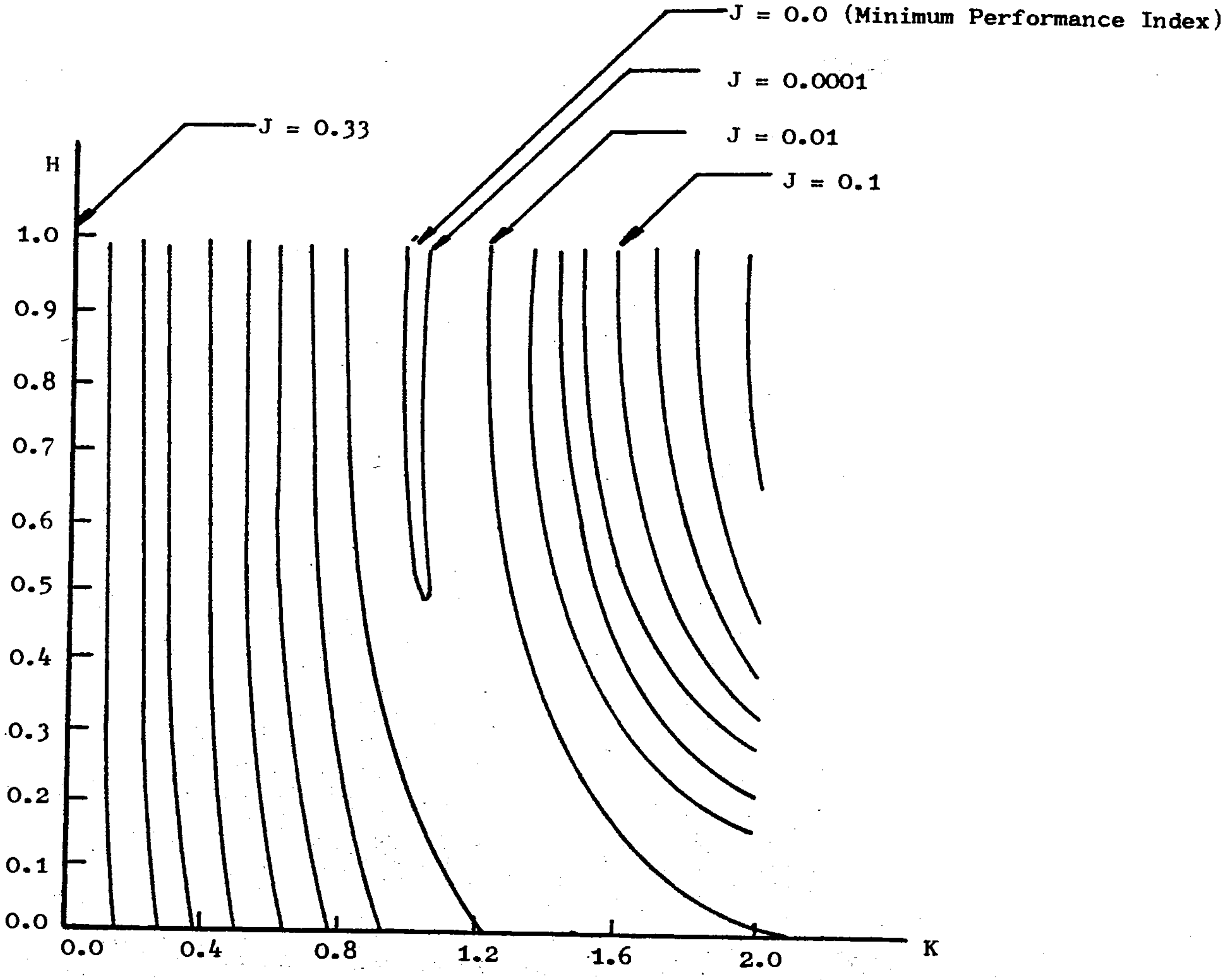

Figure 7. Contours of Constant Performance Index

This system was compared with the ideal response and was found to be a good approximation. Even though the example is trivial, further work may find that this method of identification or parameter optimization may be feasible.

Steel Rolling Mill

This example illustrates how state-dependent time delays may occur in a realistic control problem. Also, the mathematical equations necessary to obtain a particular optimal control for this system are developed. Computer runs were not made for this example because of its complexity; however, the equations necessary to solve this problem are shown, thus illustrating the use of the necessary conditions in a realistic problem.

This example illustrates control of a steel mill. The steel is rolled between two rolls, one of which is stationary and the other is moved by a control system that controls the variation in the required thickness of the steel plate, and also controls the velocity of the plate as it passes under the rolls. Due to physical limitations the measurement device for obtaining the thickness deviation has to be placed several feet from the process. Thus, a time delay occurs in measuring the deviation since the device measures $x(t-h)$ and not $x(t)$ (where x denotes the deviation of the thickness).

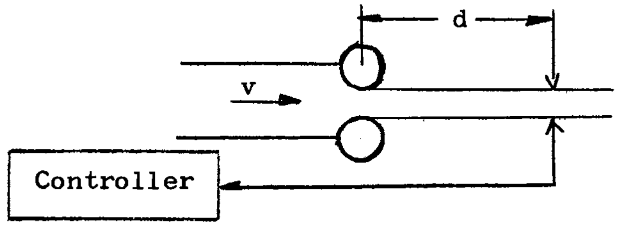

Figure 8. Steel Rolling Mill

The time delay, h, is equal to d/v. A current proportional to the deviation is fed back to the controller,

$$i(t) = k_1 x(t - d/v). \tag{15}$$

The roll is suspended on two heavy duty linear springs and is moved up or down by application of a control force proportional to the deviation feedback plus a force (to be determined) that will give the required response for the performance index chosen.

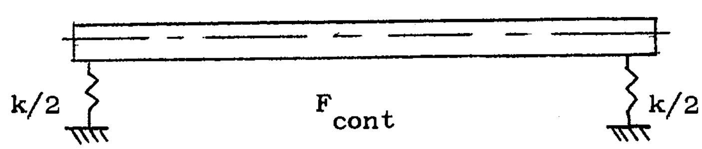

Figure 9. Roll Suspension

$$m_{roll}\ddot{x} = F_{cont} - kx$$

$$F_{cont} = F - ai(t) \quad (16)$$

$$= F - bx(t-d/v).$$

Therefore, the equation for the roll can be formulated as

$$m_{roll}\ddot{x} = F - kx(t) - bx(t-d/v). \quad (17)$$

As the rollers increase pressure on the steel plate, the velocity of the plate will change. The equation of motion for the plate is approximated as

$$m_{plate}\dot{v} = F_{vc} - c\,F_{cont} \quad (18)$$

where F_{vc} is the force that controls the velocity of the plate and c is a proportionality constant; i.e., the force on the plate is assumed proportional to the thickness control force plus the velocity control force.

The equation may be written as

$$m_{plate}\dot{V} = F_{vc} - cF + bcx(t-d/v). \quad (19)$$

Let $x_1 = x$, $x_2 = \dot{x}$, $x_3 = V$, $u_1 = F$, $u_2 = F_{vc}$, $m_p = m_{plate}$, and $m_r = m_{roll}$, then the state equations may be written as

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = 1/m_r[u_1 - kx_1(t) - bx_1(t-d/x_3)] \quad (20)$$

$$\dot{x}_3 = 1/m_p[u_2 - cu_1 + bcx_1(t-d/x_3)]$$

The following performance index is chosen in order to minimize the thickness and velocity deviations over a finite period of time with minimum effort.

$$J = \int_0^{t_f} [q_1 x_1^2 + q_2(x_3 - x_{3_d})^2 + q_3 u_1^2 + q_4 u_2^2] dt \qquad (21)$$

Initial conditions are assumed to be

$$x_1(t) = \epsilon_1, \ \forall\, t \leq 0$$

$$x_2(o) = 0 \qquad (22)$$

$$x_3(o) = x_{3_d} + \epsilon_2$$

where ϵ_1 and ϵ_2 are constants and x_{3_d} represents the desired velocity. The problem consists of finding the controls $u_1(t)$ and $u_2(t)$, $\forall\, t \in [0, t_f]$ in order to minimize the performance index. The necessary conditions will now be applied to this problem.

The velocity, x_3, must be constrained to be greater or equal to zero by the following state variable inequality constraint.

$$S = -x_3 \leq 0 \qquad (23)$$

This is a first order inequality constraint since $\frac{\partial}{\partial u}\left(\frac{dS}{dt}\right) \neq 0$.

The procedure adopted earlier to eliminate state variable inequality constraints will now be utilized.

Equation (23) may be changed into an equality constraint.

$$-x_3 + \tfrac{1}{2} Z^2 = 0$$

$$(24)$$

$$-\dot{x}_3 + Z \dot{Z} = 0$$

The new state component, v, and a pseudo control component, w, may be defined and utilized in (24).

$$-\dot{x}_3 + vw = 0 \tag{25}$$

or

$$-1/m_p[u_2 - cu_1 + bcx_1(t-d/x_3)] + vw = 0 \tag{26}$$

The control u_2 may be eliminated by solving for u_2 in terms of v, w, u_1, and $x_1(t-d/x_3)$

$$u_2 = m_p vw + cu_1 - bcx_1(t-d/x_3). \tag{27}$$

This equation may now be used to eliminate u_2 from the problem. After the augmented optimization problem has been solved, u_2 can be obtained by use of this equation. The new state equations are

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 1/m_r[u_1 - kx_1 - bx_1(t-d/x_3)] \\ \dot{x}_3 &= vw \\ \dot{v} &= w \end{aligned} \tag{28}$$

The new performance index is

$$J = \int_0^{t_f} [q_1 x_1^2 + q_2(x_3 - x_{3_d})^2 + q_3 u_1^2 + q_4(m_p vw + cu_1 - bcx_{1_h})^2]dt \tag{29}$$

where $x_{1_h} = x_1(t-d/x_3)$.

The boundary conditions are

$$
\begin{aligned}
x_1(t) &= \epsilon_1, \quad \forall t \leq 0 \\
x_2(o) &= 0 \\
x_3(o) &= x_{3d} + \epsilon_2 \\
v(o) &= \pm \sqrt{2x_3(0)}
\end{aligned}
\tag{30}
$$

Either sign of the initial condition for v may be used.

The new problem to be solved is that of finding $u_1(t)$ and $w(t)$, for all $t \epsilon [0, t_f]$ such that the performance index (29) is minimized. Note that the state variable inequality constraint has now been eliminated.

The Hamiltonian for the new problem is

$$
\begin{aligned}
H = {} & q_1 x_1^2 + q_2(x_3 - x_{3_d})^2 + q_3 u_1^2 \\
& + q_4(m_p vw + cu_1 - bcx_{1_h})^2 \\
& + \lambda_1 x_2 + \lambda_2/m_r(u_1 - kx_1 - bx_{1_h}) \\
& + \lambda_3 vw + \lambda_4 w.
\end{aligned}
\tag{31}
$$

The necessary conditions are derived in the following discussion.

The state equations are

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= 1/m_r(u_1 - kx_1 - bx_{1_h}) \\
\dot{x}_3 &= vw \\
\dot{v} &= w
\end{aligned}
\tag{32}
$$

The new state vector is $x = [x_1 \ x_2 \ x_3 \ v]^T$. Also, the new control vector is $u = [u_1 \ w]^T$.

The equations for the Lagrange multipliers may now be derived by use of Equations (42) and (47) in Chapter III.

$$\dot{\lambda}_1 = \begin{cases} -(2q_1 x_1 - k\lambda_2/m_r - 1/1-\dot{h}(s)\ [-2bcq_4(m_p vw \\ + cu_1 - bcx_{1_h}) - b\lambda_2/m_r]_{t=s}, \ \forall t \in [0, t_f - d/x_3(t_f)] \\ -(2q_1 x_1 - k/m_r \lambda_2), \ \forall t \in [t_f - d/x_3(t_f), t_f] \end{cases}$$

$$\dot{\lambda}_2 = -\lambda_1, \ \forall t \in [0, t_f]$$

$$\dot{\lambda}_3 = -2q_2(x_3 - x_{3_d}) - (1/1-\dot{h}(t))(d\dot{x}_{1_h}/x_3^2)[-2bcq_4(m_p vw + cu_1 - bcx_{1_h}) - b\lambda_2/m_r], \ \forall t \in [0, t_f]$$

$$\dot{\lambda}_4 = -2q_4 m_p w(m_p vw + cu_1 - bcx_{1_h}) - \lambda_3 w, \ \forall t \in [0, t_f].$$

Boundary conditions are

$$x_1(t) \ \epsilon_1, \ \forall t \leq 0$$

$$x_2(0) = 0$$

$$x_3(0) = x_{3_d} + \epsilon_2$$

$$v(0) = \sqrt{2x_3(0)}$$

$$\lambda_1(t_f) = \lambda_2(t_f) = \lambda_3(t_f) = \lambda_4(t_f) = 0$$

Also,

$$\partial H/\partial u = \begin{bmatrix} 2q_3 u_1 + 2q_4 c(m_p vw + cu_1 - bcx_{1_h}) + \lambda_2/m_r \\ \\ 2q_4 m_p v(m_p vw + cu_1 - bcx_{1_h}) + \lambda_3 v + \lambda_4 \end{bmatrix}.$$

These equations may be used in an algorithm as defined in Chapter III to find the optimal control as a function of time. This problem shows how to utilize the necessary conditions developed in Chapter III in a realistic problem. However, as in the majority of all optimal control problems, the computational burden of solving this problem is large. Thus, a less complicated problem will be solved in order to illustrate the character of the optimal controller.

Scalar Example

Consider the scalar example problem described by the following differential-difference equation

$$\dot{x}(t) = -.5x(t) - .5x(t - x^2) + u(t) \tag{33}$$

with initial condition function

$$x(t) = 1, \ \forall t \leq 0. \tag{34}$$

The problem is to find the control, $u(t)$, such that the quadratic performance index

$$J = ½\int_0^1 (20x^2 + u^2)\,dt \tag{35}$$

is minimized.

The Hamiltonian for the problem may be written as

$$H = 1/2(20x^2 + u^2) + \lambda(-.5x - .5x_h + u). \tag{36}$$

Minimizing the Hamiltonian with respect to u yields

$$u(t) = -\lambda(t). \tag{37}$$

The equation for the Lagrange multiplier, λ, is given by

$$\dot{\lambda} = \begin{cases} -20x + .5\lambda + (1/1-\dot{h})x\lambda\dot{x}_h + .5\lambda(s)/1-\dot{h}(s), \ \forall t \in [0, \\ 1-h(1)] \\ -20x + .5\lambda + (1/1-\dot{h})x\lambda\dot{x}_h, \ \forall t \in [1-h(1), \ 1] \end{cases} \tag{38}$$

with the boundary condition

$$\lambda(1) = 0 \tag{39}$$

and where

$$\dot{h}(t) = 2x(t)\dot{x}(t) \tag{40}$$

Figures 10 and 11 are plots of the optimal response and control as functions of time.

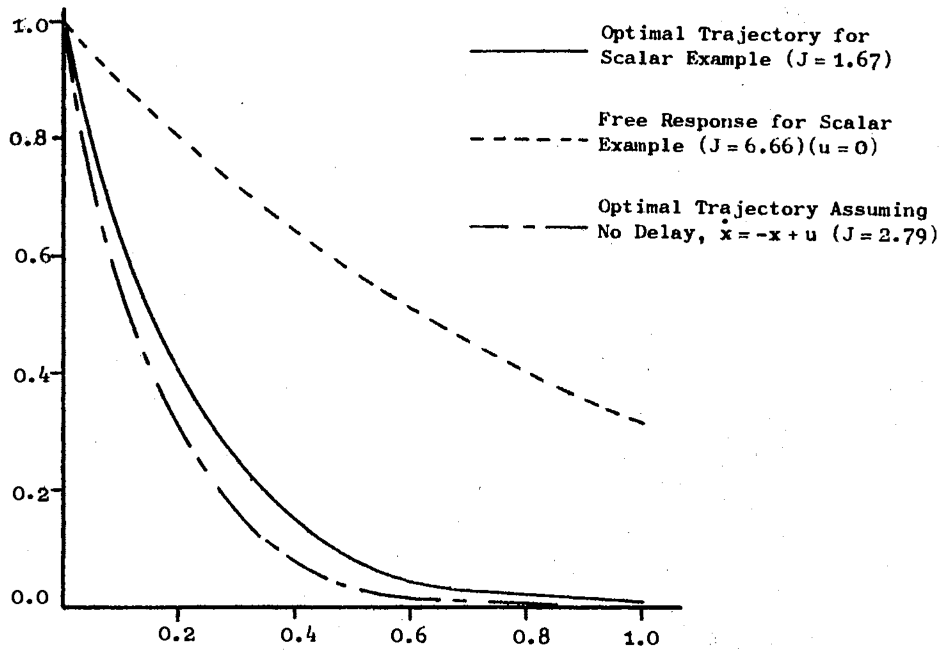

Figure 10. Optimal State Trajectory

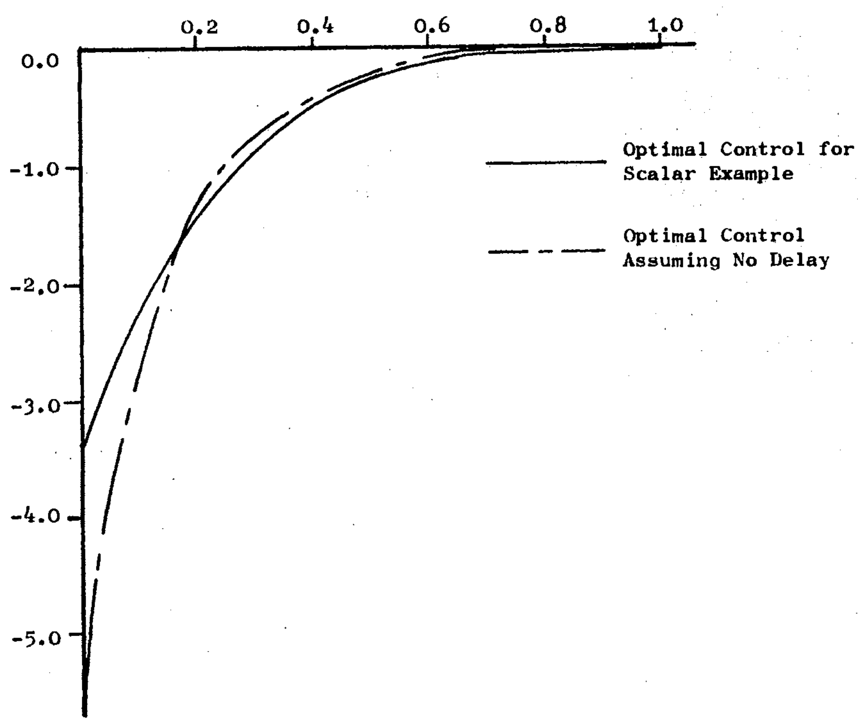

Figure 11. Optimal Control Trajectory

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

Necessary conditions for optimal control of systems containing a time delay that is a function of the state of the system and of time were derived by utilizing calculus of variations. The time delay was allowed to be in the state vector and in the control vector. The state vector and the control vector were considered to be in general constrained by inequality constraints. A transformation to eliminate state variable inequality constraints was extended to systems with time delays. Applications of the necessary conditions were shown by treating, in part, a realistic example and, in detail, a scalar example. Also, a gradient algorithm for systems with state-dependent time delays was outlined.

The results derived should benefit the engineer who is interested in applications in that he may find what the optimal control and optimal performance cost is so that he can use these as a comparison for an actual design. Also, the theoretically inclined engineer can utilize the results obtained as a starting point, for example, in an analysis to find suboptimal controls. This could be done by assuming the form of the control to be a linear transformation of the observed variables, or by trying to find approximations to the necessary conditions such that a suboptimal control can be found.

Areas Recommended for Further Study

The following salient points should be pointed out as future research problems:

1. Further investigations into computational algorithms to solve the optimal control problems should be conducted.
2. Investigations into stochastic time delay problems should be conducted.
3. The singular control problem should be investigated.
4. Differential game problems where the dynamics of the competing systems have time delays should be investigated.
5. Research in optimal and suboptimal schemes for feedback control of systems with time delays has been extremely limited. References (2) and (17) have obtained optimal feedback control laws for linear systems with a constant time delay in the state; i.e.,

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-h) + u(t)$$

with a quadratic performance index.

$$J = \tfrac{1}{2}x^T(t_f)Sx(t_f) + \tfrac{1}{2}\int_0^{t_f}[x^T Qx + u^T Ru]\,dt.$$

However, the preceding problem where the delay, h, is a function of time has not been solved for an optimal feedback control. Sebesta (61) has obtained an approximate feedback control law for sufficiently small time delay. However, no results are available when the time delay is large. The lemma proved by Eller et al. (17)

possibly could be used in this problem to obtain sufficient conditions for an optimal control feedback form.

6. The use of the necessary conditions for the identification of the time delay should be investigated more thoroughly.

7. A fourth type of time delay has not appeared in the literature, yet it is the most realistic form for a time delay. The form is that of a time delay that is a random variable or a stochastic process. In reality, in all places where time delays occur, the time delay cannot be said to be a particular value of function of time with absolute certainty. There is error in making this assumption just as there is error in assuming a particular parameter value with certainty. This error may in fact be large enough such that a system based on absolute certainty may actually be unstable when implemented. Consider a time delay system where the system stability characteristics are due to a particular measurement sensor being placed a particular distance from the process. A design may have been based on the assumption that the time delay is within the stability boundary by some epsilon and, thus, stable. However, if the time delay happens to actually be just a small amount larger or smaller, then the system may fall within the unstable region. Thus, if it is known or is evidenced by taking experimental data that the time delay can vary significantly and in a random fashion, then this fact must be taken into account when designing a system or modifying an existing system.

The necessary conditions derived in this research may lead to a solution for the problem where the time delay is a random variable by augmenting the state vector with an additional element similar to the first example in Chapter V. The initial conditions for this new state element is random (the statistics may, however, be known a priori).

8. Suboptimal feedback control laws should be derived for the linear quadratic problem. This might include assuming a form of the feedback law as a linear transformation of the available system states (or the terminus of the state) and solving for the optimal linear transformation (gain) matrix.

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