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## Optimal Decision Rules

for the E Model

of Chance-Constrained Programming


# Optimal Decision Rules 

 for the E Model of Chance-Constrained Programmingby<br>Abraham Charnes<br>Northwestern University

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## FORE WORD

This is the first in a series of papers dealing with the problem of finding the optimal decision rules for $n$-period chance-constrained programming models. The significant feature of this paper is that the admissible class of decision rules is not required to possess any specific analytic property, as it is in all previous work in this field. Instead the admissible class is the largest possible class of decision rules consistent with the interpretation given to $n$-period problems. It is shown that in this case the optimal decision rules are piecewise linear functions of certain conditional fractile points and the decision rules of all preceding periods.

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Optimal Decision Rules for the E Model of Chance-Constrained Programming


#### Abstract

The first five sections of this paper contain an introduction to the topic of chance-constrained programming. Then the general $n$-period expectation-objective model of chance-constrained programming is presented and certain necessary conditions are established for decision rules to be optimal for such a model. The question of the consistency of the constraints and the finiteness of the optimal value of the objective function for such problems is discussed and several methods of resolving these questions are presented. The simplification that results when the chance-constrained problem is treated as a problem of linear programming under uncertainty is also discussed. The paper is concluded by solving two two-stage problems.


## 1. INTRODUCTION

As an introduction to the topic of chance-constrained programming, the various kinds of objective functions and admissible classes of decision rules for chance-constrained problems that have appeared in the literature are discussed in this paper, and remarks are illustrated by using the stochastic lieat-ing-oil problem. The general $n$-period expectation-objective model is then presented and optimal decision rules for such a model are shown to be piecewise linear functions of certain conditional fractile points. Using these results, the simplification obtained by treating the problem as one of linear programming under uncertainty is discussed. The paper is concluded by solving two rathes simple two-stage problems.

The topic of chance-constrained programming is, perhaps, best introduced by first examining an ordinary linear-programming problem in its most general form, i.e.,
maximize

$$
c^{T}
$$

subject to

$$
\begin{equation*}
A x \leq h \tag{1}
\end{equation*}
$$

where $A$ is an $m \times n$ matrix of constants, and $b, c$ are, respectively, $m \times i$ and $n \times 1$ constant vectors. The ofject of problem 1 is to find the $n \times 1$ vector $X$ from the set of all $X$ satisfying $A X \leqslant b$, which maximizes $\boldsymbol{r}^{\top} X$.

If a chance-constrained formulation were to replace problem 1 a problem of the following type would be obtained:
maximize

$$
f(s, N)
$$

subject to

$$
\begin{equation*}
\Gamma(A) \leq b) \geq a, \tag{2}
\end{equation*}
$$

where $P$ means probability. Here $A, b, c$ are no longer constant matrices as they were in problem 1 ; rather, they can have some or all of their elements as random variables. The $m \times 1$ vector $\alpha$ contains a prescribed set of constants that are probability measures of the extent to which constraint violations are permitted. In other words, an element $\alpha_{1} \subset \alpha$ and satisfying $0 \leq \alpha_{1} \leqslant 1$ is associated with the constraint

$$
\sum_{1=1}^{n} a_{11} x_{1} \leq b_{1}
$$

to give

$$
\begin{equation*}
P\left(\sum_{1=1}^{n} a_{11} x \leq b_{1}\right) \geq a_{1} \tag{3}
\end{equation*}
$$

This means that the ith constraint may on occasion be violated, but it can be violated, at most, $100\left(1-\alpha_{1}\right)$ percent of the time.

As yet nothing has been said about the nature of the functional of problem 2. As might be expected, a fairly wide range of reasonable choices can be considered for the function $f(c, X)$. In Section 3 some of these choices will be discussed and the work that has been done on them will be summarized.

An important question in chance-constrained programming concerns the classes of decision rules that should be admissible for the problem. Because the object in problem 2 is to find an optimal vector of stochastic decision rules $X \equiv \varphi(A, b, c)$ with the function $\varphi$ chosen from some prescribed class of functions, a fundamental question is how to choose the admissible class of functions for o. In all the literature published thus far, it is assumed that $\varphi$ is a linear function of the elements of $b$, i.e., $X$ is restricted to a linear decision rule. In the results of this paper the admissible class of decision rules is greatly enlarged and then the optimal decision rules are shown to be piesewise linear. This result is of particular importance because it shows that there exists good reason, other than the fact that it is mathematically more manageable, to limit oneself to finding the optimal linear rule.

In some cases the class from which $\varphi$ is to be chosen may be restricted deliberately, so that at the time of application of the decision rules $X$ is not a function of some random variables; rather, the resulting $X$ values are assigned numerical, i.e., deterministic, values. When this can be done it is said that the resulting function provides a set of certainty-equivalert relations because it specifies the decision exactiy, i.e., with certainty.

In general, however, we permit a choice of ofor which the resulting $X$ vectors are also random. Thus problem 2 is solved for a set of stochastic decision rules imputing action for each manifestation of the random variables involved in the problem. This is in contrast to other possible solution notions such as mixed strategies in game theory or Manne's problem ${ }^{2}$-which, it might be noted, corresponds to the solution of only the first problem of the two into which the chance-constrained problem was factored in Charnes and Cooper ${ }^{2}$ or Tintner's stochastic programming. ${ }^{3}$

In either the stochastic or certainty equivalent case it may also be possible to develop a deterministic equivalent for problem 2. That $: s$, it may be possitle to find a probler: that is equivalent and yet does not contain any of the random variables explicitly. Hence it is a deterministic (often convex) programming problem.

These deterministic equivalents, when attainaible, of ten provide a means of convenient analysis and characterization of the solutions of the given chanceconstrained problem. In addition they may also be valuable in their own right for purposes of iheoretical analysis and policy review. How to obtain deterministic equivalents for certain choices of $f(c, \lambda)$ when the admissible class of functions for $\varphi$ is the class of linear functions of $b$ is illustrated in Section 3.

## 2. A CHANCE-CONSTRADNED EXAMPLE: THE STOCHASTIC hEATING-OIL PROBLEM

In order to elaborate iurther on the ideas developed above, an example of a chance-constrained problem will be given by using a simplified version of
the problem that first gave rise to the chance-constrained prograniming formulation and application. ${ }^{4}$ The problem is one of multiperiod scheduling of heatingoil production to meet all uncertain demand that depends heavily on the weather. In the original problem the constraints included marketing constraints, which required production to be planned in such a way that customer demands were met as they materialized, and storage constraints, which restricted the amount of oil that could be kept in inventory. The objective of the problem was to maximize expected piofits over the entire planning horizon.

As Charnes, Conper, and Symonds showed,' a deeper consideration of the constraints suggested that the problem was, in fact, one of minimizat on of expected costs rather than maximization of the expected profits. Because the constraints of the problem were such that customer demand was to be taken as given, i.e., stochastically determined, the real objective was to supply whatever femands arose at a minimum total expected cost.

A simplified version of this problem is:
minimize

$$
E\left[\sum_{1=1}^{n}\left(c, R, \cdot k_{1} 1,\right)\right]
$$

subject iu
$P\left(1_{0}+\sum_{1}^{f} R_{1}=\sum_{1=1}^{p} S_{1}\right) \geq a_{p} p-1 . \ldots n$
$P\left(R_{1}-0\right) \geq 1$
where the symbol $E$ refers to the expectation operator, so the objective is expected value minimization over the $n$ periods of the planning horizon. The $c_{1}$ and $k_{1}$ represent, respectively, the refinery costs and the inventory carrying charges for period 1 ; both $c_{1}$ and $k_{1}, l=1, \ldots, n$, may be random variables. $I_{1}$ is the initial inventory at the start of period $I$. The random variable $S_{\text {, }}$ represents the anticipated sales for period, $R_{1}$ represents the production rate to be scheduled in period, and is the quantity for which a decision rule is to be found.

Thus problem 4 shows that minimization of the total expected production costs over periods $J=1, \ldots, n$ is desired, subject to meeting customer demand at least 1000 , percent of the time in each period.

The probability $a_{\text {, }}$ is known as the "risk level" associated with the $\ell$ th constraint. The quantity

$$
b_{i} \quad \vdots_{i=1}^{i},
$$

is called the "quality level." Although in problem 4 these quality levels include all sales, it is important to note that this need not always be the case. For example, we may adjust the original random variables so that be refers to a certain fraction of the original sales variables. This would not change the problem in any mathematically essential way but it would alter our interpretation of the problem. Thus the constraints could be changed tc mean that in period $1,100 \beta$, percent of all sales demands, plus or minus a safety margin, must be met at least 1000 percent of the time. This flexibility in interpreta-
tion of mathematically similar models is one of the significant features of chance-constrained programming. In solving problem 4 it is convenient to find decision rules such that the actual refinery production rates will be known exactly at the start of the period to which they apply. In other words, it is desired that the stochastic decision rules $R_{1}, j=1, \ldots, n$, be such that $R_{j}$ is a known number at the start of the $j$ th period. This means that the admissible class of decision rules for $R_{f}$ can involve random variables only insofar as their values will be known, i.e., will have been observed, at the time of application of $R_{1}$. Hence $R_{1}$ can be an explicit function of only the random variables of periods $1, \ldots, 1-1$, and it cannot be a function of the random variables of the $f$ th or future periods.

In particular, if consideration is limited to linear rules, the rules will be of the form $R=\Gamma S+\delta$, where $\Gamma$ is a lower triangular matrix. Specifically, we can write

$$
\begin{equation*}
R_{p}=\sum_{1=1}^{P-1} r_{p} S_{1}+\delta_{p} \quad l=1 \ldots, n \tag{5}
\end{equation*}
$$

where each $\gamma_{l_{1}}, \delta_{1}$ is a scalar and $\gamma_{10}=0=\gamma_{11}$.
In their work, Charnes, Cooper, and Symonds ${ }^{4}$ were able, by using Eq 5, to convert problem 4 into a deterministic equivalent, which was then solved to give the optimal values $\gamma_{l}{ }^{*}$ and $\delta_{l}{ }_{l}$ for $\gamma_{l}$ and $\delta_{l}$, respectively. Using these values, the optimal decision rules $R_{1}, j=1, \ldots, n$, for problem 4 are given by

$$
\begin{aligned}
& R_{1}^{*}=\delta_{1}^{*} \\
& R_{2}^{*}=\gamma_{21}^{*} S_{1}+\delta_{2}^{*}, \\
& R_{3}^{*}=\gamma_{31}^{*} S_{1}+\gamma_{32}^{*} S_{2}+\delta_{3}^{*},
\end{aligned}
$$

and, in general,

$$
R_{i}^{*}=\sum_{k=1}^{1-1} y_{1 k}^{*} S_{k}+\delta_{1}^{*}
$$

It is important to observe that different $\gamma_{{ }_{i}^{*}}^{*}$ weights apply to the same observed value of $S_{1}$. Thus $S_{1}$ need not (and in general will not) receive the same weight in $R_{j}^{*}$ as it did in $R_{2}^{*}$. But once $\gamma_{j}^{*}, \delta_{j}^{*}, j>k, j=1, \ldots, n$, have been found, Eq 5 supplies a set of relations (i.e., decision rules) that can be used to generate the required numbers (i.e., decisions) at the time these numbers are needed. Thus, as the observations on the preceding $S_{\text {, }}$ values are obtained, the resulting $R_{j}$ values will be known with the certainty the problem requires.

The certainty-equivalent relations discussed above may supply all that is required for the conduct of operations. They may not, however, meet all the needs of management, since management may want to evaluate different alternatives before committing itself to a given set of policies or actions. Since Eq 5 cannot be completed until the $S_{\text {, }}$ values have been observed, this evaluation of different alternatives cannot be achieved. However, when a deterministic equivalent that assumes the form of a linear or convex programming problem is available, the duality relations of such problems can be used to supply dual evaluators. Then, prior to obtaining the data needed to achieve the certaintyequivalent relations, it is possible to study the overall effects of variations in
risk levels, quality levels, etc., and in other forms of constraint alto fation and data testing.

This completes the discussion of the stochastic heating-oil prot lem. It has been examined at some length in order to illustrate one possible irea of application of the results developed in Sections 6-10 of this paper an 1 in Ref 5. However, it is emphasized that this is not the only area to which charie-constrained programming has been successfully applied. Charnes, Coop r, Devoe, and Learner ${ }^{6}$ consider a problem of selecting studies and statistical © stimators in new-product marketing studies. Chance-constrained programmins is also very useful in problems of financial budgeting and planning for liquid ty. This is illustrated by Charnes and Thore ${ }^{7}$ when they consider the problen, of planning for liquidity in a savings and loan association. Their results are extended further in a RAC paper now in preparation. ${ }^{8}$ Also, the investigation of the uses of chance-constrained programming in critical path analysis was begun recently by Charnes, Cooper, and Thompson. ${ }^{9}$

## 3. SOME CHOICES OF OBJECTIVES IN CHANCE-CONSTRAINED PROGRAMMING

As mentioned in the "Introduction" section, there are many possible choices for the function $f(c, X)$ in problem 2. In fact, it must be emphasized that the very concept of optimization under risk and uncertainty immediately raises important questions about the choice of rational objectives. Such questions arise, for example, concerning the reasonableness of an expected-value optimization. Why not choose some other measure of value? The inportance of this question becomes clear when it is noted that the decision rules that are optimal for one problem and a given objective will not, in general, be optımal for the same problem under a different objective.

Charnes and Cooper ${ }^{10,11}$ investigate three different classes of objectives; specifically, they examine (a) an expected-value optimization, (b) a minimumvariance (or mean-square error) objective, and (c) a maximum-probability model. These models are called, respectively, the "E model," "V model," and "P model" of chance-constrained programming.

It is of interest to distinguish between the first two models and the third by reference to what H. A. Simon ${ }^{12}$ calls the "satisficing," as opposed to the "optimizing," objective. ${ }^{+}$Simon originally proposed this objective as an Iternative to (a) and (b) in order to try to resolve some of the inadequacies of optimizing objectives for characterizing certain aspects of human behavior. In particular, he suggested that human beings do not always seek an absolute extremum before taking action in a given situation; rather, they try to maximize the probability of being better off than some given point of reference.

In the $\mathbf{P}$ model, vectors $\mathrm{c}^{\prime \prime}, \mathrm{X}^{\prime}$, are specified relative to some set of values that an organism or human being will regard as satisfactory whenever these levels are achieved. Of course the organism confronted by an environment subject to risk cannot be certain of achieving the given level $c^{\prime \prime} \backslash \prime$ when effecting its choire from what it believes to be the available alternatives. Therefore

[^0]it tries to maximize the probability of obtaining at least its desired level $c^{\circ T} X^{0}$ subject to its feasible alternatives of action.

In their papers, Charnes and Cooper ${ }^{10,11}$ express the $E$ model as follows:
maximize
subject to

$$
E\left(c^{T} X\right)
$$

$$
\begin{equation*}
P(A X \leq b) \geq a, \tag{6}
\end{equation*}
$$

where $A$ is a matrix of constants and $b, c$ are vectors of random variables. They solve problem 6 for the optimal linear rule given by $X \equiv D b$, and so convert the problem to one of finding the elements of the matrix $D$. Under suitable assumptions concerning the symmetry, and the existence of second moments, of certain distributionst they show that the deterministic equivalent of problem 6 is
minimize

$$
-\mu_{c}^{\tau} D \mu_{0}
$$

subject to

$$
\begin{align*}
-a_{1}^{I} D \mu_{b}-v_{1}=-\mu_{b_{1}}, & 1=1, \ldots m, \\
-\kappa_{a_{1}}^{2} E\left|\hat{b}-a_{1}^{T} D \hat{b}\right|^{2}+v_{1}^{2} \geq 0, & 1=1, \ldots m,  \tag{7}\\
v_{1} \geq 0, & 1=1, \ldots m,
\end{align*}
$$

where $\left.\mu_{c}^{T} \equiv[E(c)] T, \mu_{b}^{T} \equiv E(b)\right]^{T}, n_{i}^{T}$ is the $i$ th row of $A, \vec{b} \equiv b-\mu_{b}$, and $-K_{\alpha_{1}} \equiv$ $F_{1}^{-1}\left(a_{1}\right)$ where $F_{1}($.$) is the cumulative dis'ribution function of the random$ variable $\left.z_{1} \equiv\left(\hat{b}_{1}-a_{1}^{I} D \hat{b}\right) / E \hat{b}_{1}-a_{1}^{I} D \hat{b}\right]^{2}$. This is a deterministic convex programming problem in the variables $v_{1}, i=1, \ldots, m$, and the elements $d_{1 j}$ of $D$.

Using the same kinds of techniques and limiting themselves to finding linear decision rules, Charnes and Cooper ${ }^{10,11}$ found deterministic equivalents for the $V$ model:
minimize

$$
E\left(c^{I} x-c^{0 \top} x^{0}\right)^{2}
$$

subject to

$$
\begin{gathered}
p(+x \leq b) \geq u \\
x=D b,
\end{gathered}
$$

where $\mathfrak{c}^{\circ T}$, $\mathrm{X}^{0}$ are given vectors; and for the $\mathbf{P}$ model:
maximize

$$
P\left(c^{1} x \geq c^{01} x^{0}\right)
$$

subject to

$$
\begin{gathered}
P(A X-b) \geq a \\
Y=D b .
\end{gathered}
$$

$\dagger$ For similar developments for other kinds of distributions and where Dis specialized in various ways see Ben-Israel ${ }^{17}$ and Ben-Israel and Charnes. ${ }^{18}$

In both these models the resulting deterministic equivalents contain the constraints $r^{〔}$ problem 7 as part of their system of constraints, although they also have additional constraints and a different objective function.

In the problem to be considered, starting in Section 6, the expectationobjective formulation will be used. No answer is attempted to the question of whether it is the "right" objective to use, but it seems to be a "reasonable" objective for problems of planning over an $n$-period horizon.

## 4. ChOICE OF DECLSION RULES

The developments of the following sections are concerned with establishing certain properties of the optimal decision rules for the general $n$-period $E$ objective model. In Section 6, the admissible class of decision rules for the problem is defined as the most general class of rules consistent with this interpretation of $n$-period problems. However, in order to illustrate clearly how closely the choice of an admissible class is related to this interpretation of the constraints of the problem, problem 4 must be considered again. There it was seen that a choice of the admissible class to be $R=\Gamma S+\delta$, where $\Gamma$ is lower triangular, led to a constraint interpretation of the following kind:

$$
\begin{equation*}
P\left(I_{p}+R_{\ell} \geq S_{p}\left|I_{0}+\sum_{j=1}^{P-1} R_{1}-\sum_{j=1}^{P-1} S_{j} \equiv\right|_{p}\right) \geq a_{p} \cdot l=1 \ldots, n, \tag{8}
\end{equation*}
$$

that is, given the observations $S_{1}, 1=1, \ldots, 1-1$, and decisions $R_{1}, 1=1, \ldots$, $1-1$, on the right of the vertical stroke in Eq 8 (the stroke is used to suggest conditional probability), and, before $S$, has been observed, a value must be found for $R_{l}$ that, when added to the f th period's beginning inventory $l_{l}$, will be sufficient to meet the unknown demand $S$, with at least the specified probability $a_{1}$.

Clearly then other choices of the matrix $\Gamma$ will lead to (or be associated with) different ways of interpreting the constraints. This would be true if, for example, $R_{\ell}$ were permitted to be an explicit function of $S_{\ell}$.

This problem can be illuminated further by considering some of the work done by Ben-Israel ${ }^{17}$ on what will be called the "zero-order decision rule." Here the decision maker wants to know all his program values in advance, i.e., at the start of the planning period. Such a situation can arise, for example, in some aspects of short-term country development planning as well as in certain kinds of budgetary planning practice.

For this particular case Ben-lsrael's theorem ${ }^{17}$ is employed to obtain the result

$$
P\left(\sum_{1=1}^{n} a_{11} x_{1} \leq b_{1}\right)=a_{1}
$$

if and only if

$$
\sum_{1=1}^{n} a_{11} x_{1} \leq F_{1}^{-1}\left(1-a_{1}\right),
$$

where $F_{1}($.$) is the cumulative marginal distribution function of thie random$ variable $b_{1}$ and $y$ is identical with $F_{1}^{-1}\left(1-\alpha_{1}\right)$ if and only if $y$ equals maximum $\left.y: F_{1}(y)<1-\alpha_{1}\right\}$. Thus the chance-constrained problem
maximize

$$
E\left(c^{\top} X\right)
$$

subject to

$$
\begin{aligned}
P(A X & \leq b) \geq a . \\
X & \geq 0
\end{aligned}
$$

has as its deterministic equivalent
maximize

$$
\mu_{r}^{I} x
$$

subject to

$$
\begin{align*}
H & \leq F^{-1}(1-a) .  \tag{9}\\
X & \geq 0 .
\end{align*}
$$

where $u^{I} \equiv E\left(c^{\mathrm{J}}\right)$.
Ir: a similar manner the vector $G^{-1}(\beta)$ can be defined as such that

$$
\left.P\left(w^{I}\right) \geq c^{I}\right) \geq B
$$

if and only if

$$
\omega^{\top} A \geq\left|G^{-1}(\beta)\right|^{\top}
$$

Thus if $G_{1}($.$) is the cumulative marginal distribution function of c_{1}$, the th component of $\sigma^{\prime}$, then $y \equiv G_{1}^{-1}(\beta)$ if and only if $y$ min $\left\{y: G_{1}(y) \sim \beta\right\}$.

Therefore the deterministic equivalent of
minimize

$$
E\left(\omega^{I} b\right)
$$

subject to

$$
\begin{gathered}
P\left(\omega^{\mathrm{I}} \mathrm{~A} \leq \sigma^{:}\right) \leq \beta . \\
\omega^{\mathrm{I}} \geq 0
\end{gathered}
$$

is: minimize

$$
\omega^{\mathrm{I}} \mu_{b}
$$

subject to

$$
\begin{aligned}
\omega^{\prime} A & \geq\left|G^{-1}(B)\right|^{1} . \\
\omega & =0 .
\end{aligned}
$$

where $\mu_{h} \equiv E(b)$.
The result obtained is the same as that of Bea-Israel, i.e., each of the following two problems is the dual of the other:
maximize

$$
\left\{\left.0^{-1}(\beta)\right|^{\mathrm{I}}\right.
$$

subject to

$$
\begin{gathered}
P(A X \leq b) \geq a \\
x \geq 0
\end{gathered}
$$

and minimize

$$
\omega^{1} F^{-1}(1-a)
$$

subject to

$$
\begin{aligned}
P\left(\omega^{\mathrm{T}} A\right. & \left.\geq c^{\mathrm{T}}\right) \geq \beta, \\
\omega^{\mathrm{T}} & \geq 0 .
\end{aligned}
$$

This example clearl; illustrates that the choice of decision rules here (or, alternatively, the interpretation of the constrain';) warkedly affects the resulting mathematical treatment of the model. For exampic, in the case of zero-order decision rules it is possible to develop a duality theory along the lines of the foregoing work, making the solution and evaluation of such problems much simpler than when much larger classes of decision rules are considered.

It would be possible to extend this discussion further by investigating other special and interesting cases. Instead, however, $n$-period problems will be considered again and interpreted. In the following developments the interpretation first suggested in Section 2 will be used. That is, the admissible class of rules for the, th-period decision rule can include explicitly only the random variables wiose values will have been observed when the time comes to put the 1 th decision into effect. Thus, as in the heating-oil example, the fthperiod decision rile will be a known number at the start of the th period. However, unlike the developments outlined in Section 3, no other restrictions will be placed on the rules. Hence a much larger class of rules than the set of linear rules already discussed will be considered, and yet the admissible class will include the class of linear rules.

## 5. MULTIPERIOD CHANCE -CONSTRAINED PROBLEMS

Before proceeding with the solution of the general n-period $\overline{\text { - }}$ objective problem, a brief survey will be presented of the literature on $n$-period problems as it pertains to this discussion. As has been mentioned, the work of Charnes, Cooper, and Symonds ${ }^{4}$ is fundamental in the field. In addition the work of Symonds ${ }^{19}$ on stochastic scheduling by the horizon method and Charnes and Thore ${ }^{7}$ on liquidity planning are mentioned.

If the special case of chance-constrained programming, which is obtained by making all the constraints hold with probability one, is considered, the problem obtained is one known in the field as "linear programming under uncertainty." Such problems werefirst discussed by Dantzig ${ }^{20}$ in relation to what he called the " $n$-stage problem." This emphasis on staging does not, however, provide a wholly adequate characterization of the problem, for-as was shown by Charnes, Cooper, and Thompson ${ }^{21}$-the $n$-stage problem of Dantzig can be converted into
a one-stage ordinary minimization problem. Thus, whereas the staging emphasis may appear to have a certain appeal in distinguishing between this and other approaches to programming, when statistical errors are present in the matrices (as, for example, in Madansky ${ }^{22,23}$ ), it also tends to conceal the simpler characterizations of the problem.

Specificaily, Charnes, Cooper, and Thompson ${ }^{21}$ established the following important result:

## Theorem 1

Consider the problem
maximize

$$
E\left[f\left(x_{1}, \ldots, x_{m} ; b_{1} \ldots, b_{m}\right)\right. \text {; }
$$

subject to

$$
\begin{aligned}
& \sum_{1=1}^{1} r_{1}=b_{1} \quad 1=1, \ldots, m \\
& r_{1} \geq 0
\end{aligned}
$$

where the $X_{1}$ are piecewise analytic functions of the $b_{1}$ and $X_{1}$, and $f$ is a linear function of $X_{1}, \ldots, X_{n}$. Then the optimal decision rule for $X_{i}$ i: a piecewise linear function of $b_{1}, \ldots, b_{1}$ and $X_{1}$.

In the following work a similar result will he obtained as an immediate corollary to the main theorem.
6. THE GENERAL $n$-PERIOD E MODEI.

The problem to be discussed is the following:
maximize

$$
E\left(c_{1}^{\top} \boldsymbol{\lambda}_{1}+c_{2}^{\top} \boldsymbol{\lambda}_{2} \cdot c_{3}^{\top} \boldsymbol{\lambda}_{3}+\ldots \cdot c_{n}^{\top} \lambda_{n}\right)
$$

subject to

$$
\begin{align*}
& P\left(b_{1} \leq \|_{11} X_{1}\right) \leq a_{1} . \\
& P\left(b_{2} \leq f_{21} x_{1} \cdot f_{22} x_{2}\right) \leq a_{2} \\
& P\left(b_{1}=f_{31} x_{1} \cdot t_{32} x_{2} \cdot f_{33} x_{3}\right)=a_{3} \\
& P\left(b_{n}=t_{n 1} x_{1} \cdot A_{n 2} \lambda_{2} \cdot A_{13} \lambda_{3} \quad . A_{n n} x_{n}\right) \geq a_{n}  \tag{10}\\
& 1,0 \quad 1=1 \ldots, n
\end{align*}
$$

where $P$ stands for probability and E denotes the expected-value operator. (A more detailed explanation of these follows.)

In problem 10 it is assumed that -
(a) $A_{11}, i>1,1,1=1, \ldots, n$, is an $m_{1}, n_{1}$ matrix all the elements of which are known constants.
(b) $\alpha_{1}, 1=1, \ldots, n$, is an $m_{1}, 1$ vector in which each element is a prescribed probability, i.e., $\alpha_{1}=\left(\alpha_{11}, \ldots, \alpha_{1 m_{1}}\right)$ is such that $0 \cdot \alpha_{1 k} \cdot 1$ for $k=1, \ldots, m_{1}, 1=1, \ldots, n$.
(c) $c_{1}^{\mathrm{T}} \equiv\left(c_{11}, \ldots, c_{n_{1}}\right), 1=1, \ldots, n$, is a 1 , $n_{1}$ vector of random variables.
(d) $b_{1} \equiv\left(b_{11}, \ldots . b_{1 m_{1}}\right), 1=1, \ldots, n$, is a $1 \times m_{1}$ vector of random variables.
(e) the joint distribution of the $b_{1 k}$ and $c_{1}, k=1, \ldots, m_{1}, i, \ldots, n_{1,1}$, $1=1, \ldots, n$, is known, i.e., the multivariate distribution function of the $b_{i k}$ and $c_{1} \ell$ is assumed to be a known function.

In addition to these five assumptions on the properties of the matrices involved in the problem, some conditions on the way in w'ich the problem is to be solved will be imposed. The nature of these additional assumptions is dictated by the interpretation of the problen, in this paper. Problem 10 will be treated as ain $n$-period, or $n$-stage, problem in which $\rangle_{1}$, the vector of decision rules for the th stage, is selected with knowledge of all decisions $\lambda_{1}, \ldots, \lambda_{1-1}$ and observations of the vectors of random variables $b_{1}, c_{1}, 1$, $\ldots 1^{-1}$, but before $b_{1}, i_{1}$, and all random variables of periods succeeding the th are observed. Thus $i$, will be known exactly (i.e.. will be a known number) at the start of the th period.

In other words, we must select $\Lambda_{1}$, our vector of first-period decisions, before observing the value of the first-period random vector $b_{1}$ and the firstperiod cost vector $r_{1}$. Then, having selected $\lambda_{1}$ and observed $b_{1}$ and $r_{1}$, the second-period decision rules $l_{\text {: }}$ must be chosen before observing the values of $b_{2}$ and $c_{2}$. This process continues for $n$ periods with $\lambda_{1}$ depending only on $l_{1} h_{1}$ $c_{1}, 1=1, \ldots, 1-1$, but not on $b_{1}, c_{1}$ or $l_{1}, b_{1}, c_{1}, 1=1+1, \ldots, n$. This interpretation means that the information at any stage laside from a knowiedge of the joint distribution of $\left.b_{1}, c_{1}, 1,1=1, \ldots, n\right)$ is limited to a knowledge of the decisions and observations of the preceding stages. Thus it is assumed that-
(f) the $n_{1}, 1$ vector $\lambda_{1}, 1=1, \ldots, n$, is a function of $\lambda_{1}, b_{1}, c_{1}$, for $1,1, \ldots$, $1-1$, but is not a lunction of the remaining decisions $1_{1}, 1-1+1, \ldots, n$, or of random vectors $b_{1}, c_{1}, 1=1, \ldots, n$.
$X$ will, of course, depend implicitly on the decisions and observations of succeeding stages because of the coupling nature of the constraints of problem 10. Assumption $f$ means that $i_{1}$ is not an explicit function of $b_{1}, c_{1}$, or $\rangle_{1}, b_{1}$, $c_{1}, \quad 1+1, \ldots, n$, so that $\backslash$ will be known exactly (i.e., will be a known number) when the th period decision is made.

As a consequence of this assumption it can be seen that the choice of $X_{1}$ affects the objective function of problem 10 not only through the $c_{1}^{I}$, term but also through the effect of $X_{1}$ on $X_{1,1}-2, \ldots, n$. In on the other hand enters the functional only in the $\sigma_{r}^{T} t_{n}$ term.

It will now be shown how assumptions $e$ and $f$ enable the constraints of problem 10 to be written in a more convenient form. The $P$ operator in the
constraints means that the probability is computed using the joint distribution of $b_{1}, c_{1}, i, i=1, \ldots, n$. Thus, by using a Stieltjes integral, the following can be written:

$$
P\left(b_{1} \geq A_{11} X_{1}+\ldots+A_{11} X_{1}\right) \geq \int_{A} d F(.)
$$

where $F($.$) is the joint distribution function of the b_{1}, c_{1}, i, 1=1, \ldots, n$, and $A$ is the set where $b_{1}>A_{11} X_{1}+\ldots+A_{11} X_{1}$.

It is important that it is clearly understood what is meant by the above identity. This is most easily seen by writing out the 1 th of the $m$ equations (as $b_{j}$ is $m_{j} \times 1$ ) in detail:

$$
\begin{aligned}
& P\left(b_{1} p=a_{11}^{(l)} x_{1}+\ldots a_{11}^{(f)} x_{1}+\ldots+a_{1!}^{(\ell)} x_{1}\right)
\end{aligned}
$$

where $a_{11}^{(p)}$ is the ${ }^{\text {t th }}$ row of $A_{11}$ and $a^{(P)}$ is the region in the sample space of the random variables involved in the problem over whic: $b_{1 p}>a_{11}^{(p)} X_{1}+\ldots+c_{11}^{(f)} X_{1}$ $+\ldots+a_{11}^{(p)} X_{1}$. Hence the compact notation

$$
P\left(b, \geq A_{11} x_{1}+\cdots+A_{11} x_{1}\right)=\int_{A} d F(.)
$$

-epresents $m_{1}$ identities of the type shown.
By assumption e, $F($. ) is a known function. Because of assumption $f$, the set A depends only on $b_{1}, c_{1}, i=1, \ldots, j$, and $b_{1}$ and not on the remaining $b_{1}$ and $c_{1}$. Consequently tise integration in the above identities can be performed by integrating first with respect to $b_{1+1}, \ldots, b_{n}$, and $c_{1}, \ldots, c_{n}$, and then integrating with respect to $b_{1}, \ldots, b_{1}$, and $c_{1}, \ldots, c_{1-1}$. But whin integrated with respect to $b_{1}, c_{1}, 1=1+1, \ldots, n$, and $c_{1}$, the region of integration is over the entire range of possible values of these vectors of random variables (i.e., the range of all values that the random variables can assume with non-zero probability) and hence the resulting value of the integral is 1 . Thus . $\mathrm{A}_{\mathrm{A}}^{\mathrm{d} F(.)=}$ . $A^{d} F_{1}($.$) , where F_{1}($.$) is the joint distribution function of b_{1}, \ldots, b_{1}$ and $c_{1}, \ldots, c_{1-1}$. Moreover $F_{1}($.$) is a known function because F($.$) is known.$

Through the use of conditional probabilities. . $A^{d} F_{1}($.$) can be written in$ terms of $\bar{F}_{1}($.$) , the conditional distribution function of b_{1}$ given $b_{1}, c_{1}, 1=1$, $\ldots, 1-1$, and $G_{1}(\ldots)$, the joint distribution function of $b_{1}, c_{1}, 1=1, \ldots, 1-1$. We have, by definition, i $A^{d F_{1}}()=.\cdots\left(A^{d} \bar{F}_{1}().\right) d G_{1}($.$) . Again F_{1}($.$) and G_{1}($. are known because they can be computed from a knowledge of $F($.).

Now $\bar{\alpha}_{1,1}=1, \ldots, n$ is defined as an $m_{1}, 1$ vector of probabilities such that

$$
\int_{-\infty}^{\infty}\left(\int_{i} d \bar{F}_{1}(.)\right) d\left(G_{1}, 1 \leq a_{1} \text { ll and onlv if } \int_{4} d \bar{F}_{1}, 1 \geq \bar{a}_{1}\right.
$$

The $\bar{\alpha}$, will be conditional probabilities depending on the given (i.e., conditional) values of $b_{1}, c_{1,1}=1, \ldots, 1$, which are used in determining $\vec{F}_{1}(\ldots)$. In other words, $\bar{\alpha}_{1}$ is such that

$$
\begin{equation*}
P\left(b_{1}=1_{11} 1_{1} \cdot \quad 1_{11} 1_{1}\right)=a_{1} \tag{11}
\end{equation*}
$$

if and only if

$$
\overline{\mathbf{P}}\left(b_{1} \geq A_{11} x_{1}+\ldots+A_{11} x_{1}\right) \geq \bar{a}_{1}
$$

where the $\overline{\mathrm{P}}$ operator means that the probability is computed using the conditional distribution of $b$, given $b_{1}, c_{1}, 1=1, \ldots, 1-1$, i.e., using $\bar{F}_{1}$ ( .). Thus, for example, if $b_{1}$ is independent of $b_{1}, c_{1}, i=1, \ldots, 1-1$, the $\bar{\rho}$ represents probability using the marginal distribution of $b_{1}$.

It is clear from the above development that if, instead of hiving $P$ and $\alpha_{\text {, }}$ as in the constraints of problem 10, a similar problem is given whose constraints involve $\overline{\mathrm{P}}$ and some set of cenditional probabilities $\bar{\alpha}_{1}, 1=1, \ldots, n$, the corresponding total probabilities P , along with their associated $\alpha_{1}$, can be obtained by integrating $P$ and $\vec{\alpha}_{1}$ with respect to $G_{1}($.$) .$

It is important to note in this development that $\bar{\sigma}_{1}=\alpha_{1}$ because $F_{1}($.$) is$ the marginal distribution function of $b_{1}$ and so is $\bar{F}_{1}\left(\right.$.). In other words, $\bar{\alpha}_{1}$ is a vector of constants and its components do not depend on the observed values of any of the random variables involved in the problem. This agrees with the interpretation of the problem in this paper because $X_{1}$ must be selected before any of the random variables are observed. Consequently the first-period constraints must have a completely deterministic equivalent. This is precisely the type of interpretation that led to Ben-Israel's theor $\mathrm{m}^{17}$ and its associated results (see Section 4). In brief, $X_{1}$ is a zero-order decision rule.

The question of how to determine $\bar{\alpha}_{1}$ for given $\alpha_{1}$ and distribution functions $\bar{F}_{1}($.$) and G_{1}($. ) is extremely difficult. Here, however, determining analytical properties of $X$ in terms of $\bar{\alpha}$, is of chief concern; hence the problem of determining $\bar{\alpha}_{,}$will be left for a subsequent paper. However, it must be emphasized that $\bar{\alpha}_{1}$ can (and in general will) depend on the given values of $b_{1}, c_{1}, 1=1, \ldots$, 1-1. This is of particular importance because it greatly enlarges the types of co.straints that are permissible in such a model. Elsewhere in the literature it is always assumed that the probabilities with which the constraints must hold are given constants no matter what values of the random variables are observed. Thus a more general problem is being treated than any previously considered in the literature.

Relation 11 and assumption ican be used to prove the following:
Lemma 1. The constraint $P\left(b_{1}>A_{11} x_{1}+\ldots+A_{\| l} t_{1}\right)>\alpha_{1}$ in problem 10 can be replaced by the equivalent constraint

$$
1_{11} 1_{1}+1_{12} 1_{2} \cdot \quad A_{11} \vdots_{1}-\bar{F}_{1}^{-1}\left(1-\bar{a}_{1}\right)
$$

where $\bar{F}^{-1}\left(1-\bar{\alpha}_{1}\right)$ is an $m_{1} \times 1$ vector of the $1-\bar{\alpha}_{1}$ percentile (or fractile) points of the conditional distribution of $b_{1}$ given $b_{1}, c_{1}, 1,1 \cdots$. The ith component of $\bar{F}_{1}^{-1}\left(1-\bar{\alpha}_{1}\right), \bar{F}_{1}^{-1}\left(1-\bar{\alpha}_{11}\right)$, is defined by $\bar{F}_{1}^{-1}\left(1-\bar{\alpha}_{11}\right) \equiv \operatorname{maximum}\left(y: \bar{F}_{1}(y) \times 1-\bar{\alpha}_{11}\right\}$.

The use of a maximum in the definition of $\bar{F}_{1}^{-1}\left(1-\bar{\alpha}_{11}\right)$ is justified because the assumption is that $\bar{F}_{1}($.$) is left continuous. +$ The reason the vector inequality $\bar{F}_{3}(Y) \cdot 1-\alpha$, does not necessarily define $Y$ uniquely is due to the fact that some of the random variables involved in $F_{1}$ (.) may be discrete and that some of the components of $1-\bar{\alpha}_{1}$ may be zero. In general each component, $y_{1}$,
$\dagger$ For a further discussion of left continuous distribution functions see Gnedenko, (Ref 24, Ch 4).
of $Y$ can take on any value in an interval. If this interval is denoted by $y_{1}\left(1-\bar{a}_{11}\right)$ and $\bar{y}_{1}\left(1-\bar{\alpha}_{11}\right), \bar{y}_{1}\left(1-\bar{\alpha}_{11}\right)$ respectively represent the smallest and largest numbers in the interval $y_{1}\left(1-\bar{\alpha}_{11}\right)$, then the $i$ th component of $\bar{F}_{1}^{-1}\left(1-\bar{\alpha}_{1}\right)$ is given by $\bar{y}_{1}\left(1-\bar{a}_{1}\right)$.

It is assumed throughout this discussion that $\bar{F}_{1}^{-1}\left(1-\bar{\alpha}_{11}\right)$ is well defined for all 1,1 . Thus, for example, if $\bar{F}_{1}($. ) is the distribution function of a normal random variable then it is implicitly assumed that no component of $\bar{\alpha}_{\text {, }}$ is 1 , so that $\bar{F}^{-1}\left(1-\bar{\alpha}_{11}\right)$ does not take on the value of $-\infty$ for any $i$. Similarly it can be seen that $\bar{\alpha}_{11}$ cannot be zero for any 1 or 1 , since in this case $1-\bar{\alpha}_{\mu}=1$ and so $y_{1}\left(1-\bar{\alpha}_{11}\right)=+\infty$, no matter what distribution $\bar{F}_{1}($.$) is used. This is, however,$ no restriction on our problem, since $\bar{\alpha}_{11}=0$ means that the corresponding constraint can be ignored because it will be satisfied for any choice of decision rules $X_{1}, \ldots, Y_{j}$.

Proof. Assume that $X_{1}, 1 \leq 1<1$ are decision rules that satisfy $P\left(b_{1}>A_{1} X_{1}+\ldots+A_{1} X_{1}\right)>\alpha_{1}$. Then

$$
\bar{P}\left(b_{1} \geq f_{11} x_{1} \cdots+f_{1} x_{1}\right) \geq \bar{a}_{1} \text { (bv relation } 11 \text { ), }
$$

or

$$
1-\bar{P}\left(b_{1} \quad A_{1} x_{1}+\ldots+A_{11} x_{1}\right) \geq \bar{a}_{1}
$$

or

$$
\bar{P}\left(b_{1} \cdot f_{11} x_{1}+\ldots+t_{11} x_{1}\right) \leq 1-\bar{a}_{1} .
$$

But $\bar{P}\left(b_{1}<A_{11} X_{1}+\ldots+A_{1} X_{1}\right) \equiv{ }^{\wedge} B^{d} \bar{F}_{1}($.$) where B$ is the set for which $b_{1}=A_{1} X_{1}+\ldots+A_{11} X_{1} \equiv 2$. Also, by assumptions a and $f, A_{1} X_{1}+\ldots+A_{11} X_{1}$ is not a function of $b_{1}$, hence $Z$ does not depend on the still-to-be-observed value of $b_{1}$. Consequently we have

$$
\int_{B} d \bar{F}_{1}()=\int_{-}{ }^{2} d \bar{F}_{1}(.)=\bar{F}_{1}(Z)=\bar{F}_{1}\left(1_{11} \boldsymbol{x}_{1}+\ldots 1_{11} \|_{1}\right) .
$$

Thus the fth constraint of problem 10 requires that

$$
\bar{F}_{1}\left(A_{11} x_{1}+\ldots A_{11} x_{1}\right)=1-\bar{a}_{1} .
$$

But any distribution function is a nondecreasing function of its argument, so that $\bar{F}_{1}\left(A_{1}, X_{1}+\ldots+A_{11} X_{1}\right) \leq 1-\bar{\alpha}_{1}$ implies

$$
\begin{equation*}
A_{11} x_{1} \cdot \cdot A_{11} 1_{1}=\bar{r}_{1}^{-1}\left(1-\bar{a}_{1}\right) \tag{12}
\end{equation*}
$$

Since all the steps of the proof are completely reversible, relation 12 implies

$$
P\left(b_{1} \geq A_{11} X_{1}+\quad \cdot A_{11} X_{1}\right) \geq a_{1}
$$

Thus the lemma is proved. ${ }^{\dagger}$
Using Lemma 1 , problem 10 is written in the following equivalent form:

[^1]maximize
$$
E\left(c_{1}^{\top} X_{1}+c_{2}^{\top} \lambda_{2}+c_{3}^{\dagger} X_{3}+\ldots+c^{\top} \boldsymbol{X}_{n}\right)
$$
subject to
\[

$$
\begin{aligned}
& 1_{11} x_{1} \quad-\bar{F}_{1}^{-1}\left(1-\bar{a}_{1}\right) . \\
& A_{21} X_{1}+A_{22} X_{2} \\
& -\bar{F}_{2}^{-1}\left(1-\bar{a}_{2}\right) . \\
& A_{31} X_{1}+A_{32} X_{2}+A_{33} X_{3} \quad=\bar{F}_{3}^{-1}\left(1-\bar{a}_{3}\right) . \\
& \text { - - - - - - - - - - - - - - - - - - - - - - - } \\
& A_{n 1} \lambda_{1}+A_{n 2} \lambda_{2}+A_{n 3} \lambda_{3}+\ldots+t_{n n} \lambda_{n}=\bar{F}_{n}^{-1}\left(1-a_{n}\right) . \\
& 1,20 \quad 1=1 \ldots n .
\end{aligned}
$$
\]

The E operator in the functional of problems 13 and 10 means that the expected value is being taken with respect to the joint distribution of $b_{1}, c_{1,1}$, $1=1, \ldots, n$. By using the linearity of the objective function in $\_{1,1}, 1, \ldots, n$, the following may be written:

$$
E\left(\sum_{1=1}^{n} \delta_{1}{ }_{1} X_{1}\right)=\sum_{1=1}^{n} \int_{-\infty}^{\infty} i_{1}^{I} X_{1} d F(.)
$$

However, this expected-value operation can be simplified by making use of assumption $f$. Since $X_{1}$ is a function of $X_{1}, b_{1}, c_{1}, 1-1, \ldots,-1$, only,

$$
\begin{aligned}
E\left(c_{1}{ }^{\top} X_{1}\right) & =\int_{-\infty}^{\infty} c_{1}^{I}\left\|_{1} d F(.)=\int_{-\infty}^{\infty} c_{1}^{I}\right\|_{1}\left[\int_{-\infty}^{\infty} d \tilde{F}_{1}(.)\right] d \tilde{F}_{1}() \\
& =\int_{-\infty}^{\infty} c_{1}^{l} \_{1} d \tilde{F}_{1}(.)
\end{aligned}
$$

where $d \hat{F}_{1}($.$) is the conditional distribution function of b_{1}, \ldots, b_{n}, c_{1,1}, \ldots, c_{n}$. given $b_{1}, \ldots, b_{1-1}, c_{1}, \ldots, c_{1}$, and $F_{1}(\ldots)$ is the joint distribution function of $b_{1}, \ldots, b_{1-1}, c_{1}, \ldots, c_{1}$. Thus, in computing $E\left(c_{1}^{I} X_{j}\right), c_{1}^{T}$, is integrated by using only $F_{1}($.$) rather than F($.$) .$

## 7. MATRDX THEORY

In the following discussion of the optimal decision rules use is made of a generalized inverse of a matrix. The properties required of such inverses are listed below. Proofs may be found in Charnes and Kirby ${ }^{25}$ and Rao. ${ }^{26}$

Definition. Let $A$ be any $m \times n$ matrix. Then any $n, m$ matrix $A^{n}$ such that $11^{\prime \prime} A$ is called a generalized inverse of $A$.

It can be shown that such an inverse always exists and is unique if and only if $m=n$.

Lemma 2. Let $A$ be an $m \times n$ matrix. Then there exists an $i^{n}$ such that $A A^{\prime \prime}$ If and only if rank $(A)=m$, i.e., $A$ has a right inverse if and only if it has full row rank.

Lemma 3. The matrix equation $A \mathcal{A}-b$ is consistent if and only if $A A^{\prime \prime} b=b$.
Lemma 4. Let the matrix equation $\|=b$ be consistent. Then the general solution is given by $1-A^{\prime \prime} b+\left(1-A^{\prime \prime} A\right) Y$ where $Y$ is an arbitrary $n \times 1$ vector and $l$ is the $n$, $n$ identity matrix.

Using these lemmas, a necessary and sufficient condition can be given for the existence of a vector $X$ satisfying the set of inequalities $1 \times<b, 1>0$. For, there exists $1>0$ such that $A X \cdot b$.
$\propto$ there exists $X, S,>0$ such that $A X+S-b$ where $S$ is an $n \cdot 1$ vector
$\Leftrightarrow$ there exists $X, S,>0$ such that $A X=b-S, A A^{\prime \prime}(b-S)-b-S$ by using
Lemma 3.
$\omega$ there exists an arbitrary vector $Y$ and a vector $S>0$ such that

$$
\begin{aligned}
& 1 A^{\prime \prime}(b-5)=b-5 \\
& 4^{\prime \prime}(b-5) \geq\left(A^{*}:(-1)\right)
\end{aligned}
$$

by using Lemma 4. In this case $X-A^{n}(b-S)+\left(1-A^{\#} A\right) Y$. Therefore there exists a finite $n \times 1$ vector 1 satisfying the set of inequalities $A \mathbb{A} \cdot b, 1>0$ if and only if there exists a finite $n \times 1$ vector $Y$ and $n \times 1$ vector $S, S>0$ such that $A A^{*}(b-S)=b-S$ and $A^{\prime \prime}(b-S), ~\left(A^{\prime \prime} A-1\right) Y$. From Lemma 2 it can be seen that if $A$ has full row rank, the condition $A A^{\prime \prime}(b-S)-b-S$ becomes an identity and it is only required that there exist vectors $Y$ and $S>0$ such that $A^{\prime \prime}(b-S)$ $>\left(A^{\prime \prime} A-1\right) Y$.

## 8. REMARKS ABOITT THE CONSISTENCY OF THE CONSTRAINTS

A significant property of chance-constrained programming models is that the constraints generally represent the "intentions* of management rather than hard and fast "rules." In other words, the constraints rep.esent "bounds" inside which management would like to operate "most of the time ${ }^{\text {e }}$ rather than *all the time." The decision rules resuling from solving a chance-constrained problem are designed to give guidelines rather than definite plans of action for management operation and decision, subject to qualifications surrounding the controls involved in implementing the rules. Thus the optimal decision rules for our problem may impute an action that, because of exceptional circumstances, cannot actually be taken.

Another major feature of the chance-constrained programming concept is that, in contrast to so-cailed "linear programming under uncertanty," the constraints (e.g., $\sum_{1}^{n} a_{1} x_{1} \cdot b_{1}$, as in relation 3) need not hold with probability 1. This generalization is important on several rnalistic grounds. First, it may not be possible, or even desirable, to specify actions in every conceivable circumstance, i.e., for every possible value of the random variables. For exa mple, in the heating-oil problem (see Section 2 ), certain combinations of events (which really did occur once or twice) were impossible for the company, acting alone, to handle either physically or economically. Second, constraints that have the nature of "policies" are not really intended to apply in every instance but only "almost all the time. Hence the constraints of the problem in Section 2 say that the supply of heating oil must exceed the customer demand, not all the time but $100 a_{\text {, }}$ percent of the time, where $a_{\text {, }}$ is some number close to but not equal to 1 . Third, by their nature, policies las opposed to definite rules) need not spell out in advance the acticns that will be taken in exceptional circumstances.

Chance-constrained programming allows for such exceptional circumstances in a very easy, natural manner, particularly when the admissible classes are defined in stme straightforward analytical manner, e.g., as linear decision rules. However, questions arise when one wishes to consider "all possible" or "all possibly desirable" stochastic actions in time sequence. It may then be seen, on reflection, that an essential feature of the analytically specified admissible class is that the analytic description carries one through sets of random events that imply constraint violations or the impossibility of actual, as opposed to imputed, further actions in the time sequence. For example, in the heating-oil problem the linear decision rules defined a value of $R_{1}$ no matter how large the values observed for $S_{1}, 1=1, \ldots, 1$. Clearly then, if extremely large demands were observed in periods $1, \ldots, j^{-1}$, it would be physically impossible for the company to produce enough oil to meet the a mount given by $R_{1}$. This would represent an exceptional circumstance and the company would not (and could not) follow the decision for period, imputed by the linear decision rule. Thus it is important to consider the question of what analytic specifications should be made to carry one through such situations in solving for optimal rules.

In addition this example shows that if constraints that place an upper bound on $R_{1}, 1=1, \ldots, n$ are added to problem 4 it is possible that the constraints will be inconsistent for some values of $S_{1,1}-1, \ldots, n$. Therefore discussion of the difficulties involved when the constraints of the problem are inconsistent must precede the establishment of certain properties of the decision rules that are uptimal for this problem.

The question of inconsistency is of significance because, as has been illustrated, one of the major (and frequently overlooked and misunderstood) properties of chance-constrained models is that points for which the constraints are inconsistent may well exist in the sample space of the random variables involved in the problem. In other words there may be points ( $b_{1}, \ldots, b_{n}$ $c_{1}, \ldots, c_{n}$ ) in the sample space of the random variables $b_{1}, c_{1}, 1,1=1, \ldots, n$, for which there do not exist decision rules $X_{1}, 1=1, \ldots, n$ that satisfy the constraints of problem 13. Thus it is possible that for some values of the random variables $b_{1}, c_{1}, 1, \ldots, 1$, and first,-1 decisions, $x_{1}, 1=1, \ldots, 1$, there will not exist an $n_{1}>1$ vector $X_{1}>0$ satisfying $A_{11} X_{1} \leq \bar{F}_{1}^{-1}\left(1-\bar{\alpha}_{1}\right)-A_{11-1} X_{1-1}$ $-\ldots-A_{11} x_{1}$. In such cases the question arises as to the meaning of the expectation operator in the functional. How is $E\left(c_{1}^{\Gamma} X_{,}\right)$calculated over those sample points for which no feasible X , exists?

One method of resolving this difficulty is to replace $\sum_{j=1}^{n} E\left(c_{l}^{I} X_{1}\right)$ in the functional bv $\left.\sum_{1=1}^{n} E_{1}\left(c_{1}^{I}\right\rangle_{1}\right)$, where $E_{1}$ means that the expected value is taken over only those sample points for which there exist decision rules $\lambda_{1}, \ldots, X_{1}$, satisfying the first । constraints of problem 13. Thus

$$
E\left(c_{1}^{\top} \psi_{1}\right)=\int_{-\infty}^{\infty} c_{1}^{\top} v_{1} d \tilde{F}_{1}(.)
$$

but

$$
\left.\left.E_{1}\left(c_{1}^{T}\right\rangle_{1}\right)=\int_{1_{1}} c_{1}^{I}\right\rangle_{1} d \tilde{F}_{1}(.)
$$

where $\Delta_{\text {, }}$ denotes the region where the first, constraints of problem 13 are consistent.

If $\bar{\Delta}_{j}$ denotes the complement of $\Delta_{1}$, it can be seen that $E\left(c_{j}^{I} X_{j}\right)$ and $E_{j}\left(c_{j} X_{j}\right)$ are related by

$$
E_{1}\left(c_{1}^{\top} x_{1}\right)=E\left(c_{1}^{\top} x_{1}\right)-\int_{\Lambda_{1}} c_{1}^{T} x_{1} d \tilde{F}_{1}(.)
$$

By applying the mean value theorem we obtain the equation

$$
E\left(c_{1}^{T} x_{1}\right)-E_{1}\left(c_{1}^{\top} x_{1}\right)-k P\left(\bar{X}_{1}\right)
$$

where $k$ is a constant and

$$
P\left(\bar{\lambda}_{1}\right)=\int_{j_{1}} d \tilde{F}_{1}(.) .
$$

Thus we see that whenever $P\left(\bar{\Delta}_{j}\right)$ is small, the use of $E$ provides a close approximation to $E$.

The concept of an analytic decision-rule class under chance constraints originated by Charnes and Cooper ${ }^{2,10,29,28}$ provides a direct and simple means of specfying the decision rules across the inconsistent points of the problem. In this case the analytic description of the class permits $X$, to be defined throughout all points of the sample space of the random variables (see the preceding discussion and the example of Section 2). Therefore, when a class of decision rules is specified, $X$, becomes defined throughout $\delta_{\text {, }}$ and the use of the $E$ operator again becomes meaningiul.

One way of defining $X_{1}$ in $\bar{\Delta}_{L}$ is to specify $X_{1}=0$ in this region. The effect of this definition is to make $E_{1}\left(c_{1}^{J} x_{1}\right)=E\left(c_{1}{ }^{T} X_{1}\right)$. Moreover use can be made of the material developed in Section 7 to specsfy, a priori, the region $\Delta_{1}$ over which $X_{1}=0$ will be put. It is known from Section 7 that there exists a decision rule $X_{1}>0$ satisfying $A_{11} X_{1} \leqslant F_{1}^{-1}\left(1-\bar{\alpha}_{1}\right)-A_{11-1} X_{1-1}-\ldots-A_{11} X_{1}$ if and only if there exist vectors $Y_{1}$ and $S_{1}>0$ satisfying

$$
\begin{array}{r}
A_{11} A_{11}^{\prime \prime}\left(\bar{F}^{-1}\left(1-\bar{a}_{1}\right)-A_{11-1} Y_{1-1}-\cdots-A_{11} r_{1}-s_{1}\right) \\
=\bar{F}_{1}^{-1}\left(1-\bar{a}_{1}\right)-A_{11-1} r_{1-1}-\cdots-1_{11} 1_{1}-s_{1}
\end{array}
$$

and

$$
4_{11}^{\prime \prime}\left(\bar{F}_{1}^{-1}\left(1-\bar{a}_{1}\right)-A_{11-1}!_{1-1}-\cdots-1_{11} 1_{1}-s_{1} \geq\left(1_{11}^{\prime \prime} 1_{11}-1\right) 1_{1}\right.
$$

and in this region

$$
1_{1}=1_{11}^{*}\left(\bar{f}_{1}^{-1}\left(1-\bar{a}_{1}\right)-1_{11-1} 1_{1-1}-\quad-1_{11} 1_{1}-1_{1} 1+\left(1-1_{11}^{*} 1_{11} r_{1}\right.\right.
$$

If $D_{1}$ denotes the region where a feasible $\lambda_{1}$ exists, it can be seen that $\Delta_{1}$, the set of points for which the first $\mid$ constrain's are consistent, is given by $\int_{1=1} D_{1}$. Hence $\bar{\Delta}_{1}$, the complement of $\Delta_{1}$, equals $i_{i=1} \bar{D}_{1}$, where $\bar{D}_{1}$ is the complement of D.

The specification of 1 as constant $(-0)$ over its region of inconsistency makes it possible to extend the results obtained below. In Theorem 2 it will be proved that in the region where the constraints are consistent the optimal decision rule for, , gives iy as a piecewise linear function of $\bar{F}_{1}^{-1}\left(1-\bar{a}_{1}\right)$, and
$x_{1-1}, \ldots, x_{1}$. Therefore specifying $X_{1}=0$ in $\bar{\Delta}_{1}$ means that the optimal decision rules become piecewise linear everywhere.

Another method of resolving the difficulty of inconsiscency that could yield values of $X_{1} \neq 0$ in $\bar{\Delta}_{1}$ is to modify the formulation of the constraints of the problem so that the right-hand side of the constraints in problem 13 is oper:ted on by a projection into the range of the $A_{l l}$ operator. This projection would be the identity operator when consistency holds and would yield some near point in the range of $A_{\|}$when the th constraint is inconsistent (if the Penrose-Moore inverse ${ }^{29}$ were used in the projection operator, the near point would be the nearest feasible point in the Euclidean sense). $\dagger$ With this change in the constraints, the E operator could then be applied without modification and the resu!ting functional value of $E\left(c_{1}^{\top} X_{1}\right)$ would differ from $E_{1}\left(c_{1}^{\top} X_{1}\right)$ by, at most, something on the order of $P\left(\bar{\Delta}_{1}\right)$.

Regardless of the procedure used to resolve the difficulty of inconsistent sample points, the problem of determining the optimal lecision rule for $X_{1}$ is reduced to finding the optimal rule when the constraints are consistent. At those sample points where sufficient consistency exists for the constraints and the functional to be meaningful, the argument of the following section is valid.

## 9. optimal Class of decision rules

Let ( $b_{1}, \ldots, b_{n}, c_{1}, \ldots, c_{n}$ ) be any point in the sample space of the vectors of random variables $b_{1}, c_{1}, 1,1=1, \ldots, n$, for which there is consistency, i.e., let this sample point be in $\prod_{i=1}^{n} D_{1}$. Suppose that for eac'l such consistent point the decision rules are found to be $X_{1}^{*}, \ldots, X_{n}^{*}$, maximizing $\sum_{i=1}^{n} c_{1}^{I} X_{1}$ subject to the constraints of problem 13. Then $X_{j}^{*}, 1=1, \ldots, n$, are the optimal decision rules for the problem. This follows from the fact that for any other feasible decision rules, $x_{1}, 1=1, \ldots, n$, for problem $13, \sum_{1=1}^{n} c_{1}^{I} x_{1}>\sum_{1=1}^{n} c_{1}^{T} x_{j}$ results from the definition of $X_{1}^{*}, 1=1, \ldots, n$. This implies that $E\left(\sum_{1=1}^{n} c_{1}, f x_{1}^{*}\right)$ , $E\left(\sum_{l=1}^{n} c_{1}^{I} \chi_{1}\right)$, thus establishing the optimal ity of $Y_{i}^{*}, 1=1, \ldots, n$, for points where consistency is present.

## Theorem 2

The optimal decision rule for $\mathrm{X}_{1}$ in problem 13 is of the form

$$
1_{1} \text { - a prece enise linear function if the } \bar{F}_{1}^{-1}\left(1-\bar{a}_{1}\right) \text { and } 1_{1-1} \ldots \boldsymbol{1}_{1}
$$

Proof. By assumption $f, l_{1}, 1-1, \ldots, n$, influences the objective function through the terms $s_{1}^{J} x_{1}, c_{1,1}^{J} x_{1,1}, \ldots, s_{n}^{I} t_{n}$. Hence the effect of the choice of $\rangle_{n}$ appears in the functional only in the $c_{n}^{I} h_{n}$ term. Consequently the first step is to find the optimal decision rule for $i_{1}$ as a function of $i_{n-1}, \ldots, x_{1}$ and then proceed recursively by finding the optimal $\ell_{n-1}$ as a function of $t_{n-2}, \ldots, l_{1}$ until the problem is ultimately reduced to finding the optimal $t_{1}$.

[^2]To avoid the detailed computation for each stage the theorem will be proved by induction on $\ell$ where $1=n+1-\ell$. After proving the theorem for $\ell=1$, i.e., after proving that the optimal $X_{n}$ is given as a piecewise linear function of $\bar{F}_{n}^{-1}\left(1-\bar{\alpha}_{n}\right), X_{n-1}, \ldots, X_{1}$, it is then assumed for induction that the theorem is true for $\ell=k$, i.e., for $X_{n-k+1}, \ldots, X_{n}$. It is then proved true for $\ell=k+1$, i.e., for $X_{n-k}$.

Let ( $b_{1}, \ldots, b_{n}, c_{1}, \ldots, c_{n}$ ) be any consistent point in the space of possible values for the vectors of random variables, $b_{1}, c_{1}, i, j=1$, . . $n$. For this sample point the problem of maximizing $\sum_{j=1}^{n} c_{1}^{I} X$ subject to the constraints of problem 13 will be solved and thus the optimal decision rules $X_{1}^{*}, 1=1 \ldots, \ldots$, will be obtained.

Let $X_{1}, \ldots, X_{n-1}$ be specific values for the first $n-1$ decision rules. Then our problem becomes one of determining a decision rule for $X_{n}$ that solves
maximize

$$
c_{n}^{\top} Y_{n}
$$

subject to

$$
\begin{align*}
A_{n n} X_{n} & \leq \bar{F}_{n}^{-1}\left(1-\bar{a}_{n}\right)-A_{n n-1} X_{n-1}-\cdots-A_{n 1} X_{1} .  \tag{14}\\
X_{n} & \geq 0 .
\end{align*}
$$

But because $X_{1}, 1=1, \ldots, n-1$ is specified and the focus is on a single point in the sample space, the right-hand side of the constraints in problem 14 is a known vector. Moreover by assumption a all $A_{i j}$ are constant matrices; hence problem 14 is a linear programming problem and so has a dual. lts dual is
minimize

$$
\omega_{n}^{I}\left[\bar{F}_{n}^{-1}\left(1-\bar{a}_{n}\right)-A_{n n-1} X_{n-1} \cdots-A_{n 1} x_{1}\right]
$$

subject to

$$
\begin{align*}
\omega_{n}^{\top} A_{n n} & \leq c_{n}^{\top}  \tag{15}\\
\omega_{n} & \geq 0 .
\end{align*}
$$

If problem 14 has a finite optimal solution $X_{n}^{*}$, then the convex set of feasib'e solutions to problem 15 has a finite number of extreme points, denoted by $\tilde{u}_{n 1}^{T}, 1=1, \ldots, N_{n}$; and, from the dual theorem of linear programming, it is concluded that

$$
\begin{align*}
r_{n}^{\top} \forall_{n}^{*} & =\min \quad\left\{\tilde{\omega}_{n 1}^{1}\left[\bar{F}_{n}^{-1}\left(1-\bar{a}_{n}\right)-A_{n n-1} \forall_{n-1} \ldots-i_{n 1} \forall_{1}\right]\right\}  \tag{16}\\
1 & =1, \quad V_{n}
\end{align*}
$$

Since there is only a finite number $V_{n}$ of extreme points, problem 16 implies that $\mathcal{V}_{n}^{*}$ is a piec wise linear function of $\bar{F}_{n}^{-1}\left(1-\bar{\sigma}_{n}\right)-A_{n-1} x_{n-1} \ldots-A_{n}, X_{1}$ with, at most, $V_{n}$ pieces. Hence $\mathcal{Y}_{n}$ is piecewise linear in $\vec{F}_{n}^{-1}\left(1-\bar{\alpha}_{n}\right)$ and $X_{n-1}, \ldots, X_{1}$. Thus the theorem is true for $\ell=1$.

Now assume for induction that the theorem is true for $1=k$. Then there exist functions $\mathbb{H}_{1} ; \bar{F}_{j}^{-1}\left(1-\bar{\sigma}_{1}\right), Y_{1-1}, \ldots, X_{1}, 1=n-k+1, \ldots, n$, which are piecewise linear in $F_{1}^{-1}\left(1-\vec{a}_{1}\right), X_{1-1}, \ldots, X_{1}$, and are such that $X_{1}^{*}=H_{1}, 1=n-k+1, \ldots, n$. Thus problem 13 can be written as
maximize

$$
\left.E \mid c_{1}^{I} X_{1}+\ldots+c_{n \rightarrow k}^{I} X_{n \rightarrow \infty}+c_{n \rightarrow+1}^{I} H_{n \rightarrow+1}+\ldots+c_{n}^{I} H_{n}\right)
$$

subject to

$$
\begin{align*}
& t_{1} \lambda_{1}=\bar{F}_{1}^{-1}\left(1-\bar{a}_{1}\right) . \\
& t_{21} l_{1}+f_{22} l_{2} \leq \bar{F}_{2}^{-1}\left(1-\bar{a}_{2}{ }^{\prime}\right. \text {. } \tag{17}
\end{align*}
$$

$$
\begin{aligned}
& 1, \therefore 0,1-1, \ldots, n+
\end{aligned}
$$

Again let $\left(b_{1}, \ldots, b_{n}, c_{1}, \ldots, c_{n}\right)$ be a particular sample point and let $\ell_{1}, \ldots, l_{n-k-1}$ be specified values for the first $n-k-1$ decision rules. Then $\forall_{n-k}^{*}$ is obtained by solving the following:
maximize
subject to

$$
\begin{align*}
& 1_{n \rightarrow k} \geq 0 . \tag{18}
\end{align*}
$$

The functional in problem 18 is piecewise linear in $\lambda_{n-k}$ because of the piecewise linearity of the $H_{1}, 1=n-k+1$, .., $n$. Moreover the optimal $k_{n-k}$ can be obtained by first solving problem 18 over each piece where the functional is linear in $X_{n-k}$ and then selecting the piece that is optimal. That is, the solution space of problem 18 can be divided into sets $K_{1}, \ldots, k_{1}$, such that in any set $K_{r}$ the functional in problem 18 is linear in $x_{1-k}$. Problem 18 is then solved for $X_{n-k,}^{*}$, the optimal $X_{n-k}$ in $k_{p}, r=1, \ldots, N$, and finally $X_{n-k}^{*}$ is selected such that

$$
\begin{aligned}
& \text { r-1..... }
\end{aligned}
$$

Since for any set $k_{r}$, the functional in problem 18 is linear in $\psi_{n-k}$ the problem in $k$, is
maximize

$$
\begin{align*}
& X_{n \rightarrow \infty}^{\prime \prime} X_{n \rightarrow} \\
& \text { subject to } \\
& A_{n \rightarrow, n \rightarrow} i_{n \rightarrow k}-\bar{r}_{n \rightarrow( }^{-1}\left(1-\bar{a}_{n \rightarrow k}\right)-A_{n \rightarrow n \rightarrow-1} A_{n \rightarrow-1}-A_{n \rightarrow, 1} \lambda_{1} .  \tag{19}\\
& x_{n \rightarrow-} \geq 0
\end{align*}
$$

where $\mathrm{l}_{n-k}$ is a $1 \times n_{n-k}$ matrix whose components depend on the sample point $\left(b_{1}, \ldots, b_{n}, c_{1}, \ldots, c_{n}\right)$ and the given values of $i_{1}, \ldots, b_{n-k-1}$.

Because problem 19 is a linear programming problem like problem 14, by writing its dual and proceeding exactly as was done in going irom problem 14 to problem 16 , it can be seen that $t_{n-r}^{*}$ is a piecewise linear function of
$\bar{F}_{n-k}^{-1}\left(1-\bar{\alpha}_{n-k}\right), x_{n-k-1}, \ldots, x_{1}$. Since this is true for each region $k_{p}$ over which the functional in problem 18 is linear, it is concluded that $X_{n-k}^{*}$ is indeed piecewise linear in $\bar{F}_{n-k}^{-1}\left(1-\bar{\alpha}_{n-k}\right), x_{n-k-1}, \ldots, X_{1}$.

Thus, by induction, the theorem is proved.
Corollary 1. The optimal decision rule for $\lambda_{1}, 1=1, \ldots, n$, in problem 13 is of the form $\boldsymbol{I}_{1}=$ a piecewise linear function oi the $\bar{F}_{1}^{-1}\left(1-\bar{\alpha}_{1}\right), 1-1, \ldots$,

## 10. OPTIMAL DECISION RULES FOR SOME SPECIAL CASES

The results of the preceding section apply to any model satisfying assumptions a through f. These assumptions are, however, very general and so it is instructive to look in closer detail at certain special cases that yield particularly interesting results. More specifically the chief concern will be the case in which problem 10 is an $n$-stage linear-programming-under-uncertainty problem (i.e., $\alpha_{1 p}=1,1,1, \ldots, m_{1}, 1=1, \ldots, n$ ).

However, another important special case of problem 10 is considered in Ref 5. This is the case in which each $A_{11}$ matrix, $1>1,1,1=1, \ldots, n$, is a 1 * 1 matrix. In other words, each period generates exactly one new constraint (instead of $m_{1}$ new constraints as in problem 10). Charnes and Kirby have shown that there is much more to say about the solution of such problems than has been said in Theorem 2. In particular an explicit method of obtaining the optimal decision rules under certain circumstances is shown.

To begin with, however, it is observed that Corollary 1 alsc implies the following.

Corollary 2. The optimal decision rule for $X_{1}, 1=1, \ldots, n$, in problem 13 is piecewise linear in the $b_{1}, c_{1}, 1=1, \ldots, 1-1$, if and only if $\bar{F}_{p}^{-1}\left(1-\bar{\alpha}_{q}\right), 1=1$, ..., , is piecewise linear in $b_{1}, c_{1}, 1=1, \ldots, t-1$.

As an example of a situation in which Corollary 2 is applicable, consider the case where $b_{1}, \ldots, b_{n}, c_{1}, \ldots, c_{n}$ are jointly normally distributed, and $\bar{\alpha}_{18}$ is a constant for all $1, \ell$, and so does not depend on the given values of $b_{1}, c_{1}, 1-1, \ldots, 1-1$. Then $\bar{F}_{1}($.$) is the distribution function of a nornal random$ variable with mean $m$ and variance $\delta^{2}$. Moreover $m$ is a linear function of the random variables that art given in the conditional distribution. Thus $m$ is a linear function of $b_{1}, c_{1}, 1=1, \ldots, 1-1$. Also $\delta^{2}$ is a constant. Therefore $\bar{F}_{1}\left(y_{p}\right)=\Phi\left(y_{p}-m / \delta\right)$ can be written where $\phi($.$) is the distribution function of a$ normal random variable with mean 0 and variance 1. Thus

$$
\bar{f}_{1}\left(p_{p}\right)=1-\bar{a}_{, p}=\eta=\bar{f}_{1}^{-1}\left(1-\bar{a}_{, p}\right)=\delta \Phi^{-1}\left(1-\bar{a}_{, p}\right)+m .
$$

But $\bar{\alpha}_{1 p}$ is, by hypothesis, a constant. Hence $\left.\Phi-41-\bar{\alpha}_{1 p}\right)$ is a constant, and the linearity of $m$ implies the linearit" of $\bar{F}_{1}-4\left(1-\bar{a}_{1 p}\right)$. Therefore each component of $\bar{F}_{1}^{-1}\left(1-\bar{a}_{1}\right), 1, \ldots, n$, is a linear function of $b_{1}, c_{1}, 1,1, \ldots, 1$.

To give examples where the fractile points are linear in the given sample points and the $\bar{\alpha}_{j}$ are not constant is fairly difficult. In fact it is clear that $\boldsymbol{X}_{1}$ * will not, in general, be piecewise linear in $b_{1}, i_{1}, 1, \ldots, 1$. 1 .

Another special case of this corollary falls into the category of dinear programmang under uncertainty.t Suppose that $\alpha_{1} 1,1=1, \ldots, n$. Then by

[^3]our definition of $\bar{\alpha}_{1}$ there must also be $\bar{\alpha}_{1}-1,1=1, \ldots, n$. Assume also that the conditional distribution of $b_{1}, 1=1, \ldots, n$, given $b_{1}, c_{1}, 1, \ldots, \ldots-1,1 s$ such that its 0 th fractile point does not depend on the given values of $b_{1}, c_{1}$, $1=1, \ldots, 1-1$. Then $\bar{F}_{1}^{-1}\left(1-\bar{\alpha}_{1}\right)=\bar{F}_{1}^{-1}(0), 1=1, \ldots, n$, becomes a consti nt vector the components of which are known before any observations have been made. The solutions space of problem 13 is therefore a fixed convex set whose bounding hyperplanes do not depend on the random variables involved in the problem Thus we are led to the following:

## Theorem 3

Assume $\alpha_{1}-1,1=1, \ldots, n$.
Assume $\bar{F}_{1}^{-1}(0), 1=1, \ldots, n$, is a vector of constants.
Then the optimal decision rule for $\ell_{\text {, }}$ in problem 13 is such that $\rangle$, is piecewise constant.

Proof. By Corollary 1 the optimal $X_{1}$ is piecewise linear in $\bar{F}_{1}^{-1}\left(1-\bar{\alpha}_{1}\right)$, 1 1...., Therefore $X_{1}$ is piecewise linear in the $\bar{F}_{1}^{-1}(0), 11, \ldots, 1-1$. But $\bar{F}_{1}^{-1}(0)$ is a constant by assumption. Hence, $\rangle$ is piecewise constant.

This result was also obtained in a different manner by Charnes, Cooper and Thompson ${ }^{21}$ for the $n$-stage linear-programming-under-uncertainty model (see Theorem 1).

The assumption used above that $\bar{F}_{1}^{-1}(0)$ be a constant vector is not as restrictive as it might appear at first glance. Indeed this assumption is true for a very large class of distribution functions $\bar{F}_{j}($. ). In fact in Theorem 3 only such distributions are eliminated as $\bar{F}_{1}($.$) , the distribution function of a$ random variable that is uniform over $(a, b)$ where $a$ depends on $b_{1}, c_{1}, 1=1, \ldots$ 1-1. However, if $a$ is a constant, $b$ can depend on $b_{1}, c_{1}, 1, \ldots, \ldots, 1$, and $\bar{F}_{1}$ (.) will still satisfy the requirement that $\bar{F}_{1}^{-1}(0)$ be a constant.

This discussion also illustrates one of the major weaknesses of linear programming under uncertainty, i.e., that for many distributions the constiaints of problem 13 are inconsistent everywhere. Suppose again that $b_{1}, c_{1}, 1=1$, . . . , $n$, are jointly normally distributed. Then $\bar{F}_{1}($. ) is the distribution function of a normal random variable, hence $\bar{F}_{1}^{-1}(0)-\infty$. Consequently there do not exist decision rules $X_{1}, 1, \ldots, \ldots$, that satisiy the constraints of problem 13. no matter what sample points are observed.

Further results are obtained on the linear-programming-under-uncertainty problem by restricting the above model still further. To do this it is assumed that $c_{1,1}=1, \ldots, n$ is a vector of constants rather than random variables and then problem 13 is shown to reduce to an ordinary linear programming problem. When $c_{1}, 1=1, \ldots, n$, are constants, the $E$ operator means that the expected value of $\leq_{i=1}^{n} c_{1} \mathrm{X}_{1}$ is taken, using the joint distribution of $b_{1}, \ldots, b_{n}$. Therefore if problem 13 is solved for a given sample point ( $b_{1}, \ldots, b_{n}$ ), the optimal de cision rules $\mathbb{V}_{1}, 1,1, \ldots, n$ will be obtained for this particular sample point. But $\bar{a}_{1} 1,1^{-1}, \ldots, n$, and by assumption $\bar{F}_{1}^{-1}(0)$ is a vector of constants; hence the constraint set of problem 13 does not depend on the sample point ( $b_{1}, \ldots, b_{n}$ ). Moreover since the $c_{1}$ art constant, problem 13 is now completely deterministic. Thus $\ell_{j}^{*}, 11, \ldots, n$ will be optimal for all sample points and these optimal decision rules can be found by solving the ordinary line ar programming problem

## maximize

$$
\sum_{i=1}^{n} c_{1}^{\top} Y_{1}
$$

subject to

$$
\begin{aligned}
& A_{1} X_{1} \leq \bar{F}_{1}^{-1}(0) \text {. } \\
& A_{21} X_{1}+A_{22} X_{2} \leq \bar{F}_{2}^{-1}(0) \text {. } \\
& A_{n 1} X_{1}+A_{n 2} X_{2}+\ldots+A_{n n} X_{n} \leq \bar{F}_{n}^{-1}(0) . \\
& \Lambda_{1}=0.1-1 \ldots n
\end{aligned}
$$

Another example of a situation in which the solutions spare of problem 13 does not depend on the observed values of $b_{1}, c_{1}, 1, \ldots, n$ can be obtained by supposing that $b_{1}, 1, \ldots, n$ is stochastically independent of $b_{1}, c_{1}, 11$, $\ldots, 1^{-1}$ and that $\bar{\alpha}_{1}, 1,1, \ldots, n$ is a constant vector. Then $\bar{F}_{1}^{\prime}\left(1-\bar{\alpha}_{1}\right)$ is a constant vector, since it represents the $1-\bar{a}_{1}$ th fractile point of the marginal distribution of $b_{1}$. But by assumption e this marginal distribution is known, so $F_{1}^{-1}(1-\bar{\alpha})$ can be computed before taking any observations. Thus the solutions space of problem 13 does not depend on the observed random variables.

Even when $b_{1}$ is dependent on the $b_{1}, c_{1}, 1,1, \ldots, 1,1, \bar{F}_{1}^{-1}\left(1-\bar{a}_{1}\right)$ is a vector of constants because it represents the $m_{1} \times 1$ vector of $1-\bar{a}_{1}$ percentile points of the marginal distribution of $b_{1}$, and $\bar{\alpha}_{1} \quad \alpha_{1}$ is constant.

It is interesting to note that if $c_{1}, 1,1, \ldots, n$ is a constant vector, if $b_{1}, 1, \ldots, \ldots$ is independent of the joint distribution of $b_{1}, 1,1, \ldots, 1$, and if $\bar{\alpha}_{1}, 11, \ldots, n$ is constant, then, by an argument analogous to that used above, it can be concluded that problem 13 becomes an ordinary linear programming problem.

## 11. EXAMPLES

To illustrate the results obtained two examples are presented. In the first example, problem 20, the constraints are consistent for all sample points and all values of $\bar{\alpha}_{1}$ in $.0,1 \%$. In the second case, problem 23 , it can be seen how a well-posed problem can have sample points for which the constraints are inconsistent.

In both these examples the various $\bar{\alpha}_{1}$ will not be explicitly written as functions of $b_{1}, r_{1}, 1=1, \ldots, 1-1$. Thus the problem will be solvedtreating $\overline{a_{1}}$ as an arbitrary function of its conditional random variables, subject only to the restriction that $0 \times \bar{\alpha}_{1 p} \cdot 1$ for all 1 and $\ell$. In addition it will be assumed that our problems are given in the form of problem 13 rather than problem 10, so that the constraints involve $\bar{p}$ and $\bar{\alpha}_{1}$ and not $P$ and $\alpha_{1}$.

## Example 1

As our first example the following two-stage problem in which $X_{1} \equiv x_{1}$, and $X_{2}=x_{2}$ are 1,1 vectors (i.e., $x_{1}, x_{2}$ are single variables) will be solved:

## maximize

$$
E_{1} c_{1} x_{1} \cdot c_{2} x_{2}
$$

subject to

$$
\begin{gather*}
\bar{p}\left(n_{1}-x_{1}\right) \quad-\vec{a}_{1} . \\
\bar{p}\left(b_{2} \leq x_{1}+x_{2}\right)=\bar{a}_{2} .  \tag{20}\\
x_{1} x_{2}=0 .
\end{gather*}
$$

where $c_{1}$ is uniformly distributed over $.15,25$, and $c_{2}$ is unformly distributed over $c_{1}-6, c_{1}+4$. Thus the mean of $c_{2}$, given $c_{1}$, is $c_{1}-1$; but $c_{2}$, like $c_{1}$, has a range of ten units. $b_{1}$ is uniformly distributed over $100,200^{\circ}$ and $b_{2}$ is uniformly distributed over $.75,2 b_{1}-75 \%$. Thus the mean of $b_{2}$, giver $b_{1}$, equals $b_{1}$, but the range of $b_{2}$ is not the same as the range of $b_{1}$.

In addition it is assumed that $c_{1}$ and $s_{2}$ are independent of $b_{1}$ and $b_{2}$. The solution of problem 20 is begun by computing $\bar{F}_{1}^{-1}\left(1-\bar{o}_{1}\right)$ and $\bar{F}_{2}^{-1}\left(1-\bar{o}_{2}\right)$.

Let $y$ be a random variable that is uniformiy distributed over $a, b$. Let $\bar{y}$ be the $1-\bar{\alpha}$ th fractile point of $y$. Then ${ }^{v}(1 / b-a) d y 1-\bar{a}$. Hence $\hat{y}-(b-a)(1-\bar{a})+a$. Thus the equivalent form ${ }^{\text {a }}$ of problem 20 is:
maximize

$$
\left[\left(c_{1} x_{1} \cdot \varepsilon_{2} x_{2}\right)\right.
$$

subject to

$$
\begin{align*}
& x_{1}=100\left(1-\bar{a}_{1}\right) \cdot 100  \tag{21}\\
& x_{1}+x_{2} \leq\left(2 h_{1}-(\bar{x}) \mid\left(1-\bar{a}_{2}\right) \cdot \because .\right. \\
& x_{1}+x_{2}=0 .
\end{align*}
$$

Or, equivalently, maximize

$$
E\left(r_{1} x_{1} \cdot r_{2} x_{2}^{\prime}\right.
$$

subject to

$$
\begin{aligned}
& x_{1} \leq 200-100 \bar{a}_{1} \cdot \\
& x_{1}+x_{2} \leq 2\left(1-\vec{a}_{2} 1 h_{1} \cdot 1 \bar{x}\right) \bar{a}_{2}-\therefore . \\
& x_{1}+x_{2}=0
\end{aligned}
$$

Let $\left(b_{1}, b_{2}, c_{1}, c_{2}\right)$ be any sample point and let $x_{1}$ be given. Then to determine $x_{2}^{*}$ the following must be solved:
maximize

$$
c_{2} x_{2}
$$

subject to

$$
\begin{aligned}
& x_{2} \leq 2\left(1-\bar{a}_{2}\right) b_{1}+150 \bar{a}_{2}-5-x_{1} . \\
& x_{2} \geq 0 .
\end{aligned}
$$

But $: 0$ always, so that $x \geqslant 2\left(1-\bar{\alpha}_{2}\right) b_{1}+150 \bar{\alpha}_{2}-75-x_{1}$. To find the optimal $x$, the following must now be solved:

## maxinize

$$
f\left(x_{1} x_{1} \cdot i_{2} x_{2}^{*}\right)
$$

subject to

$$
\begin{align*}
& \left.x_{1}-100\left(1-\bar{a}_{1}\right)+100\right)  \tag{22}\\
& s_{1}=0
\end{align*}
$$

In problem 22 I means that the expected value is taken with respect to the joint distribution of $h_{1}, b_{2}, i_{1}, c_{2}$. But $x_{1}$ is independent of these random variables by assumption $f^{\dagger}$ and $i_{1}$ and $i_{\text {: }}$ are independent of $b_{1}$ and $b_{\text {. }}$. Hence $\left.\left.t\left(i_{1} x_{1}\right) \quad E\left(i_{1}\right)_{x_{1}} \quad E_{1}, l_{i}\right) x_{1} \quad E_{1}, l_{1}, l_{1}\right) x_{1} \quad E_{1}\left(i_{1}\right) x_{1} \quad 20 x_{1}$, where the subscript of the $\{$ denotes the distribution used in computing the expectation and $\sigma_{:} r_{1}$ means that the conditional distribution of $\varepsilon_{2}$, given $c_{1}$, is used.

Also
where the expectation is computed using the joint distribution of $b_{1}, b_{i}, c_{1}, c_{i}$. However, in order to find $x$ it is necessary to know explacitly the expression
 $1,(1,-1) \quad 19$, to obtain $x_{1}$ the following must be solved:
Maximize

$$
\therefore 0 x_{1}+\left|-10 x_{1}+h\right|
$$

subject to

$$
\begin{aligned}
& 1_{1}-10011-\bar{a}_{1}, 100 . \\
& 1_{1}-11 .
\end{aligned}
$$

where $h 21\left(, h_{1}\right)-21\left(1, \bar{a}_{( } h_{1}\right)+1501\left(i_{i}\right)-751\left(c_{2}\right)$.
Therefore

$$
\dot{i}_{1}=||M||-\bar{a},|\cdot|(M \mid
$$

and

$$
\therefore=1-\bar{a}_{1} r_{1} \cdot 13 \bar{a}_{0} \cdot 1 \therefore-\left||x| \bar{a}_{1}\right.
$$

Smee $\mathrm{F}^{\prime}(1-\bar{a})$ was linear in $b_{1}, \therefore$. $\operatorname{s}$ seen to be also lmear in $h_{1}$, whoch agrees whth Corollary 2. Also of $\bar{\sigma}_{1}, \overrightarrow{0}$. $1,:^{*}$ is poceewise constant (in fact it is a constant), as was predicted by the discussion of the linear-programming-underuncertamty case.

[^4]
## Example 2

For our second example the following problem is considered:
maximize

$$
E^{\prime} s_{1} x_{1}+f_{2} x_{2}^{\prime}
$$

subject to

$$
\begin{gather*}
\left.\vec{P}^{\prime} b_{1}=a_{11} x_{1}\right) \geq \bar{a}_{1} . \\
\bar{P}_{\left(b_{2}\right.}=a_{21} x_{1}+a_{22} x_{2} \geq \bar{a}_{2},  \tag{23}\\
x_{1}, x_{2} \geq 0,
\end{gather*}
$$

where $f_{1}$ and $c_{\text {: are }}$ defined as in the previous example; $b_{1}$ and $b_{\text {: }}$ are jointly normally distributed with means $m_{1}$ and $m_{2}$ and variances $\delta_{i}^{2}$ and $\delta_{i}$, respectively, and correlation coefficient $n>0 ; a_{11}$ and $a_{2:}$ are positive constants and $a_{21}$ is a negative constant; and $c_{1}$ and $c_{2}$ are independent of $b_{1}$ and $b_{2}$.

From our definition of $b_{1}$ and $b_{\text {. It can be seen that their joint frequency }}$ function is given by

$$
\left\{n_{1} n_{2}^{\prime}=\frac{1}{2 \pi \delta_{1} \delta_{2} T-n^{2}} \times \frac{1}{21-n_{1}}\left[\left(\frac{n_{1}-m_{1}}{\delta_{i}}\right)^{2}-\therefore\binom{n_{1}-m_{1}}{\delta_{1}}\left(\frac{n_{2}-m_{2}}{\delta_{2}}\right) \cdot\binom{n_{2}-m_{2}}{\delta_{2}^{2}}^{2}\right]\right\}
$$

and $f\left(b_{2}, b_{1}\right)$, the conditional frequency function of $b_{:}$given $b_{1}$, satisfies

$$
\left\lvert\,\left(n_{2} n_{1} \left\lvert\,=\frac{\left|n_{1}, n_{2}\right|}{f_{1}\left(n_{1}\right)} .\right.\right.\right.
$$

where $f_{1}\left(b_{1}\right)$ is the marganal frequency function of $b_{1}$. Therefore

$$
\| n_{2} n_{1}^{\prime}=\frac{1}{\delta_{2}, \frac{\square}{2 \pi}, \overline{1-\theta^{2}}} e q\left\{-\frac{1}{2 \delta_{2}^{\prime}\left(1-a^{2}\right)}\left[n_{2}-m_{2}-\rho \frac{\delta_{2}}{\delta_{1}}\left(n_{1}-m_{1}\right)\right]:\right\}
$$

so that $h_{\text {, , given }} b_{1}$, is normally distribuied with mean $m_{:}+\rho\left(\delta_{-} / \delta_{1}\right)\left(b_{1}-m_{1}\right)$ and variance or (1-p).
 $\phi($. ) is the dist ribution function of a normal random variable with mean 0 and variance 1 . Let $k_{1}, k$ be defined by

$$
\begin{aligned}
& k_{1}=\| \frac{s_{2}}{s_{1}} n
\end{aligned}
$$

Then a problem equivalent to problem 23 is
maximize
‘', ', • ' '
sullect to

$$
\begin{align*}
& \left.u_{11} 1_{1}-\delta_{1} \Phi^{-1} 1-\bar{a}_{1}\right) \cdot m_{1} \\
& u_{1} 1_{1} \cdot u_{2}:-k_{1} n_{1} \cdot k_{0} \tag{24}
\end{align*}
$$

$$
1 \quad 1: 0
$$

Let ( $b_{1}, b_{1}, c_{1}, c_{2}$ ) be any sample point in the space of random variables $b_{1}, b_{2}, c_{1}, c_{2}$. Let $x_{1}$ be a given first-period decision rule. Then to find $x$ * the following must be solved:
maximize
subject to

$$
\therefore x_{2}
$$

$$
\begin{aligned}
a_{22} x_{2} & -k_{1} b_{1}+k_{2}-a_{21} x \\
x_{2} & =0
\end{aligned}
$$

or, equivalently,
maximize

$$
r_{2} x_{2}
$$

subject to

$$
\begin{aligned}
& x_{2} \leq \frac{k_{1}}{a_{22}} b_{1}+\frac{k_{2}}{a_{22}}-\frac{a_{21}}{a_{22}} x_{1} \\
& x_{2}=0
\end{aligned}
$$

as $a_{2:} 0$ by hypothesis. Since $c_{:} 0$, also by assumption, the following is obtained:

$$
\begin{aligned}
x_{2}^{*}= & \frac{k_{1}}{a_{22}} n_{1} \cdot \frac{k_{2}}{a_{22}}-\frac{a_{21}}{a_{22}} x_{1} \text { if } k_{1} b_{1}+k_{2}-a_{21} x_{1} \leq 0 . \\
& \text { nin feasible siluturnif } k_{1} b_{1}+k_{2}-a_{21} x_{1}+0 .
\end{aligned}
$$

Thus region $\bar{D}_{2}$ discussed above is seen to be defined by $k_{1} b_{1}+k_{2}-a_{21} x_{1} \cdot 0$. Moreover since $a_{11} \quad 0$ it can be seen from problem 24 that $\bar{D}_{1}$ is the region for which $\delta_{1} \Phi^{-1}\left(1-\bar{\alpha}_{1}\right)+m_{1} \cdot 0$, so that the union of $D_{1} D_{2}$ gives the set $\bar{\Delta}_{2}$.

Using the rule $x_{:}^{*} 0$ in $\bar{\Delta}_{2}$, the optimal $x_{1}$ is found by solving the following:
maximize

$$
\left.E_{1} \mid f_{1} x_{1}\right) \cdot E_{2}\left(\varepsilon_{2} x_{2}^{*}\right.
$$

subject to

$$
\begin{align*}
a_{11} x_{1} & -\delta_{1} \Phi^{-1}\left(1-\bar{a}_{1}\right) \cdot m_{1}  \tag{25}\\
x_{1} & -0 .
\end{align*}
$$

If $\delta_{1} \Phi^{-1}\left(1-\bar{\alpha}_{1}\right)+m_{1}>0$, then a feasible $x_{1}$ exists and

But $D_{:}$is the region where $k_{1} b_{1} \cdot k_{1}-a_{11} x_{1} \cdot 0$, that is, $b_{1}>\left(a_{21} / k_{1}\right) x_{1}-\left(k_{i} / k_{1}\right)$ as $k_{1} \cdot 0$ and $k_{1}$ is a constant because $\bar{o}_{1}$ is a constant (see Section 6). If it is now assumed that $k$ is also a constant, i.e., that $\bar{a}_{i}$ is a constant, the following is obtained:

$$
\begin{aligned}
E_{2}\left(c_{2} x_{2}^{*}\right) & =19 \int_{\frac{a_{21}}{k_{1}} x_{1}-\frac{k_{2}}{k_{1}}\left(\frac{k_{1}}{a_{22}} b_{1}+\frac{k_{2}}{a_{22}}-\frac{a_{21}}{a_{22}} x_{1}\right) \frac{1}{\delta_{1} \sqrt{2 \pi}} e^{-\frac{1}{2}\left(\frac{b_{1}-m_{1}}{\delta_{1}}\right)^{2}} d b_{1}} \\
= & 19 \int_{21}^{+\infty} x_{1}-k_{2}-m_{1} k_{1} \\
k_{1} \delta_{1} & \left.\frac{k_{1} \delta_{1}}{a_{22}} z+\frac{k_{1}}{a_{22}} m_{1}+\frac{k_{2}}{a_{22}}-\frac{a_{21}}{a_{22}} x_{1}\right) e^{\frac{-z^{2}}{2}} d z
\end{aligned}
$$

Finally, using the fact that $x_{1}$ is to be independent of $b_{1}$ and $c_{1}$, problem 25 can be written:
maximize

$$
\begin{aligned}
20 x_{1}-\frac{19 a_{21} x_{1}}{a_{22}}\left[1-\Phi\left(\frac{a_{21} x_{1}-k_{2}-m_{1} k_{1}}{k_{1} \delta_{1}}\right)\right] & +19\left(\frac{k_{1} m_{1}}{a_{22}}+\frac{k_{2}}{a_{22}}\right)\left[1-\Phi\left(\frac{a_{21} x_{1}-k_{1}-m_{1} k_{1}}{k_{1} \delta_{1}}\right)\right] \\
& +\frac{19 k_{1} \delta_{1}}{a_{22}} \int_{\frac{a_{21} x_{1}-k_{2}-m_{1} k_{1}}{k_{1} \delta_{1}}}
\end{aligned}
$$

subject to

$$
\begin{align*}
a_{11} x_{1} & \leq \delta_{1} \Phi^{-1}\left(1-\bar{a}_{1}\right)+m_{1} \\
x_{1} & \geq 0 \tag{26}
\end{align*}
$$

where $\varphi($.$) is the frequency function of a normal random variable with mean$ 0 and variance 1. Solving this problem (which can be done easily by applying the Lagrange-multiplier technique to problem 26), the following is obtainec:

$$
x_{1}^{*}=\frac{\delta_{1} \Phi^{-1}\left(1-\bar{a}_{1}\right)+m_{1}}{a_{11}}
$$

Thus the optimal decision rules are

$$
\begin{aligned}
& x_{1}=\left\{\begin{array}{l}
\frac{\delta_{1} \Phi^{-1}\left(1-\bar{a}_{1}\right)+m_{1}}{a_{11}}, \text { if } \delta_{1} \Phi^{-1}\left(1-\bar{a}_{1}\right)+m_{1} \geq 0 \\
0, \text { i.e., no feasible } x_{1} \text { otherwise. }
\end{array}\right. \\
& x_{2}=\left\{\begin{array}{l}
\frac{k_{1}}{a_{22}} b_{1}+\frac{k_{2}}{a_{22}}-\frac{a_{21}}{a_{22}} x_{1}, \text { if } b_{1} \geq \frac{a_{21}}{k_{1}} x_{1}-\frac{k_{2}}{k_{1}} \text { and } x_{1} \text { is feasible } \\
0, \text { i.e., no feasible } x_{2} \text { otherwise. }
\end{array}\right.
\end{aligned}
$$

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[^0]:    ${ }^{\dagger}$ See also March and Simon, ${ }^{13}$ and the extensions and applications to problems of budgetary management given in Cooper and Savvas, ${ }^{14}$ Stedry, ${ }^{15}$ and Stedry and Charnes. ${ }^{16}$

[^1]:    ${ }^{\dagger}$ Note the similarity between Lemma 1 and Ben-Israel's theorem. ${ }^{17}$

[^2]:    ${ }^{\dagger}$ See Ben-Israel and Charnes ${ }^{30}$ and other references on Penrose-Moore inverses given in Section: 2.

[^3]:    ${ }^{1}$ See Charnes, Cooper and Thompson ${ }^{21}$ and G. B. Dantzig et al. ${ }^{20,3134}$

[^4]:    1 In fact. $f$ rom the interpretation of the gencral problem, it is clear that to the wecor of first-period decosions, will alwavs havel( $\left.\rangle_{1}\right) \quad X_{f}$. This is another direct
    

