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Optimal Control of Vibration Suppression in Flexible Systems via Dislocated Sensor/Actuator Positioning

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ABSTRACT: *In this study, we present a new controller design method that not only effectively suppresses vibration in flexible systems, but also has the ability to save control energy. The proposed method allows integrated determination of sensor/actuator locations and feedback gain by minimizing the sum of the integral flexible system energy and the integral control energy. Also, the cost function is characterized by an effective representation of control systems, and is determined via an efficient solution of the Lyapunov equation. The optimization problem is solved by a recursive quadratic programming algorithm. The feasibility of applying this method to a simple and flexible structure confirms the direct relationship between our optimization criterion and effectiveness in vibration suppression.*

I. Introduction

The positioning of sensors/actuators has recently become a challenging task in controlling vibration in flexible systems, particularly since large, lightweight, lightly-damped structures are frequently used in current aerospace applications. Owing to the property of space distribution for flexible systems, determining the number and placement of control devices as well as feedback gain is inherently an integral part of controller design. Moreover, additional freedom offered by adjusting the placement of control devices can affect the control effect in addition to the feedback controller. Therefore, in this study, we develop an optimal controller that allows integrated determination of the placement of control devices and feedback gain to achieve the ultimate optimum desired performance in controlling flexible systems.

Much research has centered on the problem of selecting optimal locations for sensors and actuators (**1**, **2**). The solution to this problem depends primarily on the optimization criterion selected. Most of the criteria are used to determine sensor locations and actuator locations separately; while in some research, the actuator's location is emphasized since these investigators treated the feedback state as accessible. Therefore, selecting sensor locations would not be essential to determining the control effect. One method, as proposed by Schulz and Heimbold (**3**), has aroused great interest. This method allows integrated determination of sensor/actuator locations and feedback gain by maximizing the dissipation energy extracted by the action of the feedback system. Recently, Kondoh *et al.* (**4**) presented another optimization criterion based on minimizing the weighted sum of the integral square regulating error and the integral square input. Their approach attempted to efficiently control as well as effectively suppress vibration. The weighting matrices used are variables so as to have freedom for the solution of the optimal positioning of sensors/actuators. The optimization criterion is determined via Riccati equations. However, the primary limitation of this approach is that selecting the weighting matrices **Q** and **R** is quite arbitrary; in addition, the problem of determining the moderate weighting matrices remains unresolved.

In this study, we propose a new controller design method for optimal control of the vibration in flexible systems, thereby allowing integrated determination of dislocated sensor/actuator locations and feedback gain. The proposed method aims to simultaneously obtain efficient control and effective suppression of vibration. The optimization criterion is based on minimizing the sum of the integral flexible system energy, i.e. the flexible system energy is the sum of kinetic and potential energy of the flexible systems, and the integral control energy. Determining the cost function is transformed into an efficient solution of the Lyapunov equation, instead of the Riccati equation. This can subsequently eliminate approximation errors due to the pseudo-inverse operation in determining the direct output feedback gain. For illustrative purposes, a simple cantilever beam is adopted to evaluate our approach through direct velocity output feedback design (DVOFB).

II. System Modeling and Control

Consider a generalized undamped continuous flexible system described by the partial differential equation (**5**)

$$m(x)\ddot{w}(x,t)+\mathscr{L}w(x,t) = \mathbf{F}(x,t) \tag{1}$$

where $w(x,t)$ represents the space/time-dependent deformation of the structure, $\mathscr{L}$ a symmetric positive definite stiffness operator, $m(x)$ the positive mass density function, and $\mathbf{F}(x,t)$ the distributed control force. The space coordinate x is subject to given boundary conditions. Although some small internal damping is present in lightly-damped structures, this structure with no internal damping is used to examine a worst-case design. The control force distribution is provided by discrete force actuators at m points

$$\mathbf{F}(x,t) = \sum_{i=1}^{m} \delta(x-x_{ai})f_i(t) \tag{2}$$

where $f_i(t)$ is the actuator force amplitude and $\delta(x-x_{ai})$ represents the Dirac delta function.

By using the separation principle, an approximate solution of the homogeneous part of Eq. (1) is represented by

$$\begin{aligned} w(x,t) &= \sum_{i=1}^{n} \phi_i(x)q_i(t) \\ &= \boldsymbol{\phi}^{\mathrm{T}}(x)\mathbf{q}(t) \end{aligned} \tag{3}$$

where $\boldsymbol{\phi}(x)$ is a vector of the space-dependent admissible function solving the eigenvalue problem (mode shapes or eigenfunctions) and $\mathbf{q}(t)$ is the vector of the time-dependent generalized coordinates (modal coordinates). The value n, which represents the number of modeled modes, is chosen to be a sufficiently large finite number.

By substituting Eqns (2) and (3) into Eq. (1) and then using the orthonormality conditions of the eigenfunction $\phi_i(x)$, a system of second-order differential equations can be obtained as

$$\mathbf{M}\ddot{\mathbf{q}}(t)+\mathbf{K}\mathbf{q}(t) = \mathbf{B}(\mathbf{x}_a)\mathbf{f} \tag{4}$$

where $\mathbf{M} = \mathrm{diag}\,[1]$ represents the $n \times n$ normalized mass matrix, $\mathbf{K} = \mathrm{diag}\,[\omega_i^2]$ the $n \times n$ normalized stiffness matrix, $\mathbf{B}(\mathbf{x}_a) = [\boldsymbol{\phi}(x_{a1}), \ldots, \boldsymbol{\phi}(x_{am})]$ the $n \times m$ input matrix, and $\mathbf{f}$ the m-dimensional force input vector. Also, the r-dimensional velocity measurement vector $\mathbf{v}$ is denoted as

$$\mathbf{v} = \mathbf{C}_v(x_s)\dot{\mathbf{q}}(t) \tag{5}$$

with velocity measurement matrix

$$\mathbf{C}_v(\mathbf{x}_s) = \begin{bmatrix} \boldsymbol{\phi}^{\mathrm{T}}(x_{s1}) \\ \vdots \\ \boldsymbol{\phi}^{\mathrm{T}}(x_{sr}) \end{bmatrix} \tag{6}$$

where the superscript T represents the transposition of the matrix. Let the output feedback be defined by

$$\mathbf{f} = -\mathbf{G}_v\mathbf{v} \tag{7}$$

where $\mathbf{G}_v$ is an $m \times r$ constant gain matrix. Substituting Eqns (5) and (7) into Eq. (4) yields the closed-loop equation

$$\mathbf{M}\ddot{\mathbf{q}}(t)+\mathbf{B}(x_a)\mathbf{G}_v\mathbf{C}_v(\mathbf{x}_s)\dot{\mathbf{q}}(t)+\mathbf{K}\mathbf{q}(t)+0. \tag{8}$$

The control problem investigated here entails how to effectively suppress vibration when the system is initially subject to a disturbance force. The control system design attempts to determine actuator locations $\mathbf{x}_a$, sensor locations $\mathbf{x}_s$ and feedback gain $\mathbf{G}_v$ in one integrated design procedure. The motivation behind the design problem is to select an appropriate criterion that involves the effects of sensor/actuator placement and feedback gain.

III. Formulation of the Optimization Criterion

Selecting an appropriate criterion is essential in optimal control problems to find the most desirable performance. This criterion depends on how the system's physical requirements are translated into a mathematical form. To effectively suppress vibration in flexible systems, the total energy stored in the flexible system can be considered to be a good representation of the vibration response. The primary feature of this representation is that vibration measurement can be expressed as a scalar. The time behavior of this scalar function can be used to evaluate the effectiveness of vibration suppression. Furthermore, in light of practical and economic considerations, control energy must be saved in the controller design. Therefore, we propose a criterion which minimizes the sum of the integral flexible system energy and the integral control energy with respect to sensor/actuator locations and feedback gain. The quadratic cost function $\mathbf{J}$ is given by

$$\begin{aligned}\mathbf{J} &= \int_0^\infty \left([\mathbf{q}^T \quad \dot{\mathbf{q}}^T]\mathbf{Q}\begin{bmatrix}\mathbf{q}\\ \dot{\mathbf{q}}\end{bmatrix} + \mathbf{f}^T\mathbf{R}\mathbf{f}\right)\mathrm{d}t \\ &= \int_0^\infty \left([\mathbf{q}^T \quad \dot{\mathbf{q}}^T]\begin{bmatrix}\mathbf{K} & 0\\ 0 & \mathbf{M}\end{bmatrix}\begin{bmatrix}\mathbf{q}\\ \dot{\mathbf{q}}\end{bmatrix} + \mathbf{f}^T\mathbf{f}\right)\mathrm{d}t \end{aligned} \tag{9}$$

where $\mathbf{Q} = \mathrm{diag}\,[\mathbf{K}, \mathbf{M}]$ and $\mathbf{R} = \mathbf{I}$ are the weighting matrices. The first term of this equation represents the integral flexible system energy, and the second term represents the integral square input (the integral control energy). The objective is to make both values as small as possible. Substituting the control input [Eq. (7)] into Eq. (9) yields

$$\mathbf{J} = \int_0^\infty \left([\mathbf{q}^T \quad \dot{\mathbf{q}}^T]\begin{bmatrix}\mathbf{K} & 0\\ 0 & \mathbf{M} + \mathbf{C}_v^T(\mathbf{x}_s)\mathbf{G}_v^T\mathbf{G}_v\mathbf{C}_v(\mathbf{x}_s)\end{bmatrix}\begin{bmatrix}\mathbf{q}\\ \dot{\mathbf{q}}\end{bmatrix}\right)\mathrm{d}t. \tag{10}$$

The sum of integral flexible system energy and integral control energy $\mathbf{J}$ explicitly depends on the sensor locations $\mathbf{x}_s$ and feedback gain $\mathbf{G}_v$ and implicitly depends on the actuator locations $\mathbf{x}_a$. The optimization criterion with respect to $\mathbf{x}_s$, $\mathbf{x}_a$ and $\mathbf{G}_v$ is therefore formulated as

$$\min_{\mathbf{x}_s, \mathbf{x}_a, \mathbf{G}_v} \mathbf{J}(\mathbf{x}_s, \mathbf{x}_a, \mathbf{G}_v) \rightarrow \mathbf{x}_s^*, \mathbf{x}_a^*, \mathbf{G}_v^* \tag{11}$$

with constraints

$$\mathbf{x}_s \in X_s, \quad \mathbf{x}_a \in X_a, \quad \mathbf{G}_v \in G_v \tag{12}$$

where X_s and X_a are the regions on the flexible structure in which the sensors/actuators are allowed to be placed, and G_v poses upper bounds on the feedback gain $\mathbf{G}_v$.

In fact, the above-proposed criterion can be extended to an undamped conservative system with both displacement and velocity feedback. Let the measurement originate from m locally discrete displacement and velocity sensors. To simplify the underlying analysis, the locations of the displacement sensors and velocity sensors are assumed to be coincident. Therefore, the measurement output $\mathbf{y}(t)$,

$$\mathbf{y}(t) = [\mathbf{y}_d^T \quad \mathbf{y}_v^T]^T, \tag{13}$$

consists of two parts:

$$\mathbf{y}_d = \mathbf{C}_d(\mathbf{x}_s)\mathbf{q}(t) \tag{14}$$

and

$$\mathbf{y}_v = \mathbf{C}_v(\mathbf{x}_s)\dot{\mathbf{q}}(t) \tag{15}$$

where $\mathbf{C}_d$ is the displacement measurement matrix and $\mathbf{C}_d = \mathbf{C}_v$ by assumption.

Let the output feedback be defined by

$$\mathbf{f} = -[\mathbf{G}_d \quad \mathbf{G}_v]\mathbf{y}(t) \tag{16}$$

where $\mathbf{G}_d$ and $\mathbf{G}_v$ are both $m \times r$ constant non-negative definite gain matrices. Thus, the closed-loop equation becomes

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{B}(\mathbf{x}_a)\mathbf{G}_v\mathbf{C}_v(\mathbf{x}_s)\dot{\mathbf{q}}(t) + (\mathbf{K} + \mathbf{B}(\mathbf{x}_a)\mathbf{G}_d\mathbf{C}_d(\mathbf{x}_s))\mathbf{q}(t) = 0. \tag{17}$$

From Eq. (9), the quadratic cost function $\mathbf{J}$ can be written as

$$\mathbf{J} = \int_0^\infty \left([\mathbf{q}^T \quad \dot{\mathbf{q}}^T] \begin{bmatrix} \mathbf{K} + \mathbf{C}_d^T(\mathbf{x}_s)\mathbf{G}_d^T\mathbf{G}_d\mathbf{C}_d(\mathbf{x}_s) & \mathbf{C}_d^T(\mathbf{x}_s)\mathbf{G}_d^T\mathbf{G}_v\mathbf{C}_v(\mathbf{x}_s) \\ \mathbf{C}_v^T(\mathbf{x}_s)\mathbf{G}_v^T\mathbf{G}_d\mathbf{C}_d(\mathbf{x}_s) & \mathbf{M} + \mathbf{C}_v^T(\mathbf{x}_s)\mathbf{G}_v^T\mathbf{G}_v\mathbf{C}_v(\mathbf{x}_s) \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} \right) \mathrm{d}t. \tag{18}$$

IV. Determination of the Cost Function

Transforming the uncontrolled system of Eq. (4) into state space form yields

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & 0 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0 \\ \mathbf{B}(\mathbf{x}_a) \end{bmatrix}\mathbf{f} \tag{19}$$

$$\mathbf{v} = [0 \quad \mathbf{C}_v(\mathbf{x}_s)]\mathbf{z} \tag{20}$$

where $\mathbf{v}$ is velocity sensor output and $\mathbf{z} = [\mathbf{q}(t) \quad \dot{\mathbf{q}}(t)]^T$. Equations (19) and (20) can be rewritten more briefly as

$$\dot{\mathbf{z}} = \mathbf{A}'\mathbf{z} + \mathbf{B}'\mathbf{f} \tag{21}$$

$$\mathbf{v} = \mathbf{C}'\mathbf{z}. \tag{22}$$

For the active damping design [see Eq. (7)], the closed-loop system matrix is given as

$$\tilde{\mathbf{A}} = \mathbf{A}' + \mathbf{B}'\mathbf{G}_v\mathbf{C}'$$
$$= \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{B}(\mathbf{x}_a)\mathbf{G}_v\mathbf{C}_v(\mathbf{x}_s) \end{bmatrix}. \tag{23}$$

We define

$$\tilde{\mathbf{Q}} = \begin{bmatrix} \mathbf{K} & 0 \\ 0 & \mathbf{M}+\mathbf{C}_v^T\mathbf{G}_v^T\mathbf{G}_v\mathbf{C}_v \end{bmatrix}. \tag{24}$$

The quadratic cost function **J** can be written as

$$\mathbf{J} = \int_0^\infty \mathbf{z}^T\tilde{\mathbf{Q}}\mathbf{z}\,d\tau. \tag{25}$$

The application of standard state transformation techniques to Eq. (25) yields

$$\mathbf{J} = \mathbf{z}^T(0)\int_0^\infty e^{\tilde{\mathbf{A}}^T\tau}\cdot\tilde{\mathbf{Q}}\cdot e^{\tilde{\mathbf{A}}\tau}\,d\tau\mathbf{z}(0). \tag{26}$$

We now define

$$\mathbf{P} = -\int_0^\infty e^{\tilde{\mathbf{A}}^T\tau}\cdot\tilde{\mathbf{Q}}\cdot e^{\tilde{\mathbf{A}}\tau}\,d\tau, \tag{27}$$

which is a solution of the following Lyapunov equation

$$\tilde{\mathbf{A}}^T\mathbf{P}+\mathbf{P}\tilde{\mathbf{A}} = \tilde{\mathbf{Q}}. \tag{28}$$

The quadratic cost function is then determined as

$$\mathbf{J} = -\mathbf{z}^T(0)\mathbf{P}\mathbf{z}(0). \tag{29}$$

The determination of the quadratic cost function **J** of a flexible system with n modes requires the solution of the Lyapunov equation (28) for $2n \times 2n$ matrices. The solution of Eq. (28) exists and is unique as long as the eigenvalues of $\tilde{\mathbf{A}}$ have negative real parts (**6**). The numerical solution of the Lyapunov equation can be obtained efficiently using Bartels and Stewart's algorithm (**7**), as symmetry of $\tilde{\mathbf{Q}}$ is guaranteed. During optimization, the search path is restricted to matrices $\tilde{\mathbf{A}}$ having eigenvalues with negative real parts.

Guaranteeing the stability for dislocated control systems still remains an unresolved problem. In general, the control system is designed on the basis of a low-order system with few modes because of accuracy and calculation time restrictions. Instability may occur as soon as truncated residual modes are implemented in the model and have a zero crossing between actuator and sensor positions. An extension of our optimization method can treat this unavoidable spillover effect by adding an additional constraint. Under this constraint, the controller-induced damping matrix for the closed-loop higher-order system must be at least semidefinite. Schulz and Heimbold (**3**) proposed similar extensions for the energy dissipation method.

The optimization criterion can also be determined while the structure of controller is selected as the displacement and velocity feedback. Since its configuration in the controller design for vibration suppression is quite similar to the DVOFB and much computational effort is involved to obtain the minimum value of cost function, this case is not provided with illustrative examples here. However, determination of the cost function is presented in the Appendix for further reference.

V. Application of the Proposed Method to a Cantilever Beam

In this section, we apply the proposed method through the DVOFB controller to a relatively simple cantilevered gravity-free beam, for which characteristic data taken from Schulz and Heimbold (**3**) are listed in Table I. As in (**3**), an impulse surface load acting on the beam was chosen as a disturbance force; this force is inversely proportional to the mode amplitude. In the following design example, the first four modes are considered to be the controlled modes. The initial conditions of the generalized coordinate vector are given by

$$\mathbf{q}(0)^{T} = [0 \quad 0 \quad 0 \quad 0]$$

$$\dot{\mathbf{q}}(0)^{T} = [0.525 \quad 0.292 \quad 0.171 \quad 0.122]$$

with the initial total energy $H_0 = 0.203$.

The optimization problem as formulated earlier is a nonlinear optimization with constraints. The constraints are inequality constraints in the actuator locations $\mathbf{x}_a$, sensor locations $\mathbf{x}_s$ and the feedback gain $\mathbf{G}_v$. An inherent problem with the optimization procedures used is that a theoretically global optimum is difficult to find. The optimal parameters achieved depend on their initial values, i.e. only several locally optimal solutions can be obtained. Consequently, when searching for a possible globally optimal solution, various initial guesses for the sensor/actuator positioning and feedback gain are given in each case. In this study, the optimization program provided by IMSL Inc. (**8**) is used to compute the optimization.

In the case where only one sensor/actuator pair is used for the proposed design, the constraints are bounds on x_a, x_s and $G_{v_{11}}$:

$$0 < x_a, \quad x_s \leqslant L$$

$$-100 \leqslant G_{v_{11}} \leqslant 100.$$

To treat spillover effects discussed in Section 4, a constraint is added:

$$\mathbf{D}_c^* \geqslant 0$$

TABLE I

Flexible structure specifications: characteristic data for a cantilevered gravity free beam of length L

Mass/length	$\bar{m} = 1.49$ kg/m
Length	$L = 3.81$ m
Modulus of elasticity	$E = 2.07 \times 10^5$ N/mm^2
Moments of inertia	$J = 6.35$ mm^4
Stiffness	$EJ = 1.31$ N m^2
Eigenfrequency of the first four modes:	
$w_1 = 0.227$ rad/s	$w_2 = 1.42$ rad/s
$w_3 = 3.99$ rad/s	$w_4 = 7.85$ rad/s

TABLE II
Optimal positions and feedback gain for an impulse surface load

Solution	Sensor (x_s/L)	Actuator (x_a/L)	Gain	Quadratic cost	Integral control energy
1	0.075	0.336	28.808	3.152	1.0078
2	0.300	0.338	2.989	2.335	0.1069
3	0.448	0.469	1.441	1.364	0.0597
4	0.597	0.616	0.985	0.954	0.0179
5	0.685	0.700	1.081	0.792	0.0069
6	0.894	0.893	0.478	2.423	0.0033
7	1.0	0.999	0.184	1.108	0.002

where

$$\mathbf{D}_c^* = \tfrac{1}{2}\{\mathbf{B}^*(x_a)G_{v_{11}}\mathbf{C}^*(x_s) + [\mathbf{B}^*(x_a)G_{v_{11}}\mathbf{C}^*(x_s)]^T\}.$$

Thereby, $\mathbf{B}^*(x_a)$ and $\mathbf{C}^*(x_s)$ represent the corresponding $\mathbf{B}$ and $\mathbf{C}$ matrices (with n number of modes) for a system with n^* number of modes, where $n^* > n$. In this study, n^* is equal to 6 and n is equal to 4.

The subroutine NCONF in the IMSL program is called for. This subroutine minimizes a function using the successive quadratic programming algorithm and a finite-difference gradient. Table II lists the optimization results with a set of 50 initial values for x_a, x_s, $G_{v_{11}}$ and integral control energy. Solution 5 is the smallest cost value that the authors could find among those local optima. Figure 1 illustrates these seven locally optimal positions. Moreover, Figs 2 and 3 display the time responses at points $x_1 = 0.25L$, $x_2 = 0.5L$, $x_3 = 0.75L$ and $x_4 = 1.0L$ of the beam and control input for each solution as well as that of the open loop case (without control input); every curve of the time response in each figure is normalized with respect to the largest deviation from the equilibrium occuring at x_4 of the open loop case. Because decay time of the

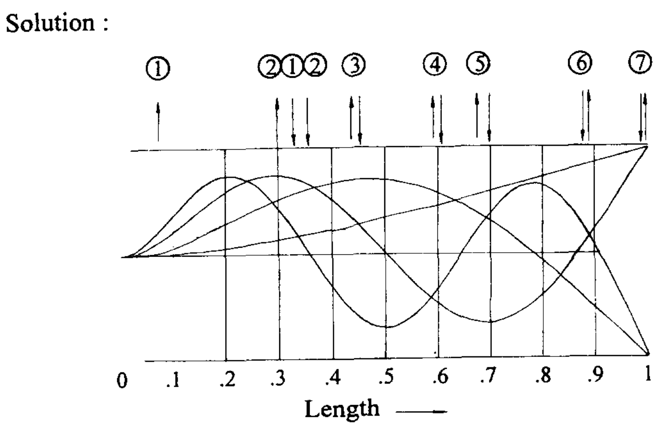

FIG. 1. Mode shapes and optimal actuator (↓) and sensor (↑) positions.

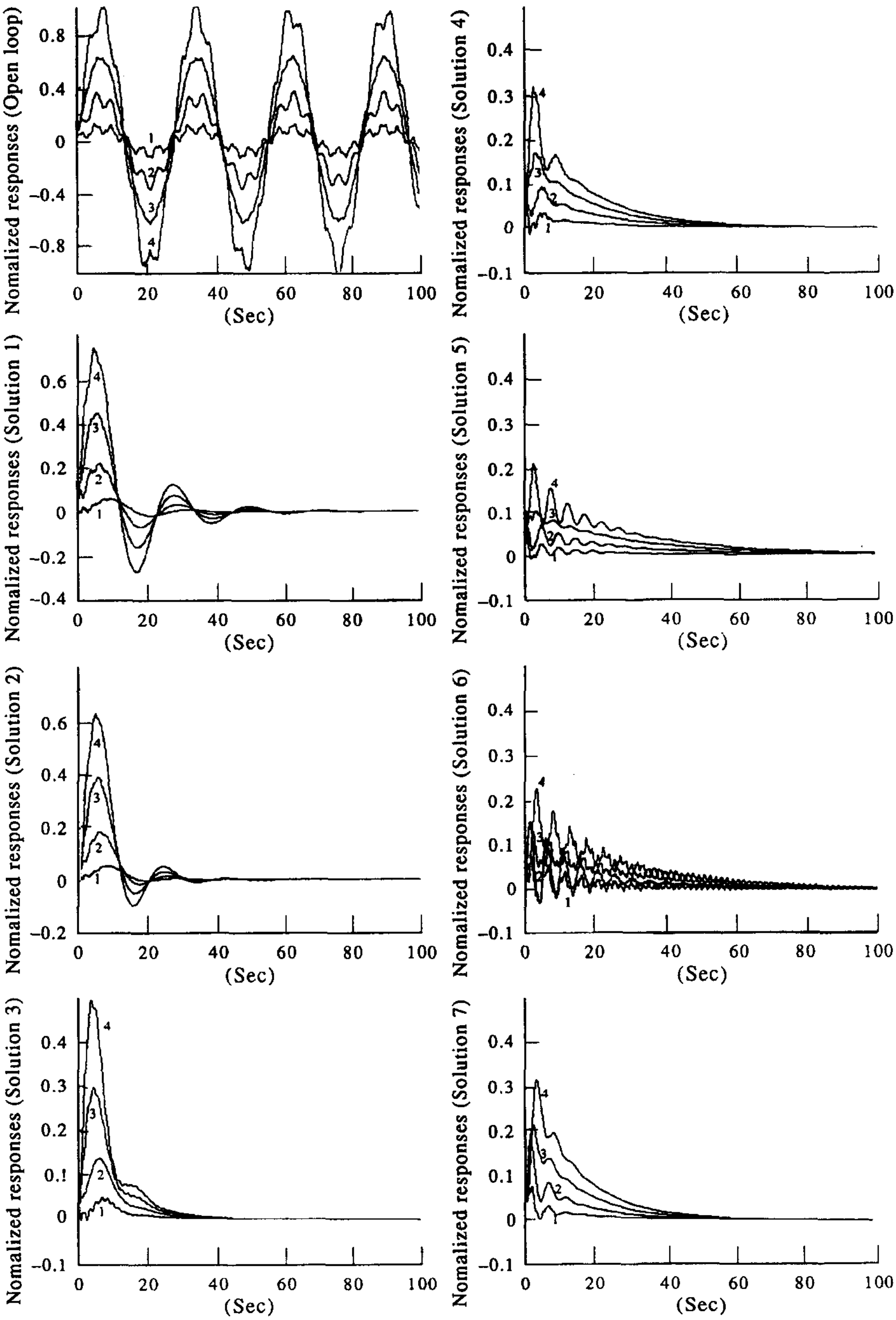

FIG. 2. Time responses for one sensor/actuator pair at (1) $x_1 = 0.25L$, (2) $x_2 = 0.5L$, (3) $x_3 = 0.75L$, (4) $x_4 = 1.0L$.

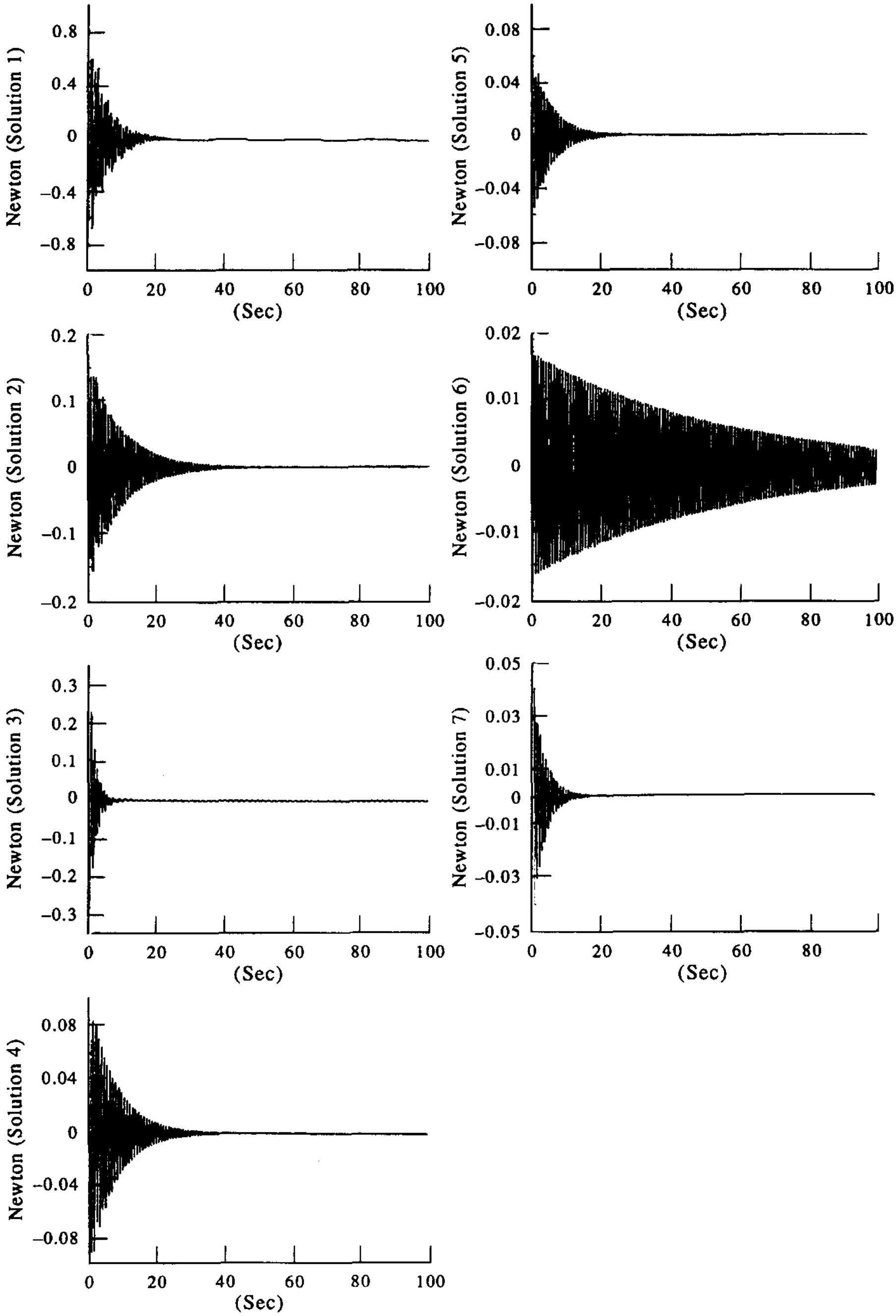

FIG. 3. Control input force for one sensor/actuator pair.

system response is not constrained here, the convergence rate of the responses in Solution 5 is not the fastest; however, the maximum amplitude of responses is the smallest among those local optima. Furthermore, because of the constraint of integral control energy, an economic control input can be obtained. Thus, it is a trade-off between the integral system energy and integral control energy.

Next, we show the case where two pairs of sensors/actuators are used for the proposed design; the constraints are bounds on $\mathbf{x}_a$, $\mathbf{x}_s$ and $\mathbf{G}_v$:

$$0 < x_{s1}, x_{s2}, x_{a1}, x_{a2} \leqslant L$$
$$0 < G_{v_{11}}, G_{v_{12}}, G_{v_{21}}, G_{v_{22}} \leqslant 50.$$

Similarly, to treat spillover effects, a constraint is also added:

$$\mathbf{D}_c^* \geqslant 0, \quad \text{where } n^* = 6.$$

Here, only one solution of optimization is presented with a set of 20 initial values for $\mathbf{x}_a$, $\mathbf{x}_s$, $\mathbf{G}_v$, quadratic cost and integral control energy:

Actuators: $\mathbf{x}_a^T = [0.399 \quad 0.826]$
Sensors: $\mathbf{x}_s^T = [0.506 \quad 0.848]$

Feedback gain: $\mathbf{G}_v = \begin{bmatrix} 1.35 & 0.00 \\ 0.00 & 1.35 \end{bmatrix}$

Optimal quadratic cost: $\mathbf{J} = 0.4593$
Integral control energy: $F_1 = 0.3699$.

The quadratic cost for this case is less than that for the case of one sensor/actuator pair. The number of sensors/actuators obviously plays an influential role on the effectiveness of vibration suppression. Figure 4(a) shows the time responses of the beam in this design. Although the integral control energy for this case is larger than that for the case of one sensor/actuator pair, the control system performance is much better because of low quadratic cost $\mathbf{J}$.

If we chose the cost function $\hat{\mathbf{J}}$ = integral flexible system energy, the optimal problem becomes

$$\min_{\mathbf{x}_s, \mathbf{x}_a, \mathbf{G}_v} \hat{\mathbf{J}}(\mathbf{x}_s, \mathbf{x}_a, \mathbf{G}_v) \rightarrow \mathbf{x}_s^*, \mathbf{x}_a^*, \mathbf{G}_v^* \tag{30}$$

with constraints

$$\mathbf{x}_s \in X_s, \quad \mathbf{x}_a \in X_a, \quad \mathbf{G}_v \in G_v. \tag{31}$$

One locally optimal solution is illustrated as follows:

Actuators: $\mathbf{x}_a^T = [0.399 \quad 0.826]$
Sensor: $\mathbf{x}_s^T = [0.506 \quad 0.848]$

Feedback gain: $\mathbf{G}_v = \begin{bmatrix} 7.798 & 5.636 \\ 2.580 & 9.630 \end{bmatrix}$

Optimal quadratic cost: $\hat{\mathbf{J}} = 1.246 \times 10^{-2}$
Proposed quadratic cost: $\mathbf{J} = 15.3577$
Integral total control energy: $F_2 = 15.3565$.

Figure 4(b) displays the time response. The normalized response is excellent in this

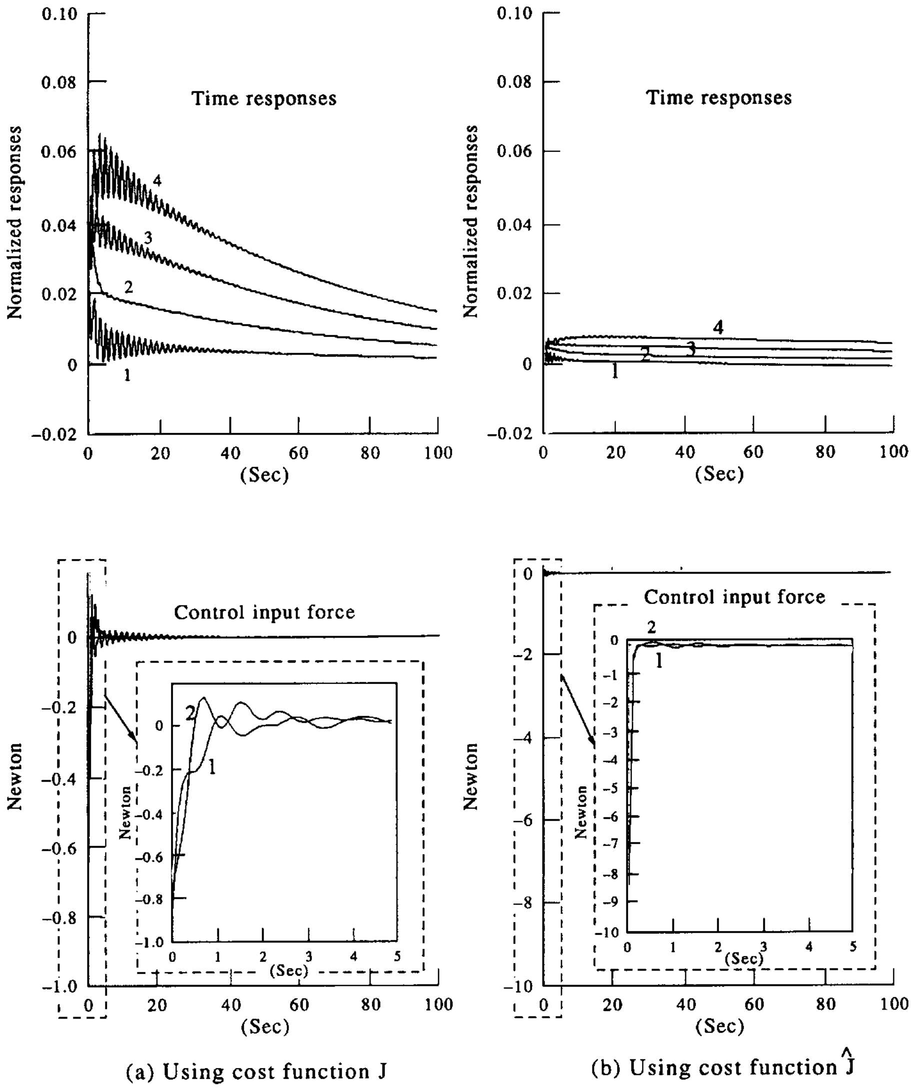

FIG. 4. Time responses and control input force for two sensor/actuator pairs using different cost functions.

case (the integral flexible system energy is 1.246×10^{-2}); however, the integral control energy is extremely large ($F_2 \gg F_1$) and the maximum absolute value of control input is also large. Hence, no guarantee is made of economic control input in this case.

VI. Conclusions

This paper presents an optimal controller design method which effectively suppresses vibration in flexible systems and saves control efforts. This method allows integrated determination of sensor/actuator locations and feedback gain. In the structure of direct velocity feedback design, the quadratic cost function of the infinite-time linear quadratic problem is reduced to an efficient solution of a Lyapunov equation. In numerical calculations, the optimization is performed by a recursive quadratic programming algorithm. Applying this method to a simple flexible structure demonstrates that the vibration can be effectively suppressed with an economic control input. Numerical results also reveal that, in addition to the placement of sensors/actuators, the number of sensors/actuators plays an influential role in vibration suppression.

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Appendix—Determination of the Cost Function with Displacement and Velocity Feedback

Here, we reconsider the direct output feedback controller with displacement and velocity feedback in Section III. Transforming the uncontrolled system of Eq. (4) into a state space form yields

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & 0 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0 \\ \mathbf{B}(\mathbf{x}_a) \end{bmatrix}\mathbf{f} \tag{A1}$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{C}_d(\mathbf{x}_s) & 0 \\ 0 & \mathbf{C}_v(\mathbf{x}_s) \end{bmatrix}\mathbf{z} \tag{A2}$$

where $\mathbf{y}$ is displacement and velocity sensor output. Equations (A1) and (A2) can be rewritten in a brief form as

$$\dot{\mathbf{z}} = \mathbf{A}'\mathbf{z} + \mathbf{B}'\mathbf{f} \tag{A3}$$

$$\mathbf{y} = \mathbf{C}'\mathbf{z}. \tag{A4}$$

For the displacement and velocity feedback design [Eq. (16)], the closed-loop system matrix is given as

$$\tilde{\mathbf{A}} = \mathbf{A}' + \mathbf{B}'[\mathbf{G}_d \quad \mathbf{G}_v]\mathbf{C}'$$
$$= \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} - \mathbf{M}^{-1}\mathbf{B}(\mathbf{x}_a)\mathbf{G}_d\mathbf{C}_d(\mathbf{x}_s) & -\mathbf{M}^{-1}\mathbf{B}(\mathbf{x}_a)\mathbf{G}_v\mathbf{C}_v(\mathbf{x}_s) \end{bmatrix}. \tag{A5}$$

According to Eq. (18), we define

$$\tilde{\mathbf{Q}} = \begin{bmatrix} \mathbf{K} + \mathbf{C}_d^T(\mathbf{x}_s)\mathbf{G}_d^T\mathbf{G}_d\mathbf{C}_d(\mathbf{x}_s) & \mathbf{C}_d^T(\mathbf{x}_s)\mathbf{G}_d^T\mathbf{G}_v\mathbf{C}_v(\mathbf{x}_s) \\ \mathbf{C}_v^T(\mathbf{x})\mathbf{G}_v^T\mathbf{G}_d\mathbf{C}_d(\mathbf{x}_s) & \mathbf{M} + \mathbf{C}_v^T(\mathbf{x}_s)\mathbf{G}_v^T\mathbf{G}_v\mathbf{C}_v(\mathbf{x}_s) \end{bmatrix}. \tag{A6}$$

The quadratic cost function $\mathbf{J}$ can be written as

$$\mathbf{J} = \int_0^\infty \mathbf{z}^T\tilde{\mathbf{Q}}\mathbf{z}\,d\tau. \tag{A7}$$

Applying standard state transformation techniques to Eq. (A7) yields

$$\mathbf{J} = \mathbf{z}^T(0)\int_0^\infty e^{\tilde{\mathbf{A}}^T\tau}\cdot\tilde{\mathbf{Q}}\cdot e^{\tilde{\mathbf{A}}\tau}\,d\tau\,\mathbf{z}(0). \tag{A8}$$

We now define

$$\mathbf{P} = -\int_0^\infty e^{\tilde{\mathbf{A}}^T\tau}\cdot\tilde{\mathbf{Q}}\cdot e^{\tilde{\mathbf{A}}\tau}\,d\tau, \tag{A9}$$

which is a solution of the following Lyapunov equation

$$\tilde{\mathbf{A}}^T\mathbf{P} + \mathbf{P}\tilde{\mathbf{A}} = \tilde{\mathbf{Q}}. \tag{A10}$$

The quadratic cost function is then determined as

$$\mathbf{J} = -\mathbf{z}^T(0)\mathbf{P}\mathbf{z}(0). \tag{A11}$$

Determining the cost function $\mathbf{J}$ requires solving the Lyapunov equation (A10) for $2n \times 2n$ matrices. The procedure of solution is the same as in Section IV.