

Optimal control with absolutely continuous strategies for spectrally negative Lévy processes*

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Abstract

In the last few years there has been renewed interest in the classical control problem of de Finetti [10] for the case that underlying source of randomness is a spectrally negative Lévy process. In particular a significant step forward is made in [25] where it is shown that a natural and very general condition on the underlying Lévy process which allows one to proceed with the analysis of the associated Hamilton-Jacobi-Bellman equation is that its Lévy measure is absolutely continuous, having completely monotone density.

In this paper we consider de Finetti's control problem but now with the restriction that control strategies are absolutely continuous with respect to Lebesgue measure. This problem has been considered by Asmussen and Taksar [2], Jeanblanc and Shiryaev [19] and Boguslavskaya [9] in the diffusive case and Gerber and Shiu [16] for the case of a Cramér-Lundberg process with exponentially distributed jumps. We show the robustness of the condition that the underlying Lévy measure has a completely monotone density and establish an explicit optimal strategy for this case that envelopes the aforementioned existing results. The explicit optimal strategy in question is the so-called refraction strategy.

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1 Introduction and main result

Recently there has been a growing body of literature which explores the interaction of classical models of ruin and fluctuation theory of Lévy processes; see for example [4, 8, 12, 17, 18, 20, 22, 23, 25–28]. Of particular note in this respect is the application of the theory of scale functions for spectrally negative Lévy processes. This article adds to the aforementioned list by addressing a modification of de Finetti’s classical dividend problem through the theory of scale functions. Before turning to our main results, let us first attend to the basic definitions of the mathematical objects that we are predominantly interested in.

Recall that a spectrally negative Lévy process is a stochastic process issued from the origin which has càdlàg paths and stationary and independent increments such that there are no positive discontinuities. To avoid degenerate cases in the forthcoming discussion, we shall additionally exclude from this definition the case of monotone paths. This means that we are not interested in the case of a deterministic increasing linear drift or the negative of a subordinator.

Henceforth we assume that $X = \{X_t : t \geq 0\}$ is a spectrally negative Lévy process under \mathbb{P} with Lévy triplet given by (γ, σ, ν) , where $\gamma \in \mathbb{R}$, $\sigma \geq 0$ and ν is a measure concentrated on $(0, \infty)$ satisfying

$$\int_{(0, \infty)} (1 \wedge z^2) \nu(dz) < \infty.$$

The Laplace exponent of X is given by

$$\psi(\lambda) = \log \mathbb{E} [e^{\lambda X_1}] = \gamma\lambda + \frac{1}{2}\sigma^2\lambda^2 - \int_{(0, \infty)} (1 - e^{-\lambda z} - \lambda z \mathbf{1}_{\{0 < z \leq 1\}}) \nu(dz),$$

which is well defined for $\lambda \geq 0$. Here \mathbb{E} denotes expectation with respect to \mathbb{P} . The reader will note that, for convenience, we have arranged the representation of the Laplace exponent in such a way that the support of the Lévy measure is positive even though the process experiences only negative jumps. As a strong Markov process we shall endow X with probabilities $\{\mathbb{P}_x : x \in \mathbb{R}\}$ such that under \mathbb{P}_x we have $X_0 = x$ with probability one. Note that $\mathbb{P}_0 = \mathbb{P}$.

It is well-known that X has paths of bounded variation if and only if $\sigma = 0$ and $\int_0^1 z\nu(dz) < \infty$. In this case X can be written as

$$X_t = ct - S_t, \quad t \geq 0, \tag{1.1}$$

where $c = \gamma + \int_0^1 z\nu(dz)$ and $\{S_t : t \geq 0\}$ is a driftless subordinator. Note that necessarily $c > 0$ and $\nu \not\equiv 0$, since we have ruled out the case that X has monotone paths. Moreover, when $\nu(0, \infty) < \infty$, then X is known in the actuarial mathematics literature as the classical Cramér-Lundberg risk process. This process is often used to model the surplus wealth of an insurance company.

The classical theory of ruin concerns itself with the path of the stochastic risk process until the moment that it first passes below the level zero; the event corresponding to ruin. An offshoot of the classical ruin problem was introduced by de Finetti [10]. His intention was to make the study of ruin more realistic by introducing the possibility that dividends are paid out to shareholders up to the moment of ruin. Further, the payment of dividends should be made in such a way as to optimize the expected net present value of the total dividends paid to the shareholders from time zero until ruin. Mathematically speaking, de Finetti's dividend problem amounts to solving a control problem which we state in the next paragraph. Although de Finetti's dividend problem has its origin in insurance mathematics, there are several papers [6, 7, 30] that have considered this problem into the context of corporate finance.

Let $\pi = \{L_t^\pi : t \geq 0\}$ be a dividend strategy, meaning that it is a left-continuous non-negative non-decreasing process adapted to the (completed and right continuous) filtration $\mathbb{F} := \{\mathcal{F}_t : t \geq 0\}$ of X . The quantity L_t^π thus represents the cumulative dividends paid out up to time t by the insurance company whose risk process is modelled by X . An additional constraint on π is that $L_{t+}^\pi - L_t^\pi \leq \max\{U_t^\pi, 0\}$ for $t \geq 0$ (i.e. lump sum dividend payments are always smaller than the available reserves). The π -controlled Lévy process is thus $U^\pi = \{U_t^\pi : t \geq 0\}$ where $U_t^\pi = X_t - L_t^\pi$. Write $\sigma^\pi = \inf\{t > 0 : U_t^\pi < 0\}$ for the time at which ruin occurs when the dividend payments are taken into account. Suppose that Π denotes some family of admissible strategies, which we shall elaborate on later. Then the expected net present value of the dividend policy $\pi \in \Pi$ with discounting at rate $q > 0$ and initial capital $x \geq 0$ is given by

$$v_\pi(x) = \mathbb{E}_x \left[\int_{[0, \sigma^\pi]} e^{-qt} dL_t^\pi \right],$$

where \mathbb{E}_x denotes expectation with respect to \mathbb{P}_x and $q > 0$ is a fixed rate. *De Finetti's dividend problem* consists of characterising the optimal value function,

$$v_*(x) := \sup_{\pi \in \Pi} v_\pi(x), \tag{1.2}$$

and, further, if it exists, establish a strategy π^* such that $v_*(x) = v_{\pi^*}(x)$.

In the case that Π consists of all strategies as described at the beginning of the previous paragraph, de Finetti's dividend problem belongs to the class of *singular* stochastic control problems; the term 'singular' refers to the property that the controls are allowed to be singular (with respect to Lebesgue measure) in time. For this case there are now extensive results in the literature, most of which have appeared in the last few years. Initially this problem was considered by Gerber [14] who proved that, for the Cramér-Lundberg model with exponentially distributed jumps, the optimal value function is the result of a *reflection strategy*. That is to say, a strategy of the form $L_t^a = (a \vee \bar{X}_t) - a$ for some, optimally chosen,

barrier $a \geq 0$ where $\bar{X}_t := \sup_{s \leq t} X_s$. In that case the controlled process $U_t^a = X_t - L_t^a$ is a spectrally negative Lévy process reflected at the barrier a . However, a sequence of innovative works [4, 5, 23, 25, 26, 28] have pushed this conclusion much further into the considerably more general setting where X is a spectrally negative Lévy process. Of particular note amongst these references is the paper of Loeffen [25] in which the optimality of the reflection strategy is shown to depend in a very subtle way on the shape of the so-called *scale functions* associated to the underlying Lévy process. Indeed Loeffen's new perspective on de Finetti's control problem leads to very easily verifiable sufficient conditions for the reflection strategy to be optimal. Loeffen shows that it suffices for the Lévy measure ν to be absolutely continuous with a completely monotone density. Though this assumption seems quite restrictive at first sight, there are actually plenty of examples of general spectrally negative Lévy processes that are used in risk theory and satisfy this assumption, see p.1677-1678 in [25]. Through largely technical adaptations of Loeffen's method, this sufficient condition was relaxed in [23, 28]. It is important to note that in general a barrier strategy is not always an optimal strategy; an explicit counter-example was provided by Azcue and Muler [5].

In this article we are interested in addressing an adaptation of de Finetti's dividend problem by considering a smaller class of admissible strategies. Specifically, we are interested in the case that, in addition to the assumption that strategies are non-decreasing and \mathcal{F} -adapted, Π only admits absolutely continuous strategies $\pi = \{L_t^\pi : t \geq 0\}$ such that

$$L_t^\pi = \int_0^t \ell^\pi(s) ds, \quad (1.3)$$

and for $t \geq 0$, $\ell^\pi(t)$ satisfies

$$0 \leq \ell^\pi(t) \leq \delta, \quad (1.4)$$

where $\delta > 0$ is a ceiling rate. Moreover, we make the assumption that

$$\delta < \gamma + \int_0^1 z \nu(dz) \quad \text{if } X \text{ has paths of bounded variation.} \quad (\mathbf{H})$$

Note that for a reflection strategy, when X has paths of unbounded variation the corresponding dividend process is supported by increase times that are singular with respect to Lebesgue measure and when X has paths of bounded variation, the dividend process is supported by increase times which are absolutely continuous with respect to Lebesgue measure with rate c (cf. Section 6.1 of [21]). Recalling the decomposition (1.1) and assumption **(H)**, we see that the reflection strategy is therefore not included into the smaller class of admissible controls and we are left with a truly different control problem. A particular motivation for studying this kind of modification is that, if a reflection strategy is applied, then the company will get ruined in finite time with probability one, which is seen as an undesired consequence, cf. Gerber [15]. By restricting the set of admissible strategies in the way described above and

provided $\mathbb{E}[X_1] > \delta$, we make sure that there is a strictly positive probability that ruin will never occur no matter which admissible dividend strategy is applied.

The reader familiar with optimal control problems of this kind, will recognize that the optimal strategy should be of bang-bang type, i.e. depending on the value of the controlled process, dividends should either be paid out at the maximum rate δ or at the minimum rate 0. A particularly simple bang-bang strategy is the one that we refer to here as a *refraction strategy*, which, in words, is the strategy where dividends are paid out at the maximum rate when the controlled process is above a certain level $b \geq 0$ and at the minimum rate when below b . Mathematically, a refraction strategy at b is the strategy which corresponds to the controlled process taking the form of the unique strong solution to the following stochastic differential equation,

$$dU_t^b = dX_t - \delta \mathbf{1}_{\{U_t^b > b\}} dt, \quad t \geq 0. \quad (1.5)$$

In the case of absolutely continuous control strategies for X , it has been shown by Asmussen and Taksar [2], Jeanblanc and Shiryaev [19] and Boguslavskaya [9] in the diffusive case and by Gerber and Shiu [16] for the case of a Cramér-Lundberg process with exponentially distributed jumps that a refraction strategy, where $b \geq 0$ is optimally chosen, is optimal. This particular control problem is also discussed in the review papers of Avanzi [3] and Albrecher and Thonhauser [1] and in the book of Schmidli [31].

In the spirit of earlier work for the more general class of admissible strategies, the point of view we shall take here is to deal with a general spectrally negative Lévy process and give sufficient conditions under which a refraction strategy of the form (1.5) is optimal. Note that when X is a general spectrally negative Lévy process, the strong existence and uniqueness of solutions to (1.5) under (H), so called *refracted Lévy processes*, were established in Kyprianou and Loeffen [24]. Our main result is the following.

Theorem 1. *Suppose the Lévy measure has a completely monotone density. Let Π be the class of admissible dividend strategies satisfying (1.3), (1.4) and (H). Then an optimal strategy, i.e. a strategy which attains the supremum in (1.2), is formed by a refraction strategy.*

Given the special cases for which Theorem 1 is already known, the recent developments on the singular version of de Finetti's problem and recent developments concerning refracted Lévy processes, the statement of Theorem 1 (and the nature of its proof) should not come as a great surprise. However, a particular step of the proof, namely Lemma 8 below, turned out to be a quite difficult puzzle to solve. This is mainly due to the fact that the expression, written in terms of scale functions, for the value function of a refraction strategy is substantially more complicated than that of a reflection strategy; compare Equation (10.25) in [24] with Proposition 1 in [4]. The above theorem offers the same sufficient condition on the Lévy measure as Loeffen [25] for the larger, general class of admissible strategies. Although,

as alluded to above, weaker assumptions have been established in that case, the technical details of our method appears not to allow us to follow suit. To illustrate the difference between the two cases, we give in Remark 9 a specific example of X and q for which no refraction strategy can be optimal in the restricted case for a certain choice of the ceiling rate δ , whereas a reflection strategy is optimal within the general class of admissible dividend strategies. We also remark that in fact our method allows us to give a more quantitative result than Theorem 1 in the sense that we are able to characterise the threshold b^* associated with the optimal refraction strategy. As some more notation is needed to do this, it is given at the end of the paper in Corollary 10.

We close this section with a brief summary of the remainder of the paper. In the next section we show the role played by scale functions in giving a workable identity for the expected net present value of the paid out dividends until ruin in the case where a refraction strategy is applied. We also use this identity to describe an appropriate candidate for the threshold associated with the optimal refraction strategy. Then in the final section we put together a series of technical lemmas which allow us to verify the optimality of the identified threshold strategy. The assumption that ν has a completely monotone density will repeatedly play a very significant role in the aforementioned lemmas.

2 Scale functions and refraction strategies

As alluded to above, a key element of the forthcoming analysis relies on the theory of so-called scale functions. We therefore devote some time in this section reminding the reader of some fundamental properties of scale functions as well as their relevance to refraction strategies.

For each $q \geq 0$ the so called q -scale function of X , $W^{(q)} : \mathbb{R} \rightarrow [0, \infty)$, is the unique function such that $W^{(q)}(x) = 0$ for $x < 0$ and on $[0, \infty)$ is a strictly increasing and continuous function whose Laplace transform is given by

$$\int_0^\infty e^{-\theta x} W^{(q)}(x) dx = \frac{1}{\psi(\theta) - q}, \quad \theta > \Phi(q). \quad (2.1)$$

Here

$$\Phi(q) = \sup\{\lambda \geq 0 : \psi(\lambda) = q\}$$

and is well defined and finite for all $q \geq 0$ as a consequence of the well known fact that ψ is a strictly convex function satisfying $\psi(0) = 0$ and $\psi(\infty) = \infty$. Note that there is an abuse of notation here as the parameter q of the scale functions has also been used to denote the discount rate. However, since we will only need the q -scale function (and $\Phi(q)$) with the corresponding parameter q being equal to the discount rate, this should not bring too much confusion.

Shape and smoothness properties of the scale functions $W^{(q)}$ will be of particular interest to us in the forthcoming analysis. In the discussion below we shall consider the behaviour of $W^{(q)}$ at $0, \infty$ as well as describing qualitative features of its shape on $(0, \infty)$. We start with some standard facts concerning the behaviour of the scale function in the neighbourhood of the origin. Recall that we have defined the constant

$$c = \gamma + \int_0^1 z \nu(dz)$$

in the case that X has bounded variation paths.

The following result is well known and can easily be deduced from (2.1). See for example Chapter 8 of [21].

Lemma 2. *As $x \downarrow 0$, the value of the scale function $W^{(q)}(x)$ and its right derivative are determined for every $q \geq 0$ as follows*

$$\begin{aligned} W^{(q)}(0+) &= \begin{cases} 1/c & \text{when } \sigma = 0 \text{ and } \int_0^1 z \nu(dz) < \infty, \\ 0 & \text{otherwise,} \end{cases} \\ W^{(q)'}(0+) &= \begin{cases} 2/\sigma^2 & \text{when } \sigma > 0, \\ (\nu(0, \infty) + q)/c^2 & \text{when } \sigma = 0 \text{ and } \nu(0, \infty) < \infty, \\ \infty & \text{otherwise.} \end{cases} \end{aligned} \quad (2.2)$$

In general it is known that one may always write for $q \geq 0$

$$W^{(q)}(x) = e^{\Phi(q)x} W_{\Phi(q)}(x), \quad (2.3)$$

where $W_{\Phi(q)}$ plays the role of a 0-scale function of an auxiliary spectrally negative Lévy process with Laplace exponent given by $\psi_{\Phi(q)}(\lambda) = \psi(\lambda + \Phi(q)) - q$. Note the fact that $\psi_{\Phi(q)}$ is the Laplace exponent follows by an exponential tilting argument, see for example Chapter 8 of Kyprianou [21]. In the same reference one also sees that $\lim_{x \uparrow \infty} W_{\Phi(q)}(x) < 1/\psi'(\Phi(q)) < \infty$, which suggests that, when $q > 0$, the function $W^{(q)}(x)$ behaves like the exponential function $e^{\Phi(q)x}$ for large x . It is therefore natural to ask whether $W^{(q)}(x)$ is convex for large values of x . This very question was addressed in Loeffen [25, 26]. In these papers it was found that, due to quite a deep connection between scale functions and potential measures of subordinators, a natural assumption which allows one to address the issue of convexity, and, in fact, say a lot more (cf. Lemma 3 below), is that the Lévy measure ν is absolutely continuous with completely monotone density. In the next lemma we collect a number of consequences of this assumption, lifted from the aforementioned two papers. We need first some more notation. Recalling that $W^{(q)}$ is continuously differentiable on $(0, \infty)$ as soon as ν has no atoms (see for example the discussion in [11]), a key quantity in the lemma is the constant

$$a^* = \sup\{a \geq 0 : W^{(q)'}(a) \leq W^{(q)'}(x) \text{ for all } x \geq 0\},$$

whereby $W^{(q)'(0)}$ stands for $W^{(q)'(0+)}$. Further $a^* < \infty$ since, by (2.3), we have that $\lim_{x \uparrow \infty} W^{(q)'(x)} = \infty$. We record here the following result taken from [26, Theorem 2 and Corollary 1]; note that $\Phi'(q) = 1/\psi'(\Phi(q))$.

Lemma 3. *Suppose the Lévy measure has a completely monotone density and $q > 0$. Then the q -scale function can be written as*

$$W^{(q)}(x) = \Phi'(q)e^{\Phi(q)x} - f(x), \quad x > 0,$$

where f is a non-negative, completely monotone function. Moreover, $W^{(q)'}$ is strictly log-convex (and hence convex) on $(0, \infty)$. Since $W^{(q)'(\infty)} = \infty$, a^* is thus the unique point at which $W^{(q)'}$ attains its minimum so that $W^{(q)'}$ is strictly decreasing on $(0, a^*)$ and strictly increasing on (a^*, ∞) .

Let us now progress to a description of the role played by scale functions in connection with the value of a refraction strategy. In addition to the scale function $W^{(q)}$ associated to the spectrally negative Lévy process X , we shall also define for each $q \geq 0$ the scale functions $\mathbb{W}^{(q)}$ which are associated to the linearly perturbed spectrally negative Lévy process $Y = \{Y_t : t \geq 0\}$ where $Y_t = X_t - \delta t$ for $t \geq 0$; recall that δ stands for the ceiling rate. Note that because of Assumption **(H)** the aforementioned process does not have monotone paths. Further we denote by $\varphi(q)$ the right inverse of the Laplace exponent of Y , i.e.

$$\varphi(q) = \inf\{\lambda \geq 0 : \psi(\lambda) - \delta\lambda = q\}.$$

The value function of the refraction strategy at level b , henceforth denoted by v_b , can now be written explicitly in terms of $W^{(q)}$, $\mathbb{W}^{(q)}$ and $\varphi(q)$ with the parameter $q > 0$ being the discount rate. Indeed it was shown in Equation (10.25) of [24] that

$$v_b(x) = -\delta \int_0^{x-b} \mathbb{W}^{(q)}(y)dy + \frac{W^{(q)}(x) + \delta \int_b^x \mathbb{W}^{(q)}(x-y)W^{(q)'(y)}dy}{h(b)}, \quad x \geq 0, \quad (2.4)$$

where $h(b)$ is given by

$$h(b) = \varphi(q)e^{\varphi(q)b} \int_b^\infty e^{-\varphi(q)y} W^{(q)'(y)} dy = \varphi(q) \int_0^\infty e^{-\varphi(q)u} W^{(q)'(u+b)} du. \quad (2.5)$$

Note that

$$v_b(x) = \frac{W^{(q)}(x)}{h(b)} \quad \text{for } x \leq b. \quad (2.6)$$

We need also to have a candidate optimal threshold, say b^* , in combination with the expression for v_b if we are to check for optimality. To this end define b^* as the largest argument at which h attains its minimum. That is to say,

$$b^* = \sup\{b \geq 0 : h(b) \leq h(x) \text{ for all } x \geq 0\}.$$

Under the same conditions as Theorem 1, we are able to say some more about b^* .

Lemma 4. *Suppose that ν has a completely monotone density. Then $b^* \in [0, a^*)$ and it is the unique point at which h attains its minimum. Moreover, $b^* > 0$ if and only if one of the following three cases hold,*

- (i) $\sigma > 0$ and $\varphi(q) < 2\delta/\sigma^2$,
- (ii) $\sigma = 0$, $\nu(0, \infty) < \infty$ and $\varphi(q) < \delta(\nu(0, \infty) + q)/c(c - \delta)$ or
- (iii) $\sigma = 0$ and $\nu(0, \infty) = \infty$.

Proof. We begin by showing that $b^* < \infty$. Note that

$$\begin{aligned} h(b) &= (\varphi(q))^2 \int_0^\infty e^{-\varphi(q)y} (W^{(q)}(y+b) - W^{(q)}(b)) dy \\ &= (\varphi(q))^2 e^{\Phi(q)b} \int_0^\infty e^{-\varphi(q)y} (e^{\Phi(q)y} W_{\Phi(q)}(y+b) - W_{\Phi(q)}(b)) dy \\ &\geq (\varphi(q))^2 e^{\Phi(q)b} W_{\Phi(q)}(b) \int_0^\infty e^{-\varphi(q)y} (e^{\Phi(q)y} - 1) dy \\ &= W^{(q)}(b) \frac{\varphi(q)\Phi(q)}{\varphi(q) - \Phi(q)}, \end{aligned}$$

where we have used a change of variables and an integration by parts for the first equality and (2.3) for the second and third. Since $W^{(q)}(\infty) = \infty$ and $\varphi(q) > \Phi(q)$, it follows that $\lim_{b \rightarrow \infty} h(b) = \infty$. The latter implies that $b^* < \infty$ as b^* is defined as the supremum of all the global minimizers of h .

From (2.5), we see that h is continuously differentiable and that

$$h'(b) = \varphi(q) (h(b) - W^{(q)'(b)}). \quad (2.7)$$

It follows immediately that $h'(b) > (<)0$ if and only if $h(b) > (<)W^{(q)'(b)}$. Thanks to Lemma 3, we know that $W^{(q)'}$ is a strictly convex function satisfying $W^{(q)'(\infty)} = \infty$ and since $h(\infty) = \infty$, it follows that, there is a unique $b^* \in [0, \infty)$ for which the minimum of h is attained and $h(b) < W^{(q)'(b)}$ for $b < b^*$ and $h(b) > W^{(q)'(b)}$ when $b > b^*$. Moreover, when $b^* > 0$, we have that $h(b^*) = W^{(q)'(b^*)}$.

Let us now show that $b^* < a^*$. Suppose for contradiction that $b^* > a^*$. In that case, since $W^{(q)''(b)} > 0$ for all $b \geq b^*$ and $h'(b^*) = 0$, it follows that there exists a sufficiently small $\epsilon > 0$ such that $W^{(q)'(b)} > h(b)$ for all $b \in (b^*, b^* + \epsilon)$. However this last statement contradicts the earlier conclusion that $W^{(q)'(b)} < h(b)$ for all $b > b^*$. Now suppose, also for contradiction, that $b^* = a^*$. Considering the second equality in (2.5), since $W^{(q)'(u+a^*)} > W^{(q)'(a^*)}$ for all $u > 0$, it is straightforward to show that $h(a^*) > W^{(q)'(a^*)}$ which again contradicts our earlier conclusion that $h(b^*) = W^{(q)'(b^*)}$.

Finally, given that b^* characterises the single crossing point of the function h over the function $W^{(q)'}$, we have that $b^* > 0$ if and only if $h(0) < W^{(q)'}(0+)$. Note from (2.5) that

$$h(0) = \varphi(q) \left(\frac{\varphi(q)}{\psi(\varphi(q)) - q} - W^{(q)}(0) \right) = \varphi(q) \left(\frac{1}{\delta} - W^{(q)}(0) \right) \quad (2.8)$$

where we have used the fact that for $q > 0$, that by integration by parts in (2.1),

$$\int_{[0, \infty)} e^{-\theta x} W^{(q)}(dx) = \frac{\theta}{\psi(\theta) - q}, \quad \theta > \Phi(q). \quad (2.9)$$

and that $\varphi(q) > \Phi(q)$. The three cases that are equivalent to $b^* > 0$ now follow directly from the right hand side of (2.8) compared against the expression given for $W^{(q)'}(0+)$ in Lemma 2. \square

3 Verification

For the remainder of the paper, we will focus on verifying the optimality of the refraction strategy at threshold level b^* under the condition that ν has a completely monotone density.

Given the spectrally negative Lévy process X , we call a function f (defined on at least the positive half line) sufficiently smooth if f is continuously differentiable on $(0, \infty)$ when X has paths of bounded variation and is twice continuously differentiable on $(0, \infty)$ when X has paths of unbounded variation. We let Γ be the operator acting on sufficiently smooth functions f , defined by

$$\Gamma f(x) = \gamma f'(x) + \frac{\sigma^2}{2} f''(x) + \int_{(0, \infty)} [f(x-z) - f(x) + f'(x)z \mathbf{1}_{\{0 < z \leq 1\}}] \nu(dz).$$

The following lemma constitutes standard technology as far as optimal control is concerned. For this reason its proof, which requires only a technical modification of Lemma 1 in [26], is deferred to the appendix.

Lemma 5. *Suppose $\hat{\pi}$ is an admissible dividend strategy such that $v_{\hat{\pi}}$ is sufficiently smooth on $(0, \infty)$, right-continuous at zero and for all $x > 0$*

$$\sup_{0 \leq r \leq \delta} \Gamma v_{\hat{\pi}}(x) - qv_{\hat{\pi}}(x) - rv'_{\hat{\pi}}(x) + r \leq 0. \quad (3.1)$$

Then $v_{\hat{\pi}}(x) = v_(x)$ for all $x \geq 0$ and hence $\hat{\pi}$ is an optimal strategy.*

As we wish to work with this lemma for the case that $v_{\hat{\pi}} = v_{b^*}$, we show next that v_{b^*} is sufficiently smooth.

Lemma 6. *Under the assumption of Theorem 1, the value function v_{b^*} is sufficiently smooth.*

Proof. Recall from Lemma 3 that when ν has a completely monotone density it follows that both $W^{(q)}$ and $\mathbb{W}^{(q)}$ are infinitely differentiable.

Now suppose that $b^* = 0$. Then from (2.4) it follows that

$$v_0(x) = -\delta \left(\int_0^x \mathbb{W}^{(q)}(y) dy - \frac{1}{\varphi(q)} \mathbb{W}^{(q)}(x) \right), \quad x \geq 0, \quad (3.2)$$

which is clearly sufficiently smooth.

Next suppose that $b^* > 0$. By differentiating (2.4), we get

$$v'_{b^*}(x) = -\delta \mathbb{W}^{(q)}(x - b^*) + \frac{(1 + \delta \mathbb{W}^{(q)}(0))W^{(q)'}(x) + \delta \int_{b^*}^x \mathbb{W}^{(q)'}(x - y)W^{(q)'}(y) dy}{W^{(q)'}(b^*)}. \quad (3.3)$$

Using an integration by parts in (3.3) leads to

$$v'_{b^*}(x) = \frac{W^{(q)'}(x) + \delta \int_{b^*}^x \mathbb{W}^{(q)}(x - y)W^{(q)''}(y) dy}{W^{(q)'}(b^*)}, \quad (3.4)$$

which is continuous in x . Differentiating (3.4) leads us to

$$v''_{b^*}(x) = \frac{W^{(q)''}(x) + \delta \mathbb{W}^{(q)}(0)W^{(q)''}(x) + \delta \int_{b^*}^x \mathbb{W}^{(q)'}(x - y)W^{(q)''}(y) dy}{W^{(q)'}(b^*)}. \quad (3.5)$$

The expression on the right hand side is clearly continuous in x when X has paths of unbounded variation as $\mathbb{W}^{(q)}(0) = 0$. \square

Inspired by the cases that X is diffusive or a Cramér-Lundberg process with exponentially distributed jumps, for which a solution to the control problem at hand is known, we move next to the following two lemmas which convert the Hamilton-Jacobi-Bellman inequality in Lemma 5 into a more user friendly sufficient condition.

Lemma 7. *Under the assumption of Theorem 1 the value function v_{b^*} satisfies (3.1) if and only if*

$$\begin{cases} v'_{b^*}(x) \geq 1 & \text{if } 0 < x \leq b^*, \\ v'_{b^*}(x) \leq 1 & \text{if } x > b^*. \end{cases} \quad (3.6)$$

Proof. We first establish the following two equalities:

$$\begin{aligned} (\Gamma - q)v_{b^*}(x) &= 0 & \text{for } 0 < x \leq b^*, \\ (\Gamma - q)v_{b^*}(x) - \delta v'_{b^*}(x) + \delta &= 0 & \text{for } x > b^*. \end{aligned} \quad (3.7)$$

Recalling (2.6) and the fact that v_{b^*} is sufficiently smooth, the first part of (3.7) is proved in Lemma 4 of [4] (see also [8]). In a similar way, the second part follows after we show that $M = \{M_t, t \geq 0\}$ given by

$$M_t = e^{-q(t \wedge \tau_{b^*}^-)} \left(v_{b^*}(Y_{t \wedge \tau_{b^*}^-}) - \frac{\delta}{q} \right), t \geq 0$$

is a \mathbf{P}_x -martingale for $x > b^*$; here $\tau_{b^*}^-$ stands for $\tau_{b^*}^- = \inf\{t > 0 : Y_t < b^*\}$ and \mathbf{P}_x is the law of Y when $Y_0 = x$. Indeed, the martingale property follows by the following two computations and the tower property of conditional expectation (cf. [33, Section 9.7]). First we have for $x > b^*$ by the strong Markov property,

$$\begin{aligned} & \mathbf{E}_x \left[e^{-q\tau_{b^*}^-} \left(v_{b^*}(Y_{\tau_{b^*}^-}) - \frac{\delta}{q} \right) \middle| \mathcal{F}_t \right] \\ &= \mathbf{E}_x \left[\mathbf{1}_{\{t < \tau_{b^*}^-\}} e^{-q\tau_{b^*}^-} \left(v_{b^*}(Y_{\tau_{b^*}^-}) - \frac{\delta}{q} \right) \middle| \mathcal{F}_t \right] + \mathbf{E}_x \left[\mathbf{1}_{\{t \geq \tau_{b^*}^-\}} e^{-q\tau_{b^*}^-} \left(v_{b^*}(Y_{\tau_{b^*}^-}) - \frac{\delta}{q} \right) \middle| \mathcal{F}_t \right] \\ &= \mathbf{1}_{\{t < \tau_{b^*}^-\}} e^{-qt} \mathbf{E}_{Y_t} \left[e^{-q\tau_{b^*}^-} \left(v_{b^*}(Y_{\tau_{b^*}^-}) - \frac{\delta}{q} \right) \right] + \mathbf{1}_{\{t \geq \tau_{b^*}^-\}} M_t. \end{aligned}$$

Here \mathbf{E}_x denotes expectation with respect to \mathbf{P}_x . Recall that U^{b^*} is the refracted Lévy process given by (1.5) with threshold b^* . Let $\sigma^{b^*} = \inf\{t > 0 : U_t^{b^*} < 0\}$, $\kappa_{b^*}^- = \inf\{t > 0 : U_t^{b^*} < b^*\}$, then

$$\begin{aligned} \mathbf{1}_{\{t < \tau_{b^*}^-\}} M_t &= \mathbf{1}_{\{t < \tau_{b^*}^-\}} e^{-qt} \mathbf{E}_{Y_t} \left[\delta \int_0^{\sigma^{b^*}} e^{-qs} \mathbf{1}_{\{U_s^{b^*} \in (b^*, \infty)\}} ds - \frac{\delta}{q} \right] \\ &= \mathbf{1}_{\{t < \tau_{b^*}^-\}} e^{-qt} \mathbf{E}_{Y_t} \left[\delta \int_{\kappa_{b^*}^-}^{\sigma^{b^*}} e^{-qs} \mathbf{1}_{\{U_s^{b^*} \in (b^*, \infty)\}} ds - \frac{\delta}{q} e^{-q\kappa_{b^*}^-} \right] \\ &= \mathbf{1}_{\{t < \tau_{b^*}^-\}} e^{-qt} \mathbf{E}_{Y_t} \left[e^{-q\tau_{b^*}^-} \left(v_{b^*}(Y_{\tau_{b^*}^-}) - \frac{\delta}{q} \right) \right], \end{aligned}$$

where we used in the last line that given $Y_0 = U_0$, $\{Y_t, 0 \leq t \leq \tau_{b^*}^-\}$ is equal in law to $\{U_t^{b^*}, 0 \leq t \leq \kappa_{b^*}^-\}$.

We now continue with the proof of the lemma. It is easily seen that condition (3.1) is equivalent to

$$\begin{cases} (\Gamma - q)v_{b^*}(x) \leq 0 & \text{if } v'_{b^*}(x) \geq 1, \\ (\Gamma - q)v_{b^*}(x) - \delta v'_{b^*}(x) + \delta \leq 0 & \text{if } v'_{b^*}(x) < 1. \end{cases} \quad (3.8)$$

Suppose now (3.6) holds. If $v'_{b^*}(x) > 1$, then (3.6) implies $x \leq b^*$ and so by (3.7) $(\Gamma - q)v_{b^*}(x) = 0$. If $v'_{b^*}(x) < 1$, then (3.6) implies $x > b^*$ and so by (3.7) $(\Gamma - q)v_{b^*}(x) - \delta v'_{b^*}(x) + \delta = 0$. If $v'_{b^*}(x) = 1$, then we have by (3.7) $(\Gamma - q)v_{b^*}(x) = 0$. Hence (3.8) holds.

Suppose now (3.8) holds. Let $0 < x \leq b^*$ and suppose $v'_{b^*}(x) < 1$. Then (3.7) and (3.8) implies $-\delta v'_{b^*}(x) + \delta \leq 0$ which implies $v'_{b^*}(x) \geq 1$ which forms a contradiction. Hence we deduce $v'_{b^*}(x) \geq 1$. Now let $x > b^*$ and suppose $v'_{b^*}(x) > 1$. Then (3.7) and (3.8) implies $\delta v'_{b^*}(x) - \delta \leq 0$ which implies $v'_{b^*}(x) \leq 1$ which forms a contradiction. Hence we deduce $v'_{b^*}(x) \leq 1$. \square

The following lemma forms the most difficult part of the proof of the main theorem. It is here that Lemma 3 and thus the assumption of complete monotonicity on the density of the Lévy measure is most crucially needed. In particular, we need here the representation of the scale function as the difference of an exponential term and a completely monotone function (cf. Lemma 3) instead of the weaker property that the derivative of the scale function is log-convex (see also the discussion below Theorem 1.2 in [28]) or the even weaker property that the derivative of the scale function is increasing on (a^*, ∞) (see also Theorem 2 in [25] and Remark 9 below); note that the latter is already sufficient for optimality of the barrier strategy for the control problem with general admissible strategies.

Lemma 8. *Suppose the Lévy measure has a completely monotone density. Then the function v_{b^*} satisfies (3.6).*

Proof. Suppose first that $b^* = 0$. In other words, from Lemma 4, assume that either

- (i) $\sigma > 0$ and $\varphi(q) \geq 2\delta/\sigma^2$, or
- (ii) $\sigma = 0$, $\nu(0, \infty) < \infty$ and $\varphi(q) \geq \delta(q + \nu(0, \infty))/c(c - \delta)$.

Then for $x > 0$, we deduce from (3.2) that

$$v'_{b^*}(x) = v'_0(x) = -\delta \left(\mathbb{W}^{(q)}(x) - \frac{1}{\varphi(q)} \mathbb{W}^{(q)'}(x) \right).$$

By the decomposition of the scale function given in Lemma 3, v'_0 is completely monotone and thus in particular decreasing on $(0, \infty)$. Hence if $b^* = 0$, it is enough to show that $v'_0(0+) \leq 1$ or equivalently

$$\frac{\delta \mathbb{W}^{(q)'}(0+)}{1 + \delta \mathbb{W}^{(q)}(0)} \leq \varphi(q). \quad (3.9)$$

Taking account of Lemma 2 we see that this requirement is automatically satisfied in cases (i) and (ii). Hence we have proved (3.6) if $b^* = 0$.

Assume now $b^* > 0$. Then for $x \leq b^*$, $v'_{b^*}(x) = W^{(q)'}(x)/W^{(q)'}(b^*)$. From this it follows that $v'_{b^*}(x) \leq 1$ since, by Lemma 4, $b^* \leq a^*$ and, by Lemma 3, $W^{(q)'}$ is decreasing for $x \leq b^*$.

Suppose now $x > b^*$. Differentiating twice the first displayed equation in Section 8 of [24] gives us the identity

$$\delta \int_0^x \mathbb{W}^{(q)'}(x-y)W^{(q)'}(y)dy = (1 - \delta W^{(q)}(0))\mathbb{W}^{(q)'}(x) - (1 + \delta \mathbb{W}^{(q)}(0))W^{(q)'}(x).$$

Hence revisiting (3.3) we obtain the expression

$$v'_{b^*}(x) = -\delta \mathbb{W}^{(q)}(x - b^*) + \frac{(1 - \delta W^{(q)}(0))\mathbb{W}^{(q)'}(x) - \delta \int_0^{b^*} \mathbb{W}^{(q)'}(x-y)W^{(q)'}(y)dy}{W^{(q)'}(b^*)}.$$

Appealing to Lemma 3 and writing $\mathbb{W}^{(q)}(x) = \varphi'(q)e^{\varphi(q)x} - f(x)$ where f is completely monotone we get

$$\begin{aligned} v'_{b^*}(x) = & \delta f(x - b^*) + \frac{-(1 - \delta W^{(q)}(0))f'(x) + \delta \int_0^{b^*} f'(x-y)W^{(q)'}(y)dy}{W^{(q)'}(b^*)} - \delta \varphi'(q)e^{\varphi(q)(x-b^*)} \\ & + \frac{1}{W^{(q)'}(b^*)} \left((1 - \delta W^{(q)}(0))\varphi'(q)\varphi(q)e^{\varphi(q)x} - \delta \int_0^{b^*} \varphi'(q)\varphi(q)e^{\varphi(q)(x-y)}W^{(q)'}(y)dy \right). \end{aligned}$$

Using (2.9) and recalling that $\varphi(q) > \Phi(q)$ we also have that

$$\begin{aligned} \int_0^{b^*} e^{-\varphi(q)y}W^{(q)'}(y)dy &= - \int_{b^*}^{\infty} e^{-\varphi(q)y}W^{(q)'}(y)dy + \int_0^{\infty} e^{-\varphi(q)y}W^{(q)'}(y)dy \\ &= - \int_{b^*}^{\infty} e^{-\varphi(q)y}W^{(q)'}(y)dy + \frac{1}{\delta} - W^{(q)}(0). \end{aligned}$$

Hence this gives us

$$\begin{aligned} v'_{b^*}(x) = & \delta f(x - b^*) + \frac{-(1 - \delta W^{(q)}(0))f'(x) + \delta \int_0^{b^*} f'(x-y)W^{(q)'}(y)dy}{W^{(q)'}(b^*)} \\ & - \delta \varphi'(q)e^{\varphi(q)(x-b^*)} + \frac{\delta e^{\varphi(q)x}\varphi'(q)\varphi(q) \int_{b^*}^{\infty} e^{-\varphi(q)y}W^{(q)'}(y)dy}{W^{(q)'}(b^*)} \end{aligned}$$

and recalling that

$$W^{(q)'}(b^*) = h(b^*) = \varphi(q)e^{\varphi(q)b^*} \int_{b^*}^{\infty} e^{-\varphi(q)y}W^{(q)'}(y)dy,$$

we get the simpler expression

$$v'_{b^*}(x) = \delta f(x - b^*) + \frac{-(1 - \delta W^{(q)}(0))f'(x) + \delta \int_0^{b^*} f'(x-y)W^{(q)'}(y)dy}{W^{(q)'}(b^*)}.$$

Since f is completely monotone, by Bernstein's theorem (cf. [13, Theorem 1a in Section XIII.4]), it can be written in the form $f(x) = \int_0^\infty e^{-xt} \mu(dt)$ for some measure μ . Therefore using Tonelli's Theorem (cf. [32, Theorem 4.1.5]), we come to rest at the identity

$$v'_{b^*}(x) = \int_0^\infty e^{-xt} \left\{ \delta e^{b^*t} + \frac{1 - \delta W^{(q)}(0)}{W^{(q)'(b^*)}} t - \frac{\delta}{W^{(q)'(b^*)}} \int_0^{b^*} t e^{yt} W^{(q)'(y)} dy \right\} \mu(dt).$$

Denote for $t > 0$ by $g(t)$ the expression between curly brackets. We have

$$g''(t) = \delta(b^*)^2 e^{b^*t} - \frac{2\delta}{W^{(q)'(b^*)}} \int_0^{b^*} y e^{yt} W^{(q)'(y)} dy - \frac{\delta t}{W^{(q)'(b^*)}} \int_0^{b^*} y^2 e^{yt} W^{(q)'(y)} dy.$$

Since $W^{(q)'(y)} \geq W^{(q)'(b^*)}$ for $y \in (0, b^*)$, we have using also an integration by parts,

$$\frac{\delta t}{W^{(q)'(b^*)}} \int_0^{b^*} y^2 e^{yt} W^{(q)'(y)} dy \geq \delta t \int_0^{b^*} y^2 e^{yt} dy = \delta y^2 e^{yt} \Big|_0^{b^*} - 2\delta \int_0^{b^*} y e^{yt} dy$$

and hence using again that $W^{(q)'(y)} \geq W^{(q)'(b^*)}$ for $y \in (0, b^*)$,

$$g''(t) \leq -\frac{2\delta}{W^{(q)'(b^*)}} \int_0^{b^*} y e^{yt} W^{(q)'(y)} dy + 2\delta \int_0^{b^*} y e^{yt} dy \leq 0.$$

In conclusion, g is a concave function and in particular there exists $0 \leq p_1 \leq \infty$ such that g is increasing on $(0, p_1)$ and decreasing on (p_1, ∞) . Since $g(0) = \delta$, it follows that there exists $0 < p_2 \leq \infty$ such that g is positive on $(0, p_2)$ and negative on (p_2, ∞) . Consequently $e^{-(x-b^*)t} g(t) \geq e^{-(x-b^*)p_2} g(t)$ for all $t > 0$ and thus (noting that we are allowed to switch the derivative and the integral)

$$v''_{b^*}(x) = - \int_0^\infty e^{-(x-b^*)t} e^{-b^*t} t g(t) dt \leq -e^{-(x-b^*)p_2} \int_0^\infty e^{-b^*t} t g(t) dt = e^{-(x-b^*)p_2} v''_{b^*}(b^*+). \quad (3.10)$$

From (3.5) we easily deduce that

$$v''_{b^*}(b^*+) = \frac{(1 + \delta \mathbb{W}^{(q)}(0)) W^{(q)''(b^*)}}{W^{(q)'(b^*)}} \leq 0$$

where the inequality is a result of the fact that $b^* < a^*$ and hence by Lemma 3, $W^{(q)''(b^*)} \leq 0$. In combination with (3.10), it follows that v'_{b^*} is decreasing on (b^*, ∞) and since $v'_{b^*}(b^*) = 1$, we deduce that $v'_{b^*}(x) \leq 1$ for $x > b^*$ as required. \square

Finally we can put all the pieces together to establish our main result.

Proof of Theorem 1. By Lemmas 8 and 7, it follows that v_{b^*} satisfies (3.1). Since in addition v_{b^*} is right-continuous at zero and is sufficiently smooth by Lemma 6, it follows by Lemma 5 that v_{b^*} equals the optimal value function v_* and thus the refraction strategy with threshold level b^* is an optimal strategy. \square

Remark 9. Here we give the example mentioned at the end of Section 1. Let $\nu(dz) = 10ze^{-z}dz$, $\gamma = 20.67 - \int_0^1 z\nu(dz)$ and $q = 0.1$. That is, X is a Cramér-Lundberg risk process with premium rate 20.67, jump arrival rate 10 and with a Gamma(2, 1) claim distribution. The scale function for X , via the method of partial fraction, is given by

$$W^{(q)}(x) = \sum_{i=1}^3 D_i e^{\theta_i x}, \quad x \geq 0,$$

where $\{\theta_i : i = 1, 2, 3\}$ are the (distinct) roots of $\lambda \mapsto \psi(\theta) - q$ with $\theta_1 > 0$ and $\theta_2, \theta_3 < 0$ and where $\{D_i : i = 1, 2, 3\}$ are given by $D_i = 1/\psi'(\theta_i)$. Similarly, the scale function of Y can be calculated. If one plots the second derivative of $W^{(q)}$, then one sees that $W^{(q)''}(x) > 0$ on $(0, \infty)$ and therefore by Theorem 2 of Loeffen [25], the reflection strategy at the barrier 0 is optimal for the control problem with general admissible dividend strategies. If we now consider the case with absolutely continuous dividend strategies and choose the upper bound δ equal to 20.59, then one can check that the function h is strictly increasing and thus $v_0(0) = W^{(q)}(0)/h(0) > W^{(q)}(0)/h(b) = v_b(0)$ for all $b > 0$. This means that the only refraction strategy that can be optimal is the one with threshold level equal to zero. However, one can calculate that $v_0'(3.15) = 1.0005 > 1$ and so v_0 does not satisfy (3.6) and consequently by Lemma 7, v_0 does not satisfy (3.1). Since we are in the Cramér-Lundberg setting with a continuous claim distribution, we can deduce from Theorem 2.32 of Schmidli [31] that the optimal value function v_* has to satisfy (3.1) and therefore $v_0 \neq v_*$. We conclude that for this particular example we have the remarkable property that the optimal strategy in the case with no extra restrictions on the controls, is to always pay out the maximum amount of dividends that is allowed, whereas in the restricted case with this particular value of δ , it is not optimal to always pay out dividends at the maximum rate. Further, as for this example $W^{(q)'}$ is increasing on (a^*, ∞) , it follows that this latter condition is not sufficient for a refraction strategy to be optimal amongst the class of dividend strategies satisfying (1.3) and (1.4); this is in contrast with the singular optimal dividends problem, cf. Theorem 2 of [25].

On a final note, we present the following corollary of the proof of Theorem 1, giving a characterisation of the optimal threshold, cf. Lemma 4.

Corollary 10. *The threshold for the optimal refraction strategy in Theorem 1 is characterised as the unique point $b^* \in [0, a^*)$ such that*

$$b^* = \sup\{b \geq 0 : h(b) \leq h(x) \text{ for all } x \geq 0\}$$

where

$$h(b) = \varphi(q)e^{\varphi(q)b} \int_b^\infty e^{-\varphi(q)y} W^{(q)'}(y) dy.$$

Moreover $b^* > 0$ if and only if

- (i) $\sigma > 0$ and $\varphi(q) < 2\delta/\sigma^2$,
- (ii) $\sigma = 0$, $\nu(0, \infty) < \infty$ and $\varphi(q) < \delta(\nu(0, \infty) + q)/c(c - \delta)$ or
- (iii) $\sigma = 0$ and $\nu(0, \infty) = \infty$;

otherwise $b^* = 0$.

Appendix: proof of Lemma 5

By definition of v_* , it follows that $v_{\hat{\pi}}(x) \leq v_*(x)$ for all $x \geq 0$. We write $w := v_{\hat{\pi}}$ and show that $w(x) \geq v_\pi(x)$ for all $\pi \in \Pi$ for all $x \geq 0$. First we suppose $x > 0$. We define for $\pi \in \Pi$ the stopping time σ_0^π by $\sigma_0^\pi = \inf\{t > 0 : U_t^\pi \leq 0\}$ and denote by Π_0 the following set of admissible dividend strategies

$$\Pi_0 = \{\pi \in \Pi : v_\pi(x) = \mathbb{E}_x \left[\int_0^{\sigma_0^\pi} e^{-qt} dL_t^\pi \right] \text{ for all } x > 0\}.$$

We claim that any $\pi \in \Pi$ can be approximated by dividend strategies from Π_0 in the sense that for all $\epsilon > 0$ there exists $\pi_\epsilon \in \Pi_0$ such that $v_\pi(x) \leq v_{\pi_\epsilon}(x) + \epsilon$ and therefore it is enough to show that $w(x) \geq v_\pi(x)$ for all $\pi \in \Pi_0$. Indeed, we can take π_ϵ to be the strategy where you do not pay out any dividends until the stopping time $\kappa := \inf\{t > 0 : L_t^\pi \geq \epsilon\}$, then from that time point κ onwards follow the same strategy as π until ruin occurs for the latter strategy at which point you stop paying out dividends. Note that $\pi_\epsilon \in \Pi_0$ because if $\sigma_0^{\pi_\epsilon} < \kappa$, then $\sigma_0^{\pi_\epsilon} = \sigma_0^\pi$ since until the first dividend payment is made, the process U^{π_ϵ} is equal to X and for the spectrally negative Lévy process X , the first entry time in $(-\infty, 0]$ is equal almost surely to the first entry time in $(-\infty, 0)$, provided $X_0 > 0$. Further if $\sigma_0^{\pi_\epsilon} \geq \kappa$ and $\kappa < \infty$, then $\sigma_0^{\pi_\epsilon} \geq \sigma_0^\pi$ and so by construction there are no dividends paid out in the time interval $(\sigma_0^{\pi_\epsilon}, \sigma_0^\pi)$.

We now assume without loss of generality that $\pi \in \Pi_0$. Let $(T_n)_{n \in \mathbb{N}}$ be the sequence of stopping times defined by $T_n = \inf\{t > 0 : U_t^\pi > n \text{ or } U_t^\pi < \frac{1}{n}\}$. Since U^π is a semimartingale and w is sufficiently smooth, we can use the change of variables/Itô's formula

(cf. [29, Theorem II.31 & II.32]) on $e^{-q(t \wedge T_n)} w(U_{t \wedge T_n}^\pi)$ to deduce under \mathbb{P}_x ,

$$\begin{aligned} e^{-q(t \wedge T_n)} w(U_{t \wedge T_n}^\pi) - w(x) &= - \int_{0+}^{t \wedge T_n} e^{-qs} q w(U_{s-}^\pi) ds + \int_{0+}^{t \wedge T_n} e^{-qs} w'(U_{s-}^\pi) d(X_s - L_s^\pi) \\ &\quad + \sum_{0 < s \leq t \wedge T_n} e^{-qs} [\Delta w(U_{s-}^\pi + \Delta X_s) - w'(U_{s-}^\pi) \Delta X_s], \end{aligned}$$

where we use the following notation: $\Delta U_s^\pi = U_s^\pi - U_{s-}^\pi$, $\Delta w(U_s^\pi) = w(U_s^\pi) - w(U_{s-}^\pi)$. Rewriting the above equation leads to

$$\begin{aligned} e^{-q(t \wedge T_n)} w(U_{t \wedge T_n}^\pi) - w(x) &= \int_{0+}^{t \wedge T_n} e^{-qs} (\Gamma - q) w(U_{s-}^\pi) ds - \int_{0+}^{t \wedge T_n} e^{-qs} w'(U_{s-}^\pi) dL_s^\pi \\ &\quad + \left\{ \int_{0+}^{t \wedge T_n} e^{-qs} w'(U_{s-}^\pi) d[X_s - (c - \int_0^1 x \nu(dx))s] - \sum_{0 < u \leq s} \Delta X_u \mathbf{1}_{\{|\Delta X_u| > 1\}} \right\} \\ &\quad + \left\{ \sum_{0 < s \leq t \wedge T_n} e^{-qs} [\Delta w(U_{s-}^\pi + \Delta X_s) - w'(U_{s-}^\pi) \Delta X_s \mathbf{1}_{\{|\Delta X_s| \leq 1\}}] \right. \\ &\quad \left. - \int_{0+}^{t \wedge T_n} \int_{0+}^{\infty} e^{-qs} [w(U_{s-}^\pi - y) - w(U_{s-}^\pi) + w'(U_{s-}^\pi) y \mathbf{1}_{\{0 < y \leq 1\}}] \nu(dy) ds \right\}. \end{aligned}$$

By the Lévy-Itô decomposition the expression between the first pair of curly brackets is a zero-mean martingale and by the compensation formula (cf. [21, Corollary 4.6]) the expression between the second pair of curly brackets is also a zero-mean martingale. Hence we derive at

$$\begin{aligned} w(x) &= - \int_0^{t \wedge T_n} e^{-qs} [(\Gamma - q) w(U_{s-}^\pi) - \ell^\pi(s) w'(U_{s-}^\pi) + \ell^\pi(s)] ds \\ &\quad + \int_0^{t \wedge T_n} e^{-qs} \ell^\pi(s) ds + e^{-q(t \wedge T_n)} w(U_{t \wedge T_n}^\pi) + M_t, \end{aligned}$$

where $\{M_t : t \geq 0\}$ is a zero-mean \mathbb{P}_x -martingale. Using $w \geq 0$ and (3.1) leads to (since $0 \leq \ell_s^\pi \leq \delta$)

$$w(x) \geq \int_0^{t \wedge T_n} e^{-qs} \ell^\pi(s) ds + M_t.$$

Now taking expectations and letting t and n go to infinity and using the monotone convergence theorem we get, noting that $T_n \nearrow \sigma_0^\pi$ \mathbb{P}_x -a.s. and that $\pi \in \Pi_0$,

$$w(x) \geq \mathbb{E}_x \left[\int_0^{\sigma^\pi} e^{-qs} \ell^\pi(s) ds \right] = v_\pi(x).$$

Hence we proved $w(x) \geq v_*(x)$ for all $x > 0$.

To finish the proof, note that v_* is an increasing function (in the weak sense) and hence because w is right-continuous at zero, $v_*(0) \leq \lim_{x \downarrow 0} v_*(x) \leq \lim_{x \downarrow 0} w(x) = w(0)$.

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