Int. J. Appl. Comput. Math (2016) 2:137–152
DOI 10.1007/s40819-015-0051-9

ORIGINAL PAPER

Optimal Control of Time Delay Systems via Hybrid of Block-Pulse Functions and Orthonormal Taylor Series

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Published online: 17 April 2015

Abstract A new method to find the optimal control of time delay systems with quadratic performance index is discussed. The method is based on hybrid functions. The properties of the hybrid functions which consists of block-pulse functions and orthonormal Taylor series are presented. The operational matrices of integration, delay, dual and product are used to reduce the solution of optimal control time delay system to the solution of algebraic equations. Numerical examples are included to illustrate the effectiveness and validity of the technique.

Keywords Time delay system · Orthonormalization · Operational matrices · Hybrid functions

Introduction

The dynamics of many control systems may be expressed by time-delay equations. The delay(s) may appear in the system state, control input and/or output. Delays occur frequently in incubation periods, mechanics, viscoelasticity, physics, physiology, population dynamics, communication, information technologies, stability of networked control systems, maturation times, age structure, blood transfusions, biological, chemical, electronic and transportation systems [1–3]. Therefore the control of time-delay systems has been interested by many engineers and scientists, due to its variety presence in realistic models of phenomena. Since the analytical methods, especially in optimal control of time-delay systems, have less implementation ability and the application of Pontryagin's maximum principle to the optimization of control systems with time-delays as outlined by Kharatishvili [4] results in a system of

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coupled two-point boundary-value (TPBV) problem involving both delay and advance terms whose exact solution, except in very special cases, is very difficult [5], different numerical methods have been devised to overcome the problems arising from the application of analytical methods. In other word, the main object of all computational aspects of optimal time-delay systems have been to produce a new method to avoid the solution of the mentioned (TPBV) problem. Orthogonal functions (OFs) and polynomial series have received considerable attentions in dealing with various problems of dynamic systems [6]. For such kind of problem, the approach is that of converting the underlaying differential equations govering the dynamical system to an algebraic form through the use of an operational matrix of integration which can be uniquely determined based on the particular OFs. Special attentions have been given to applications of Walsh functions [7], block-pulse functions [8–10], Laguerre polynomials [11], Legendre polynomials [12–14], Chebyshev polynomials [15,16], Taylor series [17,18] and Fourier series [19,20].

The aim of present paper is to introduce a new numerical method to solve the quadratic optimal control problem with delay systems. This method consists of reducing the optimal control problem to a set of algebraic equations by expanding the state and control vectors as hybrid functions with unknown coefficients. These hybrid functions, which consist of block-pulse functions and orthonormal Taylor series are given. The operational matrices of integration and delay are introduced. The necessary conditions of optimality are derived as a system of algebraic equations in the unknown coefficients of state and control vectors and Lagrange multipliers. These coefficients are determined in such a way that the necessary conditions for extremization are imposed. In this paper, we show a novel strategy by using hybrid functions to find the approximate solutions of time delay optimal control problems. In this method, we divided the time interval into N subintervals and approximate the trajectory and control functions by hybrid of block-pulse functions and orthonormal Taylor series. Indeed in applying the method, by increasing the accuracy of the approximate solutions, but the CPU time and computer needed memory reduce nevertheless, since the operational matrices have large number of zero elements and they are mostly sparce.

Hybrid Functions and Their Properties

Hybrid functions $H_{n,m}(t), n = 1, 2, \ldots, N, m = 0, 1, \ldots .M - 1$; have three arguments; n and m are the order of block-pulse functions and orthonormal Taylor series, respectively, and t is the normalized time. They are defined on the interval [0, 1) as follows (since any interval $[a, b)$ can be shifted to [0, 1) therefore we consider [0, 1) here)

$$H_{n,m}(t) = \begin{cases} \sqrt{N}\,OT_m(Nt - n + 1), & \left(\frac{n-1}{N}\right) \leq t < \frac{n}{N}, \\ 0, & otherwise, \end{cases} \tag{1}$$

where $OT_m(t)$'s are orthonormal Taylor series govered by the Gram–Schmidt orthonormalization process on $T_m = \{1, t, t^2, t^3, \ldots, t^m\}$ and the time interval is [0, 1] with the weight function $w(t) = 1$. For example we have

$$\begin{aligned} OT_0(t) &= 1, \\ OT_1(t) &= (2t - 1)\sqrt{3}, \\ OT_2(t) &= (6t^2 - 6t + 1)\sqrt{5}, \\ OT_3(t) &= (2t - 1)(10t^2 - 10t + 1)\sqrt{7}, \\ OT_4(t) &= (70t^4 - 140t^3 + 90t^2 - 20t + 1)\sqrt{9}. \end{aligned}$$

A function $f(t)$ belongs to the space $L^2[0, 1]$ may be expanded by hybrid functions as follows:

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} H_{n,m}(t). \tag{2}$$

By truncating the series (2), we can abtain an approximation for $f(t)$ as follows:

$$f(t) \simeq \sum_{n=1}^{N} \sum_{m=0}^{M-1} c_{n,m} H_{n,m}(t) = C^T H(t), \tag{3}$$

where

$$C = [c_{1,0}c_{1,1} \dots c_{1,M-1}c_{2,0}c_{2,1} \dots c_{2,M-1} \dots c_{N,0} \dots c_{N,M-1}]^T,$$

and

$$H(t) = [H_{1,0}(t)H_{1,1}(t) \dots H_{1,M-1}(t)H_{2,0}(t)H_{2,1}(t) \dots H_{2,M-1}(t) \dots \\ H_{N,0}(t)H_{N,1}(t) \dots H_{N,M-1}(t)]^T, \tag{4}$$

where, $c_{n,m}, n = 1, 2, \dots, N, m = 0, 1, \dots, M - 1$, are the coefficients expansion of the function $f(t)$ in the n-th subinterval $[\frac{(n-1)}{N}, \frac{n}{N})$.

We have $c_{n,m} = \langle f(t).H_{n,m}(t)\rangle$ and $\langle . \rangle$ is the standard inner product on $L^2[0, 1)$.

Operational Matrix of Integration

We can approximate the integration of $H(t)$ defined in (4) as follows:

$$\int_0^t H(s)ds \simeq P_h H(t), \tag{5}$$

where P_h is $MN \times MN$ operational matrix for integration and is given as:

$$P_h = \begin{pmatrix} A_1 & B_1 & B_1 & \dots & B_1 \\ 0 & A_1 & B_1 & \dots & B_1 \\ 0 & 0 & A_1 & \dots & B_1 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & A_1 \end{pmatrix}, \tag{6}$$

where

$$B_1 = \frac{1}{N} \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}_{(M \times M)},$$

and

$$A_1 = \frac{1}{2N}\begin{pmatrix} 1 & \frac{\sqrt{1}\sqrt{3}}{1\times 3} & 0 & 0 & 0 & \dots & 0 \\ -\frac{\sqrt{1}\sqrt{3}}{1\times 3} & 0 & \frac{\sqrt{3}\sqrt{5}}{3\times 5} & 0 & 0 & \dots & 0 \\ 0 & -\frac{\sqrt{3}\sqrt{5}}{3\times 5} & 0 & \frac{\sqrt{5}\sqrt{7}}{5\times 7} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \dots & 0 \\ 0 & 0 & 0 & \dots & \vdots & \dots & \frac{\sqrt{2M-1}\sqrt{2M-3}}{(2M-1)\times(2M-3)} \\ 0 & 0 & 0 & \dots & \vdots & -\frac{\sqrt{2M-1}\sqrt{2M-3}}{(2M-1)\times(2M-3)} & 0 \end{pmatrix}_{(M\times M)}.$$

Dual and Product Operational Matrices

Since $H_{n,m}(t)$'s are disjoint and orthonormal sets on [0, 1), so the dual operational matrix of $H(t)$ is

$$L = \int_0^1 H(t)H^T(t)dt = \mathbf{I},$$

where $\mathbf{I}$ is MN identity matrix. Also the following property of the product of two hybrid function vectors will be used. Let

$$H(t)H^T(t)C \simeq \tilde{C}H(t), \tag{7}$$

where $\tilde{C}$ is a $MN \times MN$ product operational matrix. To show the calculation procedure we may choose $N = 4$ and $M = 3$. Thus we have

$$C = [c_{1,0}c_{1,1}c_{1,2}\dots c_{4,0}c_{4,1}c_{4,2}]^T, \tag{8}$$

$$H(t) = [H_{1,0}(t)H_{1,1}(t)H_{1,2}(t)\dots H_{4,0}(t)H_{4,1}(t)H_{4,2}(t)]^T, \tag{9}$$

where

$$\left.\begin{array}{ll} H_{1,0} = & 2 \\ H_{1,1} = & 2(8t-1)\sqrt{3} \\ H_{1,2} = & 2(96t^2-24t+1)\sqrt{5} \end{array}\right\} 0 \le t < \frac{1}{4}, \quad \left.\begin{array}{ll} H_{2,0} = & 2 \\ H_{2,1} = & 2(8t-3)\sqrt{3} \\ H_{2,2} = & 2(96t^2-72t+13)\sqrt{5} \end{array}\right\} \frac{1}{4} \le t < \frac{2}{4}, \tag{10}$$

and

$$\left.\begin{array}{ll} H_{3,0} = & 2 \\ H_{3,1} = & 2(8t-5)\sqrt{3} \\ H_{3,2} = & 2(96t^2-120t+37)\sqrt{5} \end{array}\right\} \frac{2}{4} \le t < \frac{3}{4}, \quad \left.\begin{array}{ll} H_{4,0} = & 2 \\ H_{4,1} = & 2(8t-7)\sqrt{3} \\ H_{4,2} = & 2(96t^2-168t+73)\sqrt{5} \end{array}\right\} \frac{3}{4} \le t < 1. \tag{11}$$

For example from (1) one can obtain

$$H_{1,1}(t) = \sqrt{4}OT_1(4t) = 2(8t-1)\sqrt{3}, \quad 0 \le t < \frac{1}{4}$$
$$H_{2,1}(t) = \sqrt{4}OT_1(4t-1) = 2(8t-3)\sqrt{3}, \quad \frac{1}{4} \le t < \frac{2}{4}$$
$$H_{3,1}(t) = \sqrt{4}OT_2(4t-3) = 2(8t-5)\sqrt{3}, \quad \frac{2}{4} \le t < \frac{3}{4}$$
$$H_{4,1}(t) = \sqrt{4}OT_2(4t-5) = 2(8t-7)\sqrt{3}, \quad \frac{3}{4} \le t < 1,$$

We also have

$$H(t)H^T(t) = \begin{pmatrix} H_{1,0}H_{1,0} & H_{1,0}H_{1,1} & H_{1,0}H_{1,2} & & & \\ H_{1,1}H_{1,0} & H_{1,1}H_{1,1} & H_{1,1}H_{1,2} & & \bigcirc & \\ H_{1,2}H_{1,0} & H_{1,2}H_{1,1} & H_{1,2}H_{1,2} & & & \\ & & & \ddots & & \\ & \bigcirc & & H_{1,0}H_{1,0} & H_{1,0}H_{1,1} & H_{1,0}H_{1,2} \\ & & & H_{1,1}H_{1,0} & H_{1,1}H_{1,1} & H_{1,1}H_{1,2} \\ & & & H_{1,2}H_{1,0} & H_{1,2}H_{1,1} & H_{1,2}H_{1,2} \end{pmatrix},$$

where $\bigcirc$ denoted zero matrix. By using the vector C in (8) the 12×12 matrix $\tilde{C}$ in (7) is

$$\tilde{C} = \begin{pmatrix} \tilde{C}_1 & 0 & 0 & 0 \\ 0 & \tilde{C}_2 & 0 & 0 \\ 0 & 0 & \tilde{C}_3 & 0 \\ 0 & 0 & 0 & \tilde{C}_4 \end{pmatrix}, \tag{12}$$

where $\tilde{C}_i$'s, $i = 1, 2, 3, 4$ are 3×3 matrices given by

$$\tilde{C}_i = \begin{pmatrix} 2c_{i,0} & 2c_{i,1} & 2c_{i,2} \\ 2c_{i,1} & 2c_{i,0} + 4\frac{\sqrt{5}}{5}c_{i,2} & 4\frac{\sqrt{5}}{5}c_{i,1} \\ 2c_{i,2} & 4\frac{\sqrt{5}}{5}c_{i,1} & 2c_{i,0} + 4\frac{\sqrt{5}}{7}c_{i,2} \end{pmatrix}. \tag{13}$$

Delay Operational Matrix

The delay function $H(t - \tau)$ is the shifted of the function $H(t)$ defined in (4), along the time axis by τ. In other word we have

$$H(t - \tau) = D_\tau H(t), \quad t > \tau, \ \ 0 \leq t \leq 1, \tag{14}$$

where D_τ is the delay operational matrix of hybrid functions. To find D_τ, we first choose N the order of block-pulse functions, as the following manner [5]:

$$N = \begin{cases} \frac{1}{\tau}, & \frac{1}{\tau} \in \mathbb{Z}, \\ \left[\frac{1}{\tau}\right] + 1 & otherwise, \end{cases} \tag{15}$$

where [.] denotes greatest integer value.

Note that in the interval $\tau \leq t \leq 2\tau$, the terms $H_{1,m}(t)$ for $m = 0, 1, \ldots, M - 1$ are nonzero and all other terms are zero. So if we expand $H_{1,m}(t)$ in terms of $H_{2,m}(t)$ then the coefficients form an $M \times M$ identity matix since we have $H_{1,m}(t - \tau) = H_{2,m}(t)$. Similar manner can be used to all other intervals. Thus if we expand $H(t - \tau)$ in terms of $H(t)$ we find $NM \times NM$ matrix D_τ as

$$D_\tau = \begin{pmatrix} 0 & I & 0 & 0 & \ldots & 0 \\ 0 & 0 & I & 0 & \ldots & 0 \\ 0 & 0 & 0 & I & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \\ 0 & 0 & 0 & 0 & \ldots & I \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \tag{16}$$

where I is $M \times M$ identity matrix.

Problem Statement

Consider the following quadratic time-independent delay control system:

$$\min \quad J = \frac{1}{2}x^T(1)Fx(1) + \frac{1}{2}\int_0^1 \{x^T(t)Sx(t) + u^T(t)Wu(t)\}dt, \tag{17}$$

$$s.t \quad \dot{x}(t) = Ax(t) + Bx(t-\tau_1) + Eu(t) + Du(t-\tau_2), \quad 0 \le t \le 1, \tag{18}$$

$$x(0) = x_0, \tag{19}$$

$$x(t) = \theta(t), \quad -\tau_1 \le t < 0, \tag{20}$$

$$u(t) = \psi(t), \quad -\tau_2 \le t < 0, \tag{21}$$

where W is symmetric positive definite and F, S are positive semi-definite matrices [21], $x(t) \in \mathbb{R}^p$, $u(t) \in \mathbb{R}^q$ are state and control vectores respectively and A, B, E, D are matrices of appropriate dimensions, x_0 is a constant specified vector, and $\theta(t)$, $\psi(t)$ are arbitrary known functions. We choose W as a symmetric positive definite and F, S as positive semi-definite to show that the cost functional J is quadratic and convex, so the necessary conditions for existence of solution, is also sufficient. The problem is to find $x(t)$ and $u(t)$, $0 \le t \le 1$, satisfying (18)–(21) while minimizing (17).

Assume that

$$x(t) = [x_1(t)x_2(t)\ldots x_p(t)]^T, \tag{22}$$

$$u(t) = [u_1(t)u_2(t)\ldots u_q(t)]^T, \tag{23}$$

$$\hat{H}(t) = I_p \otimes H(t), \tag{24}$$

$$\hat{H}_1(t) = I_q \otimes H(t), \tag{25}$$

where I_p and I_q are the p and q dimensional identity matrices and $\otimes$ denotes Kronecker product [22]. Here

$$\hat{H}(t) = I_p \otimes H(t) = \begin{pmatrix} H(t) & \ldots & H(t) \\ \vdots & \vdots & \vdots \\ H(t) & \ldots & H(t) \end{pmatrix}_{p\times p},$$

$$\hat{H}_1(t) = I_q \otimes H(t) = \begin{pmatrix} H(t) & \ldots & H(t) \\ \vdots & \vdots & \vdots \\ H(t) & \ldots & H(t) \end{pmatrix}_{q\times q}.$$

$\hat{H}(t)$ and $\hat{H}_1(t)$ are $pMN \times p$ and $qMN \times q$ matrices respectively while $H(t)$ is the vector function defined in (4). Assume that each of $x_i(t)$ and each of $u_j(t)$, $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, q$, can be written in terms of hybrid functions as

$$x_i(t) = H^T(t)X_i,$$

$$u_j(t) = H^T(t)U_j.$$

Using Eqs. (22)–(25) we have

$$x(t) = \hat{H}^T(t)X, \tag{26}$$

$$u(t) = \hat{H}_1^T(t)U, \tag{27}$$

where

$$X = [X_1 X_2 \dots X_p]^T,$$
$$U = [U_1 U_2 \dots U_q]^T.$$

Similarly we have

$$x(0) = \hat{H}^T(t)G, \tag{28}$$
$$\theta(t - \tau_1) = \hat{H}^T(t)K, \tag{29}$$
$$\psi(t - \tau_2) = \hat{H}^T(t)R, \tag{30}$$

where

$$G = [g_1 g_2 \dots g_p]^T,$$
$$K = [k_1 k_2 \dots k_p]^T,$$
$$R = [r_1 r_2 \dots r_q]^T.$$

We can also write $x(t - \tau_1)$ and $u(t - \tau_2)$ in terms of hybrid functions as

$$x(t - \tau_1) = \begin{cases} \hat{H}^T(t)K, & 0 \le t \le \tau_1, \\ \hat{H}^T(t)\hat{D}_1^T X, & \tau_1 \le t \le 1, \end{cases}$$
$$u(t - \tau_2) = \begin{cases} \hat{H}^T(t)R, & 0 \le t \le \tau_2, \\ \hat{H}^T(t)\hat{D}_2^T U, & \tau_2 \le t \le 1, \end{cases}$$

where $\hat{D}_1 = I_p \otimes D_{\tau_1}$ and $\hat{D}_2 = I_q \otimes D_{\tau_2}$ and D_{τ_1}, D_{τ_2} are respectively delay operational matrices given in (16).

Moreover we have

$$\int_0^t \hat{H}^T(s)ds = (I_p \otimes H^T(t))(I_p \otimes P_h^T) = \hat{H}(t)\hat{P}_h^T, \tag{31}$$
$$\int_0^t x(s - \tau_1)ds = \begin{cases} \hat{H}^T(t)\hat{P}_h^T K, & 0 \le t \le \tau_1, \\ \hat{H}^T(t)Z_1 K + \hat{H}^T(t)\hat{P}_h^T \hat{D}_1^T X, & \tau_1 \le t \le 1, \end{cases} \tag{32}$$
$$\int_0^t u(s - \tau_2)ds = \begin{cases} \hat{H}^T(t)\hat{P}_h^T R, & 0 \le t \le \tau_2, \\ \hat{H}^T(t)Z_2 R + \hat{H}^T(t)\hat{P}_h^T \hat{D}_2^T U, & \tau_2 \le t \le 1, \end{cases} \tag{33}$$

where P_h is the operational matrix of integration given in (6) and the constant matrices Z_1, Z_2 are

$$\int_0^{\tau_i} \hat{H}^T(t)dt = \hat{H}^T(t)Z_i, \quad i = 1, 2.$$

By integrating (18) from 0 to t and using (26)–(33) we have

$$\begin{aligned} \hat{H}^T(t)X - \hat{H}^T(t)G = A\hat{H}^T(t)\hat{P}_h^T X + B\hat{H}^T(t)\hat{P}_h^T K \\ +B\hat{H}^T(t)Z_1 K + B\hat{H}^T(t)\hat{P}_h^T \hat{D}_1^T X \\ +E\hat{H}^T(t)\hat{P}_h^T U + D\hat{H}^T(t)\hat{P}_h^T R \\ +D\hat{H}^T(t)Z_2 R + D\hat{H}^T(t)\hat{P}_h^T \hat{D}_2^T U. \end{aligned} \tag{34}$$

From Eq. (34) and by deleting $\hat{H}^T(t)$ from both sides we have

$$\begin{aligned} C^* = X - G - A\hat{P}_h^T X - B\hat{P}_h^T K - BZ_1 K - B\hat{P}_h^T \hat{D}_1^T X \\ -E\hat{P}_h^T U - D\hat{P}_h^T R - DZ_2 R - D\hat{P}_h^T \hat{D}_2^T U = 0. \end{aligned} \tag{35}$$

Similarly for J in (17), we have

$$J(X,U)=\frac{1}{2}X^T(H(1)H^T(1)\otimes F)X+\frac{1}{2}X^T(L\otimes S)X$$
$$+\frac{1}{2}U^T(L\otimes W)U, \tag{36}$$

where $L=\int_0^1 H(t)H^T(t)dt$ and indeed we know that $L=I$, where I is MN identity matrix.

The delay optimal control problem has now been reduced to a parameter optimization problem which can be stated as follows. Find X and U so that $J(X,U)$ is minimized subject to the constraints in Eq. (35).

Let

$$J^*(X,U,\lambda)=J(X,U)+\lambda^T C^*, \tag{37}$$

where the vector λ represents the unknown Lagrange multipliers, then the necessary conditions for stationary are given by

$$\frac{\partial}{\partial X}J^*(X,U,\lambda)=0,$$
$$\frac{\partial}{\partial U}J^*(X,U,\lambda)=0,$$
$$\frac{\partial}{\partial \lambda}J^*(X,U,\lambda)=0. \tag{38}$$

Illustrative Examples

In this section numerical examples are given to demonstrate the applicability, efficiency and accuracy of our proposed method.

Example 1

For a system described by [23]

$$\dot{x}(t)=-x(t)+u(t)-0.5u\left(t-\frac{2}{3}\right),\quad 0\le t\le 1,$$
$$x(0)=1.0,$$
$$u(t)=0,\quad t\in\left[-\frac{2}{3},0\right],$$

minimize

$$J(X,U)=\frac{1}{2}\int_0^1\{x^2(t)+u^2(t)\}dt.$$

Here, we solve this problem with hybrid functions. Note that in this example delay is applied on control only, and $\tau=\frac{2}{3}$. Suppose that

$$x(t)=X^TH(t),\quad u(t)=U^TH(t),\quad x(0)=C_0^TH(t),$$

Table 1 J values in Example 1 with N = 3

M	J value
3	0.195494339136585
4	0.195494339136887
5	0.195494339136887
6	0.195494339136887

where X^T, U^T, $H(t)$ are defined previously and C_0 is the coefficient vector of 1 in term of hybrid functions expansion. If we integrate $\dot{x}(t)$ from 0 to t and use (31)–(33)we have

$$\int_0^t \dot{x}(s)ds = -\int_0^t x(s)ds + \int_0^t u(s)ds - \frac{1}{2}\int_0^t u\left(s-\frac{2}{3}\right)ds. \tag{39}$$

For time delay control function one can easily find that

$$\int_0^t u\left(s-\frac{2}{3}\right)ds = \begin{cases} 0, & 0 \le t \le \frac{2}{3}, \\ U^T P_h D_{\tau_2} H(t), & \frac{2}{3} \le t \le 1. \end{cases} \tag{40}$$

So from (39) we obtain

$$X^T H(t) - C_0^T H(t) = -X^T P_h H(t) + U^T P_h H(t) - \frac{1}{2}U^T D_{\tau_2} P_h H(t). \tag{41}$$

By deleting $H(t)$ from both sides and reordering of (41) we conclude that

$$C^* = X^T - C_0^T + X^T P_h - U^T P_h + \frac{1}{2}U^T D_{\tau_2} P_h = 0. \tag{42}$$

Now for cost functional J we have

$$\begin{aligned} J &= \frac{1}{2}\int_0^1 \left\{X^T H(t)H^T(t)X + U^T H(t)H^T(t)U\right\} dt \\ &= \frac{1}{2}\left[X^T L X + U^T L U\right], \end{aligned}$$

where L is defined previously. Thus we have reduced the system as follows

$$\begin{aligned} \min \quad & J = \frac{1}{2}\left[X^T L X + U^T L U\right], \\ s.to \quad & C^* = X^T - C_0^T + X^T P_h - U^T P_h + \frac{1}{2}U^T D_{\tau_2} P_h = 0. \end{aligned}$$

We have solved this example with $N = 3$ and $M = 3, 4, 5, 6$ by Maple 15 software with the CPU time of a core i5 in 2.45 s. The values of J are presented in Table 1. The curves of state and control functions for M = 6 are shown in Fig. 1.

Example 2

Consider the following delay optimal control system [14,19,24]

$$\min \quad J = \frac{1}{2}\int_0^2 \left\{x^2(t) + u^2(t)\right\} dt, \tag{43}$$

$$s.t \quad \dot{x}(t) = x(t-1) + u(t), \quad 0 \le t \le 2, \tag{44}$$

$$x(t) = 1 \quad -1 \le t \le 0. \tag{45}$$

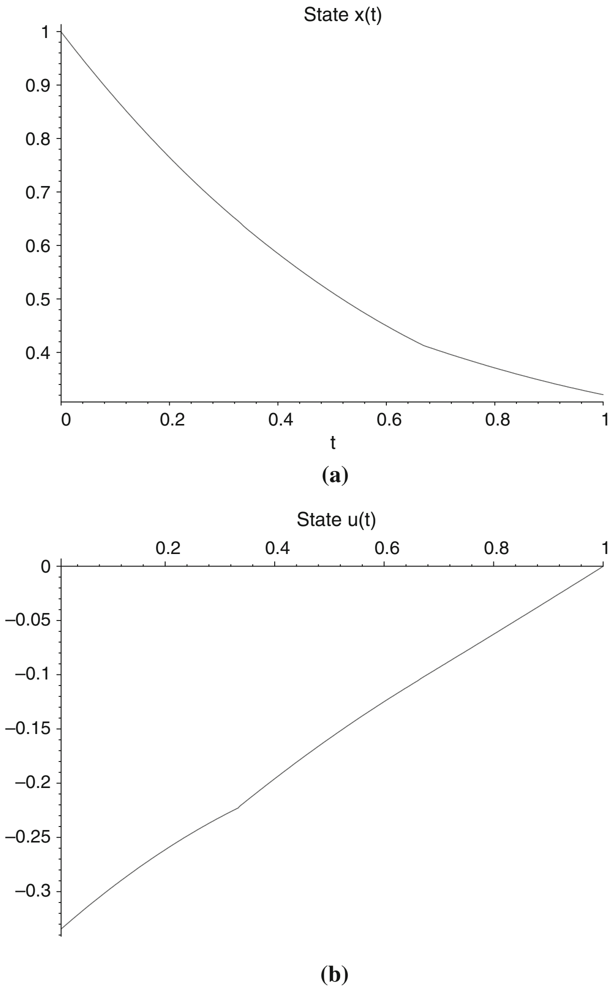

Fig. 1 State vector x(t) and control u(t) in Example 1. **a** State x(t). **b** Control u(t)

Suppose that

$$x(t) = X^T H(t), \quad u(t) = U^T H(t), \quad x(0) = C_0^T H(t).$$

We solve this example with $N = 2$ and $M = 4$. By expanding $x(0)$ in terms of hybrid functions we get

$$x(0) = [1, 0, 0, 0, 1, 0, 0, 0]^T = k_1^T H(t). \tag{46}$$

Also we have

$$\int_0^t x(s-1)ds = \begin{cases} k_2^T H(t), & 0 \le t \le 1, \\ k_3^T H(t) + X^T D_\tau P_h H(t), & 1 \le t \le 2, \end{cases} \tag{47}$$

Table 2 Results of Example 2 with $N = 2$ and $M = 6$

Time	State x(t)		Control u(t)	
	Method of [23]	Peresent method	Method of [23]	Peresent method
0.00	1.0000	1.000000	−1.9870	−1.987936
0.20	0.8364	0.836465	−1.6566	−1.657582
0.41	0.7299	0.729501	−1.3691	−1.370340
0.61	0.6794	0.678789	−1.1143	−1.143743
0.81	0.6703	0.669425	−0.9547	−0.955914
1.02	0.6971	0.695945	−0.7947	−0.795585
1.22	0.7321	0.730322	−0.6525	−0.652911
1.43	0.7716	0.770880	−0.5031	−0.495490
1.63	0.8310	0.826640	−0.3362	−0.336116
1.83	0.9163	0.910498	−0.1631	−0.162960
2.00	1.0189	1.011921	0.0000	0.00000

where

$$k_2 = \left[\frac{1}{2}, \frac{\sqrt{2}\sqrt{6}}{12}, 0, 0, 0, 0, 0, 0\right]^T, \tag{48}$$

$$k_3 = [0, 0, 0, 0, 1, 0, 0, 0]^T, \tag{49}$$

and P_h is the operational matrix of integration given in (6) and D_τ is the delay operational matrix given by

$$D_\tau = \begin{pmatrix} 0 & I_4 \\ 0 & 0 \end{pmatrix}, \tag{50}$$

where I_4 is 4-dimensional identity matrix. If we integrate (44) from 0 to t and use (45)–(50) we have

$$X^T - X^T D_\tau P_h - U^T P_h - k^T = 0, \tag{51}$$

where we have $k = k_1 + k_2 + k_3$.

The cost functional J in (43) now changes to the following form

$$J(X, U) = \frac{1}{2}(X^T LX + U^T LU). \tag{52}$$

Now we have reduced the system as follows

$$\min \quad J(X, U) = \frac{1}{2}(X^T LX + U^T LU),$$
$$s.t \quad C^* = X^T - X^T D_\tau P_h - U^T P_h - k^T = 0.$$

The results obtained are given in Table 2. The results are compared well with the solutions obtained in [23].

Values of cost functional J in [23], is reported as 1.6497 with $m = 100$ and 1.6504 with $m = 10$. Moreover in [14], J value is reported as $2\frac{2226}{2615}$. Approximate values of the cost function J with $N = 2$ and for $M = 4, 5, 6, 8$ are given in the Table 3. The curves of state and control functions for M = 6 are shown in Fig. 2.

Table 3 The cost functional J in Example 2 with N = 3

M	J
4	1.64787431
5	1.64787419
6	1.64787419

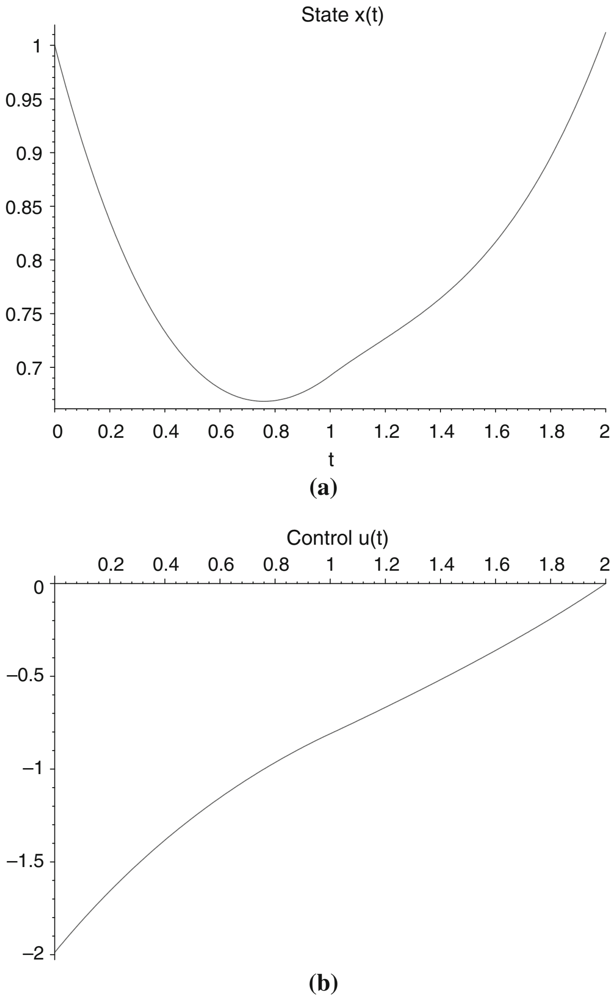

Fig. 2 State vector x(t) and control u(t) in Example 2. **a** State x(t). **b** Control u(t)

Example 3

Consider the following delay optimal control system [5,9,14,23]

$$\min \qquad J = \frac{1}{2}\int_0^1 \{x^2(t) + \frac{1}{2}u^2(t)\}dt, \tag{53}$$

$$s.t \quad \dot{x}(t) = -x(t) + x(t - 1/3) + u(t) - \frac{1}{2}u(t - 2/3), \quad 0 \le t \le 1, \tag{54}$$

$$x(t) = 1, \quad -1 \le t \le 0, \tag{55}$$

$$u(t) = 0, \quad -2/3 \le t < 0. \tag{56}$$

Here we have different delays in state($\tau_1 = 1/3$) and control($\tau_2 = 2/3$). The problem is to find the optimal control $u(t)$ which minimizes J in (53) subject to (54)-(56). The exact solution of this problem is not known, so we solve it by hybrid functions and by choosing $N = 3$ and $M = 4$. Suppose that

$$x(t) = X^T H(t), \quad u(t) = U^T H(t), \quad x(0) = C_0^T H(t).$$

By expanding $x(0)$ in terms of hybrid functions we get

$$x(0) = \left[\frac{\sqrt{3}}{3}, 0, 0, 0, \frac{\sqrt{3}}{3}, 0, 0, 0, \frac{\sqrt{3}}{3}, 0, 0, 0\right]^T = e_1^T H(t). \tag{57}$$

Also we have

$$\int_0^t x\left(s - \frac{1}{3}\right) ds = \begin{cases} e_2^T H(t), & 0 \le t \le \frac{1}{3}, \\ e_3^T H(t) + X^T D_{\tau_1} P_h H(t), & \frac{1}{3} \le t \le 1, \end{cases} \tag{58}$$

$$\int_0^t u\left(s - \frac{2}{3}\right) ds = \begin{cases} 0, & 0 \le t \le \frac{2}{3}, \\ U^T D_{\tau_2} P_h H(t), & \frac{2}{3} \le t \le 1, \end{cases} \tag{59}$$

where

$$e_2 = \left[\frac{\sqrt{3}}{18}, \frac{1}{18}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right]^T, \tag{60}$$

$$e_3 = \left[0, 0, 0, 0, \frac{\sqrt{3}}{9}, 0, 0, 0, \frac{\sqrt{3}}{9}, 0, 0, 0\right]^T, \tag{61}$$

and P_h is the operational matrix of integration given in (6) and D_{τ_1}, D_{τ_2} are the delay operational matrices given by

$$D_{\tau_1} = \begin{pmatrix} 0 & I_4 & 0 \\ 0 & 0 & I_4 \\ 0 & 0 & 0 \end{pmatrix}, \tag{62}$$

and

$$D_{\tau_2} = \begin{pmatrix} 0 & 0 & I_4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{63}$$

where I_4 is 4-dimensional identity matrix. By integrating (54) from 0 to t and using (55)-(56) we have

$$\int_0^t \dot{x}(s)ds = -\int_0^t x(s)ds + \int_0^t x(s - \frac{1}{3})ds + \int_0^t u(s)ds - \frac{1}{2}\int_0^t u(s - \frac{2}{3})ds,$$

$$X^T H(t) - e_1^T H(t) = -X^T P_h H(t) + e_2^T H(t) + e_3^T H(t) + X^T D\tau_1 P_h H(t) + U^T P_h H(t) - \frac{1}{2}U^T D_{\tau_2} P_h H(t).$$

Table 4 The cost functional J in Example 3 with N = 3

BP method [9]	Peresent method
0.3723904 (N = 6)	0.373112935 (N = 3, M = 4)
0.3732373 (N = 9)	0.373112935 (N = 3, M = 5)
0.3731831 (N = 12)	0.373112935 (N = 3, M = 6)
0.3731359 (N = 21)	0.373112935 (N = 3, M = 8)
0.3731179 (N = 45)	0.373112935 (N = 3, M = 10)

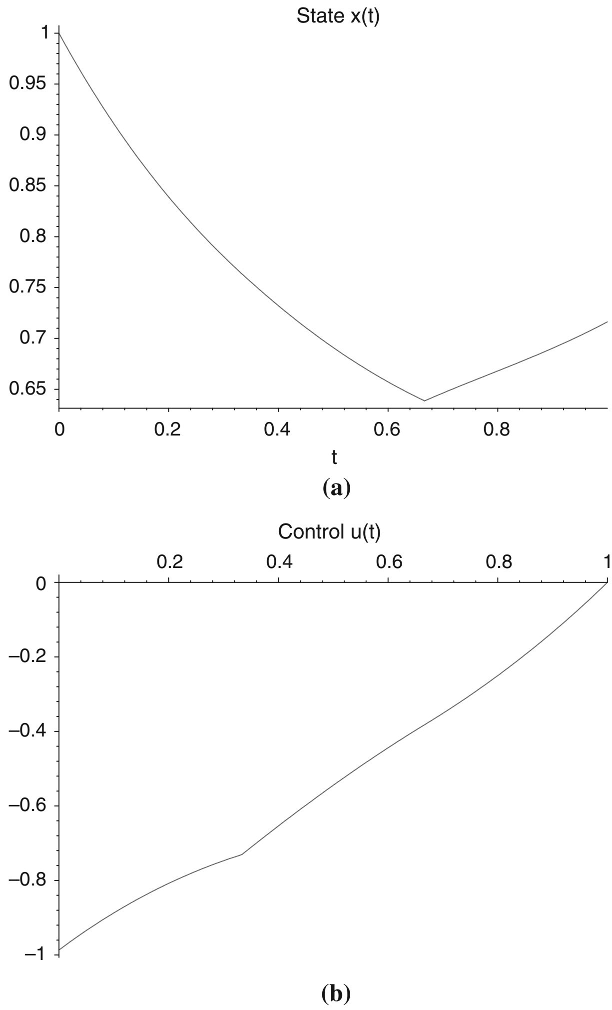

Fig. 3 State vector x(t) and control u(t) in Example 3. **a** State x(t). **b** Control u(t)

So it is easily to find that

$$X^T + X^T P_h - X^T D_{\tau_1} P_h - U^T P_h + \frac{1}{2} U^T D_{\tau_2} P_h - e^T = 0, \tag{64}$$

where we have $e = e_1 + e_2 + e_3$.

The functional J can be written as

$$J(X, U) = \frac{1}{2}\left(X^T L X + \frac{1}{2}U^T L U\right), \tag{65}$$

where L is the dual operational matrix of $H(t)$. Thus the system is reduced as follows

$$\begin{aligned} \min \quad & J(X, U) = \frac{1}{2}\left(X^T L X + \frac{1}{2}U^T L U\right), \\ s.t \quad & C^* = X^T + X^T P_h - X^T D_{\tau_1} P_h - U^T P_h + \frac{1}{2}U^T D_{\tau_2} P_h - e^T = 0. \end{aligned}$$

Approximate values of the cost function J with $N = 3$ and for $M = 4, 5, 6, 8, 10$ are given in the Table 4 and are compared with the solutions obtained in [9]. The curves of state and control functions for M = 6 are shown in Fig. 3.

Conclusion

A new approach in solving optimal control of time delay systems with quadratic performance index has been proposed using hybrid of general block-pulse functions and orthonormal Taylor series. The operational matrices of integration, dual, product and delay are obtained and used to reduce the solution of optimal control problem to the solution of algebraic equations. The operational matrices of integration and product have many zeros and so they are sparse matrices which makes hybrid functions computationally very attractive. So the computational cost is decreased. Illustrative examples demonstrate that the method is valid.

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