Optimal Derivatives Design under Dynamic Risk Measures

Pauline Barrieu and Nicole El Karoui

Abstract. We develop a methodology to optimally design a financial issue to hedge non-tradable risk on financial markets. Economic agents assess their risk using monetary risk measure. The inf-convolution of convex risk measures is the key transformation in solving this optimization problem. When agents' risk measures only differ from a risk aversion coefficient, the optimal risk transfer is amazingly equal to a proportion of the initial risk.
For dynamic risk measures defined through their local specifications using BSDE's, their inf-convolution is equivalent to that of their associated drivers. In this case, it is also possible to characterize the optimal risk transfer.

Introduction

In recent years, a new type of financial instruments (among them, the so-called "insurance derivatives") has appeared on financial markets. Even though they have all the features of financial contracts, they are very different from the classical structures. Their underlying risk is indeed related to a non-financial risk (natural catastrophe, weather event...), which may somehow be connected to more traditional financial risks. Their high level of illiquidity, deriving partly from the fact that the underlying asset is not traded on financial markets, makes them difficult to evaluate and to use. Several authors (see, for instance, D. Becherer [**Be1**], M. Davis [**Da2**] or M. Musiela and T. Zariphopoulou [**MuZ**]) have been interested in these new products, especially in their pricing. However, neither their impact on "classical" investments nor their optimal design are mentioned in the literature.
On the other hand, this accrued complexity of financial products has naturally lead to an increasing interest in quantitative methods of assessing the risk related to a given financial position.
This paper focuses on these problems in a framework where economic agents may take positions on two types of risk: a purely financial risk (or market risk) and a (non-financial) non-tradable risk. The optimal structure of a contract depending on the non-tradable risk and its price are determined.
Since the structure represents a new diversification instrument for any investor,

1991 *Mathematics Subject Classification.* Primary 60Gxx, 91B28, 91B90; Secondary 46N10, 90C39.

Key words and phrases. Static and dynamic risk measure, non-tradable risk, optimal design, illiquidity, inf-convolution, BSDE.

optimal wealth allocation becomes a more complex question and the question of an efficient quantitative risk assessment becomes crucial. Different authors have recently been interested in defining and constructing a coherent, in some sense, risk measure (see, for instance, Artzner *et al.* [**ADEH**] or Föllmer and Schied [**FS1**]), using a systematic axiomatic approach. The framework developed by these authors will be that of this study.

This paper is structured as follows: in the first section, after having recalled some basic properties of convex risk measures, we generate new risk measures as the inf-convolution of convex risk measures. Then, in the second section, we solve the problem of an optimal non-tradable risk transfer. In the third section, we introduce dynamic risk measures defined through their local specifications with the help of Backward Stochastic Differential Equations in order to propose a method to characterize completely the optimal structure.

1. Risk transfer and inf-convolution of risk measures

In this section, we first present a general class of risk measures introduced by Föllmer and Schied ([**FS1**] and [**FS2**]). Then, we generate new risk measures as the inf-convolution of different risk measures. We finally apply these results to the optimal design of a transaction based on a non-tradable risk. In particular, we obtain a necessary and sufficient condition to the existence of an optimal risk transfer.

1.1. Convex risk measures.

1.1.1. *Definition and basic properties .* We first recall the definition and some key properties of the convex risk measures introduced by Föllmer and Schied ([**FS1**] and [**FS2**]). In the following, $\mathcal{X}$ denotes a linear space of bounded functions including constant functions, defined on the measurable space $(\Omega, \mathcal{F})$.

Definition 1.1. The functional $\rho : \mathcal{X} \to \mathbb{R}$ is a *convex risk measure in the sense of Föllmer and Schied* if, for any X and Y in $\mathcal{X}$, it satisfies the following properties:
a) Convexity: $\forall \lambda \in [0,1] \quad \rho(\lambda X + (1-\lambda) Y) \leq \lambda \rho(X) + (1-\lambda)\rho(Y)$;
b) Monotonicity: $X \leq Y \Rightarrow \rho(X) \geq \rho(Y)$;
c) Translation invariance: $\forall m \in \mathbb{R} \quad \rho(X+m) = \rho(X) - m$.

Intuitively, $\rho(\Psi)$ may be interpreted as the amount the agent has to hold to completely cancel the risk associated with her risky position Ψ

$$\rho(\Psi + \rho(\Psi)) = 0$$

The risk measure ρ induces a particular set of positions: the *acceptance set*, $\mathcal{A}_\rho$, defined as the set of all acceptable positions as they carry no positive risk:

$$\mathcal{A}_\rho = \{\Psi \in \mathcal{X}, \quad \rho(\Psi) \leq 0\} \tag{1.1}$$

We now present a key result obtained by Föllmer and Schied [**FS2**] (Theorem 4.12) in the following Theorem.

Theorem 1.2. *Let $\mathcal{M}_{1,f}$ be the set of all additive measures on $(\Omega, \mathcal{F})$. Another formulation of the convex risk measure is given in terms of a penalty function, $\alpha(\mathbb{Q})$*

taking values in $\mathbb{R} \cup \{+\infty\}$*:*

$$\forall \Psi \in \mathcal{X} \quad \rho(\Psi) = \sup_{\mathbb{Q} \in \mathcal{M}_{1,f}} \{\mathbb{E}_{\mathbb{Q}}(-\Psi) - \alpha(\mathbb{Q})\} \tag{1.2}$$

By duality between $\mathcal{M}_{1,f}$ *and* $\mathcal{X}$*,*

$$\forall \mathbb{Q} \in \mathcal{M}_{1,f} \quad \alpha(\mathbb{Q}) = \sup_{\Psi \in \mathcal{X}} \{\mathbb{E}_{\mathbb{Q}}(-\Psi) - \rho(\Psi)\} \qquad (\geq \rho(0)) \tag{1.3}$$

or equivalently

$$\forall \mathbb{Q} \in \mathcal{M}_{1,f} \quad \alpha(\mathbb{Q}) = \sup_{\Psi \in \mathcal{A}_\rho} \mathbb{E}_{\mathbb{Q}}(-\Psi) \tag{1.4}$$

In the following, we are especially interested in risk measures related to probability measures. In general, the assumption of decreasing continuity from below is made and suffices to imply that the dual formulation of risk measure (Equation (1.2)) is satisfied for $\mathbb{Q} \in \mathcal{M}_1$, where $\mathcal{M}_1$ is the set of all probability measures on the considered space. In this case, the equations previously obtained concerning the penalty function (Equations 1.4 and 1.3) still hold replacing $\mathcal{M}_{1,f}$ by $\mathcal{M}_1$. Moreover, Föllmer and Schied have proven in [**FS2**] (Theorem 4.12) that there always exists a measure of $\mathcal{M}_{1,f}$ such that the supremum in Equation (1.2) is reached. When working with $\mathcal{M}_1$, the supremum is reached under some conditions presented in Theorem 4.22 of [**FS2**]. These results will be quite important in the following as they ensure the existence of an "optimal" measure (or "optimal" probability measure under some assumptions).

Example 1.3. A classical example of convex risk measure is the *entropic risk measure*

$$\forall \Psi \in \mathcal{X} \qquad e_\gamma(\Psi) = \sup_{\mathbb{Q} \in \mathcal{M}_1} \left(\mathbb{E}_{\mathbb{Q}}(-\Psi) - \gamma h(\mathbb{Q}/\mathbb{P})\right) = \gamma \ln \mathbb{E}_{\mathbb{P}}\left(\exp\left(-\frac{1}{\gamma}\Psi\right)\right) \tag{1.5}$$

where γ is the risk tolerance coefficient and $h(\mathbb{Q}/\mathbb{P})$ is the relative entropy[1] of $\mathbb{Q}$ with respect to the prior probability $\mathbb{P}$.

1.1.2. *Risk measure generated by a convex set and coherent risk measure* . We now introduce some particular convex risk measures generated by a convex set as follows

Definition 1.4. Let $\mathcal{H}$ be a non-empty convex subset of $\mathcal{X}$ such that

$$\inf\{m \in \mathbb{R}, \text{ such that } \exists \xi \in \mathcal{H},\ m \geq \xi\} > -\infty$$

Then the functional $v^{\mathcal{H}}$ defined as

$$v^{\mathcal{H}}(\Psi) = \inf\{m \in \mathbb{R}; \text{such that } \exists \xi \in \mathcal{H},\ m + \Psi \geq \xi\}$$

is a convex risk measure. The associated penalty function α is given by:

$$\forall \mathbb{Q} \in \mathcal{M}_{1,f} \quad \alpha^{\mathcal{H}}(\mathbb{Q}) = \sup_{H \in \mathcal{H}} \mathbb{E}_{\mathbb{Q}}(-H)$$

[1]When finite (i.e. if $\mathbb{Q} \ll \mathbb{P}$), the relative entropy is defined by

$$h(\mathbb{Q}/\mathbb{P}) = \mathbb{E}_{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}} \ln \frac{d\mathbb{Q}}{d\mathbb{P}}\right)$$

and the acceptance set is defined by

$$\mathcal{A}_{v^{\mathcal{H}}} = \{\Psi \in \mathcal{X}, \exists \xi \in \mathcal{H}, \quad m + \Psi \geq \xi\}$$

When $\mathcal{H}$ is a cone, the penalty function associated with $v^{\mathcal{H}}$ can only take two possible values

$$\alpha(\mathbb{Q}) = \ 0 \quad \text{ if } \mathbb{Q} \in \mathcal{Q}_{\mathcal{H}} \qquad \text{and} \qquad +\infty \qquad \text{otherwise}$$

where $\mathcal{Q}_{\mathcal{H}}$ is the set of all additive measures such that $\forall \xi \in \mathcal{H}$, $\mathbb{E}_{\mathbb{Q}}(\xi) \geq 0$. The risk measure $v^{\mathcal{H}}$ is then coherent [2] in the sense of Artzner *et al.* ([**ADEH**]) and its dual formulation is simply given by

$$\forall \Psi \in \mathcal{X} \qquad v^{\mathcal{H}}(\Psi) = \sup_{\mathbb{Q} \in \mathcal{Q}_{\mathcal{H}}} \mathbb{E}_{\mathbb{Q}}(-\Psi)$$

1.1.3. *Inf-convolution of risk measures.* Rockafellar ([**Ro**]) has given some stability properties of the inf-convolution of convex functions. The following Theorem extends these results to the inf-convolution of convex functionals:

THEOREM 1.5. *Let ρ_1 and ρ_2 be two convex risk measures with respective penalty functions α_1 and α_2. Let $\rho_{1,2}$ be the inf-convolution of ρ_1 and ρ_2 defined as*

$$\Psi \to \rho_{1,2}(\Psi) \equiv \rho_1 \Box \rho_2(\Psi) = \inf_{H \in \mathcal{X}} \{\rho_1(\Psi - H) + \rho_2(H)\}$$

and assume that $\rho_{1,2}(0) > -\infty$. Then $\rho_{1,2}$ is a convex risk measure, which is finite for all $\Psi \in \mathcal{X}$. The associated penalty function is given by

$$\forall \mathbb{Q} \in \mathcal{M}_{1,f} \quad \alpha_{1,2}(\mathbb{Q}) = \alpha_1(\mathbb{Q}) + \alpha_2(\mathbb{Q})$$

Note that the convex risk measure $\rho_{1,2}$ may also be defined as the value functional of the program

$$\rho_{1,2}(\Psi) = \inf\{\rho_1(\Psi - H), H \in \mathcal{A}_{\rho_2}\}$$

PROOF. Please refer to Barrieu-El Karoui [**BEK2**]. □

Moreover, using Subsection 1.1.2, the following result is a direct consequence of Theorem 1.5:

COROLLARY 1.6. *Let $\mathcal{H}$ be a cone of $\mathcal{X}$ and ρ be a convex risk measure with penalty function α such that*

$$\inf\{\rho(-H), H \in \mathcal{H}\} > -\infty$$

The inf-convolution of ρ and $\nu^{\mathcal{H}}$, $\rho^{\mathcal{H}} \equiv \rho \Box \nu^{\mathcal{H}}$, also defined as

$$\rho^{\mathcal{H}}(\Psi) \equiv \inf\{\rho(\Psi - H), H \in \mathcal{H}\} = \sup_{\mathbb{Q} \in \mathcal{Q}^{\mathcal{H}}} \{\mathbb{E}_{\mathbb{Q}}(-\Psi) - \alpha(\mathbb{Q})\}$$

is a convex risk measure with penalty function α on $\mathcal{Q}^{\mathcal{H}}$ and $+\infty$ otherwise.

[2] It satisfies indeed the positive homogeneity property $\forall \Psi \in \mathcal{X}, \forall \lambda \geq 0, \quad v^{\mathcal{H}}(\lambda \Psi) = \lambda v^{\mathcal{H}}(\Psi)$. (For more details, please refer to Föllmer and Schied [**FS2**], Remark 4.13). This property simply translates the fact that the size of the transaction or exposure does not have any particular impact.

REMARK 1.7. This property may be interpreted in terms of hedging strategies. The inf-convolution $\rho^{\mathcal{H}}$ is simply the residual risk measure after having optimally chosen the hedging strategy for Ψ with elements of $\mathcal{H}$. For instance, one may see $\mathcal{H}$ as the cone of all gain processes related to a given financial market. In the following, we will refer to $\rho^{\mathcal{H}}$ (sometimes denoted ρ^m) as the *modified market risk measure*, when appropriate.

1.1.4. *Market modified risk measure in the entropic framework.*

INCOMPLETE MARKET. In this entropic framework (corresponding to an exponential utility function), this problem has been widely studied as an hedging problem when $\mathcal{H}$ is the space of bounded gain processes written on locally bounded semi-martingale price processes S, at a future time T,

$$\mathcal{V}_T = \left\{ \xi_T = \int_0^T \langle \varphi_t, dS_t \rangle \quad ; \quad \varphi_t \in K_t \right\}$$

associated with financial strategies, φ, satisfying some geometric constraints described by K_t. Several authors (for instance, Frittelli [**Fr1**], Delbaen et al. [**DGRSS**], Becherer [**Be1**]) have solved the dual problem (1.6 to obtain the existence of an optimal hedging strategy. El Karoui-Rouge [**EKR**] have established the same type of results and characterized the solution using a different approach based on BSDE's techniques presented in the last section of this paper. Musiela-Zariphopoulou [**MuZ**] have also been interested in this problem and completely solved a particular example using PDE's.

PARTIAL INFORMATION. Some economic agents, unable to observe the non-tradable risk, base their financial strategies only on the information contained in the financial asset prices S. In particular, they can observe the filtration $\mathcal{F}_T^S$ generated by $\sigma(S_u; 0 \le u \le T)$. A measurability constraint is then added to the geometric constraint on the financial strategies. The space of gain processes is denoted by $\mathcal{V}_T^S$:

$$\mathcal{V}_T^S = \left\{ \xi_T = \int_0^T \langle \varphi_t, dS_t \rangle \quad ; \quad \varphi_t \in \mathcal{F}_t^S \quad \text{and} \quad \varphi_t \in K_t \right\}$$

In the entropic case, it is possible to solve the problem as previously, given that

$$\begin{aligned} e^{m,S}(\Psi) &= \inf\{\gamma \ln \mathbb{E}_{\mathbb{P}} \left(\exp(-\tfrac{1}{\gamma}(\Psi - \xi)) \right); \xi \in \mathcal{V}_T^S\} \\ &= \inf\{\gamma \ln \mathbb{E}_{\mathbb{P}} \left(\exp(-\tfrac{1}{\gamma}(\Psi^S - \xi)) \right); \xi \in \mathcal{V}_T^S\} \end{aligned}$$

where

$$\Psi^S \equiv -\gamma \ln \mathbb{E}_{\mathbb{P}} \left[\exp\left(-\frac{1}{\gamma} \Psi \right) / \Im_T^S \right]$$

is the opposite of the conditional entropic risk measure of Ψ given $\mathcal{F}_T^S$, assessing the cost of partial information.

In the so-called "filtering framework", the financial assets' prices S are associated with a risk premium depending on the non-tradable risk. Different authors (see for instance Lakner [**La1**] [**La2**], Lefèvre [**Le**], Pham-Quenez [**PhQ**]) have shown however that there exists a "risk-neutral" probability measure $\widehat{\mathbb{Q}}_T$ such that the $\Im^S$-market is complete. The set $\mathcal{Q}^{\mathcal{H}}$ is then the set of all probability measures on the considered measurable space $(\Omega, \Im)$ such that their restriction to $\Im^S$ is the risk-neutral probability measure $\widehat{\mathbb{Q}}_T$. In particular, the market modified risk measure may be written as

$$e^{m,S}(\gamma, \Psi) = \gamma \mathbb{E}_{\widehat{\mathbb{Q}}_T}\left(\ln \mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{1}{\gamma}\Psi\right)/\Im_T^S\right]\right)$$

For more details, please refer to Barrieu-El Karoui [**BEK4**].

2. Optimal design problem

In the following, we focus on the question of an optimal transaction between two economic agents. These agents, respectively denoted A and B, are evolving in an uncertain universe modeled by a probability space $(\Omega, \Im, \mathbb{P})$. At a fixed future date T, agent A is exposed towards a non-tradable risk Θ for an amount $X \equiv X(\Theta, \omega)$ in the scenario ω. A wants to issue a financial product $F \equiv F(\Theta, \omega)$ and sell it to agent B for a forward price at time T denoted by π as to reduce her exposure. We assume that X and F belong to $\mathcal{X}$.

2.1. General framework. Both agents assess the risk associated with their respective positions by a convex risk measure, denoted respectively ρ_A and ρ_B, with associated penalty functions α_A and α_B.
The issuer, agent A, wants to determine the structure (F, π) as to minimize her global risk measure

$$\min_{F \in \mathcal{X}, \pi} \rho_A(X - F + \pi)$$

while the issuer's constraint related to the buyer's interest in doing the transaction may be written as

$$\rho_B(F - \pi) \leq \rho_B(0)$$

This constraint now imposes a maximum threshold to the risk the buyer accepts to bear.
We now consider a more general framework where both agents may also invest in the financial market in order to reduce their respective exposure. They choose optimally their financial investments via two cones $\mathcal{V}_T^{(A)}$ and $\mathcal{V}_T^{(B)}$, characterizing the terminal gains associated with their respective admissible financial strategies.
This opportunity to invest optimally in a financial market has a direct impact on the risk measure of both agents as previously mentioned. As a consequence, provided the condition $\inf_{\xi_B \in \mathcal{V}_T^{(B)}} \rho_B(\xi_B) > -\infty$ and $\inf_{\xi_A \in \mathcal{V}_T^{(A)}} \rho_A(\xi_A) > -\infty$
we are exactly in the framework of Corollary 1.6 and both agents simply assess their non-tradable exposure using a market modified risk measure, denoted respectively by ρ_A^m and ρ_B^m. The optimization program related to the F -transaction simply becomes

$$\inf_{F \in \mathcal{X}, \pi} \rho_A^m(X - F + \pi) \qquad \text{subject to} \qquad \rho_B^m(F - \pi) \leq \rho_B^m(0)$$

Using the cash translation invariance property and binding the constraint at the optimum, the pricing rule of the F-structure is fully determined by the buyer as

$$\pi^*(F) = \rho_B^m(0) - \rho_B^m(F) \tag{2.1}$$

It corresponds to an "indifference" pricing rule from the point of view of agent B's market modified risk measure.

Using again the cash translation invariance property, the optimization program simply becomes

$$\inf_{F\in\mathcal{X}} \left(\rho_A^m (X-F) + \rho_B^m (F) - \rho_B^m (0)\right)$$

We are almost in the framework of Theorem 1.5, apart from the constant $\rho_B^m(0)$. To deal with it, we consider the reduced program[3]

$$R_{AB}^m (X) = \inf_{F\in\mathcal{X}} \left(\rho_A^m (X-F) + \rho_B^m (F)\right) = \left(\rho_A^m \square \rho_B^m\right)(X) \tag{2.2}$$

The value functional R_{AB}^m of this program may be interpreted as a measure of the residual risk after all transactions.
A direct consequence of Theorem 1.5 is now given:

Proposition 2.1. *The inf-convolution of the risk measures ρ_A^m and ρ_B^m, $R_{AB}^m(X)$, is a convex risk measure with the penalty function given for any $\mathbb{Q}$ in*[4] $\mathcal{Q}^{(A)} \cap \mathcal{Q}^{(B)}$ *by*

$$\alpha_{AB}^m (\mathbb{Q}) = \alpha_A (\mathbb{Q}) + \alpha_B (\mathbb{Q}) \quad , \quad +\infty \text{ otherwise}$$

2.2. Characterization of the optimal structure.

2.2.1. *Generalized entropic framework.* In the case of the entropic risk measure e_γ defined by Equation (1.5), we easily obtain the following semi-group property

$$e_\gamma \square e_{\gamma'} = e_{\gamma+\gamma'}$$

More generally, let ρ be a convex risk measure with penalty function α. The risk measure ρ_γ with penalty function $\gamma\alpha$, is equal to the "right scalar multiplication" of ρ defined by Rockafellar ([**Ro**]), more precisely:

$$\forall \Psi \in \mathcal{X} \qquad \rho_\gamma (\Psi) = \gamma \rho \left(\frac{1}{\gamma}\Psi\right) \tag{2.3}$$

In this family of convex risk measures, by duality, the inf-convolution defines a new convex risk measure of the same family: for any (γ, γ'), strictly positive, indeed, the following stability property holds

$$\rho_\gamma \square \rho_{\gamma'} = \rho_{\gamma+\gamma'}$$

In this case, the optimal structure F^* realizing the inf-convolution (2.2) may be explicitely obtained:

Proposition 2.2. *When both agents have the same access to the financial market and have market modified risk measures of the type described above by* (2.3), *the optimal structure F^* is given by:*

$$F^* = \frac{\gamma_B}{\gamma_A + \gamma_B} X \qquad \mathbb{P} \ a.s.$$

Proof. The result is immediately obtained by checking that:

$$\begin{aligned} \rho_A^m(X-F^*) + \rho_B^m(F^*) &= \gamma_A \rho^m\left(\tfrac{X}{\gamma_A+\gamma_B}\right) + \gamma_B \rho^m\left(\tfrac{X}{\gamma_A+\gamma_B}\right) \\ &= (\gamma_A+\gamma_B)\rho^m\left(\tfrac{X}{\gamma_A+\gamma_B}\right) \\ &= \left(\rho_A^m \square \rho_B^m\right)(X) \end{aligned}$$

Hence, the optimality of F^* is deduced. $\square$

---

[3]The value functional obtained in this case should be translated by the constant $-\rho_B^m(0)$ in order to obtain the value function of the previous program.

[4]Note that $\mathcal{Q}^{(A)} \cap \mathcal{Q}^{(B)}$ is the set of all additive measures $\mathbb{Q}$ such that $\forall \xi \in \mathcal{V}_T^{(A)} \cap \mathcal{V}_T^{(B)}$, $\mathbb{E}_{\mathbb{Q}}(\xi) \geq 0$.

Interpretation: when both agents have the same access to the financial market, the underlying logic of this new asset class is that of insurance and is far away from that of speculation. The issuer has an interest to sell a structure if and only if she is initially exposed (or, more precisely, if her initial exposure differs from that of the buyer). The underlying logic is that of insurance and hedging. It is by no way a speculative logic and the sale of this type of contract aims to hedge a real exposure towards a non-financial risk.

2.2.2. *Characterization of the optimal structure in a general framework.* We now consider a more general case and find some conditions to have an optimal structure F^* realizing inf-convolution $R_{AB}^m(X)$ for a given X. First let us give two definitions of optimality and precise the dual relationship between exposure and additive measure:

DEFINITION 2.3. Given a convex risk measure ρ and its associated penalty function α, we say
$i)$ that the measure $\mathbb{Q}_\rho^\Psi$ is optimal for (Ψ,ρ) if $\rho(\Psi)=\mathbb{E}_{\mathbb{Q}_\rho^\Psi}(-\Psi)-\alpha\left(\mathbb{Q}_\rho^\Psi\right)$.
$ii)$ that the exposure Ψ is optimal for $(\mathbb{Q},\alpha)$ if $\alpha(\mathbb{Q})=\mathbb{E}_{\mathbb{Q}}(-\Psi)-\rho(\Psi)$.

Let $\mathbb{Q}_{AB}^X$ be the optimal measure for for (X,R_{AB}^m), the existence of which has been mentioned in Subsection 1.1.1.
The following Theorem presents a necessary and sufficient condition to have an optimal structure F^* in terms of this optimal measure $\mathbb{Q}_{AB}^X$.

THEOREM 2.4. *Let*

$$\begin{aligned}\mathcal{C}_A^*\left(\mathbb{Q}_{AB}^X\right) &= \left\{F;\alpha_A^m\left(\mathbb{Q}_{AB}^X\right)=\mathbb{E}_{\mathbb{Q}_{AB}^X}\left(-(X-F)\right)-\rho_A^m(X-F)\right\}\\ \mathcal{C}_B^*\left(\mathbb{Q}_{AB}^X\right) &= \left\{F;\alpha_B^m\left(\mathbb{Q}_{AB}^X\right)=\mathbb{E}_{\mathbb{Q}_{AB}^X}(-F)-\rho_B^m(F)\right\}\end{aligned}$$

The necessary and sufficient condition to have an optimal solution to the inf-convolution problem described in the Program (2.2) *is*

$$\mathcal{C}_A^*\left(\mathbb{Q}_{AB}^X\right)\cap\mathcal{C}_B^*\left(\mathbb{Q}_{AB}^X\right)\neq\emptyset$$

Moreover, denoting by F^* an optimal solution for the inf-convolution problem, the following relationships prevail:

$$\mathbb{Q}_{AB}^X \text{ "}=\text{" } \mathbb{Q}_A^{X-F^*} \text{ "}=\text{" } \mathbb{Q}_B^{F^*}$$

PROOF. Please refer to Barrieu-El Karoui [**BEK3**]. □

This Theorem gives a procedure to obtain the optimal structure. For a given X, there exists an optimal measure $\mathbb{Q}_{AB}^X$ for R_{AB}^m. This measure is necessary and sufficient to have an optimal structure F^*. Hence, the solutions of the inf-convolution problem are determined by the dual formulation of the residual risk measure R_{AB}^m. Note also that the last equalities translate the fact that both agents valuate their respective residual risk using the same measure $\mathbb{Q}_{AB}^X$. This enables the transaction.

3. Solving the inf-convolution problem in a dynamic framework

We come back in this section to the inf-convolution problem when considering various classes of convex functionals. We adopt dynamic programing techniques, in particular Backward Stochastic Differential Equations (BSDE's), to study risk measures defined by their local specifications and propose a method to characterize the optimal solution of the inf-convolution problem. This approach leads to a particular definition of dynamic risk measures.

3.1. Localization of convex risk measures.

3.1.1. *Introduction.* On the probability space $(\Omega, \Im, \mathbb{P})$, let consider a Brownian filtration $(\Im_t = \sigma(W_s; 0 \leq s \leq t); t \geq 0)$. This enables us to extend the notion of static entropic risk measure to a more local and dynamic one

$$e_{\gamma,t}(X) = \gamma \ln \mathbb{E}_{\mathbb{P}}\left(\exp\left(-\frac{1}{\gamma}X\right)/\Im_t\right) \quad , \qquad X \in \mathcal{X}$$

with the terminal condition

$$e_{\gamma,T}(X) = -X$$

The dynamics of the adapted process $(e_{\gamma,t}(X); t \in [0,T])$ is given by the following BSDE with the quadratic driver $f(t,z) = \frac{1}{2\gamma}\|z\|^2$, using stochastic calculus arguments:

$$-de_{\gamma,t}(X) = \frac{1}{2\gamma}\|z_t\|^2 dt - \langle z_t, dW_t\rangle \qquad \text{with the terminal condition } e_{\gamma,T}(X) = -X$$

The idea is then to introduce families of solutions driven by convex generators $f(t,z)$ of the same kind. As we will see, they generate local convex risk measures. Under some regularity assumptions[5] for the function f, implying the existence and uniqueness of a solution to this BSDE, the key tool is the so-called *Comparison Theorem*. It corresponds to the maximum principle when working with PDEs.

THEOREM 3.1. *Consider the general BSDE with the solution* (Y_t, z_t)

$$-dY_t = f(t, z_t)\,dt - \langle z_t, dW_t\rangle \quad \textit{with terminal condition } Y_T = X \tag{3.1}$$

Let X_1 and X_2 be two elements of $\mathcal{X}$ and f^1 and f^2 two "regular" drivers. We denote by (Y^1, z^1) and (Y^2, z^2) the associated solutions.
We assume that $X_1 \geq X_2$ $\mathbb{P}$ a.s. and $f^1(t, z_t^2) \geq f^2(t, z_t^2)$ $dt \times d\mathbb{P}$ a.s.. Then we have

$$\forall t \geq 0 \qquad Y_t^1 \geq Y_t^2$$

3.1.2. *Local specification.* We are now able to generalize the notion of static convex risk measure to a more dynamic notion, by considering the BSDE's solutions as functional of their terminal condition. More precisely, thanks to the Comparison Theorem 3.1, properties as monotonicity, convexity of the drivers are also satisfied by the solution.

THEOREM 3.2. *Suppose that the regular driver $f(t,z)$ is convex w.r. to z. The solution $(\rho_t(X))t \leq T$ of the BSDE* (3.1) *with terminal conditon $-X$.*

$$-d\rho_t(X) = f(t, z_t)\,dt - \langle z_t, dW_t\rangle >, \qquad \rho_T(X) = -X$$

is for any time t a convex risk measure.

The convexity result has been proved in the study of pricing functionals with constraints (see, for instance, El Karoui-Quenez [**EKQ**], Peng [**P**] or Gianin [**Gi**]). The cash invariance property is a consequence of the independence between f and ρ.

[5] For example, f uniformly Lipschitz or with quadratic growth, as shown by El Karoui-Quenez [*EKQ*] , Kobylanski [*Kob*] or Lepeltier-San Martin [*LSMa*].

3.2. Inf-convolution. In this subsection, given f^1 and f^2 two regular convex drivers, we compare the solution of different BSDE's related to

$$\begin{array}{lll} (1) & -d\rho_t^1(\Psi) = f^1\left(t, z_t^1\right) dt - \left\langle z_t^1, dW_t\right\rangle, & \rho_T^1(\Psi) = -\Psi \\ (2) & -d\rho_t^2(\Psi) = f^2\left(t, z_t^2\right) dt - \left\langle z_t^2, dW_t\right\rangle, & \rho_T^2(\Psi) = -\Psi \end{array}$$

In particular, we will study for any t the inf-convolution of the convex functionals ρ_t^1 and ρ_t^2 defined as

$$\left(\rho^1 \square \rho^2\right)_t (X) = \inf_F \left\{\rho_t^1 (X - F) + \rho_t^2 (F)\right\} \tag{3.2}$$

The first step is to introduce the following BSDE

$$-d\rho_t^{1,2}(X) = \left(f^1 \square f^2\right)(t, z_t) dt - \langle z_t, dW_t\rangle, \qquad \rho_T^{1,2}(X) = -X \tag{3.3}$$

where the driver $f^1 \square f^2(t,z)$ is the inf-convolution between both convex functions $f^1(t,z)$ and $f^2(t,z)$. In that follows, we assume its regularity.
The next step is to verify that, under some additional assumptions, the solution of BSDE generated by $f^1 \square f^2$ is the inf-convolution $\rho^1 \square \rho^2$ of the dynamic risk measures ρ^1 and ρ^2.

THEOREM 3.3. *For a given $X \in \mathcal{X}$, let $\left(\rho_t^{1,2}(X), z_t\right)$ be the solution of (3.3) with a regular driver $f^1 \square f^2$ and terminal condition $-X$.*
Then, the following results hold:
i) For any $(F,t) \in \mathcal{X} \times [0,T]$, $\qquad \rho_t^{1,2}(X) \le \rho_t^1(X - F) + \rho_t^2(F) \quad \mathbb{P}$ *a.s.*
ii) If there exists an admissible $\widehat{z}_t^2$ such that

$$\forall t \ge 0 \qquad f^1 \square f^2 (t, z_t) = f^1\left(t, z_t - \widehat{z}_t^2\right) + f^2\left(t, \widehat{z}_t^2\right).$$

then

$$\forall t \ge 0 \qquad \rho_t^{1,2}(X) = \left(\rho^1 \square \rho^2\right)_t (X) \quad \mathbb{P}\ a.s$$

iii) Under this assumption, let F^* the structure defined by the forward equation

$$F^* = \int_0^T f^2\left(t, \widehat{z}_t^2\right) dt - \int_0^T \left\langle \widehat{z}_t^2, dW_t\right\rangle$$

Then F^* is an optimal solution for the inf-convolution problem (3.2) of the dynamic risk measures.

PROOF. i) By definition of inf-convolution,

$$\forall (t,z,y) \qquad \left(f^1 \square f^2\right)(t,z) \le f^1(t, z - y) + f^2(t,y)$$

On the other hand, for any $F \in \mathcal{X}$, $\rho_t^1(X-F) + \rho_t^2(F)$ satisfies

$$\begin{array}{rcl} -d\left(\rho_t^1(X-F) + \rho_t^2(F)\right) & = & \left(f^1(t,z_t^1) + f^2(t,z_t^2)\right) dt - \langle z_t^1 + z_t^2, dW_t\rangle \\ & = & \left(f^1(t, z_t - z_t^2) + f^2(t,z_t^2)\right) dt - \langle z_t, dW_t\rangle \end{array}$$

with terminal condition $-X$.
The second formulation expresses that $\rho_t^1(X-F) + \rho_t^2(F)$ is solution of a BSDE with terminal condition $-X$ and a driver written as $f^1(t, z_t - z_t^2) + f^2(t, z_t^2)$ where z_t^2 is fixed as the solution of the BSDE (2) with terminal condition $-F$.
This driver is always greater than that of the BSDE (3.3) and their respective terminal conditions are identical. Thanks to the Comparison Theorem 3.1, the result is obtained.
ii) and iii) Let us assume that there exists an admissible $\widehat{z}_t^2$ such that

$$\forall t \ge 0 \qquad \left(f^1 \square f^2\right)(t, z_t) = f^1\left(t, z_t - \widehat{z}_t^2\right) + f^2\left(t, \widehat{z}_t^2\right)$$

where z_t is the solution of (3.3) with the terminal condition $-X$.
We now introduce the structure F^* defined by the forward equation

$$F^* = \int_0^T f^2\left(t, \widehat{z}_t^2\right) dt - \int_0^T \left\langle \widehat{z}_t^2, dW_t \right\rangle$$

Let us observe that $-R_t^2 \equiv \int_0^t f^2\left(u, \widehat{z}_u^2\right) du - \int_0^t \left\langle \widehat{z}_u^2, dW_u \right\rangle$ satisfies

$$R_t^2\left(F^*\right) = -F^* + \int_t^T f^2\left(u, \widehat{z}_u^2\right) dt - \int_t^T \left\langle \widehat{z}_u^2, dW_u \right\rangle$$

Then $R_t^2\big(F^*\big)$ is solution of the BSDE with driver f^2 and terminal condition $-F^*$. By uniqueness, this process is the dynamic risk measure $\rho_t^2(F^*)$.
We have seen in i) that $\rho_t^1(X - F^*) + \rho_t^2(F^*)$ is solution of the BSDE with driver written as $f^1(t, z_t - \widehat{z}_t^2) + f^2(t, \widehat{z}_t^2)$ and terminal condition $-X$. Given that $\left(f^1 \Box f^2\right)(t, z_t) = f^1\left(t, z_t - \widehat{z}_t^2\right) + f^2\left(t, \widehat{z}_t^2\right)$, by uniqueness, the following equality holds

$$\forall t \geq 0 \qquad \rho_t^{1,2}(X) = \left(\rho^1 \Box \rho^2\right)_t(X) \quad \mathbb{P}\ a.s$$

The proof also gives the optimality for the Problem (3.2) of the structure

$$F^* = \int_0^T f^2\left(t, \widehat{z}_t^2\right) dt - \int_0^T \left\langle \widehat{z}_t^2, dW_t \right\rangle$$

□

Given the results of Proposition 2.2, it is natural to study which assumptions on the driver of such dynamic risk measures lead to a non-speculative logic[6].
To simplify the arguments, we now consider normalized risk measures, i.e.

$$\forall t \quad \rho_t^1(0) = \rho_t^2(0) = 0$$

COROLLARY 3.4. *Assume that* $f^1(t,0) = f^2(t,0) = 0$ *and* $\partial_z f^1(t,0) = \partial_z f^2(t,0) = 0$, *then:*
i) The inf-convolution $\left(f^1 \Box f^2\right)(t,0)$ *and that of the associated risk measure* $\left(\rho_1 \Box \rho_2\right)(t,0)$ *are identically null.*
Moreover, $F^* \equiv 0$ *is an optimal solution for the inf-convolution problem (3.2).*
ii) If both drivers f^1 *and* f^2 *are strictly convex, then* $F^* \equiv 0$ *is the unique optimal solution for the inf-convolution problem (3.2).*

PROOF. i) Since $f^1(t,0) = f^2(t,0) = 0$, we have

$$\left(f^1 \Box f^2\right)(t,0) = \inf_y \{f^1(t,-y) + f^2(t,y)\} \leq 0$$

On the other hand,

$$\forall y \qquad f^1(t,-y) + f^2(t,y) \geq (-\partial_z f^1(t,0) + \partial_z f^2(t,0))y = 0$$

using the assumption $\partial_z f^1(t,0) = \partial_z f^2(t,0) = 0$.
Hence $\left(f^1 \Box f^2\right)(t,0) = 0$ and as a consequence $\left(\rho_1 \Box \rho_2\right)(t,0) = 0$.
Moreover, as $y = 0$ is a solution of $f^1(t,-y) + f^2(t,y) = 0$, by construction, $F^* \equiv 0$

[6]By non-speculative logic, we simply mean that the issuer has an interest to sell a structure if and only if she is initially exposed. The underlying logic is that of insurance and hedging.

is an optimal solution for the inf-convolution problem (3.2).
ii) When both drivers f^1 and f^2 are strictly convex, then $y = 0$ is the unique solution of $f^1(t,-y) + f^2(t,y) = 0$ and consequently $F^* \equiv 0$ is the *unique* optimal solution for the inf-convolution problem (3.2). □

4. Conclusion

Standard diversification will occur in exchange economies as soon as agents have non-speculative risk measures. The regulator has to impose very different rules on agents as to generate risk measures, which are not non-speculative, if she wants to increase the diversification in the market. In other words, diversification occurs when agents are very different one from the other. This result supports for instance the intervention of reinsurance companies on financial markets in order to increase the diversification on the reinsurance market.

References

[ADEH] P. Artzner, F. Delbaen, J.M. Eber, D. Heath, *Coherent Measures of Risk* Mathematical Finance **9** (1999), 203–228.

[B] P. Barrieu, *Structuration Optimale de Produits Financiers en Marché Illiquide et Trois Excursions dans d'autres Domaines des Probabilités* Thèse de doctorat, Université de Paris VI (2002).

[BEK1] P. Barrieu, N. El Karoui, *Reinsuring Climatic Risk using Optimally Designed Weather Bonds* Geneva Papers, Risk and Insurance Theory **27** (2002), 87–113.

[BEK2] P. Barrieu, N. El Karoui, *Optimal Design of Derivatives in Illiquid Markets* Quantitative Finance **2** (2002), 1–8.

[BEK3] P. Barrieu, N. El Karoui, *Optimal Derivatives Design and Diversification in Financial Market with non-tradable Risk* Working Paper (2003).

[BEK4] P. Barrieu, N. El Karoui, *Structuration optimale de produits financiers et diversification en présence de sources de risque non-négociables* Comptes Rendus de l'Académie des Sciences, Série I **336** (2003), 493-498.

[Be1] D. Becherer, *Rational Hedging and Valuation with Utility-Based Preferences*, PhD Thesis, Berlin University (2001).

[Be2] D. Becherer, *Rational Hedging and Valuation of Integrated Risks under Constant Absolute Risk Aversion*, To appear in Insurance: Mathematics and Economics.

[BelF] F. Bellini, M. Frittelli, *On the Existence of Minimax Martingale Measures*, Mathematical Finance **12** (2002), 1–21.

[Bo] K. Borch, *Equilibrium in a Reinsurance Market*, Econometrica **30** (1962), 424–444.

[Bu] H. Bühlmann, *Mathematical Methods in Risk Theory* Springer Verlag (1970).

[BuDES] H. Bühlmann, F. Delbaen, P. Embrechts, A. Shiryaev, *On Esscher Transforms in Discrete Finance Models* Working Paper (www.math.ethz.ch/˜delbaen/ftp/preprints/Esscher-BDES-ASTIN.pdf).

[Da1] M. Davis, *Option Pricing in Incomplete Markets* in *Mathematics of derivative securities* (eds: M.A.H. Dempster and S.R. Pliska), Cambridge University Press (1997), 227–254.

[Da2] M. Davis, *Pricing Weather Derivatives by Marginal Value* Quantitative Finance **1** (2001), 1–4.

[DGRSS] F. Delbaen, P. Grandits, T. Rheinländer, D. Samperi, M. Schweizer, C. Stricker, *Exponential Hedging and Entropic Penalities*, Mathematical Finance **12** (2002), 99–123.

[EG] L. Eeckhoudt, C. Gollier, *Risk: Evaluation, Management and Sharing* Harvester Wheatsheaf (1995).

[EKQ] N. El Karoui, M.C. Quenez, *Non-linear Pricing Theory and Backward Stochastic Differential Equations, Financial Mathematics* (ed: W.J. Runggaldier), Lecture Notes in Mathematics 1656, Springer Verlag (1996), 191–246.

[EKPQ] N. El Karoui, N. Peng, M.C. Quenez, *Backward Stochastic Differential Equations in Finance* Mathematical Finance **7** (1997), 1–71.

[EKR] N. El Karoui, R. Rouge, *Pricing via Utility Maximization and Entropy* Mathematical Finance **10** (2000), 259–276.

[FS1] H. Föllmer, A. Schied, *Convex Measures of Risk and Trading Constraints* Finance and Stochastics **6** (2002), 429–447.

[FS2] H. Föllmer, A. Schied, *Stochastic Finance: An Introduction in discrete Time* De Gruyter Studies in Mathematics (2002).

[Fr1] M. Frittelli, *The Minimal Entropy Martingale Measure and the Valuation in Incomplete Markets* Mathematical Finance **10** (2000), 39–52.

[Fr2] M. Frittelli, *Introduction to a Theory of Value Coherent with the No-Arbitrage Principle* Finance and Stochastics, **4** (2000), 275–297.

[Gi] E.R. Gianin, *Some Examples of Risk Measures via g-Expectations* Working Paper (2003).

[HN] S.D. Hodges, A. Neuberger, *Optimal Replication of Contingent Claims under Transaction Costs* Review of Futures Markets **8** (1989), 222–239.

[K] G. Kallianpur, *Stochastic Filtering Theory* Springer Verlag (1980).

[KaKo] I. Karatzas, S.G. Kou, *On the Pricing of Contingent Claims under Constraints* Annals of Applied Probability, **6** (1996), 321–369.

[Kob] M. Kobylanski, *Backward Stochastic Differential Equations and Partial Differential Equations with Quadratic Growth* Annals of Probability, **28** (2000), 558-602.

[La1] P. Lakner, *Utility Maximization with Partial Information* Stochastic processes and their applications, **56** (1995), 247-273.

[La2] P. Lakner, *Optimal Trading for an Investor: The Case of Partial Information* Stochastic processes and their applications, **76** (1998), 77-96.

[Le] D. Lefèvre, *An Introduction to Utility Maximization with Partial Observation* Rapport de recherche n°4183, INRIA (2001).

[LSMa] J.P. Lepeltier, J. San Martin, *Existence of BSDE with Superlinear Quadratic Coefficient* Stochastics and Stochastic Report, **63** (1998), 227-240.

[LiS] R.S. Lipster, A.N. Shiryayev, *Statistics of Random Processes* Springer Verlag (2001).

[Me] R. Merton, *Optimum Consumption and Portfolio Rules in a Continuous Time Model* Journal of Economic Theory **3** (1971), 373–413.

[MuZ] M. Musiela, T. Zariphopoulou, *Pricing and Risk Management of Derivatives Written on Non-Traded Assets* Working Paper (2001).

[P] S. Peng, *Backward SDE and Related g-Expectations* in *Backward Stochastic Diffential Equations* (eds: N. El Karoui and L. Mazliak), Pitman Res. Notes Math. Ser. Longman Harlow, 364 (1997), 141-159.

[PhQ] H. Pham, M.C. Quenez, D*ynamic portfolio optimization in the case of partially observed drift process* Annals of Probability, **11** (2001), 210-238.

[QV] J.P. Quadrat, M. Viot, *Introduction à la commande stochastique* Cours polycopié de l' Ecole Polytechnique (1996).

[Ra] A. Raviv, *The Design of an Optimal Insurance Policy* American Economic Review **69** (1979), 84–96.

[Ro] R.T. Rockafellar, *Convex Analysis* Princeton Landmarks in Mathematics (1970).

Pauline Barrieu, L.S.E., Statistics Department, Houghton Street, London, WC2A 2AE, United Kingdom

E-mail address: `p.m.barrieu@lse.ac.uk`

Nicole El Karoui, C.M.A.P., Ecole Polytechnique, 91128 Palaiseau Cédex, France

E-mail address: `elkaroui@cmapx.polytechnique.fr`