Optimal Control with Mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Performance Objectives

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Abstract

This paper considers the analysis and synthesis of control systems subject to two types of disturbance signals: signals with bounded power spectral density and signals with bounded power. The resulting control problem involves minimizing a mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norm of the system. It is shown that the controller shares a separation property similar to those of pure $\mathcal{H}_2$ or $\mathcal{H}_\infty$ controllers. It is also shown that the mixed problem reduces naturally to $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problems in special cases. Some necessary and sufficient conditions are obtained for the existence of a solution to the mixed problem. Explicit state space formulae are given for the optimal controllers.

1 Introduction

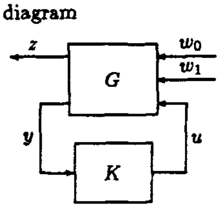
Two performance measures in optimal control theory which have been the focus of much recent research are the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms, defined in the frequency-domain for a stable transfer matrix $G(s)$ as

$$\|G\|_2 := \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Trace}[G(j\omega)^* G(j\omega)]d\omega\right)^{1/2}$$

$$\|G\|_\infty := \sup_\omega \sigma_{max}[G(j\omega)] \quad (\ \sigma_{max} := \text{maximum singular value}\ )$$

It is beyond the scope of this paper to review the vast literature associated with the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ theory. The interested reader might consult [Francis and Doyle, 1987], or the recent paper by Doyle, Glover, Khargonekar, and Francis, [DGKF, 1989], and the references therein.

The $\mathcal{H}_\infty$ results of [DGKF] suggest the possibility of a single theory that has the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ results as special cases, and this encourages us to consider a more general problem. The basic system we use has the block diagram



where G is the generalized plant and K is the controller. Only finite dimensional linear time-invariant (LTI) systems and controllers will be considered in this paper. The generalized plant G contains what has been called the plant in traditional control problems as well as any weighting functions. The signals w_0 and w_1 represent all external inputs, including disturbances, sensor noise and commands.

The signal w_0 is assumed to have a fixed or bounded power spectrum, while w_1 is assumed to be bounded in power; z is an output error signal with power as the performance objective; y represents the measured variables; and u is the control input. Let the transfer function from w_0 and w_1 to z be T_{zw}. The analysis problem is, given G and K, determine the induced norm of T_{zw}. The synthesis problem is, given G, find a controller K which stabilizes the plant and makes the norm of T_{zw} less than a pre-specified performance level. Both the analysis and the synthesis problems are referred to as "mixed" $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problems.

Note that if only w_0 is present, then the problem reduces to the standard $\mathcal{H}_2$ problem. Similarly, if only w_1 is present we obtain the standard $\mathcal{H}_\infty$ problem. Often we compare the results of this paper with those for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problems as presented in [DGKF], which are referred to as the "pure" $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problems. The major motivation of this paper is to begin providing more flexibility in the modeling assumptions required in order to use optimal control methods.

The main results of this paper are presented in sections 3 and 4. Specifically, section 3 presents the analysis results and section 4 presents the synthesis results. The proofs of the synthesis results exploit the "separation" structure of the controller, which is reminiscent of the classical $\mathcal{H}_2$ controller and the $\mathcal{H}_\infty$ theory in [DGKF]. Of course, there are significant differences that reflect the mixed criterion used in the problem. These differences are similar to the differences between the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ separation principle discussed in [DGKF].

If full state feedback is available, then the central controller is simply a gain matrix F_∞, obtained by solving a single Riccati equation, which is the same as in the pure $\mathcal{H}_\infty$ problem. Also, the optimal estimator is an observer whose gain is obtained as a solution to three coupled equations; this reflects the complexity of the mixed problem. In the general output feedback case the central controller can be interpreted as an optimal estimator for $F_\infty x$. This paper does not present a complete solution, as there is a small difference between the necessary and sufficient conditions for the estimation problem. We believe that this difference can be removed, and present a series of conjectures which do so. Although we strongly believe that these conjectures are correct, we do not have proofs at this time and more work must be done.

To make the results more accessible, we have chosen to treat only a special case of the general mixed problem in this paper. This problem is similar to the problem treated in [DGKF], and captures the essential features of the general problem. While there is some loss of generality in doing this, it relieves the proofs of serious algebraic encumberment, and makes the formulae much easier to interpret. In addition, the assumptions are common in the standard presentation of the $\mathcal{H}_2$ problem. Although the theory developed here follows [DGKF], important motivation came from the work of Bernstein and Haddad (1989), which uses Lagrange multiplier techniques to solve a different mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problem. Due to space limitations, all the proofs are left as exercises; they can be found in [Doyle, 1989].

2 Preliminaries

This section reviews some elementary mathematical and system theoretic results, and presents the notation, which is fairly standard.

2.1 Notation

The Hardy space $\mathcal{H}_2$ consists of square-integrable functions on the imaginary axis with analytic continuation into the right half-plane. The Hardy space $\mathcal{H}_\infty$ consists of bounded functions with analytic continuation into the right half-plane. The Lebesgue spaces $\mathcal{L}_1(-\infty,\infty)$ and $\mathcal{L}_2(-\infty,\infty)$ consist, respectively, of absolute and square-integrable functions on $(-\infty,\infty)$; $\mathcal{L}_\infty$ consists of bounded functions on $(-\infty,\infty)$.

All integrals are Lebesgue integrals. In general, $u(t): \mathbf{R} \to \mathbf{R}^m$ and $w_i(t): \mathbf{R} \to \mathbf{R}^{m_i}$ will be used to denote signals which are inputs to systems, $z(t): \mathbf{R} \to \mathbf{R}^q$ and $y(t): \mathbf{R} \to \mathbf{R}^p$ denote signals which are the outputs of a system, and $x(t): \mathbf{R} \to \mathbf{R}^n$ denotes a signal which is the state of a system. Let $\otimes$ denote the convolution operator, and $\langle x, y \rangle$ the usual inner product on $\mathbf{C}^n$ or $\mathbf{R}^n$.

A transfer matrix in terms of state-space data is denoted

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := C(sI - A)^{-1}B + D.$$

For a matrix $M \in \mathbf{C}^{p\times r}$ or $\mathbf{R}^{p\times r}$, M^T denotes its transpose, M^* denotes its conjugate transpose, $\sigma_{max}(M) = \rho(M^*M)^{1/2}$ denotes its maximum singular value, and $\rho(M)$ denotes its spectral radius (if $p = r$). The prefix $\mathcal{B}$ denotes the unit ball and the prefix $\mathcal{R}$ denotes real-rational. The unsubscripted norm $\|\cdot\|$ will denote the standard Euclidean norm on vectors. We will omit all vector and matrix dimensions throughout, and assume that all quantities have compatible dimensions.

2.2 Signals and Norms

All signals considered in this paper are assumed to be deterministic. For a given signal $u(t) \in \mathbf{R}^m$, its autocorrelation matrix is defined as

$$R_{uu}(\tau) := \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} u(t+\tau)u^T(t)dt,$$

if the limit exists and is finite. If $u(t)$ is a signal such that $R_{uu}(\tau)$ exists and is finite for all τ, then $u(t)$ is called a power signal, or, more formally, a signal of bounded power. The set of all signals having bounded power is denoted by

$$\mathcal{P} := \{u(t) \in \mathbf{R}^m : R_{uu}(\tau) \text{ exists and is finite for all } \tau.\}$$

Note that not every signal having finite ∞-norm has bounded power; however, if $u \in \mathcal{P}$ and $\|u(t)\|_\infty < \infty$, then $\|u\|_P \le \sqrt{m}\|u\|_\infty$, where m is the dimension of u.

A semi-norm can be defined on the space of signals of bounded power, i.e.,

$$\|u\|_P = \left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\|u(t)\|^2\,dt\right)^{1/2} = (\text{Trace}\,[R_{uu}(0)])^{1/2}.$$

The capital "P" is used to differentiate this power semi-norm from the usual Lebesgue $\mathcal{L}_p$ norm. Note that signals of bounded power are persistent signals in time such as sines or cosines. Any $\mathcal{L}_2$ signal has zero power, and thus $\|\cdot\|_P$ is only a semi-norm, not a norm.

The spectral density matrix of u is the Fourier transform of its autocorrelation:

$$S_{uu}(j\omega) := \int_{-\infty}^{\infty} R_{uu}(\tau)e^{-j\omega\tau}d\tau.$$

$R_{uu}(\tau)$ can be obtained from $S_{uu}(j\omega)$ by inverse Fourier transform as

$$R_{uu}(\tau) := \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{uu}(j\omega)e^{j\omega\tau}d\omega.$$

Now suppose $u \in \mathcal{P}$, then

$$\|u\|_P^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\text{Trace}[S_{uu}(j\omega)]d\omega$$

A signal $u(t)$ is said to have bounded spectral density if $\|S_{uu}(j\omega)\|_\infty < \infty$. The set of signals having bounded spectral density is denoted as

$$\mathcal{S} := \{u(t) \in \mathbf{R}^m : \|S_{uu}(j\omega)\|_\infty < \infty\}.$$

The quantity $\|u\|_s := \|S_{uu}(j\omega)\|_\infty^{1/2}$ is called the spectral density semi-norm of $u(t)$.

The engineering relevance of the set $\mathcal{S}$ is that it can be used to model signals with fixed spectral characteristics by passing the signals through a weighting filter. Similarly, $\mathcal{P}$ could be used to model signals whose spectrum is not known but which are bounded in power.

output\input	$\|u\|_2$	$\|u\|_\infty$	power $\|u\|_P$	spectrum $\|u\|_S$
$\|z\|_2$	$\|G\|_\infty$	∞	∞	∞
$\|z\|_\infty$	$\le \|G\|_2\dagger$	$\|G\|_1\dagger$	∞	∞
$\|z\|_P$	0	$\le \|G\|_\infty$	$\|G\|_\infty$	$\|G\|_2$
$\|z\|_S$	∞	∞	∞	$\|G\|_\infty$

Table 1: Systems input/output gains

System Input/Output Gains

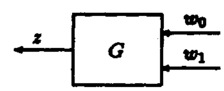
For the linear system shown in the previous diagram, assume G is strictly proper. G considered as an operator from the input space to the output space induces a norm on G, which, loosely speaking, measures the size of the output for a given input u. Table 1 summarizes some of the input and output relations we are interested in this paper.

2.3 The Riccati Operator

We assume the reader is familiar with basic material on the Riccati equation and the Riccati operator, such as can be found in section 2.2 of [DGKF]. We will adopt the conventions of [DGKF] when discussing these.

3 Systems Performance Analysis with Mixed Inputs

In this section we look at the norms induced on G when G has two different types of inputs. In particular, let $w_0(t) \in \mathcal{S}$ and $w_1(t) \in \mathcal{P}$. Pictorially, we have



The "size" of the output signal $z(t)$ is measured by the power semi-norm. Assume G is stable and partition G compatibly with w_0 and w_1 as $[G_0\ G_1]$, where G_0 is assumed strictly proper. In terms of the state-space matrices, this can be represented as

$$G(s) = \left[\begin{array}{c|cc} A & B_0 & B_1 \\ \hline C & 0 & D_1 \end{array}\right].$$

We wish to compute

$$\sup_{w_1\in\mathcal{BP}, w_0\in\mathcal{BS}} \|z\|_P^2 \tag{1}$$

This problem is referred to as the "mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$" problem because, from the earlier tables, if we ignore w_1 then the norm induced on G from w_0 to z is the $\mathcal{H}_2$ norm; similarly, if we ignore w_0 then the norm induced on G from w_1 to z is the $\mathcal{H}_\infty$ norm.

The remainder of this section is devoted to the solution of this problem. For clarity and simplicity, we assume $D_1 = 0$. Assuming D_1 non-zero only complicates the formulae; it does not present additional theoretical difficulties.

Theorem 1. *Suppose $\gamma > \|G_1\|_\infty$. Then*

$$\sup_{w_0\in\mathcal{BS}}\{\|z\|_P^2 - \gamma^2\|w_1\|_P^2\} = \text{Trace}(B_0^T X_\gamma B_0) - \gamma^2\|w_1 - \gamma^{-2}B_1^T X_\gamma x\|_P^2$$

and

$$\sup_{w_1\in\mathcal{P}, w_0\in\mathcal{BS}}\{\|z\|_P^2 - \gamma^2\|w_1\|_P^2\} = \text{Trace}(B_0^T X_\gamma B_0)$$

with a worst-case signal $\tilde{w}_1 = \gamma^{-2}B_1^T X_\gamma x$, where X_γ is the solution to the Riccati equation

$$A^T X_\gamma + X_\gamma A + \gamma^{-2}X_\gamma B_1 B_1^T X_\gamma + C^T C = 0$$

and $A + \gamma^{-2}B_1B_1^T X_\gamma$ is stable.

Assume now that the input to the system is $\tilde{w}_1$, for fixed γ. Then the system equations become

$$\begin{aligned} \dot{x} &= \left(A + \frac{1}{\gamma^2}B_1B_1^TX_\gamma\right)x + B_0w_0(t), \qquad \|x(-\infty)\| < \infty \\ z &= Cx \end{aligned}$$

Let P_γ be the solution to the Lyapunov equation

$$\left(A + \frac{1}{\gamma^2}B_1B_1^TX_\gamma\right)^T P_\gamma + P_\gamma\left(A + \frac{1}{\gamma^2}B_1B_1^TX_\gamma\right) + B_0B_0^T = 0$$

then

$$\|z\|_P^2 = \text{Trace}\left(CP_\gamma C^T\right).$$

Note that for small γ, $\tilde{w}_1$ may be outside of $\mathcal{BP}$. Hence to compute (1), we have to find a suitable γ such that $\tilde{w}_1 \in \mathcal{BP}$, this is given in the following theorem.

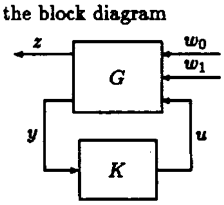
Theorem 2. *Let γ_0 be such that $\|\gamma_0^{-2}B_1^TX_{\gamma_0}x\|_P = 1$. Then*

$$\sup_{w_1\in\mathcal{BP}, w_0\in\mathcal{BS}} \|z\|_P^2 = \text{Trace}\left(CP_{\gamma_0}C^T\right).$$

Hence computing the power norm of z involves iterations on γ, as in the pure $\mathcal{H}_\infty$ case. As an aside, note that the optimal γ level almost always satisfies $\gamma > \|G_1\|_\infty$ when $w_0 \neq 0$. Also, the worst-case signals in $\mathcal{BS}$ are always white noise, which in this context is simply the limit of signals in $\mathcal{BS}$ when $S_{w_0w_0} \to I$. White noise signals do not have bounded power, and in this treatment, only exist as limits. Nevertheless, we will use the shorthand convention of treating them as being in $\mathcal{BS}$.

4 Mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Synthesis

In this section, we consider the synthesis problem when the system subjected to mixed disturbance signals. Specifically, consider the system described by the block diagram



where again the plant G and controller K are assumed to be real-rational and proper.

Problem (G) *Given the plant G and a constant γ, assume that the exogenous signals w_0 and w_1 are as follows*

$$w_0 \in \mathcal{S}$$
$$w_1 \in \mathcal{P}$$

the mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal control problem is to find a controller K such that

$$\min_K \sup_{w_1\in\mathcal{P}, w_0\in\mathcal{BS}} \left\{\|z\|_P^2 - \gamma^2\|w_1\|_P^2\right\}$$

is solved, where the minimization is constrained to those K such that the internal stability is guaranteed.

As mentioned earlier, when $w_0 = 0$ or $w_1 = 0$, the induced norm becomes the the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm, respectively. Thus, Problem (G) is solvable only if the corresponding pure $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problems are solvable. In this paper, we do not usually address the issue of the optimal mixed controller and only discuss optimality in terms of a given γ, restricting γ to be greater than the corresponding $\mathcal{H}_\infty$ optimal γ level. Thus, optimal controller means optimal for a given γ level. Clearly, any mixed optimal controller is a sub-optimal pure $\mathcal{H}_\infty$ controller, but the converse need not be true. However, if a sub-optimal pure $\mathcal{H}_\infty$ controller exists, then it is an admissible mixed controller, hence an optimal mixed controller exists. We have just shown

Lemma 1. *Problem (G) is solvable if and only if there exists a K such that $\|T_{zw_1}\|_\infty < \gamma$, i.e., the corresponding $\mathcal{H}_\infty$ problem ($w_0 = 0$) is solvable.*

Note that although Lemma 1 gives necessary and sufficient conditions for the existence of a solution to Problem (G), it does not give a method for finding it. Finding an explicit solution to this problem is the focus of the rest of this paper.

Assumptions on the Plant G

The system G(s) has the following realization

$$G(s) = \left[\begin{array}{c|ccc} A & B_0 & B_1 & B_2 \\ \hline C_1 & 0 & 0 & D_{12} \\ C_2 & D_{20} & D_{21} & 0 \end{array}\right]$$

and the following assumptions are made:

(i) (A, B_1) is stabilizable and (C_1, A) is detectable

(ii) (A, B_2) is stabilizable and (C_2, A) is detectable

(iii) $D_{12}^T\left[\begin{array}{cc} C_1 & D_{12}\end{array}\right] = \left[\begin{array}{cc} 0 & I \end{array}\right]$

(iv) $\left[\begin{array}{c} B_1 \\ D_{21}\end{array}\right] D_{21}^T = \left[\begin{array}{c} 0 \\ R_1 \end{array}\right]$, $R_1 \geq 0$

(v) $\left[\begin{array}{c} B_0 \\ D_{20}\end{array}\right] D_{20}^T = \left[\begin{array}{c} 0 \\ R_0 \end{array}\right]$, $R_0 > 0$

Similar comments made in [DGKF] about assumptions on the $\mathcal{H}_\infty$ problem apply here for (i)–(iii). Note that in (iv) we do not require positive definiteness of R_1, but instead require $R_0 > 0$. A more desirable assumption would be that $R_0 + R_1 > 0$; this, however, complicates the treatment substantially.

4.1 Separation Principle for Mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Problems

The following theorem is one of the main results in this paper. It shows that the solution to Problem (G) shares a kind of separation principle, i.e., state feedback and optimal estimation of the feedback signal.

Theorem 3. *There exists an admissible controller which solves the following optimization problem*

$$\min_K \sup_{w_1\in\mathcal{P}, w_0\in\mathcal{BS}} \left\{\|z\|_P^2 - \gamma^2\|w_1\|_P^2\right\}$$

iff the following conditions hold:

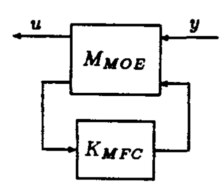
(i) $H_\infty \in dom(Ric)$ and $X_\infty := Ric(H_\infty) \geq 0$

(ii) There exists a controller K_{MFC} which solves Problem (G) with $G = \hat{G}_{MFC}$ (called the Mixed Full Control (MFC) Problem)

$$\hat{G}_{MFC}(s) = \left[\begin{array}{c|ccc} A_{tmp} & B_0 & B_1 & \left[\begin{array}{cc} I & 0 \end{array}\right] \\ \hline -F_\infty & 0 & 0 & \left[\begin{array}{cc} 0 & I \end{array}\right] \\ C_2 & D_{20} & D_{21} & \left[\begin{array}{cc} 0 & 0 \end{array}\right] \end{array}\right]$$

where $A_{tmp} = A + \gamma^{-2}B_1B_1^TX_\infty$ and $F_\infty = -B_2^TX_\infty$.

Moreover, when these conditions hold, one such controller equals the transfer matrix from y to u in



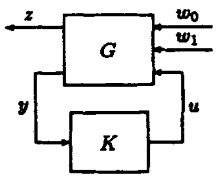
$$M_{MOE}(s) = \left[\begin{array}{c|cc} A + \gamma^{-2}B_1B_1^TX_\infty + B_2F_\infty & 0 & \left[\begin{array}{cc} I & -B_2 \end{array}\right] \\ \hline -F_\infty & 0 & \left[\begin{array}{cc} 0 & I \end{array}\right] \\ C_2 & I & \left[\begin{array}{cc} 0 & 0 \end{array}\right] \end{array}\right]$$

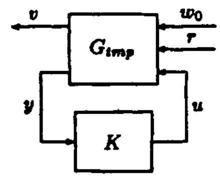
Notice that (i) corresponds to the condition for full information control and (ii) corresponds to the condition for the optimal estimation of $F_\infty x$. Thus, the separation principle of mixed controllers is now evident and is similar to the separation principle for $\mathcal{H}_\infty$ controllers given in [DGKF]: *The mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ output feedback controller is the output estimator of the full information control law in the presence of a "worst-case" disturbance $w_{1_{worst}} = \gamma^{-2}B_1^T X_\infty x$.*

(A_{tmp}, B_1) is stabilizable since (A, B_1) is and $(-F_\infty, A_{tmp})$ is detectable since $A_{tmp} + B_2F_\infty$ is stable. For the MFC Problem to be solvable, it is also necessary to require (C_2, A_{tmp}) be detectable. This condition will be satisfied implicitly if there is an admissible controller solving the MFC Problem. On the other hand, if $R_1 > 0$, then from [DGKF], $J_\infty \in \mathrm{dom}(Ric)$ and $Ric(J_\infty) \geq 0$ will guarantee (C_2, A_{tmp}) be detectable. However, we will not pose these conditions here.

The proof of Theorem 3 is given in section 4.3. The proof in section 4.3 uses following lemma and the result in the next section. At this point, we again remind the readers that *white noise is always a worst signal for w_0 in the problem considered in this paper. Hence in the subsequent development, we will drop the supremum over w_0 and assume w_0 is white with spectral density equal to the identity.*

Lemma 2. *Suppose $H_\infty \in \mathrm{dom}(Ric)$ and $X_\infty = Ric(H_\infty) \geq 0$. Then there exists an admissible controller $K(s)$ such that $K(s)$ solves Problem (G) iff $K(s)$ solves Problem (G_{tmp}),*



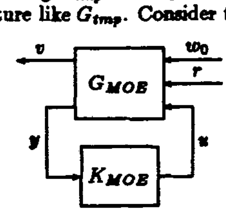


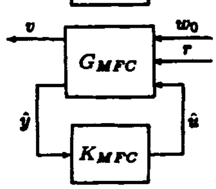
where

$$G_{tmp}(s) = \left[\begin{array}{c|ccc} A_{tmp} & B_0 & B_1 & B_2 \\ \hline -F_\infty & 0 & 0 & I \\ C_2 & D_{20} & D_{21} & 0 \end{array}\right].$$

4.2 Mixed Output Estimation

As we have mentioned earlier in Theorem 3, we will call the mixed control problem having structure like G_{MFC} the Mixed Full Control (MFC) problem, while the problem having structure like G_{tmp} will be called the Mixed Output Estimation (MOE) problem. In the following, we show how to reduce the MOE problem to the MFC problem. Instead of using G_{tmp} directly, we use an arbitrary plant G_{MOE} having structure like G_{tmp}. Consider the following diagram





where

$$G_{MOE}(s) = \left[\begin{array}{c|ccc} A & B_0 & B_1 & B_2 \\ \hline C_1 & 0 & 0 & I \\ C_2 & D_{20} & D_{21} & 0 \end{array}\right]$$

$$G_{MFC}(s) = \left[\begin{array}{c|ccc} A & B_0 & B_1 & [\,I \;\; 0\,] \\ \hline C_1 & 0 & 0 & [\,0 \;\; I\,] \\ C_2 & D_{20} & D_{21} & [\,0 \;\; 0\,] \end{array}\right]$$

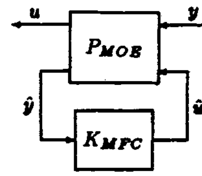
and we assume $A - B_2C_1$ in the realization of G_{MOE} is stable.

Note that assumptions (i) through (v) on Problem (G) are not needed in obtaining the reduction from the MOE problem to the MFC. This will be clear from the procedure.

Let T_{MOE} and T_{MFC} denote the closed-loop transfer matrices from w_0 and r to v for the MOE problem and the MFC problem, respectively. The proof is in the same spirit as section 8 of [DGKF].

Proposition 1. *The controller K_{MOE} internally stabilizes G_{MOE} iff $K_{MFC} = \begin{bmatrix} B_2 \\ I \end{bmatrix} K_{MOE}$ internally stabilizes G_{MFC}. Furthermore, in this case $T_{MOE} = T_{MFC}$.*

To complete the equivalence, suppose that we have a controller for the MFC problem, denoted by K_{MFC} and let K_{MOE} be the transfer function generated by



$$P_{MOE} = \left[\begin{array}{c|cc} A - B_2C_1 & 0 & [\,I \;\; -B_2\,] \\ \hline C_1 & 0 & [\,0 \;\; I\,] \\ C_2 & I & [\,0 \;\; 0\,] \end{array}\right]$$

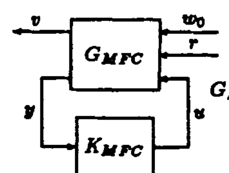
Proposition 2. *The controller K_{MFC} internally stabilizes G_{MFC} iff K_{MOE} internally stabilizes G_{MOE}. Furthermore, in this case $T_{MOE} = T_{MFC}$.*

4.3 Proof of Theorem

Since a controller solving Problem (G) is also a suboptimal $\mathcal{H}_\infty$ controller, it is obvious that (i) is necessary. Hence if the problem is solvable, then $H_\infty \in \mathrm{dom}(Ric)$. Now using Lemma 2, the original problem is equivalent to Problem (G_{tmp}). Since $X_\infty = \mathrm{Ric}(H_\infty) \geq 0$, $A_{tmp} - B_2(-F_\infty) = A + \gamma^{-2}B_1B_1^TX_\infty + B_2B_2^TX_\infty$ is stable. The theorem then follows by applying Propositions 1 and 2 to G_{tmp}. ∎

4.4 Mixed Full Control Problem

In Theorem 3, we have seen that the mixed synthesis problem, Problem (G), can be reduced to an MFC problem with $G_{MFC} = \hat{G}_{MFC}$. This section is devoted to the solution of this problem. We will give some explicit necessary and sufficient conditions for solving this problem. We will also point out some unresolved issues. Consider the following diagram



$$G_{MFC}(s) = \left[\begin{array}{c|ccc} A & B_0 & B_1 & [\,I \;\; 0\,] \\ \hline C_1 & 0 & 0 & [\,0 \;\; I\,] \\ C_2 & D_{20} & D_{21} & [\,0 \;\; 0\,] \end{array}\right]$$

The assumptions on G_{MFC} are the same as on G.

MFC Problem *Find an (or all) admissible controller(s) K_{MFC} such that K_{MFC} internally stabilizes G_{MFC} and minimizes*

$$\min_{K_{MFC}} \sup_{r \in \mathcal{P}} \{\|v\|_{\mathcal{P}}^2 - \gamma^2\|r\|_{\mathcal{P}}^2\}$$

where w_0 is white noise and has spectral density equal to the identity.

The next lemma follows from standard min-max optimization theory.

Lemma 3. *Suppose the plant G_{MFC} is given as above. Then*

$$\sup_{r \in \mathcal{P}} \left\{\|v\|_{\mathcal{P}}^2 - \gamma^2\|r\|_{\mathcal{P}}^2 \;|K_{MFC} \text{ given}\right\} \geq \min_{K_{MFC}} \sup_{r \in \mathcal{P}} \{\|v\|_{\mathcal{P}}^2 - \gamma^2\|r\|_{\mathcal{P}}^2\}$$

$$\geq \sup_{r \in \mathcal{P}} \min_{K_{MFC}} \{\|v\|_{\mathcal{P}}^2 - \gamma^2\|r\|_{\mathcal{P}}^2\} \geq \min_{K_{MFC}} \left\{\|v\|_{\mathcal{P}}^2 - \gamma^2\|r\|_{\mathcal{P}}^2 \;|r \text{ given}\right\}$$

The solution to the MFC problem involves three equations in unknowns L, Y, and P,

$$(L)\quad LR_0 + PC_2^T + \gamma^{-2}PYLR_1 = 0$$
$$(Y)\quad Y(A+LC_2) + (A+LC_2)^TY + \gamma^{-2}Y(B_1B_1^T + LR_1L^T)Y + C_1^TC_1 = 0$$
$$Y \geq 0 \text{ and } A + LC_2 + \gamma^{-2}(B_1B_1^T + LR_1L^T)Y \text{ is stable}$$
$$(P)\quad \{A + LC_2 + \gamma^{-2}(B_1B_1^T + LR_1L^T)Y\}P + P\{A + LC_2 + \gamma^{-2}(B_1B_1^T + LR_1L^T)Y\}^T + B_0B_0^T + LR_0L^T = 0$$

Note that since $A + LC_2 + \gamma^{-2}(B_1B_1^T + LR_1L^T)Y$ is stable, $(A + LC_2, \gamma^{-1}(B_1 + LD_{21})^TY)$ is detectable. This in turn implies that $A + LC_2$ is stable since $Y \geq 0$.

The following Lemma gives sufficient conditions for the inequalities in Lemma 3 to be equalities.

Lemma 4. *Suppose there exist real matrices L, $Y = Y^T$, and $P = P^T$ such that equations (L), (Y), and (P) are satisfied. Then*

$$\sup_{r\in\mathcal{P}} \left\{ \|v\|_P^2 - \gamma^2\|r\|_P^2 \;\middle|\; K_{MFC} = \begin{bmatrix} L \\ 0 \end{bmatrix} \right\}$$
$$= \min_{K_{MFC}} \left\{ \|v\|_P^2 - \gamma^2\|r\|_P^2 \;\middle|\; r = \gamma^{-2}(B_1 + LD_{21})^TY(x - \hat{x}) \right\} \quad (2)$$

where x is the state of the system G_{MFC} and $\hat{x}$ is obtained from

$$\dot{\hat{x}} = (A + LC_2)\hat{x} - Ly + \begin{bmatrix} I & 0 \end{bmatrix} u$$

The following lemma gives a necessary condition.

Lemma 5. *Suppose there exists a constant matrix L such that $K_{MFC} = \begin{bmatrix} L \\ 0 \end{bmatrix}$ solves the MFC problem. Then L satisfies equation (L_Y) where*

$$(L_Y)\quad Y(LR_0 + PC_2^T + \gamma^{-2}PYLR_1) = 0$$

and there exist $Y \geq 0$ and $P \geq 0$ satisfying equations (Y) and (P), respectively.

The main result of this section is the following theorem which follows immediately from Lemmas 4 and 5.

Theorem 4. *Suppose there exist real matrices L, $Y = Y^T$, and $P = P^T$ such that equations (L), (Y), and (P) are satisfied. Then $K_{MFC} = \begin{bmatrix} L \\ 0 \end{bmatrix}$ solves the MFC problem. On the other hand, suppose there exists a constant matrix L such that $K_{MFC} = \begin{bmatrix} L \\ 0 \end{bmatrix}$ solves the MFC problem. Then there exist $Y \geq 0$ and $P \geq 0$ which, together with L, satisfy equations (L_Y), (Y), and (P).*

Note that there is a gap between the necessary conditions and sufficient conditions in the above theorem if Y is singular. Nevertheless, we believe the following, which fill the gap, are true.

Conjecture 1. *Suppose there exists a controller in the form $K_{MFC} = \begin{bmatrix} L \\ 0 \end{bmatrix}$ which solves the MFC problem. Then there exist $\hat{L}$, $Y \geq 0$, and $P \geq 0$ which solve equations (L), (Y), and (P). Here $\hat{L}$ may not be equal to the L given by K_{MFC} if Y is singular.*

Conjecture 2. *There exists a constant matrix L such that $K_{MFC} = \begin{bmatrix} L \\ 0 \end{bmatrix}$ solves the MFC problem if and only if there exist $Y \geq 0$ and $P \geq 0$, which, together with L, satisfy equations (L_Y), (Y), and (P).*

We note that while Conjectures 1 and Conjectures 2 both fill the gap in Theorem 4, they do so in different senses. From a computational point of view, equation (L) may be more tractable than equation (L_Y). The following conjecture is concerned with the generalization of standard Kalman filter theory. It represents a start at bridging the gap.

Conjecture 3. *Given a dynamic system*

$$\dot{x} = Ax + B_0w_0$$
$$y = C_2x + D_{20}w_0$$

with assumptions as before. Then there exists a filter $K(s)$ such that $u = K(s)y$ minimizes

$$\|C_1x + u\|_P^2$$

iff there exists L_2, $P_2 \geq 0$, and $\tilde{Y}_2 \geq 0$ satisfying the following equations:

$$\tilde{Y}_2(L_2R_0 + P_2C_2^T) = 0$$
$$\tilde{Y}_2(A + L_2C_2) + (A + L_2C_2)^T\tilde{Y}_2 + C_1^TC_1 = 0, \text{ and } A + L_2C_2 \text{ is stable}$$
$$(A + L_2C_2)P_2 + P_2(A + L_2C_2)^T + B_0B_0^T + L_2R_0L_2^T = 0$$

Moreover, in this case $K = \left[\begin{array}{c|c} A + L_2C_2 & -L_2 \\ \hline -C_1 & 0 \end{array}\right]$ is an optimal filter, and

$$\|C_1x + u\|_P^2 = \operatorname{Trace}(P_2C_1^TC_1).$$

The necessity part is easy to show using Lagrange multipliers. It is also sufficient if $\tilde{Y}_2 > 0$; moreover, it is known to be sufficient in certain simple examples when $\tilde{Y}_2$ is singular, but a proof of the sufficiency for the general case needs to be found. Note that the above equations for L_2, $\tilde{Y}_2$, and P_2 are the degenerate form of equations (L_Y), (Y), and (P), when $B_1 = 0$ and $D_{21} = 0$.

Special Cases: Connections with Kalman Filter and the $\mathcal{H}_\infty$ Full Control Problem

We now examine how the MFC problem simplifies when the input disturbances are restricted to a single disturbance.

(A) Suppose $B_1 = 0, D_{21} = 0$, and $R_0 > 0$. In this case, the equations (L), (Y), and (P) reduce to

$$LR_0 + PC_2^T = 0$$
$$Y(A + LC_2) + (A + LC_2)^TY + C_1^TC_1 = 0, \text{ and } A + LC_2 \text{ is stable}$$
$$(A + LC_2)P + P(A + LC_2)^T + B_0B_0^T + LR_0L^T = 0$$

Note that equations (L) and (P) can also be written as

$$L = -PC_2^TR_0^{-1}$$
$$AP + PA^T - PC_2^TR_0^{-1}C_2P + B_0B_0^T = 0$$

This is the standard Kalman filter solution. Note also that there always exists a unique $Y \geq 0$ solving (Y) since $A + LC_2$ is stable. Hence the mixed full control problem reduces to the Kalman filter solution.

(B) Suppose $B_0 = 0, D_{20} = 0$, and $R_1 > 0$. In this case, we can assume $R_1 = I$. This problem is called Full Control Problem in [DGKF]. We show here that our solution to the MFC can be reduced to that one. It is easy to see that in this setting (P) has unique solution $P = 0$ and (L) disappears. The only equation left is (Y). Now we have a very interesting situation, since we need to find L and Y such that the single equation (Y) with a stability constraint is satisfied. To understand this equation, another interpretation of (Y) is useful in this case. From the characterization of the $\mathcal{H}_\infty$ norm of a transfer matrix, e.g., in [DGKF], we see that (Y) is true if and only if there exists an L such that $A + LC_2$ is stable and

$$\left\| \left[\begin{array}{c|c} A + LC_2 & B_1 + L_2D_{21} \\ \hline C_1 & 0 \end{array}\right] \right\|_\infty < \gamma$$

which is true if and only if

$$\left\| \left[\begin{array}{c|c} (A + LC_2)^T & C_1^T \\ \hline (B_1 + L_2D_{21})^T & 0 \end{array}\right] \right\|_\infty < \gamma$$

Now it follows from the Full Information results in [DGKF] that the above is true iff there exists a $Y_\infty \geq 0$ such that

$$AY_\infty + Y_\infty A^T + Y_\infty(\gamma^{-2}C_1^TC_1 - C_2^TC_2)Y_\infty + B_1B_1^T = 0$$

and

$$A + Y_\infty(\gamma^{-2}C_1^TC_1 - C_2^TC_2) \text{ is stable}$$

iff $J_\infty \in \operatorname{dom}(Ric)$ and $Y_\infty = Ric(J_\infty) \geq 0$. Furthermore, L can be chosen to be $L = -Y_\infty C_2^T$. This is exactly the result obtained in [DGKF].

4.5 Explicit State Space Formulae for Mixed Control

In this section, we give some explicit formulae for mixed norm synthesis. The formulae are obtained from combining Theorem 3 and Theorem 4. However, only a sufficient condition is presented here, and various other combinations can also be written down. Our purpose is to get some explicit comparisons with $\mathcal{H}_2$ and $\mathcal{H}_\infty$ results.

Theorem 5. *Given $\gamma > 0$ and plant G, there exists a controller $K(s)$ which solves Problem (G) if the following conditions hold:*

(i) $H_\infty \in dom(Ric)$ and $X_\infty := Ric(H_\infty) \geq 0$

(ii) There exist L, Y, and P which satisfy

$$\begin{aligned}
&LR_0 + PC_2^T + \gamma^{-2}PYLR_1 = 0\\
&Y(A_{tmp} + LC_2) + (A_{tmp} + LC_2)^T Y\\
&+\gamma^{-2}Y(B_1B_1^T + LR_1L^T)Y + C_1^TC_1 = 0\\
&Y \geq 0 \text{ and } A_{tmp} + LC_2 + \gamma^{-2}(B_1B_1^T + LR_1L^T)Y \text{ is stable}\\
&\{A_{tmp} + LC_2 + \gamma^{-2}(B_1B_1^T + LR_1L^T)Y\}P + P\{A_{tmp} + LC_2\\
&+\gamma^{-2}(B_1B_1^T + LR_1L^T)Y\}^T + B_0B_0^T + LR_0L^T = 0
\end{aligned}$$

Moreover, when these conditions hold, one such controller is

$$K(s) := \left[\begin{array}{c|c} A + \gamma^{-2}B_1B_1^TX_\infty + B_2F_\infty + LC_2 & -L \\ \hline F_\infty & 0 \end{array}\right]$$

where $A_{tmp} = A + \gamma^{-2}B_1B_1^TX_\infty$ and $F_\infty = -B_2^TX_\infty$.

It should be clear from the the relationship established in the last section among the MFC problem, Kalman filter, and $\mathcal{H}_\infty$ Full Control Problem that the solution in Theorem 5 reduces to the pure $\mathcal{H}_2$ and $\mathcal{H}_\infty$ solutions, respectively, in these special cases. In particular, it is easy to see that $L = L_2$ if $B_1 = 0$, $D_{21} = 0$, and $R_0 = I$; also $L = Z_\infty L_\infty$ if $B_0 = 0$, $D_{20} = 0$, and $R_1 = I$.

We have noted before that the controllers characterised here and in previous sections are only optimal for a given $\gamma > \gamma_\infty$, the pure $\mathcal{H}_\infty$ optimal γ-level. To find a truly optimal mixed controller which satisfies

$$\min_K \sup_{w_1 \in BP, w_0 \in BS} \|z\|_P ,$$

we must pick an appropriate γ_{mixed} to design for. One way of obtaining this γ_{mixed} is through the following iteration: pick $\gamma > \gamma_\infty$ and compute a controller as above. Apply the analysis in section 3 to the closed loop system and determine the power of the worst-case signal, $w_{1_{\text{worst}}}$. Increase or decrease γ according to whether $\|w_{1_{\text{worst}}}\|_P$ is greater than or less than 1, respectively, and repeat the process. The optimal γ_{mixed} occurs when $\|w_{1_{\text{worst}}}\|_P = 1$.

5 Some Unsolved Issues

In this paper, we have formulated and obtained a partial solution to a mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problem. This problem is an interesting generalization of existing $\mathcal{H}_2$ and $\mathcal{H}_\infty$ theory. We have shown that it reduces to the pure $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problems naturally in special cases. An interesting feature of this problem formulation is that no stochastic concepts have been used, i.e., the problem is approached from a completely deterministic viewpoint.

The work present here is still, however, a step away from being completed, and many important issues remained to be solved. Although we strongly believe that our sufficient conditions are also necessary, which is Conjecture 1, a proof has not been obtained. Conjecture 3 itself is very interesting. It is also an important step towards understanding a more general necessary and sufficient condition for the MFC problem, i.e., Conjecture 2. We remark that the necessary conditions given in Lemma 5 are obtained by assuming there exists a constant controller solving the MFC problem. This is true for the pure $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problems; a rigorous proof is still to be done for the mixed case. Another unresolved issue concerns the assumptions made on the system: we assumed $R_0 > 0$ whereas it is more reasonable to assume $R_0 + R_1 > 0$. We needed the assumption $R_0 > 0$ because if R_0 is singular then we must solve a singular LQG problem in the proof of Lemma 4; necessary and sufficient conditions for the solution are more difficult to characterise. However the assumption $R_0 + R_1 > 0$ makes more sense since it simply means that the measurement noise is nonsingular. We do not currently know how to solve this problem, nevertheless, we believe that all the results in this paper hold for the case where R_0 is singular.

From an applications point of view, a major problem concerns solving the coupled Riccati equations. To that end, homotopy methods such as those used in the algorithms developed by Richtor (1987) and Mariton and Bertrand (1985) may prove useful. Since our equations are much simpler than those appearing in the oblique projection method, it is possible that special properties may be exploited and an efficient algorithm developed. This is another subject for future research.

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