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OPTIMAL CONTROL WITH INTEGRAL STATE EQUATIONS

by

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M. I. Kamien and E. Muller

Methods of optimal control theory have proved useful in studying the class of dynamic economic problems that can be posed as optimization of

(1) $$\int_0^T F(t,x(t),u(t))dt$$

subject to

(2) $$\dot{x}(t) = f(t,x(t),u(t)), \quad x(0) = x_0$$

where t denotes time and $x(t)$, $\dot{x}(t)$, $u(t)$, the values of the state variable, its derivative, and the control variable respectively at time t, see [2] and [4]. The differential equation (2), sometimes referred to as the transition or state equation, links changes in the state variable $x(t)$ to contemporaneous changes in the control $u(t)$. The Maximum Principle of Pontryagin and Hestenes has facilitated economic interpretation of the necessary conditions for an optimum of (1) and (2); see [10]. The Maximum Principle also applies to problems involving nonnegativity constraints on the state and control variables as well as other interrelationships among them, to problems involving several state and control variables, and to infinite horizon problems; see [2].

Presented in this paper is a Maximum Principle for optimal control problems for which the state equation is a nonlinear Volterra integral equation of the second kind; see [7]:

$$(3) \quad x(t) = x_0 + \int_0^t f(t,x(s),u(s),s)ds$$

The class of state equations defined by (3) includes those of (2) because (2) can be written as:

$$(4) \quad x(t) = x_0 + \int_0^t f(s,x(s),u(s))ds$$

The converse, however, is not true because the integral equation (3) cannot in general be written as a differential equation. Instead differentiation of (3) yields the integro-differential equation:

$$(5) \quad \dot{x}(t) = f(t,x(t),u(t),t) + \int_0^t [\partial f(t,x(s),u(s),s)/\partial t]ds$$

providing f is a smooth function of t; see [11, p. 65]. Only in special cases such as $\partial f/\partial t = c(t)f(t,x(s),u(s),s)$ will (5) reduce to a pure differential equation.

A Maximum Principle appropriate for an optimal control problem consisting of (1) and (3), similar to Bakke's [3], is presented in the first section. An informal proof of this result along the lines of the variational approach employed by Smith [12, pp. 288-293], is presented. The theorem of Bakke applies to multivariate problems with inequality constraints involving state and control variables. A second theorem indicating circumstances under which the necessary conditions are also sufficient follows. Application of this Maximum Principle to extension of some recent work in capital replacement theory is then demonstrated.

The Maximum Principle

It is assumed that the requirements for the existence of a unique continuous solution to the integral equation (3), which include continuity of the function f over the triangular region $a \leq s \leq t \leq b$ together with a Lipschitz condition, are met; see [7, pp. 24-30] or [7, pp. 85-93] for a weaker set of conditions. In addition it is supposed that the partial derivatives $\partial f/\partial x = f_x$, $\partial f/\partial u = f_u$ exist and are continuous. Lastly let $f(t,x,u,s) \equiv 0$ for $t < s$, and $F(t,x,u)$ be a smooth function with partial derivatives denoted $\partial F/\partial x = F_x$ and $\partial F/\partial u = F_u$.

Theorem 1: Suppose the assumptions stated above obtain and the function $u^*(t)$, $a \leq t \leq b$, maximizes:

(6) $$\int_a^b F(t,x(t),u(t))\,dt$$

subject to

(7) $$x(t) = x(a) + \int_a^t f(t,x(s),u(s),s)\,ds$$

Then there exists a continuous multiplier function of time, $\lambda(t)$, and a Hamiltonian, $H(t,x,u,\lambda)$ defined by

$$(8) \quad H(t,x(t),u(t),\lambda(t)) = F(t,x(t),u(t))+\int_t^b f(s,x(t),u(t),t)\lambda(s)ds,$$

such that, for each t,

(i) $u^*(t)$ maximizes $H(t,x,u,\lambda)$; so that $\partial H/\partial u \Big|_{u = u^*(t)} = 0$, providing H is differentiable with respect to u.

(ii) $\lambda(t) = \partial H/\partial x$ evaluated at the optimal control value $u^*(t)$ and the corresponding state function $x(t)$ and multiplier $\lambda(t)$ values.

Proof: Let $u(t)$ be a fixed continuous control function. Then the solution to (7), assumed to exist, can be written as:

$$(9) \quad x = x(t,u)$$

and substituted into (6) to yield a functional in u,

$$(10) \quad J(u) = \int_a^b F(t,x(t,u),u)\,dt$$

According to a well known result of the calculus of variations, the variation of J, written

$$(11) \quad \delta J(u,\Delta u) = \frac{d}{d\epsilon} J(u+\epsilon\Delta u) \Big|_{\epsilon=0} = 0$$

at an optimizing u, for all functions $\Delta u(t)$ continuous in t where ϵ is an arbitrary small number, see [12, pp. 33-41]. Computation of the variation of (10) and combination with (11) yields:

(12) $$\int_a^b [F_x(t,x(t,u),u)\frac{\partial}{\partial\varepsilon}x(t,u+\varepsilon\Delta u)\Big|_{\varepsilon=0} + F_u(t,x(t,u),u)\Delta u]dt \equiv 0.$$

Now from (7):

(13) $$\frac{\partial}{\partial\varepsilon}x(t,u+\varepsilon\Delta u)\Big|_{\varepsilon=0} = \frac{\partial}{\partial\varepsilon}\int_a^t f(t,x(s,u+\varepsilon\Delta u),u+\varepsilon\Delta u,s)ds\Big|_{\varepsilon=0}$$

$$= \int_a^t \frac{\partial}{\partial\varepsilon} f(t,x(s,u+\varepsilon\Delta u),u+\varepsilon\Delta u,s)\Big|_{\varepsilon=0} ds$$

$$= \int_a^t [f_x(t,x(s,u),u,s)\frac{\partial}{\partial\varepsilon}x(s,u+\varepsilon\Delta u)\Big|_{\varepsilon=0} + f_u(t,x(s,u),u,s)\Delta u]ds$$

Letting

(14)
$$A(t,s) = f_x(t,x(s,u),u(s),s)$$
$$B(t,s) = f_u(t,x(s,u),u(s),s)$$
$$y(t) = \frac{\partial}{\partial\varepsilon}x(t,u+\varepsilon\Delta u)\Big|_{\varepsilon=0}$$

and substituting into (13) yields a linear integral equation

(15) $$y(t) = \int_a^t [A(t,s)y(s) + B(t,s)\Delta u(s)]ds.$$

According to a basic property of linear integral equations, see [7, pp. 189-93] the solution to (15) can be written as:

(16) $$y(t) = \int_a^t r(t,s)[\int_a^s B(s,\tau)\Delta u(\tau)d\tau]ds + \int_a^t B(t,s)\Delta u(s)ds$$

where the function r(t,s), called the resolvent kernel, satisfies

$$(17) \quad r(t,s) = A(t,s) + \int_s^t r(t,\tau)A(\tau,s)d\tau .$$

Now on a triangular region $a \le \tau \le t \le T$, and function g

$$(18) \quad \int_a^T \int_a^t g(t,\tau)d\tau dt = \int_a^T \int_t^T g(\tau,t)d\tau dt$$

so (16) can be rewritten as

$$(19) \quad y(t) = \int_a^t [B(t,s) + \int_s^t r(t,\tau)B(\tau,s)d\tau]\Delta u(s)ds$$

Recalling (14) and substituting from (19) into (12) yields:

$$(20) \quad \int_a^b \{F_x(t,x(t,u),u(t)) [\int_a^t [B(t,s) + \int_s^t r(t,\tau)B(\tau,s)d\tau]\Delta u(s)ds]$$

$$+ F_u(t,x(t,u),u(t))\Delta u(t)\}dt = 0$$

Application of (18) to (20) yields

$$(21) \quad \int_a^b \{ \int_t^b F_x(s,x(s,u),u(s))[B(s,t)+ \int_t^s r(s,\tau)B(\tau,t)d\tau]ds$$

$$+ F_u(t,x(t,u),u(t))\}\Delta u(t)dt = 0$$

Since (21) must hold for all continuous functions $\Delta u(t)$, it must obtain in the particular case when $\Delta u(t)$ equals the curly bracketed expression in (21). This implies

$$(22) \quad \int_a^b \{ \int_t^b F_x(s,x(s,u),u(s))[B(s,t)+ \int_t^s r(s,\tau)B(\tau,t)d\tau]ds$$

$$+ F_u(t,x(t,u),u(t))\}^2 dt = 0,$$

from which it follows that

$$(23)\quad \int_t^b F_x(s,x(s,u),u(s))[B(s,t)+\int_t^s r(s,\tau)\ B(\tau,t)d\tau]ds+F_u(t,x(t,u),u(t))=$$

Another application of (18), this time to (23), yields an Euler equation

$$(24)\quad \int_t^b [F_x(s,x(s,u),u(s))+\int_s^b F_x(\tau,x(\tau,u),u(\tau))r(\tau,s)d\tau]B(s,t)ds +$$

$$F_u(t,x(t,u),u(t)) = 0$$

If we define $\lambda(s)$ by

$$(25)\quad \lambda(s) = F_x(s,x(s,u),u(s))+\int_s^b F_x(\tau,x(\tau,u),u(\tau))r(\tau,s)d\tau$$

and substitute from (25) into (24) we get

$$(26)\quad F_u(t,x(t,u),u(t))+\int_t^b \lambda(s)B(s,t)ds = 0.$$

Substitution from (14) and recollection of (8) discloses that

$$(27)\quad F_u(t,x(t,u),u(t))+\int_t^b f_u(s,x,u,t)\lambda(s)ds = 0 = \partial H/\partial u.$$

verifying part (i) of the theorem.

To establish part (ii) it has to be shown that the $\lambda(t)$ defined in (25) is consistent with

(28) $\lambda(t) = \partial H/\partial x = F_x(t,x(t,u),u(t)) + \int_t^b f_x(s,x,u,t)\lambda(s)ds$

Recollection of (14) yields

(29) $\lambda(t) = F_x(t,x(t,u),u(t)) + \int_t^b A(s,t)\lambda(s)ds,$

a linear integral equation in λ. Consistency of (25) and (28) reduces to demonstrating that the former is a solution to the latter. To do this we substitute from (25) into (29) to get

(30) $\lambda(t) = \int_t^b A(s,t)[F_x(s,x(s,u),u(s)) + \int_s^b F_x(\tau,x(\tau,u),u(\tau))\,r(\tau,s)d\tau]ds$

$+ F_x(t,x(t,u),u(t))$

Application of (18) to the double integral yields

(31) $\lambda(t) = \int_t^b [A(s,t) + \int_t^s A(\tau,t)r(s,\tau)d\tau]\ F_x(s,x(s,u),u(s))ds +$

$F_x(t,x(t,u),u(t))$

Substitution for the bracketed term in (31) from (17) yields

(32) $\lambda(t) = F_x(t,x(t,u),u(t)) + \int_t^b F_x(s,x(s,u),u(s))r(s,t)ds$

which is exactly (25), with s replaced by t, and τ by s. This establishes (ii) of Theorem 1.

Thus, in place of the two differential equations that are part of the necessary conditions for the optimal control problem (1) and (2) are the two integral equations (7) and (25). To show that this Maximum Principle specializes to the one applicable to (1) and (2) let $\mu(t) = \int_t^b \lambda(s)ds$ which implies $\lambda(t) = -\dot{\mu}(t) = -d\mu(t)/dt$ and $\mu(b) = 0$. Then (27) becomes

$$(27') \quad F_u(t,x(t,u),u(t)) - \int_t^b f_u(s,x,u,t)\dot{\mu}(s)ds = 0$$

and (28) becomes

$$(28') \quad -\dot{\mu}(t) = F_x(t,x(t,u),u(t)) - \int_t^b f_x(s,x,u,t)\dot{\mu}(s)ds$$

Integration by parts of the last terms in the above expressions yield, respectively

$$(27'') \quad F_u(t,x(t,u),u(t)) + \mu(t)f_u(t,x,u,t) + \int_t^b (\partial f_u/\partial s)\mu(s)ds = 0$$

$$(28'') \quad -\dot{\mu}(t) = F_x(t,x(t,u),u(t)) + \mu(t)f_x(t,x,u,t) + \int_t^b (\partial f_x/\partial s)\mu(s)ds$$

But in accordance with the discussion of (3) and (4), if the state equations are differential equations, the integrals in (27") and (28") vanish or integrate out and the familiar necessary conditions obtain. Likewise, as in the familiar case, concavity of the maximized Hamiltonian in the state variable assures that the necessary conditions are also sufficient. This is demonstrated in:

Theorem 2: Let $u(t,x,\lambda)$ be the solution to max $H(t,x,u,\lambda)$, where H is defined by (8), and $\bar{H}(t,x,\lambda) = H(t,x,u(t,x,\lambda),\lambda)$. If $\bar{H}(t,x,\lambda)$ is a concave function of the state variable x, then the necessary conditions (27) and (28) are also sufficient for a maximum of (6) subject to (7).

Proof: The proof follows that in Kamien and Schwartz [5] for the familiar case. Let $x^*(t)$, $u^*(t)$, $\lambda(t)$ satisfy (7), (27) and (28) so that $u^*(t) = u(t,x^*,\lambda)$, and $x(t),u(t)$ satisfy (7), for $a \le t \le b$. Moreover, denote:

$$F^*(t) = F(t,x^*,u^*) \quad , \quad f^*(t,s) = f(t,x^*(s),u^*(s),s)$$

$$\bar{F}(t) = F(t,x,u(t,x,\lambda)) \quad , \quad \bar{f}(t,s) = f(t,x(s),u(s,x,\lambda),s)$$

$$F(t) = F(t,x,u) \quad , \quad f(t,s) = f(t,x(s),u(s),s)$$

The proof consists of showing that

$$D \equiv \int_a^b (F^*(t)-F(t)dt \ge 0 \tag{33}$$

Adding and subtracting

$$\bar{F}(t) + \int_a^b (s,t)\lambda(s)ds + \int_a^b \bar{f}(s,t)\lambda(s)ds$$

under the integral of

$$D = \int_a^b \{\bar{F}(t) + \int_a^b \bar{f}(s,t)\lambda(s)ds - \int_a^b f^*(s,t)\lambda(s)ds +$$

$$F^*(t) + \int_a^b f^*(s,t)\lambda(s)ds - \bar{F}(t) - \int_a^b \bar{f}(s,t)\lambda(s)ds - F(t)\}dt$$

Using the concavity of $\bar{H}$, hypothesized in the theorem, yields

$$(34)\quad D \geq \int_a^b \{ \bar{F}(t) + \int_a^b \bar{f}(s,t)\lambda(s)ds - \int_a^b f^*(s,t)\lambda(s)ds +$$

$$(x^*-x)[\partial F^*(t)/\partial x + \int_a^b \lambda(s)(\partial f^*(s,t)/\partial x)ds + (\partial F^*(t)/\partial u)\ (\partial u^*/\partial x) +$$

$$\int_a^b \lambda(s)(\partial f^*(s,t)/\partial u)(\partial u^*/\partial x)ds] - F(t)\}dt$$

Rearrangement of terms and substitution from (27) and (28) into (34) yields

$$(35)\quad D \geq \int_a^b [\bar{F}(t) + \int_a^b \bar{f}(s,t)\lambda(s)ds - \int_a^b f^*(s,t)\lambda(s)ds + (x-x^*)\lambda(t) - F(t)]dt$$

Recollection of (7) implies

$$(36)\quad \int_a^b [x(t)-x^*(t)]\lambda(t)dt = \int_a^b \int_t^b (f^*(s,t)-f(s,t))\lambda(s)dsdt$$

where the last line follows from reversing the order of integration in accordance with (18). Recollection that $f(s,t) \equiv 0$ if $s < t$

implies:

$$\int_t^b f(s,t)ds = \int_a^b f(s,t)ds$$

and substitution from (36) into (35) gives

$$(37)\quad D \geq \int_a^b [\bar{F}(t) + \int_a^b \bar{f}(s,t)\lambda(s)ds$$
$$-F(t) - \int_a^b f(s,t)\lambda(s)ds]dt \geq 0$$

since $u(t,x,\lambda)$ maximizes $H(t,x,u,\lambda)$ by assumption.

Application:

The Maximum Principle of Theorem 1 can be employed to analyze a capital replacement problem studied by Arrow [1]. The problem posed is selection of an investment plan through time, $I(t)$, that maximizes the present value of future cash flow, where current operating profit depends on current capital stock, and current capital stock depends on the history of previous investments. Analytically, the problem is to

$$(38)\qquad \text{Max} \int_0^T \alpha(t)[P(k(t),t)-I(t)]dt$$

s.t.

$$(39)\qquad \dot{k}(t) = I(t) - \int_0^t m(t-s)I(s)ds$$
$$k(0) = k_0$$

where

$P(k,t)$ = operating profit from stock of capital k at time t

$k(t)$ = stock of capital at time t

$\alpha(t)$ = discount factor at time t

$\rho(t)$ = $-\dot{\alpha}(t)/\alpha(t)$ = instantaneous rate of interest

$m(t-s)$ = mortality density, the fraction of gross investment made at time s, that disappears about time t.

Expression (39) indicates that net capital formation at time t, $\dot{k}(t)$ is the difference between contemporaneous gross investment $I(t)$ and disappearance (deterioration) of capital. This equation of capital formation specializes to the more familiar

$$\dot{k}(t) = I(t) - \delta k(t)$$

when the mortality density has the form $m(t-s) = \delta e^{-\delta(t-s)}$

Arrow employs algebraic ingenuity to convert the problem posed in (38) and (39) into one suitable for analysis by calculus of variations methods and thereby derives the necessary conditions for a maximum. To apply Theorem 1, we convert (39) into an integral equation.

$$(40) \quad k(t) = k_o + \int_0^t I(s)ds - \int_0^t [\int_s^t m(u-s)du] I(s)ds$$

Expressions (38) and (40) constitute a control problem with investment $I(s)$ the control variable and capital stock $k(t)$, the state variable. The state equation for the infinite horizon version of this problem analyzed by Arrow can be written as:

(40') $K(t) = \int_{-\infty}^{t} M(t-s)I(s)ds$

where M(t) is the mortality rate and $\dot{M}(t) = -m(t)$. The necessary conditions for this version of the problem are identical to those that are developed below.

(41) $H = \alpha(t)[P(k(t),t)-I(t)] + \int_t^T [I(t) - \int_t^s m(u-t)I(t)du]\lambda(s)ds$

and the necessary conditions are

(42) $\partial H/\partial I = -\alpha(t) \qquad + \int_t^T[1 - \int_t^s m(u-t)du]\lambda(s)ds = 0$

(43) $\partial H/\partial k = \lambda(t) = \alpha(t)P_k[k(t),t]$

Differentiation of (42) yields a linear integral equation of the convolution type in $\lambda(t)$

(44) $$\lambda(t) = -\dot{\alpha}(t) + \int_t^T[m(t-t) - \int_t^s \frac{dm(u-t)}{dt} du]\lambda(s)ds$$
$$= -\dot{\alpha}(t) + \int_0^T m(s-t)\lambda(s)ds$$

by observation that $dm(u-t)/dt = -dm(u-t)/du$, integration of the inner integral and since $m(s-t) \equiv 0$ for $s < t$. The solution of (44) can be written as

(45) $\lambda(t) = -\dot{\alpha}(t) - \int_0^T r(s)\dot{\alpha}(s-t)ds$

where r(s) denotes the resolvent kernel, see [7, p. 13-21], that can be shown to be identical to the replacement density $r(\tau)$ defined by Arrow.

Substitution into (43) and division of both sides by $\alpha(t)$ yields

$$(46)\quad P_k[k\ (t),t] = -\dot{\alpha}(t)/\alpha(t) - \int_0^T \frac{r(s)}{\alpha(t)} \dot{\alpha}(s-t)ds$$

Recollection of the definition of $\rho(t)$ gives rise to a finite horizon version of Arrow's myopic rule for optimal capital investment

$$(47)\quad P_k[k(t),t] = \rho(t) + \int_0^T [r(s)\rho(s-t)\alpha(s-t)/\alpha(t)]ds$$
$$= \rho(t) + \bar{r}(t)$$

where $\bar{r}$, implicitly defined by (47) may be interpreted as the average number of replacements from t forward. While the above analysis only demonstrates that results obtained by more traditional methods can be duplicated by application of Theorem 1, it also provides a vehicle for generalizations that could not be as conveniently treated by those techniques. For example, a mortality density that is dependent on the contemporaneous level of the capital stock $m(t-s,k(s))$ could be accomodated by the Maximum Principle, as could other generalizations recently suggested by Malcomson [6] and Nickell [9].

Before concluding we indicate an extention of the Nerlove-Arrow [8] result regarding the optimal ratio of advertising goodwill to sales. In the Nerlove-Arrow paper revenue $P(k(t),t)$ is replaced by $pq(p,k)-c(q)$, where p denotes product price, product demand $q(p,k)$ is a function of both price and advertising goodwill k, and $c(q)$

denotes production cost. We assume that advertising goodwill is accumulated through current expenditure, I(t), in accordance with (39), rather than in accordance with the special case of Nerlove-Arrow in which the mortality follows an exponential law. The maximization problem has the same form as (38) and (39) except for the addition of product price as a control variable. Necessary conditions for this problem are

(48) $\lambda(t) = -\dot{\alpha}(t) + \int_0^T m(s-t)\lambda(s)ds$

(49) $\alpha(t)\,[q+(p-c')\partial q/\partial p] = 0$

(50) $\lambda(t) = \alpha(t)(p-c')\partial q/\partial k$

If $\alpha(t) \neq 0$ then (49) implies $(p-c') = -q/\frac{\partial q}{\partial p}$ which upon substitution into (50), multiplication of both sides by k/pq, and rearrangement of terms yields.

(51) $\beta\,\alpha(t)/\eta\lambda(t) = k/pq$

where $\beta = \frac{k\ \partial q/\partial k}{q}$, $\eta = \frac{-p\ \partial q/\partial p}{q}$

are the elasticity of demand with respect to goodwill and price, respectively. Recollection of (43), (46), and (47) and substitution for $\lambda(t)/\alpha(t)$ in (51) yields the desired result:

(52) $k/pq = \beta/\eta\ [\rho(t)+\bar{r}(t)]$.

Expression (52) specializes to the marginal condition obtained by Nerlove-Arrow under the supposition of exponential decay of good will at rate δ, for then $\bar{r} = \delta$. According to (52) the ratio of goodwill to sales revenue along an optimal policy is directly related to the elasticity of demand with respect to goodwill and inversely to the price elasticity of demand, the instantaneous rate of interest, and the average anticipated decay rate from the present forward.

Summary

A simplified version of a theorem of Bakke for optimal control with an integral state equation has been presented. Usual concavity conditions have been shown to render the necessary conditions for an optimum sufficient as well as in this problem. Finally, application of this Maximum Principle to some capital replacement problems has demonstrated how they might be treated in a uniform fashion.