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Optimal convergence analysis of an immersed interface finite element method

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Abstract We analyze an immersed interface finite element method based on linear polynomials on noninterface triangular elements and piecewise linear polynomials on interface triangular elements. The flux jump condition is weakly enforced on the smooth interface. Optimal error estimates are derived in the broken H^1-norm and L^2-norm.

Keywords Immersed interface · Finite element method · Uniform grid

Mathematics Subject Classifications (2000) 65N15 · 65N30 · 35J60

1 Introduction

Second order elliptic equations with discontinuous coefficients are often used to model problems in material sciences and fluid dynamics when two or

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more distinct materials or fluids with different conductivities or densities or diffusions are involved. These interface problems must satisfy interface jump conditions due to conservation laws. If the interface is smooth enough, then the solution of the interface problem is also very smooth in individual regions where the coefficient is smooth, but due to the jump of the coefficient across the interface, the global regularity is usually low and has order of $H^{1+\alpha}(\Omega)$, $0 \leq \alpha < 1$. Because of the low global regularity and the irregular geometry of the interface, achieving accuracy is difficult with standard finite element methods, unless the elements fit with the interface of general shape.

Babuska [1] applied the fitted finite element method for the elliptic interface problem and under some approximation assumptions on finite element spaces, the energy-norm estimates were obtained. Bramble and King [2] derived a finite element method in which the smooth boundary and interface of the problem domain are approximated by polygonal domain and interface. With the boundary and interface data transferred in a natural way, they obtained the optimal order error estimates using the piecewise Sobolev norm on $H^2(\Omega^+ \cup \Omega^-)$ for linear elements on a quasi-uniform triangulation. More recently, Chen et al. [5] demonstrated some new techniques in deriving estimates for fitted grid finite element methods using standard finite element with special fitted grids. However, a method using fitted grids to the interface is costly for more complicated time dependent problems in which the interface moves with time and repeated grid generation is called for.

Finite difference methods were applied to the interface problem quite early and unfitted or immersed interface methods are natural in this context since the Cartesian grid cannot match a curved interface. LeVeque and Z. Li [18] proposed an immersed interface method for interface problems defined on a regular domain on which a uniform rectangular grid can be used. The finite difference methods were constructed based on the uniform grid and the jump conditions on the interface. They subsequently applied the same ideas to other interface problems such as the Stokes flow problem [19], the one-dimensional moving interface problem and Hele-Shaw flow [15]. The resulting linear systems from these methods are non-symmetric and indefinite even when the original problem is self-adjoint and uniformly elliptic. Although these methods were demonstrated to be very effective, convergence analysis of related finite difference methods are extremely difficult and are still open.

On the other hand, for finite element methods, Z. Li, T. Lin and X. Wu [25] recently proposed an immersed finite element method using uniform Cartesian triangular grids and their *numerical examples* demonstrated an optimal order of the errors. Once again, it is not easy to analyze this method. The best one can do is to derive the approximation ability of the interpolation finite element space. Indeed, we quote from Remark 8.1 of Li and Ito [23]. "Although we have the error estimate for the interpolation functions for the nonconforming finite element method in terms of piecewise $C^2(\Omega)$ space, the convergence analysis for FE solution is not straight. Our result indicates that the nonconforming IFE space has approximation capability similar to that of the standard conforming linear finite space based on body–fitting partitions."

In view of the fact that the element introduced in [13, 24, 25] seems to be the simplest possible immersed interface element and practically more efficient than other similar methods, it is desirable to know whether this method has the optimal convergence.

Indeed, in this paper we derive optimal H^1 and L^2-error estimates for this interesting scheme and show that the method actually converges. An optimal H^1 convergence is shown in Theorem 4.8 and the L^2 convergence shown in Theorem 5.1.

2 Preliminaries

Let Ω be a convex polygonal domain in $\mathbb{R}^2$ which is separated into two subdomains Ω^+ and Ω^- by a C^2 interface $\Gamma = \partial\Omega^- \subset \Omega$, with $\Omega^+ = \Omega \setminus \Omega^-$ as in Fig. 1. We consider the following elliptic interface problem

$$-\nabla \cdot (\beta \nabla u) = f \;\text{ in } \Omega, \qquad (2.1)$$
$$u = 0 \;\text{ on } \partial\Omega$$

with the jump conditions on the interface

$$[u] = 0, \quad \left[\beta \frac{\partial u}{\partial n}\right] = 0 \;\text{ across } \Gamma, \qquad (2.2)$$

where $f \in L^2(\Omega)$ and $u \in H_0^1(\Omega)$. We assume that the coefficient β is positive and piecewise constant, that is,

$$\beta(x) = \beta^- \;\text{ for } x \in \Omega^-; \quad \beta(x) = \beta^+ \;\text{ for } x \in \Omega^+.$$

We take as usual the weak formulation of the interface problem: Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \beta \nabla u \cdot \nabla v dx = \int_\Omega f v dx, \quad \forall v \in H_0^1(\Omega). \qquad (2.3)$$

For the analysis, we introduce the space

$$\widetilde{H}^2(\Omega) := \left\{u \in H^1(\Omega) \,:\, u \in H^2\left(\Omega^s\right), s = +, -\right\}$$

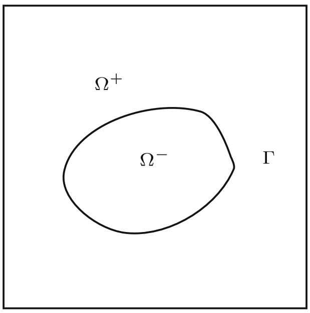


Fig. 1 Sketch of the domain Ω for the interface problem

equipped with the norm

$$\|u\|^2_{\widetilde{H}^2(\Omega)} := \|u\|^2_{H^2(\Omega^+)} + \|u\|^2_{H^2(\Omega^-)}, \quad \forall u \in \widetilde{H}^2(\Omega),$$

where $H^m(\Omega^s) = W_2^m(\Omega^s)$ is the usual Sobolev space of order m.

Then we have the following regularity theorem for the weak solution u of the variational problem (2.3); see [2, 28] and [17].

Theorem 2.1 *Assume that $f \in L^2(\Omega)$. Then the variational problem* (2.3) *has a unique solution $u \in \widetilde{H}^2(\Omega)$ which satisfies for some constant $C > 0$*

$$\|u\|_{\widetilde{H}^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}. \tag{2.4}$$

3 The immersed interface finite element space

We describe the immersed interface finite element space introduced in [24, 25]. Let $\{\mathcal{T}_h\}$ be the usual shape regular finite element triangulations of the domain Ω. We call an element $T \in \mathcal{T}_h$ an interface element if the interface Γ passes through the interior of T, otherwise we call T a noninterface element. (If one of the edges is part of the interface, then the element is a noninterface element.) Let $\mathcal{T}_h^*$ be the collection of all interface elements and Ω' be the union of them. We assume that the interface meets the edges of an interface element at no more than two points. For simplicity of presentation, we assume a rectangular domain Ω is partitioned into triangles obtained by cutting axis-parallel rectangles diagonally, but our presentation holds for general regular triangular partition $\mathcal{T}_h$ with a mesh size h on polygonal domain Ω. Let $\overline{DE}$ be the line segment connecting the intersections of the interface and the edges of a triangle T. This line segment divides T into two parts T^+ and T^- with $T = T^+ \cup T^- \cup \overline{DE}$. Note that there is a small region in T

$$T^* = T - \left(\Omega^+ \cap T^+\right) - \left(\Omega^- \cap T^-\right). \tag{3.1}$$

Since $\overline{DE}$ can be considered as an approximation of the C^2 curve $\Gamma \cap T$, the interface is perturbed by a $O(h^2)$ term. From [2, 5], one can see for the interpolation polynomial defined below, such a perturbation will only affect interpolation error to the order of h^2.

As usual, we want to construct local basis functions on each element T of the partition $\mathcal{T}_h$. For a noninterface element $T \in \mathcal{T}_h$, we simply use the standard linear shape functions on T whose degrees of freedom are functional values on the vertices of T, and use $\overline{S}_h(T)$ to denote the linear spaces spanned by the three nodal basis functions on T:

$$\overline{S}_h(T) = \text{span}\{\,\phi_i\ :\ \phi_i \text{ is the standard linear shape function}\,\}$$

This space has the following approximation property:

$$\|u - I_h u\|_{L^2(T)} + h\|u - I_h u\|_{H^1(T)} \leq Ch^2\|u\|_{H^2(T)}, \tag{3.2}$$

where $I_h : H^2(T) \to \overline{S}_h(T)$ is the interpolation operator. Finally, we use $\overline{S}_h(\Omega)$ to denote the space of conforming piecewise linear polynomials with vanishing boundary values.

3.1 Local basis functions on an interface element

Consider a typical interface element T whose geometric configuration is given in Fig. 2 in which the three vertices are given by $A_1 = (0, h_2)$, $A_2 = (0, 0)$, $A_3 = (h_1, 0)$, and the curve between points D and E is a part of the interface across which the quantity β has a jump. Here we assume that the ratio $r := h_1/h_2$ is bounded below and above by some constant $\kappa \geq 1$, i.e., $1/\kappa \leq r \leq \kappa$.

Let ϕ_i denote the usual Lagrange nodal basis function associated with the vertex A_i, i.e., $\phi_1 = y/h_2$, $\phi_2 = 1 - x/h_1 - y/h_2$, $\phi_3 = x/h_1$. For any given linear function $\phi = V_1\phi_1 + V_2\phi_2 + V_3\phi_3$ on T, we would like to construct a new function $\hat{\phi}$ which is linear on T^+ and T^- respectively and satisfies the same condition as (2.2) on $\overline{DE}$. Let the interface intersect the edges at $D(0, ah_2)$ and $E(bh_1, 0)$, where $0 < a < 1$ and $0 < b \leq 1$. Then a unit normal vector to $\overline{DE}$ is $\mathbf{n}_{\overline{DE}} = (ah_2, bh_1)/\sqrt{a^2h_2^2 + b^2h_1^2}$.

The modified basis function $\hat{\phi}$ on an interface element T can be conveniently described in the following form:

$$\hat{\phi} = \begin{cases} \hat{\phi}^- = c_1\phi_1 + V_2\phi_2 + c_3\phi_3 \text{ in } T^-, \\ \hat{\phi}^+ = V_1\phi_1 + c_2\phi_2 + V_3\phi_3 \text{ in } T^+, \end{cases} \tag{3.3}$$

$$\hat{\phi}^-(D) = \hat{\phi}^+(D), \; \hat{\phi}^-(E) = \hat{\phi}^+(E), \tag{3.4}$$

$$\beta^- \frac{\partial \hat{\phi}^-}{\partial n_{\overline{DE}}} = \beta^+ \frac{\partial \hat{\phi}^+}{\partial n_{\overline{DE}}}. \tag{3.5}$$

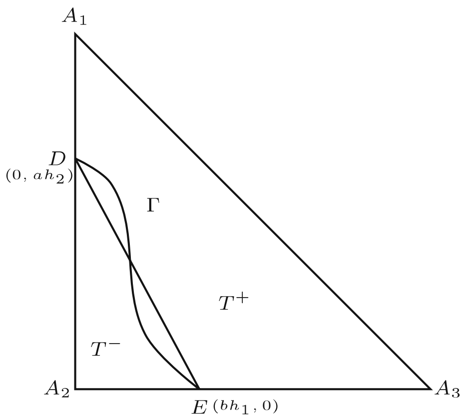


Fig. 2 Typical interface triangle T

The point continuity condition (3.4) gives

$$c_1 a + V_2(1-a) = V_1 a + c_2(1-a),$$
$$V_2(1-b) + c_3 b = c_2(1-b) + V_3 b,$$

while the flux continuity (3.5) becomes

$$\rho\,(c_1\nabla\phi_1 + V_2\nabla\phi_2 + c_3\nabla\phi_3)\cdot \mathbf{n}_{\overline{DE}} = (V_1\nabla\phi_1 + c_2\nabla\phi_2 + V_3\nabla\phi_3)\cdot \mathbf{n}_{\overline{DE}},$$

where $\rho = \beta^-/\beta^+$.

In matrix form,

$$\begin{bmatrix} -a & 1-a & 0 \\ 0 & 1-b & -b \\ -\rho\nu_1 & \nu_2 & -\rho\nu_3 \end{bmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{bmatrix} -a & 1-a & 0 \\ 0 & 1-b & -b \\ -\nu_1 & \rho\nu_2 & -\nu_3 \end{bmatrix}\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}, \tag{3.6}$$

where $\nu_i = \nabla\phi_i \cdot \mathbf{n}_{\overline{DE}}$, $i = 1, 2, 3$. Note that

$$\nu_1 = \frac{br}{\sqrt{a^2h_2^2 + b^2h_1^2}}, \quad \nu_2 = \frac{-\left(a + br^2\right)}{r\sqrt{a^2h_2^2 + b^2h_1^2}}, \quad \nu_3 = \frac{a}{r\sqrt{a^2h_2^2 + b^2h_1^2}}, \tag{3.7}$$

where $r = h_1/h_2$, and $\sum_{i=1}^{3}\nu_i = 0$.

Let us write the above equation in the form

$$M_C\, C = M_V\, V, \tag{3.8}$$

then by (3.7) the determinant of M_C is

$$\det(M_C) = (1-a)b\rho\nu_1 - ab\,\nu_2 + a(1-b)\rho\nu_3 > 0.$$

Similarly, we have

$$\det(M_V) = (1-a)b\,\nu_1 - ab\rho\nu_2 + a(1-b)\nu_3 > 0.$$

Finally, we get the following result.

Lemma 3.1 *Given a linear function ϕ on an interface element T, the modified function $\hat{\phi}$ is uniquely determined by* (3.6).

Remark 3.1 The above lemma is also valid for interface elements whose interface segment $\overline{DE}$ in Fig. 2 straddles a side and the hypotenuse.

Lemma 3.2 *The matrix norm $\|M_V^{-1}M_C\|$ is bounded below and above by constants depending on the jump of β but independent of the mesh size h and the location of the interface.*

Proof Some tedious calculation shows that $M_V^{-1} M_C$ is given by

$$\frac{1}{D'}\begin{bmatrix} a^2+\rho(br)^2+(\rho-1)a^2b & -(\rho-1)(1-a)b\,(a+br^2) & -(\rho-1)(1-a)ab \\ (\rho-1)a(br)^2 & a^2+(br)^2 & (\rho-1)a^2b \\ -(\rho-1)a(1-b)br^2 & -(\rho-1)a(1-b)(a+br^2) & \rho a^2+(br)^2+(\rho-1)a(br)^2 \end{bmatrix}, \tag{3.9}$$

where $D' = a^2 + (br)^2 + (\rho - 1)ab\,(a + br^2)$.

Since any matrix norms are equivalent, it is enough to show that the infinite norm of $M_V^{-1} M_C$ is bounded below and above independent of a, b and r.

Without loss of generality, we may assume that $r = h_1/h_2 \geq 1$. First, if $\rho \geq 1$, then $a^2 + (br)^2 \leq D' \leq \rho(a^2 + (br)^2)$ since $0 < a, b \leq 1$. Let S_i, $i = 1, 2, 3$ denote the l_1-norm of ith row of the above matrix without D' factor, then

$$\begin{aligned} S_1 &= a^2 + \rho(br)^2 + (\rho - 1)\left\{a^2b + (1 - a)b\left(a + br^2\right) + (1 - a)ab\right\} \\ &\leq \rho\left(a^2 + (br)^2\right) + (\rho - 1)\left(b\left(a + br^2\right) + ab\right) \\ &\leq \rho\left(a^2 + (br)^2\right) + (\rho - 1)\left\{br(a + br) + a(a + br)\right\} \\ &\leq \rho\left(a^2 + (br)^2\right) + (\rho - 1)(a + br)^2 \\ &\leq (3\rho - 2)\left(a^2 + (br)^2\right), \end{aligned}$$

where we used the inequality $(a + br)^2 \leq 2(a^2 + (br)^2)$, and

$$S_1 \geq a^2 + (br)^2.$$

Hence we have $1/\rho \leq S_1/D' \leq 3\rho - 2$. By the assumption $r \leq \kappa$,

$$\begin{aligned} S_3 &= \rho a^2 + (br)^2 + (\rho - 1)\left\{a(br)^2 + a(1 - b)\left(a + br^2\right) + a(1 - b)br^2\right\} \\ &\leq \rho\left(a^2 + (br)^2\right) + (\rho - 1)\left(a\left(a + br^2\right) + abr^2\right) \\ &\leq \rho\left(a^2 + (br)^2\right) + \kappa(\rho - 1)\left\{a(a + br) + (a + br)br\right\} \\ &\leq \rho\left(a^2 + (br)^2\right) + \kappa(\rho - 1)(a + br)^2 \\ &\leq \kappa(3\rho - 2)\left(a^2 + (br)^2\right) \end{aligned}$$

and the lower bound is the same as S_1. So we have $1/\rho \leq S_3/D' \leq \kappa(3\rho - 2)$. Now since

$$S_2 = a^2 + (br)^2 + (\rho - 1)ab\left(a + br^2\right),$$

we have $S_2/D' = 1$. Hence we obtain that $1 \leq \|M_V^{-1} M_C\|_\infty \leq \kappa(3\rho - 2)$ when $\rho \geq 1$.

Now if $0 < \rho < 1$, then $\rho(a^2 + (br)^2) \le D' \le a^2 + (br)^2$ since $D' \ge a^2 + (br)^2 - (1-\rho)(a^2 + (br)^2) = \rho(a^2 + (br)^2)$. Noting that $\rho - 1 \le 0$ and the first entry in the first row is positive, we see

$$\begin{aligned} S_1 &= a^2 + \rho(br)^2 - (1-\rho)a^2 b + (1-\rho)(1-a)b\left(a + br^2\right) + (1-\rho)(1-a)ab \\ &\le a^2 + (br)^2 + (1-\rho)\left\{b\left(a + br^2\right) + ab\right\} \\ &\le a^2 + (br)^2 + (1-\rho)(a+br)^2 \le (3-2\rho)\left(a^2 + (br)^2\right) \end{aligned}$$

and

$$\begin{aligned} S_1 &= a^2 + \rho(br)^2 - 2(1-\rho)a^2 b + (1-\rho)\left\{2ab + (br)^2 - ab\left(a + br^2\right)\right\} \\ &= a^2 + \rho(br)^2 + 2(1-\rho)ab(1-a) + (1-\rho)\left\{(br)^2 - ab\left(a + br^2\right)\right\} \\ &\ge a^2 + \rho(br)^2 + (1-\rho)(br)^2 - (1-\rho)ab\left(a + br^2\right) \\ &= a^2 + (br)^2 + (\rho - 1)ab\left(a + br^2\right). \end{aligned}$$

Hence we have $1 \le S_1/D' \le 3/\rho - 2$. We also see that

$$\begin{aligned} S_3 &= \rho a^2 + (br)^2 - (1-\rho)a(br)^2 + (1-\rho)a(1-b)\left(a + br^2\right) + (1-\rho)a(1-b)br^2 \\ &\le a^2 + (br)^2 + (1-\rho)\left\{a\left(a + br^2\right) + abr^2\right\} \\ &\le \kappa(3-2\rho)\left(a^2 + (br)^2\right) \end{aligned}$$

and the lower bound is the same as S_1. So we have $1 \le S_3/D' \le \kappa(3/\rho - 2)$. Now since

$$a^2 + (br)^2 \le S_2 = a^2 + (br)^2 + (1-\rho)ab\left(a + br^2\right) \le (2-\rho)\left(a^2 + (br)^2\right),$$

we have $1 \le S_2/D' \le 2/\rho - 1$. By the above results, we obtain $1 \le \|M_V^{-1} M_C\|_\infty \le \kappa(3/\rho - 2)$ when $0 < \rho < 1$.

Finally, we obtain $1 \le \|M_V^{-1} M_C\|_\infty \le \kappa \cdot \max\{3\rho - 2,\ 3/\rho - 2\}$, and this completes the proof. □

We denote by $\widehat{S}_h(T)$ the finite element space on the interface element T whose basis functions $\hat{\phi}_i$, $i = 1, 2, 3$ are defined by above construction. Furthermore, we define the *immersed interface finite element space* $\widehat{S}_h(\Omega)$. Given a function ϕ in $\overline{S}_h(\Omega)$, we use its nodal values and the above local construction to generate a new function $\hat{\phi}$. $\widehat{S}_h(\Omega)$ is the collection of such functions. Hence its member is linear on each noninterface element and belongs to $\widehat{S}_h(T)$ on each interface element $T \in \mathcal{T}_h$. We note that a function in $\widehat{S}_h(\Omega)$ is in general not continuous across an edge common to interface elements. We also note that a function in $\widehat{S}_h(\Omega)$ vanishes on the boundary edges.

Remark 3.2 Note that if $\phi(A_i) = 0$, $i = 1, 2, 3$ for some $\phi \in \overline{S}_h(T)$, then by (3.6) $\hat{\phi} \equiv 0$ on T. Moreover if $\phi(A_i) = C$, $i = 1, 2, 3$ for some constant C, then $\hat{\phi} \equiv C$ on T.

3.2 Approximation property of the immersed interface space $\widehat{S}_h(T)$

For analysis, we introduce the following spaces: For any $T \subset \Omega$,

$$\widetilde{W}_p^m(T) := \left\{ u : u|_{T\cap\Omega^s} \in W_p^m(T \cap \Omega^s), s = +, - \right\}, \quad p \geq 1, \ m \geq 0 ,$$

$$\widetilde{H}_{int}^2(T) := \left\{ u \in H^1(T) : u|_{T^s} \in H^2(T \cap \Omega^s), \ s = +, -, \ \left[\beta \frac{\partial u}{\partial n} \right] = 0 \text{ on } \Gamma \cap T \right\}$$

and for any $u \in \widetilde{W}_p^m(T)$,

$$\|u\|_{m,p,T}^2 := \|u\|_{m,p,T\cap\Omega^+}^2 + \|u\|_{m,p,T\cap\Omega^-}^2, \quad |u|_{m,p,T}^2 := |u|_{m,p,T\cap\Omega^+}^2 + |u|_{m,p,T\cap\Omega^-}^2,$$

where $\| \cdot \|_{m,p,T^s}$ is the norm of $W_p^m(T \cap \Omega^s)$, $s = +, -$. When $p = 2$, we define $\widetilde{H}^m(T) = \widetilde{W}_p^m(T)$ as usual and denote its norm by $\|u\|_{m,T}$. Furthermore, we define $H^{1/2}(e)$ as the trace space on an edge e of T of all functions in $H^1(T)$ with the norm (see [12] and [11])

$$\|v\|_{1/2,e} := \inf_{\substack{u \in H^1(T) \\ u|_e = v}} \|u\|_{1,T} \tag{3.10}$$

and $H^{-1/2}(e)$ as the dual space of $H^{1/2}(e)$, where the norm is given by

$$\|u\|_{-1/2,e} := \sup_{v \in H^{1/2}(e)} \frac{< u, v >_e}{\|v\|_{1/2,e}}. \tag{3.11}$$

Here $< \cdot, \cdot >_e$ is the duality pairing.

Although for functions in $\widehat{S}_h(T)$ the flux jump condition is enforced on line segments $\overline{DE}$, they actually satisfy a weak flux jump condition on the interface Γ. This is stated in the following lemma [24]. For completeness, we give the proof by a simple application of the divergence theorem.

Lemma 3.3 *For an interface triangle T, every function $\hat{\phi} \in \widehat{S}_h(T)$ satisfies the flux jump condition on $\Gamma \cap T$ in the following weak sense:*

$$\int_{\Gamma\cap T} \left(\beta^- \nabla \hat{\phi}^- - \beta^+ \nabla \hat{\phi}^+ \right) \cdot \mathbf{n}_\Gamma ds = 0.$$

Proof Let $\hat{\phi}$ be any function in $\widehat{S}_h(T)$. By the divergence theorem, we have

$$\begin{aligned} &\int_{\Gamma\cap T} \left(\beta^- \nabla \hat{\phi}^- - \beta^+ \nabla \hat{\phi}^+ \right) \cdot \mathbf{n}_\Gamma ds + \int_{\overline{DE}} \left(\beta^- \nabla \hat{\phi}^- - \beta^+ \nabla \hat{\phi}^+ \right) \cdot \mathbf{n}_{\overline{DE}} ds \\ &\quad = \int_{T^*} \nabla \cdot \left(\beta^- \nabla \hat{\phi}^- - \beta^+ \nabla \hat{\phi}^+ \right) ds = 0. \end{aligned}$$

By the flux continuity of $\hat{\phi}$ on $\overline{DE}$,

$$\int_{\overline{DE}} \left(\beta^- \nabla \hat{\phi}^- - \beta^+ \nabla \hat{\phi}^+ \right) \cdot \mathbf{n}_{\overline{DE}} ds = 0,$$

which completes the proof. □

For any $u \in \widetilde{H}^2_{int}(T)$, we let $I_h u \in \widehat{S}_h(T)$ be such that

$$I_h u(A_i) = u(A_i), \quad i = 1, 2, 3$$

where A_i, $i = 1, 2, 3$ are the vertices of T and we call $I_h u$ the *interpolant* of u in $\widehat{S}_h(T)$. We can naturally extend the interpolant I_h such that $I_h : \widetilde{H}^2_{int}(\Omega) \to \widehat{S}_h(\Omega)$ and $(I_h u)|_T = I_h u|_T$.

Then we have an estimate of the interpolant given in the following theorem; see Z. Li et al. [24].

Theorem 3.4 *Let T be an interface element. Then there exists a constant $C > 0$ such that the interpolation operator $I_h : \widetilde{H}^2_{int}(T) \to \widehat{S}_h(T)$ satisfies*

$$\|u - I_h u\|_{m,T} \le Ch^{2-m}\|u\|_{2,T}, \quad m = 0, 1 \tag{3.12}$$

for any $u \in \widetilde{H}^2_{int}(T)$.

4 Immersed interface finite element method and its convergence analysis

We now consider the immersed interface finite element problem: Find $\hat{u}_h \in \widehat{S}_h(\Omega)$ such that

$$a_h\left(\hat{u}_h, \hat{\phi}\right) = \left(f, \hat{\phi}\right), \quad \forall \hat{\phi} \in \widehat{S}_h(\Omega), \tag{4.1}$$

where

$$a_h(u, v) = \sum_{T \in \mathcal{T}_h} \int_T \beta \nabla u \cdot \nabla v dx, \quad \forall u, v \in H_h(\Omega),$$

$$H_h(\Omega) := H^1_0(\Omega) + \widehat{S}_h(\Omega)$$

and $H_h(\Omega)$ is endowed with the *broken H^1-seminorm* as $\|v\|^2_{1,h} := \sum_{T \in \mathcal{T}_h} |v|^2_{1,T}$. Note that the bilinear operator $a_h(\cdot, \cdot)$ is clearly bounded.

We now show the coercivity of the bilinear form $a_h(\cdot, \cdot)$ on $\widehat{S}_h(T)$. To this end, we introduce a transfer operator $\gamma : \widehat{S}_h(T) \to \overline{S}_h(T)$ as follows: For any $\hat{\phi} \in \widehat{S}_h(T)$, define $\gamma\hat{\phi} = \phi \in \overline{S}_h(T)$ such that

$$\phi(A_i) = \hat{\phi}(A_i) \text{ at vertices } A_i, \; i = 1, 2, 3.$$

We can naturally extend it to the whole of $\widehat{S}_h(\Omega)$ by $(\gamma\hat{\phi})|_T = \gamma\hat{\phi}|_T$.

Lemma 4.1 (Discrete Poincaré inequality) *There exists a constant C independent of h and the interface Γ such that*

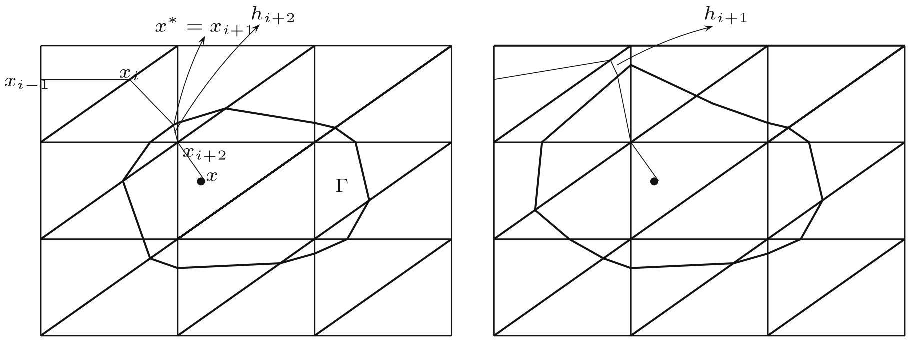
$$C\|\hat{\phi}\|^2_{L^2(\Omega)} \le a_h\left(\hat{\phi}, \hat{\phi}\right) \qquad \forall \hat{\phi} \in \widehat{S}_h(\Omega). \tag{4.2}$$

Proof The idea of the proof is very similar to the one in [6], but here we need to choose the integration path more judiciously. Let $\hat{\phi} \in \widehat{S}_h(\Omega)$ be given and we want to define a piecewise linear path $C = \cup_i [x_i, x_{i+1}]$, a union of line segments $[x_i, x_{i+1}]$, such that $\hat{\phi}$ is continuous and piecewise differentiable on the path. Since the function $\hat{\phi}$ is continuous on noninterface elements, we can choose any line segment there. However, if C meets an interface element at a point, say x_i in Fig. 3, we choose a point $x_* := x_{i+1}$ on the interface as a next node for the path. Then choose the vertex point on the other side of the interface as x_{i+2}. Here we have to choose x_*(or adjust x_i also) in such a way that if one part of the interface element is too thin, the line segment $x_i x_*$(or $x_* x_{i+2}$) is close to the shortest path reaching next node so that the area of that part of interface element is bounded below by $C|\overline{x_* x_{i+2}}|^2$(or $C|\overline{x_i x_*}|^2$).

By the above argument, for any $x \in \Omega$ there is a sequence of points $x_i, i = 0, \cdots, \ell$ such that $x_0 \in \partial\Omega$ with $\hat{\phi}(x_0) = 0$, $x_\ell = x$, and $\hat{\phi}$ is continuous along the polygonal curve C joining $x_i, i = 0, \cdots, \ell$.

Then using the mean value theorem and the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
|\hat{\phi}(x)|^2 = |\hat{\phi}(x_\ell)|^2 &= \left| \sum_{i=0}^{\ell-1} \left(\hat{\phi}\,(x_{i+1}) - \hat{\phi}\,(x_i) \right) \right|^2 \\
&= \left| \sum_{i=0}^{\ell-1} \left(\nabla\hat{\phi}(\bar{x}_i)\,(x_{i+1} - x_i) \right) \right|^2 \\
&\le \ell \sum_{i=0}^{\ell-1} |\nabla\hat{\phi}\,(\bar{x}_i)\,|^2 h_i^2 \\
&\le C\ell \left(\sum_{i=0}^{\ell-1} |\nabla\hat{\phi}|^2_{0,T_i} \right),
\end{aligned}
$$



a Case of a thin triangle **b** Case of a thin quadrilateral

Fig. 3 A path C along which $\hat{\phi}$ is continuous, piecewise differentiable and $|T_i| \ge Ch_i^2$ (**a**, **b**) where $h_i = |x_{i+1} - x_i|$, $\bar{x}_i$ is some point on $[x_i, x_{i+1}]$ and in the last inequality we used the fact that

$$|\nabla\hat{\phi}(\bar{x}_i)|^2 h_i^2 \le C|\nabla\hat{\phi}|_{0,T_i}^2,$$

which is an immediate consequence of choice of x_* and the fact that $\nabla\hat{\phi}$ is constant on each T_i. Here T_i is either a noninterface element or one of the two parts of an interface element.

Now suppose that x is in an element T, then

$$\int_T |\hat{\phi}(x)|^2 dx \le C\ell h^2 \sum_{i=0}^{l-1} |\nabla\hat{\phi}|_{0,T_i}^2$$
$$\le CC_0 h \sum_{i=0}^{l-1} |\nabla\hat{\phi}|_{0,T_i}^2.$$

Here ℓh is bounded by some constant since the number of line segments used above is bounded by C/h. Summing over T in such a way that the same T_i appears at most ℓ times and using the fact that $\ell h \le C_0$, we conclude

$$C\|\hat{\phi}\|_{L^2(\Omega)}^2 \le a_h\left(\hat{\phi}, \hat{\phi}\right).$$

This completes the proof. □

We introduce two trace spaces on an edge e of T:

$$\widehat{S}_h(e) := \left\{\hat{\phi}|_e : \hat{\phi} \in \widehat{S}_h(T)\right\}, \quad \overline{S}_h(e) := \left\{\phi|_e : \phi \in \overline{S}_h(T)\right\}. \tag{4.3}$$

Now we define $\gamma_e : \widehat{S}_h(e) \to \overline{S}_h(e)$ by $\gamma_e\hat{\phi}|_e := (\gamma\hat{\phi})|_e$, $\hat{\phi} \in \widehat{S}_h(T)$. Note that γ_e is well-defined, since $(\gamma\hat{\phi}_1)|_e = (\gamma\hat{\phi}_2)|_e$ whenever $\hat{\phi}_1|_e = \hat{\phi}_2|_e$ for any $\hat{\phi}_1, \hat{\phi}_2 \in \widehat{S}_h(T)$.

Now we show a negative norm estimate of γ_e. For this purpose, we use a reference element $\widetilde{T}$ which is typical in finite element analysis. Let e be any edge of an element T and $\tilde{e}$ the edge of $\widetilde{T}$ corresponding to e. Note that given an element T in the triangulation, there exists an affine transformation $F\tilde{x} = B\tilde{x} + b$ from $\widetilde{T}$ to T.

We next prove an approximation property in a fractional norm.

Lemma 4.2 *Let e be an edge of T and $\phi \in H^{1/2}(e)$. Then*

$$\inf_{m\in\mathbb{R}} \|\phi - m\|_{0,e} \le Ch^{1/2}|\phi|_{1/2,e}. \tag{4.4}$$

Proof *Let us first show that for $g \in H^\alpha(e)$, one has*

$$|g|_{\alpha,e}^2 = h^{1-2\alpha}|\tilde{g}|_{\alpha,\tilde{e}}^2, \ 0 \le \alpha < 1. \tag{4.5}$$

For $\alpha = 0$, this is standard scaling argument for $L^2(e)$. So assume $0 < \alpha < 1$. Let $\eta = s/h, \xi = t/h$, where $h = |e|$. Then using the fractional norm [4, 16], *one has*

$$
\begin{aligned}
|g|^2_{\alpha,e} &= \int_e \int_e \frac{|g(s) - g(t)|^2}{|s - t|^{1+2\alpha}} ds dt \\
&= h^{1-2\alpha} \int_{\tilde{e}} \int_{\tilde{e}} \frac{|\tilde{g}(\eta) - \tilde{g}(\xi)|^2}{|\eta - \xi|^{1+2\alpha}} d\eta d\xi \\
&= h^{1-2\alpha} |\tilde{g}|^2_{\alpha,\tilde{e}}.
\end{aligned}
$$

Noting (4.5) *and that on the reference element,*

$$\inf_{m \in \mathbb{R}} \|\tilde{\phi} - m\|_{0,\tilde{e}} \leq C|\tilde{\phi}|_{1/2,\tilde{e}},$$

we obtain the result. □

Lemma 4.3 *Let e be any edge of T. Then there exists a constant $C > 0$ independent of the mesh size h of T such that*

$$\inf_{m \in \mathbb{R}} \|u - m\|_{-1/2,e} \leq Ch|u|_{1,T} \tag{4.6}$$

for any $u \in H^1(T)$.

Proof By definition, we have

$$\|u - m\|_{-1/2,e} = \sup_{v \in H^{1/2}(e)} \frac{< u - m, v >_e}{\|v\|_{1/2,e}}. \tag{4.7}$$

Taking m as the average of u on e, we get for any constant c

$$< u - m, v >_e = < u - m, v - c >_e \leq \|u - m\|_{0,e} \|v - c\|_{0,e}.$$

To estimate this, we apply the Bramble-Hilbert lemma (given in the form of Lemma 3 of [9]) on a reference element. Consider the functional f on $H^1(\widetilde{T})$ defined by $f(\tilde{u}) := \|\tilde{u} - (\tilde{u})_{\tilde{e}}\|_{0,\tilde{e}}$, where $(\tilde{u})_{\tilde{e}}$ is the average of $\tilde{u}$ on $\tilde{e}$, then we have $\|\tilde{u} - (\tilde{u})_{\tilde{e}}\|_{0,\tilde{e}} \leq C|\tilde{u}|_{1,\widetilde{T}}$. Now scaling argument gives

$$\|u - m\|_{0,e} \leq Ch^{1/2}|u|_{1,T}.$$

On the other hand, by Lemma 4.2 we have that $\inf_{c \in \mathbb{R}} \|v - c\|_{0,e} \leq Ch^{1/2}\|v\|_{1/2,e}$. Hence we obtain

$$< u - m, v >_e \leq Ch|u|_{1,T} \|v\|_{1/2,e}.$$

This proves (4.6). □

Lemma 4.4 *Let T be an interface element and e an edge of T. Then there exists a constant C independent of h and interface points such that the following inequality holds for all $\hat{\phi} \in \widehat{S}_h(T)$:*

$$\|\hat{\phi}|_e - \gamma_e \hat{\phi}|_e\|_{-1/2,e} \leq Ch|\hat{\phi}|_{1,T}. \tag{4.8}$$

Proof First, let us assume that

$$\|\gamma_e\|_{-1/2,e} \le C_\gamma \tag{4.9}$$

holds for some positive constant C independent of h and the location of the interface.

Now with $\hat{\phi} \in \widehat{S}_h(T)$ and any constant m, we have

$$\begin{aligned}\|\hat{\phi}|_e - \gamma_e\hat{\phi}|_e\|_{-1/2,e} &= \|\hat{\phi}|_e - m - \gamma_e\left(\hat{\phi}|_e - m\right)\|_{-1/2,e} \le \|I-\gamma_e\|_{-1/2,e}\|\hat{\phi}|_e - m\|_{-1/2,e}\\ &\le C\|\hat{\phi}|_e - m\|_{-1/2,e}.\end{aligned}$$

Since m is an arbitrary constant, by Lemma 4.3

$$\|\hat{\phi}|_e - \gamma_e\hat{\phi}|_e\|_{-1/2,e} \le Ch|\hat{\phi}|_{1,T}.$$

This completes the proof of (4.8). □

Now we show the constant C_γ is independent of the location of interface. Notice that the interface is completely determined by the numbers a and b in Fig. 2.

Without loss of generality, we may assume that $V_1 = 0$ in (3.6) (This corresponds to edge A_2A_3) and assume for simplicity $h = 1$. Notice the basis functions have the form $\phi_2 = (1 - x)$ and $\phi_3 = x$. Since

$$\hat{\phi} = \begin{cases} \hat{\phi}^- = V_2\phi_2 + c_3\phi_3 \text{ on } A_2E,\\ \hat{\phi}^+ = c_2\phi_2 + V_3\phi_3 \text{ on } EA_3,\end{cases} \tag{4.10}$$

the maximum $\hat{\phi}$ is attained either at the vertex points or the interface point. ($\hat{\phi}(0) = V_2$, $\hat{\phi}(b) = V_2(1-b) + c_3 b$, $\hat{\phi}(1) = V_3$.)

Lemma 4.5 *We have*

$$\int_0^b |\hat{\phi}|^2 = \frac{b}{6}\left(V_2^2 + 4|(V_2\phi_2 + c_3\phi_3)\left(\frac{b}{2}\right)|^2 + |\hat{\phi}(b)|^2\right) \tag{4.11}$$

$$\int_b^1 |\hat{\phi}|^2 = \frac{1-b}{6}\left(|\hat{\phi}(b)|^2 + 4\left|(c_2\phi_2 + V_3\phi_3)\left(\frac{1+b}{2}\right)\right|^2 + V_3^2\right) \tag{4.12}$$

$$\|\hat{\phi}\|_{1/2,e}^2 \le C\left(b|V_2 - c_3|^2 + (1-b)|c_2 - V_3|^2 + \|\hat{\phi}\|_e^2\right) \tag{4.13}$$

for some $C > 0$ and

$$\|\hat{\phi}\|_{-1/2,e} \ge \frac{\|\hat{\phi}\|_e^2}{\|\hat{\phi}\|_{1/2,e}}. \tag{4.14}$$

Proof *The first two identities are clear from Simpson's rule. For* (4.13)*, we first note by a simple computation that*

$$\|\hat{\phi}\|_{1,e}^2 = \left(b|V_2 - c_3|^2 + (1-b)|c_2 - V_3|^2 + \|\hat{\phi}\|_e^2\right).$$

Hence

$$\|\hat{\phi}\|^2_{1/2,e} \le C\|\hat{\phi}\|^2_{1,e} \le C\left(b\,|V_2 - c_3|^2 + (1-b)|c_2 - V_3|^2 + \|\hat{\phi}\|^2_e\right).$$

For (4.14)*, we see*

$$\|\hat{\phi}\|_{-1/2,e} = \sup_{v\in H^{1/2}(e)} \frac{<\hat{\phi}, v>_e}{\|v\|_{1/2,e}}$$
$$\ge \sup_{v\in \hat{S}_h(e)} \frac{<\hat{\phi}, v>_e}{\|v\|_{1/2,e}} \ge \frac{\|\hat{\phi}\|^2_e}{\|\hat{\phi}\|_{1/2,e}}. \qquad \square$$

Lemma 4.6 *There are positive constants C_m and C_M independent of a and b such that*

$$C_m\left(|V_2|^2 + |V_3|^2\right) \le \|\hat{\phi}\|^2_e \le C_M\left(V_2^2 + V_3^2\right). \tag{4.15}$$

Proof For the left hand side of the above inequality, by adding (4.11) and (4.12) and underestimating, we have

$$\frac{1}{6}\left(b\,V_2^2 + |\hat{\phi}(b)|^2 + (1-b)V_3^2\right) \le \|\hat{\phi}\|^2_e. \tag{4.16}$$

Defining $f(b) := \left(b\,V_2^2 + |\hat{\phi}(b)|^2 + (1-b)V_3^2\right)$ and noting that $f(b)$ is a quadratic(convex) function of b, we see that the tangent line approximation $L(f)(t)$ satisfies $L(f)(t) \le f(t), 0 \le t \le 1$. Since $f(0) = |V_2|^2 + |V_3|^2$ and $f'(0) = (-V_2^2 - V_3^2 + 2c_3V_2)$, the tangent line approximation $L(f)(t)$ at $t = 0$ has the form

$$L(f)(t) = \left(V_2^2 + V_3^2\right) + t\left(-V_2^2 - V_3^2 + 2c_3V_2\right).$$

Since c_3 is a linear bounded(below and above independent of a, b) function of $V_i's$(Lemma 3.2) and $V_1 = 0$, there exist positive constants δ and C_0 such that

$$L(f)(t) \ge C_0\left(V_2^2 + V_3^2\right) \quad \text{for } |t| \le \delta < \frac{1}{2}. \tag{4.17}$$

Clearly for $\delta \le b \le 1-\delta$,

$$f(b) \ge \delta\left(V_2^2 + V_3^2\right).$$

When $b \ge 1-\delta$, we switch the role of $f(0)$ with $f(1)$ to get a similar inequality. Hence (4.15) holds with $C_m = \frac{1}{6}\min(C_0, \delta)$ for all b. Finally, the right hand side is trivial by Lemma 4.5 and boundedness of c_i's as functions of V_i's. $\square$

Now we proceed to derive a bound of the operator norm $\|\gamma_e\|_{-1/2,e}$ independent of a and b:

$$\begin{aligned}\|\gamma\|_{-1/2,e} = \sup_{\hat{\phi}\in\hat{S}_h(e)} \frac{\|\gamma_e\hat{\phi}\|_{-1/2,e}}{\|\hat{\phi}\|_{-1/2,e}} \\ \le \sup_{\hat{\phi}\in\hat{S}_h(e)} \frac{\|\gamma_e\hat{\phi}\|_{-1/2,e}\|\hat{\phi}\|_{1/2,e}}{\|\hat{\phi}\|_e^2} \\ \le \sup_{\hat{\phi}\in\hat{S}_h(e)} \frac{C\|\gamma_e\hat{\phi}\|_e\|\hat{\phi}\|_{1/2,e}}{\|\hat{\phi}\|_e^2}.\end{aligned} \tag{4.18}$$

where $C > 0$ is a generic constant. Note that

$$\|\gamma\hat{\phi}\|_e^2 = \frac{1}{3}\left(V_2^2 + V_2V_3 + V_3^2\right) \le V_2^2 + V_3^2.$$

Thus by Lemma 4.5, Lemma 4.6, and (4.18), we have

$$\begin{aligned}\|\gamma\|_{-1/2,e} &\le C\sup_{\hat{\phi}} \frac{\|\gamma\hat{\phi}\|_e\left(b|V_2-c_3|^2+(1-b)|c_2-V_3|^2+\|\hat{\phi}\|_e^2\right)^{1/2}}{\|\hat{\phi}\|_e^2} \\ &\le C\sup_{\hat{\phi}} \frac{\left(V_2^2+V_3^2\right)^{1/2}\left(\sqrt{b}|V_2-c_3|+\sqrt{1-b}|c_2-V_3|+\|\hat{\phi}\|_e\right)}{\|\hat{\phi}\|_e^2} \\ &\le C\sup_{\hat{\phi}} \frac{\left(\sqrt{b}|V_2-c_3|+\sqrt{1-b}|c_2-V_3|+\|\hat{\phi}\|_e\right)}{C_m^{1/2}\left(V_2^2+V_3^2\right)^{1/2}}.\end{aligned}$$

Since c_i's are bounded functions of V_i's we have shown (4.9) holds with the C_γ independent of the location of interface.

For energy-norm error estimate of the immersed interface finite element method, we need the well-known second Strang lemma, since the immersed finite element space is nonconforming.

Lemma 4.7 (Second Strang lemma) *Let $u \in \widetilde{H}^2(\Omega)$, $\hat{u}_h \in \widehat{S}_h(\Omega)$ be the solutions of* (2.3) *and* (4.1) *respectively. Then there exists a constant $C > 0$ such that*

$$\|u-\hat{u}_h\|_{1,h} \le C\left\{\inf_{\hat{v}_h\in\widehat{S}_h(\Omega)}\|u-\hat{v}_h\|_{1,h} + \sup_{\hat{\phi}\in\widehat{S}_h(\Omega)}\frac{|a_h\left(u,\hat{\phi}\right)-\left(f,\hat{\phi}\right)|}{\|\hat{\phi}\|_{1,h}}\right\}. \tag{4.19}$$

Remark 4.1 The constant C in the above lemma is guaranteed by the coercivity of $a_h(\cdot,\cdot)$ on $\widehat{S}_h(\Omega)$, a fact that can be trivially shown through the Poincaré inequality as in (4.2), since a function in $\widehat{S}_h(\Omega)$ vanishes on the boundary.

We now use the second Strang lemma to prove the following broken H^1-error estimate.

Theorem 4.8 *Let* $u \in \widetilde{H}^2(\Omega)$, $\hat{u}_h \in \widehat{S}_h(\Omega)$ *be the solutions of* (2.3) *and* (4.1) *respectively. Then there exists a constant* $C > 0$ *such that*

$$\|u - \hat{u}_h\|_{1,h} \le Ch\|u\|_{\widetilde{H}^2(\Omega)}. \tag{4.20}$$

Proof The first term in the second Strang lemma is nothing but an approximation error:

$$\inf_{\hat{v}_h \in \widehat{S}_h(\Omega)} \|u - v_h\|_{1,h} \le Ch\|u\|_{\widetilde{H}^2(\Omega)}. \tag{4.21}$$

For the consistency error estimate, we have from the definition of $a_h(\cdot,\cdot)$ and Green's formula

$$\begin{aligned} a_h\left(u, \hat{\phi}\right) - \left(f, \hat{\phi}\right) &= \sum_{T \in \mathcal{T}_h} \int_T \beta \nabla u \cdot \nabla \hat{\phi}\, dx - \int_\Omega f \hat{\phi}\, dx \\ &= \sum_{T \in \mathcal{T}_h} \int_T \beta \nabla u \cdot \nabla \hat{\phi}\, dx \\ &\quad - \left(\sum_{T \in \mathcal{T}_h} \int_T \beta \nabla u \cdot \nabla \hat{\phi}\, dx - \sum_{T \in \mathcal{T}_h} < \beta \frac{\partial u}{\partial n}, \hat{\phi} >_{\partial T} \right) \\ &= \sum_{T \in \mathcal{T}_h} < \beta \frac{\partial u}{\partial n}, \hat{\phi} >_{\partial T}, \end{aligned} \tag{4.22}$$

where $\hat{\phi} \in \widehat{S}_h(\Omega)$ and n is a unit outward normal vector on each ∂T. Note that the integral $\int_e \beta \frac{\partial u}{\partial n} \gamma_e \hat{\phi} ds$ is well defined on each edge of $\mathcal{T}_h$ so that $\sum_{T \in \mathcal{T}_h} < \beta \frac{\partial u}{\partial n}, \gamma_e \hat{\phi} >_{\partial T} = 0$, then we have

$$\begin{aligned} \sum_{T \in \mathcal{T}_h} < \beta \frac{\partial u}{\partial n}, \hat{\phi} >_{\partial T} &= \sum_{T \in \mathcal{T}_h^*} < \beta \frac{\partial u}{\partial n}, \hat{\phi} - \gamma_e \hat{\phi} >_{\partial T} \\ &= \sum_{T \in \mathcal{T}_h^*} \sum_{e \subset \partial T} < \beta \frac{\partial u}{\partial n}, \hat{\phi} - \gamma_e \hat{\phi} >_e \\ &= \sum_{T \in \mathcal{T}_h^*} \sum_{e \subset \partial T} \left\{ < \beta \frac{\partial u}{\partial n}, \hat{\phi} - \gamma_e \hat{\phi} >_{e^+} + < \beta \frac{\partial u}{\partial n}, \hat{\phi} - \gamma_e \hat{\phi} >_{e^-} \right\} \end{aligned}$$

where $e^s = e \cap \Omega^s$, $s = +, -$. Since the conclusion of Lemma 4.4 holds when e is replaced by its portion e^+ or e^-,

$$\begin{aligned} < \beta \frac{\partial u}{\partial n}, \hat{\phi} - \gamma_e \hat{\phi} >_{e^s} &\le \|\beta \frac{\partial u}{\partial n}\|_{1/2,e^s} \|\hat{\phi} - \gamma_e \hat{\phi}\|_{-1/2,e^s} \\ &\le C\|\nabla u\|_{H^{1/2}(e^s)} h |\hat{\phi}|_{1,T} \\ &\le Ch\|u\|_{\widetilde{H}^2(T)} |\hat{\phi}|_{1,T}. \end{aligned}$$

Finally summing over all T, we obtain

$$\sum_{T\in\mathcal{T}_h} < \beta\frac{\partial u}{\partial n}, \hat{\phi} >_{\partial T} \leq Ch\|u\|_{\widetilde{H}^2(\Omega)}\|\hat{\phi}\|_{1,h}.$$

Thus the approximation property of $\widehat{S}_h(\Omega)$ and second Strang lemma give the result. □

5 L^2-error estimate

We now apply the duality argument to obtain L^2-norm estimate of the error. For this purpose, let us consider an auxiliary problem: Given $g \in L^2(\Omega)$, find $\varphi \in \widetilde{H}^2(\Omega)$ such that

$$-\nabla\cdot(\beta\nabla\varphi) = g \quad \text{in } \Omega, \tag{5.1}$$
$$\varphi = 0 \quad \text{on } \partial\Omega$$

with jump conditions $[\varphi] = 0$ and $[\beta\frac{\partial\varphi}{\partial n}] = 0$ across Γ. Then by Theorem 2.1, the solution of this problem satisfies $\|\varphi\|_{\widetilde{H}^2(\Omega)} \leq C\|g\|_{L^2(\Omega)}$.

Let $\hat{\varphi}_h \in \widehat{S}_h(\Omega)$ be the solution of the corresponding variational problem:

$$a_h\left(\hat{v}_h, \hat{\varphi}_h\right) = \left(\hat{v}_h, g\right), \quad \forall \hat{v}_h \in \widehat{S}_h(\Omega). \tag{5.2}$$

Then

$$\begin{aligned}(u-\hat{u}_h, g) &= \sum_{T\in\mathcal{T}_h}\int_T \beta\nabla\left(u-\hat{u}_h\right)\cdot\nabla\varphi\,dx - \sum_{T\in\mathcal{T}_h}\int_{\partial T}\left(u-\hat{u}_h\right)\beta\frac{\partial\varphi}{\partial n}\,ds\\ &= a_h\left(u-\hat{u}_h, \varphi-\hat{\varphi}_h\right) + a_h\left(u-\hat{u}_h, \hat{\varphi}_h\right) - \sum_{T\in\mathcal{T}_h}\int_{\partial T}\left(u-\hat{u}_h\right)\beta\frac{\partial\varphi}{\partial n}\,ds\\ &= a_h\left(u-\hat{u}_h, \varphi-\hat{\varphi}_h\right) + \sum_{T\in\mathcal{T}_h}\int_{\partial T}\beta\frac{\partial u}{\partial n}\hat{\varphi}_h\,ds - \sum_{T\in\mathcal{T}_h}\int_{\partial T}\left(u-\hat{u}_h\right)\beta\frac{\partial\varphi}{\partial n}\,ds\\ &=: I + II - III.\end{aligned}$$

By continuity of $a_h(\cdot,\cdot)$ and H^1-error estimate of $\varphi - \hat{\varphi}_h$,

$$\begin{aligned}|I| &\leq C\|u-\hat{u}_h\|_{1,h}\|\varphi-\hat{\varphi}_h\|_{1,h} \leq Ch\|u-\hat{u}_h\|_{1,h}\|\varphi\|_{\widetilde{H}^2(\Omega)}\\ &\leq Ch^2\|u\|_{\widetilde{H}^2(\Omega)}\|\varphi\|_{\widetilde{H}^2(\Omega)}.\end{aligned}$$

As for II, we apply the analysis for the consistency error of H^1-error estimate (4.23). First, note that φ is the unique solution of (5.1). Now we introduce the *extended* transfer operator $\overline{\gamma} : \widehat{S}_h(T) \oplus \text{span}\{\varphi\} \to \overline{S}_h(T) \oplus \text{span}\{\varphi\}$ defined by $\overline{\gamma}(\hat{\phi} + c\varphi) := \gamma\hat{\phi} + c\varphi$ for $\hat{\phi} \in \widehat{S}_h(T),\ c \in \mathbb{R}$. Then we can formally write as $\overline{\gamma} = \gamma \oplus I$, where I is the identity operator. We also define $\overline{\gamma}_e$ as the restriction of $\overline{\gamma}$ on each edge e of T by $\overline{\gamma}_e(\hat{\phi} + c\varphi)|_e := (\overline{\gamma}(\hat{\phi} + c\varphi))|_e$. It is clear that $\|\overline{\gamma_e}\|$ is bounded above by a constant independent of h and the location of the interface for any norm $\|\cdot\|$. Applying an analysis similar to Theorem 4.8,

$$II = \sum_{T\in\mathcal{T}_h}\int_{\partial T}\beta\frac{\partial u}{\partial n}\,\hat{\varphi}_h\,ds = \sum_{T\in\mathcal{T}_h}\int_{\partial T}\beta\frac{\partial u}{\partial n}\left(\hat{\varphi}_h-\varphi\right)\,ds$$

$$= \sum_{T\in\mathcal{T}_h}\sum_{e\subset\partial T}\int_{e}\beta\frac{\partial u}{\partial n}\left\{\hat{\varphi}_h-\varphi-\overline{\gamma}_e\left(\hat{\varphi}_h-\varphi\right)\right\}\,ds.$$

Hence by Lemma 4.4, we get

$$|II| \le Ch\|u\|_{\widetilde{H}^2(\Omega)}\|\hat{\varphi}_h-\varphi\|_{1,h} \le Ch^2\|u\|_{\widetilde{H}^2(\Omega)}\|\varphi\|_{\widetilde{H}^2(\Omega)}.$$

Interchanging the role of φ and u, we obtain

$$|III| = |\sum_{T\in\mathcal{T}_h}\int_{\partial T}\left(u-\hat{u}_h\right)\beta\frac{\partial\varphi}{\partial n}\,ds| \le Ch\|u-\hat{u}_h\|_{1,h}\|\varphi\|_{\widetilde{H}^2(\Omega)}$$

$$\le Ch^2\|u\|_{\widetilde{H}^2(\Omega)}\|\varphi\|_{\widetilde{H}^2(\Omega)}.$$

Since $\|\varphi\|_{\widetilde{H}^2(\Omega)} \le C\|g\|_{L^2(\Omega)}$, we see

$$\|u-\hat{u}_h\|_{L^2(\Omega)} = \sup_{g\in L^2(\Omega)}\frac{(u-\hat{u}_h,g)}{\|g\|_{L^2(\Omega)}} \le Ch^2\|u\|_{\widetilde{H}^2(\Omega)}. \tag{5.3}$$

Thus we obtain the following L^2-error estimate.

Theorem 5.1 *Let $u\in\widetilde{H}^2(\Omega)$, $\hat{u}_h\in\widehat{S}_h(\Omega)$ be the solutions of* (2.3) *and* (4.1) *respectively. Then there exists a constant $C>0$ such that*

$$\|u-\hat{u}_h\|_{L^2(\Omega)} \le Ch^2\|u\|_{\widetilde{H}^2(\Omega)}. \tag{5.4}$$

Remark 5.1 The numerical results in [24, 25] support our error estimates of the immersed interface finite element method for broken H^1 and L^2-norms with first and second order convergence respectively.

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