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Optimal Design of a Minimum Weight Thermal Diffuser with Constraint on the Output Thermal Power Flux

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Abstract. The object of this paper is the development of efficient mathematical and numerical tools to find the optimal shape of a minimum-weight thermal diffuser with a priori specifications on the inward thermal power flux (TPF) and a bound on the outward TPF. The present problem arises in connection with the use of high-power solid state devices in future communications satellites. In a space application the thermal power must ultimately be dissipated to the environment by using heatpipes and correspondingly large radiating areas. However, heatpipes can accept only a limited TPF from a source. Hence we have the requirement of a minimum-weight thermal diffuser with a uniform bound on the outward TPF. Shape optimal design and finite elements methods are used. Complete numerical results are provided.

1. Introduction

The object of this paper is the development of efficient mathematical and numerical tools to find the optimal shape of a minimum-weight thermal diffuser with a priori specifications on the input and output thermal power flux. This paper contains a working theory, the necessary computations and numerical examples. A similar problem was studied by Ph. Destuynder [5] with the require-

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ment that the temperature at every point of the diffuser be less than a specified critical temperature.

The present problem arises in connection with the use of high-power solid state devices (HPSSD's) in future communication satellites. The specifications for this diffuser came from the Center for Research in Communication (CRC) in Canada[1].

"An HPSSD dissipates a large amount of thermal power (typ. > 50W) over a relatively small mounting surface (typ. 1.25cm^2). Yet its junction temperature is required to be kept moderately low (typ. 110°C). The thermal resistance from the junction to the mounting surface is known for any particular HPSSD (typ. 1°C/W), so that the mounting surface is required to be kept at a lower temperature than the junction (typ. 60°C).

In a space application the thermal power must ultimately be dissipated to the environment by the mechanism of radiation. However, to radiate large amounts of thermal power at moderately low temperatures, correspondingly large radiating areas are required. Thus we have the requirement to efficiently spread the high thermal power flux (TPF) at the HPSSD source (typ. 40W/cm^2) to a low TPF at the radiator (typ. .04W/cm^2) so that the source temperature is maintained at an acceptably low level (typ. < 60°C at mounting surface). The efficient spreading task is best accomplished using heatpipes, but the snag in the scheme is that heatpipes can accept only a limited maximum TPF from a source (typ. max. 4W/cm^2).

Hence we are led to the requirement for a thermal diffuser. This device is inserted between the HPSSD and the heatpipes, and reduces the TPF at the source (typ. > 40W/cm^2) to a level acceptable to the heatpipes (typ. max. 4W/cm^2). The heatpipes then sufficiently spread the heat over large space radiators, reducing the TPF from a level at the diffuser (typ. max. 4W/cm^2) to that at the radiator (typ. .04W/cm^2). This scheme of heat spreading is depicted in Fig. 1.

It is the design of the thermal diffuser which is the problem at hand. We may assume that the HPSSD presents a uniform thermal power flux to the diffuser at the HPSSD/diffuser interface. Heatpipes are essentially isothermalizing devices, and we may assume that the diffuser/heatpipes interface is indeed isothermal. Any other surfaces of the diffuser may be treated as adiabatic."

Some early results were presented in Delfour-Payre-Zolésio [4].

Notation. $\mathbb{R}$ (resp. $\mathbb{R}_+$) is the field (resp. semigroup) of all (resp. positive or zero) reals. Given an integer $n \geq 1$, $\mathbb{R}^n$ is the n-dimensional Euclidean space. The topological dual of a Banach space E will be denoted by E'. Let Ω be an open subset of $\mathbb{R}^n$. $\mathcal{D}(\Omega)$ is the vector space of all real functions defined on Ω whose partial derivatives of all orders exist and are continuous and whose support is contained in some compact subset of Ω. For $k \geq 1$ an integer, $H^k(\Omega)$ denotes the vector space of all real functions f on Ω such that f and its distributional derivatives $D^s f$ of order $|s| = \Sigma_{j=1}^n |s_j| \leq k$ all belong to $L^2(\Omega)$. For real $k \geq 0$,

[1]The statement of the problem and Figure 1 have been graciously provided by Dr. V. A. Wehrle of CRC.

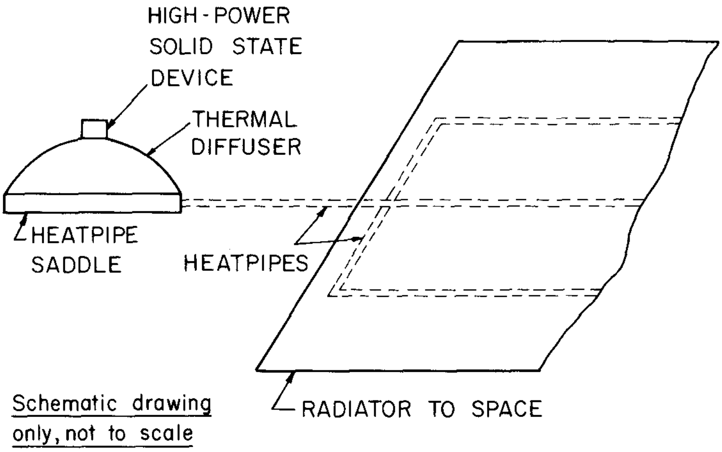

Fig. 1. Heat-spreading scheme for high-power solid state devices.

$H^k(\Omega)$ is defined by interpolation; for negative reals $s<0$, $H^s(\Omega)=(H_0^{-s}(\Omega))'$ where $H_0^s(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in the $H^s(\Omega)$-topology.

Given a Banach space E, $C^0(\Omega; E)$ is the vector space of continuous functions defined on Ω into E. For $k \geqslant 1$ an integer, $C^k(\Omega; E)$ is the vector space of all continuous functions defined on Ω into E whose partial derivatives are continuous up to order k. When $E=\mathbb{R}$ we shall write $C^0(\Omega)$ and $C^k(\Omega)$.

2. Statement of the Problem

We assume that the thermal diffuser is a volume Ω symmetrical about the z-axis (cf. Figure 2A) whose boundary surface is made up of three regular pieces: the mounting surface Σ_1 (a disk perpendicular to the z-axis with center in $(r,z)=(0,0)$), the lateral adiabatic surface Σ_2 and the interface Σ_3 between the diffuser and the heatpipes saddle (a disk perpendicular to the z-axis with center in $(r,z)=(0,L)$).

The temperature distribution over this volume Ω is the solution of the stationary heat equation $\Delta T=0$ (ΔT, the Laplacian of T) with the following boundary conditions on the surface $\Sigma=\Sigma_1\cup\Sigma_2\cup\Sigma_3$ (or boundary of Ω):

$$\left.\begin{aligned} k\frac{\partial T}{\partial n} &= q_{\text{in}} && \text{on } \Sigma_1 \\ k\frac{\partial T}{\partial n} &= 0 && \text{on } \Sigma_2 \\ T &= T_3 && \text{on } \Sigma_3,\ T_3=\text{constant}, \end{aligned}\right\} \tag{2.1}$$

where n always denotes the outward normal to the boundary surface Σ and

$\partial T/\partial n$ is the normal derivative to the boundary surface Σ,

$$\frac{\partial T}{\partial n} = \nabla T \cdot n \qquad (\nabla T = \text{gradient of } T). \tag{2.2}$$

The parameters appearing in (2.1) are:

k = thermal conductivity (typ. 1.8W/cm.×°C)

q_{in} = uniform inward thermal power flux at the source (positive constant).

The radius R_0 of the mounting surface Σ_1 is fixed so that the boundary surface Σ_1 is already given in the design problem.

For practical considerations, we assume the diffuser to be solid without interior hollows or cutouts. The class of shapes for the diffuser is characterized by the design parameter $L \geqslant 0$ and the positive function $R(z)$, $0 \leqslant z \leqslant L$, with $R(0) = R_0 > 0$. They are volumes of revolution Ω about the z-axis generated by the surface A between the z-axis and the function $R(z)$ (cf. Fig. 2B), that is

$$\Omega = \{(x, y, z) | 0 \leqslant z \leqslant L, x^2 + y^2 \leqslant R^2(z)\}. \tag{2.3}$$

So the shape of Ω is completely specified by the length L and the function R on the interval $[0, L]$.

Assuming that the diffuser is made up of a homogeneous material of uniform density (no hollows) the design objective is to minimize the volume

$$J(\Omega) = \pi \int_0^L R^2(z)\, dz \tag{2.4}$$

subject to a uniform constraint on the outward thermal power flux at the interface Σ_3 between the diffuser and the heatpipes saddle:

$$\sup_{p \in \Sigma_3} - k \frac{\partial T}{\partial z}(p) \leqslant q_{\text{out}}, \tag{2.5}$$

where q_{out} is a specified positive constant.

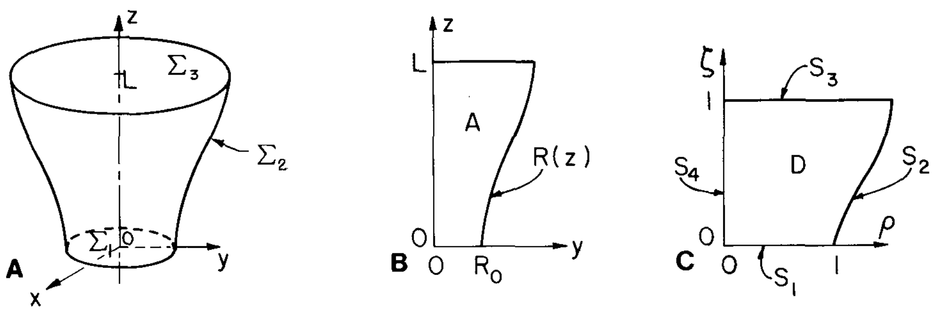

Fig. 2. (A) Volume Ω and its boundary Σ; (B) Surface A generating Ω; (C) Surface D generating $\tilde{\Omega}$.

It is readily seen that the minimization problem (2.4) subject to the constraint (2.5) (where T is the solution of the heat equation with the boundary conditions (2.1)) is independent of the fixed temperature on boundary Σ_3. In other words the optimal shape Ω, if it exists is independent of T_3. As a result, *from now on we set* T_3 *equal to* 0.

3. Reformulation of the Problem

In this problem, the *shape parameter* L and the *shape function* R are not independent of each other since the function R is defined on the interval $[0, L]$. This motivates the following change of variable on the z-axis

$$\zeta = \frac{z}{L}, \qquad 0 \leqslant \zeta \leqslant 1. \tag{3.1}$$

The length of the diffuser is now one and it is possible to work on a fixed interval $[0, 1]$. Similarly we can scale the x_1 and x_2 variables by R_0:

$$\xi_1 = \frac{x_1}{R_0}, \qquad \xi_2 = \frac{x_2}{R_0}. \tag{3.2}$$

We shall see that the shape parameter $\tilde{L} = L/R_0$ will become a design parameter in the differential equation. The shape of the transformed domain is now completely specified by the new *shape function* $\tilde{\rho}$ defined on the interval $[0, 1]$ with the conditions $\tilde{\rho}(0) = 1$ and $\tilde{\rho}(\zeta) \geqslant 0$, $0 \leqslant \zeta \leqslant 1$. The former and new shape functions are related as follows:

$$\tilde{\rho}(\zeta) = R(L\zeta)/R_0. \tag{3.3}$$

Denote by $\tilde{\Omega}$ and $\tilde{\Sigma}, \tilde{\Sigma}_1, \tilde{\Sigma}_2, \tilde{\Sigma}_3$ the transformed domain Ω and surfaces $\Sigma, \Sigma_1, \Sigma_2, \Sigma_3$ in the new coordinate system (ξ_1, ξ_2, ζ) through the transformations (3.1) and (3.2). The *transformed domain* $\tilde{\Omega}$ is solely dependent on the function $\tilde{\rho}$ and is generated by the revolution of the surface D,

$$D = \{(\rho, \zeta) | 0 < \zeta < 1, 0 < \rho < \tilde{\rho}(\zeta)\}, \tag{3.4}$$

about the ζ-axis (see Figure 2C). The surface D has four regular boundaries:

$$\left.\begin{aligned} S_1 &= \{(\rho, \zeta) | 0 < \rho < 1, \zeta = 0\} \\ S_2 &= \{(\rho, \zeta) | 0 < \zeta < 1, \rho = \tilde{\rho}(\zeta)\} \\ S_3 &= \{(\rho, \zeta) | 0 < \rho < \tilde{\rho}(1), \zeta = 1\} \\ S_4 &= \{(\rho, \zeta) | 0 < \zeta < 1, \rho = 0\}. \end{aligned}\right\} \tag{3.5}$$

3.1. *Equations for the Scaled Temperature*

Introduce the *scaled temperature*

$$y(\xi_1, \xi_2, \zeta) = \frac{k}{Lq_{\text{in}}} T(R_0\xi_1, R_0\xi_2, L\zeta) \tag{3.6}$$

in the new variables (ξ_1, ξ_2, ζ) or in cylindrical coordinates

$$y(\rho, \zeta) = \frac{k}{Lq_{\text{in}}} T(R_0\rho, L\zeta), \qquad \rho = \sqrt{\xi_1^2 + \xi_2^2}. \tag{3.6a}$$

This scaling of the original temperature is motivated by the fact that we want to define y in such a way that

$$\frac{k}{q_{\text{in}}}\frac{\partial T}{\partial z}(r, L) = \frac{\partial y}{\partial \zeta}(\rho, 1), \qquad 0 \leqslant r = R_0\rho \leqslant R(L) = R_0\tilde{\rho}(1). \tag{3.7}$$

This quantity appears in the constraint on the outward thermal power flux through the boundary surface Σ_3.

The solution of the heat equation in Ω subject to boundary conditions (2.1) with $T_3 = 0$ coincides with the solution of the following variational equation: to find T in $H_0(\Omega)$ such that

$$k\int_\Omega \left[\frac{\partial T}{\partial x_1}\frac{\partial v}{\partial x_1} + \frac{\partial T}{\partial x_2}\frac{\partial v}{\partial x_2} + \frac{\partial T}{\partial z}\right] d\Omega = q_{\text{in}}\int_{\Sigma_1} v \, d\Sigma_1 \tag{3.8}$$

for all v in $H_0(\Omega)$, where

$$H_0(\Omega) = \left\{v \in H^1(\Omega) : v|_{\Sigma_3} = 0\right\} \tag{3.9}$$

is a closed linear subspace of $H^1(\Omega)$.

In the new variables (ξ_1, ξ_2, ζ) the variational problem (3.8)–(3.9) becomes: to find y in $H_0(\Omega)$ such that

$$\int_{\tilde{\Omega}} \left[\left(\frac{L}{R_0}\right)^2\left(\frac{\partial y}{\partial \xi_1}\frac{\partial v}{\partial \xi_1} + \frac{\partial y}{\partial \xi_2}\frac{\partial v}{\partial \xi_2}\right) + \frac{\partial y}{\partial \zeta}\frac{\partial v}{\partial \zeta}\right] d\tilde{\Omega} = \int_{\tilde{\Sigma}_1} v \, d\tilde{\Sigma}_1 \tag{3.10}$$

for all v in $H_0(\tilde{\Omega})$ (same definition as in (3.9)). Equivalently, it is easy to show that y is the solution of the following boundary value problem:

$$A(y) = -\left[\left(\frac{L}{R_0}\right)^2\left(\frac{\partial^2 y}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2}\right) + \frac{\partial^2 y}{\partial \zeta^2}\right] = 0 \text{ in } \tilde{\Omega} \tag{3.11}$$

with boundary conditions

$$\left.\frac{\partial y}{\partial \nu_A}\right|_{\tilde{\Sigma}_1} = 1, \qquad \left.\frac{\partial y}{\partial \nu_A}\right|_{\tilde{\Sigma}_2} = 0, \qquad y|_{\tilde{\Sigma}_3} = 0, \tag{3.12}$$

where

$$\frac{\partial y}{\partial \nu_A} = \left(\frac{L}{R_0}\right)^2 \left(\nu_1 \frac{\partial y}{\partial \xi_1} + \nu_2 \frac{\partial y}{\partial \xi_2}\right) + \nu_\zeta \frac{\partial y}{\partial \zeta} \tag{3.13}$$

is the conormal derivative on the boundary $\tilde{\Sigma}$ associated with the operator A and the unit outward normal vector $\nu = (\nu_1, \nu_2, \nu_\zeta)$ in the (ξ_1, ξ_2, ζ) coordinate system. The cost function becomes

$$J(\Omega) = R_0^3 J(\tilde{L}, \tilde{\rho}), \qquad J(\tilde{L}, \tilde{\rho}) = \tilde{L}\pi \int_0^1 \tilde{\rho}(\zeta)^2 \, d\zeta, \qquad \tilde{L} = L/R_0, \tag{3.14}$$

where $\tilde{L}$ is now a dimensionless length; the constraint (2.5) reduces to

$$\operatorname*{Sup}_{p \in \tilde{\Sigma}_3} -\frac{\partial y}{\partial \nu_A}(p) = \operatorname*{Sup}_{p \in \tilde{\Sigma}_3} -\frac{\partial y}{\partial \zeta}(p) \leqslant \frac{q_{\text{out}}}{q_{\text{in}}}, \tag{3.15}$$

which only depends on the dimensionless *flux ratio*

$$q = q_{\text{out}}/q_{\text{in}}. \tag{3.16}$$

3.2. Variational Equations in Cylindrical Coordinates

In cylindrical coordinates the variational problem (3.8) becomes: to find T in $V_0(A)$ such that

$$k \int_0^L \int_0^{R(z)} \left[\frac{\partial T}{\partial r} \frac{\partial v}{\partial r} + \frac{\partial T}{\partial z} \frac{\partial v}{\partial z} \right] r \, dr \, dz = q_{\text{in}} \int_0^{R_0} v r \, dr \tag{3.17}$$

for all v in $V_0(A)$, where

$$V_0(A) = \left\{ v \,|\, \sqrt{r}\, v, \sqrt{r} \frac{\partial v}{\partial r}, \sqrt{r} \frac{\partial v}{\partial z} \in L^2(A) \text{ and } v(r, L) = 0, 0 \leqslant r \leqslant R(L) \right\}. \tag{3.18}$$

Similarly the variational problem (3.10) for the scaled temperature y becomes: to find y in $V_0(D)$ such that

$$\int_0^1 \int_0^{\tilde{\rho}(\zeta)} \left[\tilde{L}^2 \frac{\partial y}{\partial \rho} \frac{\partial v}{\partial \rho} + \frac{\partial y}{\partial \zeta} \frac{\partial v}{\partial \zeta} \right] \rho \, d\rho \, d\zeta = \int_0^1 v(\rho, 0) \rho \, d\rho \tag{3.19}$$

for all v in $V_0(D)$ (same definition as in (3.18), but in new coordiantes (ρ, ζ)).

3.3. *The Constrained Minimization Problem (P)*

Denote by (P) the constrained minimization problem which consists in minimizing $J(\tilde{L}, \tilde{\rho})$ in (3.14) with respect to the dimensionless *design parameter* $\tilde{L}$ and *design function* $\tilde{\rho}$ subject to the constraint

$$\sup\left\{-\frac{\partial y}{\partial \zeta}(\rho, 1) : 0 \leqslant \rho \leqslant \tilde{\rho}(1)\right\} \leqslant q, \tag{3.20}$$

where y is the solution of the variational problem (3.19).

It is readily seen that the *optimal design is only a function of* $q = q_{\text{out}}/q_{\text{in}}$. The parameter R_0 only appears as a scaling parameter. If $\tilde{L}^*$ and $\tilde{\rho}^*$ are the optimal scaled parameter and function, then the optimal parameter L^* and function R^* for the original problem (2.1)–(2.5) are given by

$$L^* = R_0\tilde{L}^*, \qquad R^*(z) = R_0\tilde{\rho}^*(z/R_0\tilde{L}^*). \tag{3.21a}$$

This corresponds to the following coordinate transformation:

$$x_1 = R_0\xi_1, \qquad x_2 = R_0\xi_2, \qquad z = R_0\tilde{L}^*\zeta = L^*\zeta. \tag{3.21b}$$

3.4. *Angles and Smoothness of the Solution*

In the forthcoming sections of this paper we shall need sufficient smoothness of the solution T to the heat equation with boundary conditions (2.1). Since the boundary conditions on each piece of boundary is constant, the global smoothness of the solution will only be affected by the smoothness of the function $R(z)$ and the angles θ_1 and θ_3 between surfaces Σ_1 and Σ_2 and Σ_3 and Σ_2, respectively (cf. Fig. 3)

In this section we assume that R (resp. ρ) is sufficiently regular, so that the smoothness of T is solely affected by the angles θ_1 and θ_3 between the curve $(z, R(z))$ and the planes $z = 0$ and $z = L$, respectively.

Away from the curves $C_1 = \Sigma_1 \cap \Sigma_2$ and $C_3 = \Sigma_2 \cap \Sigma_3$, the solution T is infinitely continuously differentiable. So we only need to study the smoothness of T in neighborhoods V_1 and V_3 of C_1 and C_3, respectively.

In the special geometry where $\theta_1 = \pi/2$ (resp. $\theta_3 = \pi/2$), we can use the "principle of symmetry" to show that the solution T belongs to $C^\infty(V_1)$ (resp. $C^\infty(V_3)$) in a neighborhood V_1 (resp. V_3) of the curve C_1 (resp. C_3).

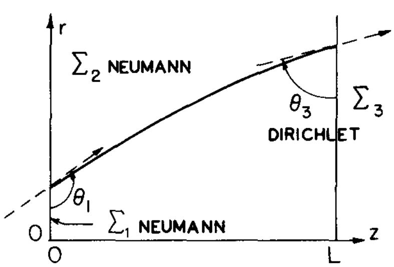

Fig. 3. Angles θ_1 and θ_3 between the surface Σ_2 and the surfaces Σ_1 and Σ_3.

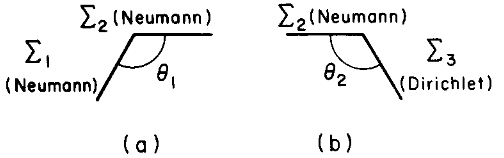

Fig. 4. Smoothness of the solution in the vicinity of a corner as a function of the angle.

When the angles θ_1 and θ_3 are between 0 and π we can use the following results from P. Grisvard [7, 8, 9, 10]. We consider the two cases of Fig. 4.

In a neighborhood V_1 of the curve C_1, the solution belongs to $H^{1+(\pi/\theta_1)-\varepsilon}(V_1)$ for all $\varepsilon > 0$; in a neighborhood V_3 of the curve C_3, the solution belongs to $H^{1+(\pi/2\theta_3)-\varepsilon}(V_3)$ for all $\varepsilon > 0$. Therefore, for the range of angles we are interested in, there exist sufficiently small $\varepsilon > 0$ and a neighborhood V_1 (resp. V_3) such that

$$\forall\theta_1, 0 < \theta_1 < \pi, \exists\varepsilon > 0, 1 + \frac{\pi}{\theta_1} - \varepsilon > 2$$

(resp. $\forall\theta_3$, $0 < \theta_3 < \pi$, $\exists\varepsilon > 0$, $1 + \pi/2\theta_3 - \varepsilon > 3/2$). Then

$$T|_{V_1} \in H^{2+\sigma}(V_1) \qquad \text{for } 0 \leqslant \sigma < \varepsilon \tag{3.22}$$

and

$$T|_{V_2} \in H^{3/2+\sigma}(V_2) \quad \text{for } 0 \leqslant \sigma < \varepsilon. \tag{3.23}$$

4. Approximation of the Solution to the Constrained Minimization Problem (P)

In the absence of existence and uniqueness result, we shall assume the existence of at least one solution to problem (P) and concentrate on the approximation of the solution.

The constraint (3.20) is completely equivalent to the new constraint

$$f(\tilde{L}, \tilde{\rho}) = 0, \tag{4.1}$$

where

$$f(\tilde{L}, \tilde{\rho}) = \int_{\tilde{\Sigma}_3} \left[\frac{\partial y}{\partial \zeta} + q\right]^- d\sigma = 2\pi \int_0^{\tilde{\rho}(1)} \left[\frac{dy}{\partial \zeta}(\rho, 1) + q\right]^- \rho \, d\rho \tag{4.2}$$

with $u^- = \sup\{-u, 0\}$.

Associate with an arbitrary family $\{\varepsilon : \varepsilon > 0\}$ of small positive numbers the penalized cost function

$$J_\varepsilon(\tilde{L}, \tilde{\rho}) = J(\tilde{L}, \tilde{\rho}) + \frac{1}{\varepsilon} f(\tilde{L}, \tilde{\rho}). \tag{4.3}$$

Replace the original constrained minimization problem (P) specified in section

3.3 by the following family of ε-indexed unconstrained minimization problems $(P_\varepsilon)_{\varepsilon>0}$: to find $(\tilde{L}_\varepsilon, \tilde{\rho}_\varepsilon)$ such that

$$J_\varepsilon(\tilde{L}_\varepsilon, \tilde{\rho}_\varepsilon) \leqslant J_\varepsilon(\tilde{L}, \tilde{\rho}) \tag{4.4}$$

for all $\tilde{L}$ and $\tilde{\rho}$ such that

$$\tilde{L} \geqslant 0 \quad \text{and} \quad \forall \zeta, \qquad \tilde{\rho}(\zeta) \geqslant 0 \text{ with } \tilde{\rho}(0) = 1. \tag{4.5}$$

It is readily seen that any limit point $\tilde{L}$, $\tilde{\rho}$ of a sequence $(\tilde{L}_{\varepsilon_n}, \tilde{\rho}_{\varepsilon_n})$ as ε_n goes to zero is a global minimum solution to problem (P). This is a consequence of the fact that the function f is nonnegative.

5. Differentiability of the Scaled Temperature with Respect to the Shape

5.1. *The Speed Method*

We make use of the techniques introduced by J. Cea [1, 2] (cf. J.-P. Zolésio [12–15]). Let V be a time-dependent regular vector field defined in a neighborhood U of $\bar{D}$ ($\bar{D}$, the closure of the domain D in Figure 2C) and for any point x in D consider the solution x to the differential equation

$$\frac{d}{dt}x(t, X) = V(t, x(t, X)), \qquad x(0, X) = X. \tag{5.1}$$

Then there exists a number $t_1 > 0$ such that, for all X in D, the solution $x(t, X)$ is defined for all t in $[0, t_1]$. Consider the transformation

$$T_t(V): X \to x(t, X).$$

It changes the domain D into a new domain

$$D_t = T_t(V)(D), \qquad 0 \leqslant t < t_1, \tag{5.2}$$

where

$$D_t = \{x | x = x(t, X), X \in D\}.$$

It can be shown that if V is chosen in $C^0(0,1; C^k(U; \mathbb{R}^3))$, the transformation $T_t(V)$ is a C^k-diffeomorphism from a neighborhood of D into its ring (cf. J.-P. Zolésio [12, 14]). The field is called the *deformation speed*.

5.2. *Choice of the Speed V*

The initial three-dimensional problem consists in specifying the domain Ω. But Ω is z-axisymmetrical and the scaled domain

$$\tilde{\Omega} = \left\{(\xi_1, \xi_2, \zeta) | \xi_1 = \frac{x_1}{R_0}, \xi_2 = \frac{x_2}{R_0}, \zeta = \frac{z}{L}, (x_1, x_2, z) \in \Omega\right\} \tag{5.3}$$

is completely specified by its boundary $\tilde{\Sigma}_2$ which is obtained by revolving the graph of the function $\tilde{\rho}$ about the ζ-axis.

When $\tilde{\Omega}$ is "deformed" into $\tilde{\Omega}_t$ by the speed V, the boundary $\tilde{\Sigma}_2$ must remain a ζ-axisymmetric graph. So we shall only consider speeds V of the following special form:

$$V(\rho,\zeta) = \begin{pmatrix} \omega(\rho,\zeta) \\ 0 \end{pmatrix}, \qquad \rho = \sqrt{\xi_1^2 + \xi_2^2}. \tag{5.4}$$

In the deformation the ζ-axis and the boundary $\tilde{\Sigma}_1$ both remain fixed, that is,

$$\omega(0,\zeta) = 0, \qquad \forall\, \zeta \text{ in } [0,1] \tag{5.5}$$

$$\omega(\rho,0) = 0, \qquad \exists\, \rho \text{ in } [0,1]. \tag{5.6}$$

5.3. *Derivation with Respect to the Shape*

The scaled temperature $y = y(\tilde{\Omega})$ is defined on the domain $\tilde{\Omega}$. In the "deformed" or "perturbed" domain $\tilde{\Omega}_t$, the new scaled temperature $y_t = y(\tilde{\Omega}_t)$ is the solution of the following variational problem: to find y_t in $H_0(\tilde{\Omega}_t)$ such that

$$\int_{\tilde{\Omega}_t} \left[\tilde{L}^2 \left(\frac{\partial y_t}{\partial \xi_1} \frac{\partial \psi}{\partial \xi_1} + \frac{\partial y_t}{\partial \xi_2} \frac{\partial \psi}{\partial \xi_2} \right) + \frac{\partial y_t}{\partial \zeta} \frac{\partial \psi}{\partial \zeta} \right] d\xi\, d\zeta = \int_{\tilde{\Sigma}_1} \psi\, d\tilde{\Sigma}_1 \tag{5.7}$$

for all ψ in $H_0(\tilde{\Omega}_t)$, where

$$H_0(\tilde{\Omega}_t) = \{\psi \in H^1(\tilde{\Omega}_t) | \psi = 0 \text{ on } \tilde{\Sigma}_3\}. \tag{5.8}$$

Assume that the angles θ_1 and θ_3 (cf. sec. 3.4) are such that the solution y_t of (3.7) is itself smooth in a neighborhood of the curves $C_1 = \tilde{\Sigma}_2 \cap \tilde{\Sigma}_1$ and $C_3 = \tilde{\Sigma}_2 \cap \tilde{\Sigma}_3$. Then the smoothness of y_t in the domain $\tilde{\Omega}_t$ is given by its smoothness in an arbitrarily small neighborhood of the surface $\tilde{\Sigma}_2$. So further assume that the boundary $\tilde{\Sigma}_2$ (that is, the functions $\tilde{\rho}$ and ω) is sufficiently smooth in order to have the solution

$$y_t = y(\tilde{\Omega}_t) \in H^2(\tilde{\Omega}_t) \cap H_0(\tilde{\Omega}_t). \tag{5.9}$$

Any function in $H^2(\tilde{\Omega}_t)$ may be extended to a function in $H^2(\mathbb{R}^3) \to H^2(\mathbb{R}^3)$. So look at $y_t = y(\tilde{\Omega}_t)$ as the restriction to $t \times \tilde{\Omega}_t$ of a smooth function Y defined on $[0, t_1] \times \mathbb{R}^3$ (for some $t_1 > 0$) such that for each t in $[0, t_1]$ the map

$$x \to Y(t,x) : \mathbb{R}^3 \to \mathbb{R}$$

belongs to $H^2(\mathbb{R}^3)$. It is known from J.-P. Zolésio [12, 14], that Y may be chosen in such a way that

$$t \to Y(t,\cdot) : [0, t_1] \to H^1(\mathbb{R}^3)$$

is differentiable. Introduce the notation

$$Y' = \left.\frac{\partial Y}{\partial t}\right|_{t=0,\, x\in\tilde{\Omega}}. \tag{5.10}$$

Again from J. -P. Zolésio [12, 14], Y' is independent of the choice of the regular extension Y and we call Y' the *derivative* of $y = y(\tilde{\Omega})$ *for the* domain $\tilde{\Omega}$ *in the direction* V.

We shall now specify the boundary value problem of which Y' is a solution in the domain $\tilde{\Omega}$. This will require the introduction of elements of tangential differential calculus on the surface $\tilde{\Sigma}_2$.

5.4. *Tangential Calculus*

Let Σ be the boundary of a smooth bounded domain Ω, n be a normal vector field defined on Σ, W be a smooth vector field defined in a neighborhood of Σ, and DW be its Jacobian matrix.

It is known (cf. J.-P. Zolésio [12, 14]) that the expression

$$\operatorname{div} W - \langle DW.n, n\rangle \tag{5.11}$$

only depends on the restriction $W|_\Sigma$ of the field W to the boundary Σ. Here "$\cdot$" and "$\langle\,,\rangle$" denote the product of a matrix by a vector and the inner product of two vectors, respectively.

So define the *tangential divergence*

$$\operatorname{div}_T W = \operatorname{div} W - \langle DW.,n\rangle \text{ on } \Sigma. \tag{5.12}$$

It defines a differential operator on the surface Σ. This tangential divergence can also be defined by transposition in the following way. Given any φ in $H^s(\Sigma)$, $s > 1$, there exists ψ in $H^{s+1/2}(\Omega)$ such that

$$\begin{cases} \psi = \varphi & \text{on } \Sigma \\ \dfrac{\partial\psi}{\partial n} = 0 & \text{on } \Sigma. \end{cases} \tag{5.13}$$

The gradient $\nabla\psi$ on Σ is a tangent vector and is in fact independent of the choice of the function ψ. So define the *tangential gradient* of φ on the surface Σ as

$$\nabla_T\varphi = \nabla\psi|_\Sigma \in H^{s-1}(\Sigma), \tag{5.14}$$

where φ is a given function only defined on Σ.

From J.-P. Zolésio [12], we have

$$\int_\Sigma \operatorname{div}_T W\varphi\, d\sigma = -\int_\Sigma \langle W, \nabla_T\varphi\rangle\, d\sigma - \int_\Sigma \langle H\varphi W, n\rangle\, d\sigma, \tag{5.15}$$

where H is the "mean curvature" of the surface Σ (here we pick $H = (k_1 + k_2)/2$, where k_1 and k_2 are the upper and lower bounds of the normal curvatures on Σ, see, for instance, Do Carmo [1]).

5.5 *Boundary-value Problem for Y'*

Again from J.-P. Zolésio [12, 14], the shape derivative Y' of the scaled temperature $y(\tilde{\Omega})$ in the direction V is the solution of the following variational problem: to find Y' in $H_0(\tilde{\Omega})$ such that

$$\int_{\tilde{\Omega}}\left[\tilde{L}^2\left(\frac{\partial Y'}{\partial \xi_1}\frac{\partial \varphi}{\partial \xi_1}+\frac{\partial Y'}{\partial \xi_2}\frac{\partial \varphi}{\partial \xi_2}\right)+\frac{\partial Y'}{\partial \zeta}\frac{\partial \varphi}{\partial \zeta}\right]d\xi\, d\zeta$$

$$= -\int_{\tilde{\Sigma}_2}\left[\tilde{L}^2\left(\frac{\partial y}{\partial \xi_1}\frac{\partial \varphi}{\partial \xi_1}+\frac{\partial y}{\partial \xi_2}\frac{\partial \varphi}{\partial \xi_2}\right)+\frac{\partial y}{\partial \zeta}\frac{\partial \varphi}{\partial \zeta}\right]\langle V, \nu\rangle\, d\sigma \tag{5.16}$$

for all φ in $H^{3/2+\sigma}(\mathbb{R}^3)$, $\sigma > 0$ arbitrary, such that $\varphi = 0$ on $\tilde{\Sigma}_3$.

Remark 5.1. It is known from J. -P. Zolésio [14] that for $s \geqslant 2$

$$y \in H^s(\tilde{\Omega}) \cap H_0(\tilde{\Omega}) \Rightarrow Y' \in H^{s-1}(\tilde{\Omega}) \cap H_0(\tilde{\Omega}). \tag{5.17}$$

It follows from (5.16) that Y' is the solution of the following boundary-value problem:

$$\left.\begin{aligned} AY' &= 0 \quad \text{in } \tilde{\Omega} \\ Y' &= 0 \quad \text{on } \tilde{\Sigma}_3 \\ \frac{\partial Y'}{\partial \nu_A} &= 0 \quad \text{on } \tilde{\Sigma}_1 \end{aligned}\right\} \tag{5.18}$$

$$\left.\begin{aligned} &\int_{\tilde{\Sigma}_2}\frac{\partial Y'}{\partial \nu_A}\varphi\, d\sigma = -\int_{\tilde{\Sigma}_2}\left[\tilde{L}^2\left(\frac{\partial y}{\partial \xi_1}\frac{\partial \varphi}{\partial \xi_1}+\frac{\partial y}{\partial \xi_2}\frac{\partial \varphi}{\partial \xi_2}\right)+\frac{\partial y}{\partial \zeta}\frac{\partial \varphi}{\partial \zeta}\right]\langle V, \nu\rangle\, d\sigma \\ &\forall \varphi \in H^{3/2+\sigma}(\tilde{\Omega}), \text{ such that } \exists \text{ a neighbourhood } V_2 \text{ of } \tilde{\Sigma}_2 \text{ where} \\ &\varphi = 0 \text{ on } V_2 \cap \tilde{\Sigma}_3. \end{aligned}\right\} \tag{5.19}$$

For $0 < \sigma < \frac{1}{2}$, the closure in $H^{3/2+\sigma}(\tilde{\Omega})$ of the set of functions

$$\{\varphi \in H^{3/2+\sigma}(\tilde{\Omega}) | \exists \text{ a neighborhood } V_2 \text{ of } \tilde{\Sigma}_2 \text{ such that } \varphi = 0 \text{ on } V_2 \cap \tilde{\Sigma}_3\} \tag{5.20}$$

coincides with

$$\Phi = \{\varphi \in H^{3/2+\sigma}(\tilde{\Omega}) | \varphi = 0 \text{ on } \tilde{\Sigma}_2 \cap \tilde{\Sigma}_3\}. \tag{5.21}$$

Recall that the injection

$$H^{3/2+\sigma}(\tilde{\Omega}) \to C^0(\bar{\tilde{\Omega}}) \tag{5.22}$$

is continuous and that the evaluation map

$$\varphi \to \varphi|_{\tilde{\Sigma}_2 \cap \tilde{\Sigma}_3} : C^0(\bar{\tilde{\Omega}}) \to C^0(\tilde{\Sigma}_2 \cap \tilde{\Sigma}_3) \tag{5.23}$$

is also continuous. For σ in $[0, \frac{1}{2}]$ there is no condition on the partial derivatives of φ on $\tilde{\Sigma}_2 \cap \tilde{\Sigma}_3$. As a result (5.19) is true for all φ in Φ as defined in (5.21):

$$\left.\begin{aligned} &\int_{\tilde{\Sigma}_2} \frac{\partial Y'}{\partial \nu_A} \varphi \, d\sigma = -\int_{\tilde{\Sigma}_2} \left[\tilde{L}^2 \left(\frac{\partial y}{\partial \xi_1} \frac{\partial \varphi}{\partial \xi_1} + \frac{\partial y}{\partial \xi_2} \frac{\partial \varphi}{\partial \xi_2} \right) + \frac{\partial y}{\partial \zeta} \frac{\partial \varphi}{\partial \zeta} \right] \langle V, \nu \rangle \, d\sigma \\ &\forall \varphi \in H^{3/2+\sigma}(\tilde{\Omega}) \text{ such that } \varphi = 0 \text{ on } \tilde{\Sigma}_2 \cap \tilde{\Sigma}_3, \qquad 0 < \sigma \leqslant \tfrac{1}{2}. \end{aligned}\right\} \tag{5.24}$$

We now show that condition (5.24) is a Neuman condition on the boundary $\tilde{\Sigma}_2$. To see that we introduce the matrix

$$\mathcal{Q} = \begin{bmatrix} \tilde{L}^2 & 0 & 0 \\ 0 & \tilde{L}^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.25}$$

and perform the following computation on the expression

$$E = -\int_{\tilde{\Sigma}_2} \left[\tilde{L}^2 \left(\frac{\partial y}{\partial \xi_1} \frac{\partial \varphi}{\partial \xi_1} + \frac{\partial y}{\partial \xi_2} \frac{\partial \varphi}{\partial \xi_2} \right) + \frac{\partial y}{\partial \zeta} \frac{\partial \varphi}{\partial \zeta} \right] \langle V, \nu \rangle \, d\sigma. \tag{5.26}$$

By rewriting

$$E = -\int_{\tilde{\Sigma}_2} \langle \langle V, \nu \rangle \mathcal{Q} \cdot \nabla y, \nabla \varphi \rangle \, d\sigma, \tag{5.27}$$

where $\mathcal{Q} \cdot \nabla y$ is the vector

$$\left(\tilde{L}^2 \frac{\partial y}{\partial \xi_1}, \tilde{L}^2 \frac{\partial y}{\partial \xi_2}, \frac{\partial y}{\partial \zeta} \right) \tag{5.28}$$

Define the field

$$W = \langle V, \nu \rangle \mathcal{Q} \cdot \nabla y \tag{5.29}$$

and rewrite the right-hand side of (5.27) as

$$\begin{aligned} E &= -\int_{\tilde{\Sigma}_2} \langle W, \nabla \varphi \rangle \, d\sigma \\ &= -\int_{\tilde{\Sigma}_2} \langle W, \nabla_T \varphi \rangle \, d\sigma - \int_{\tilde{\Sigma}_2} \left\langle W, \frac{\partial \varphi}{\partial \nu} \nu \right\rangle d\sigma. \end{aligned} \tag{5.30}$$

The second term on the right-hand side of (5.30) can be rewritten in the form

$$\int_{\tilde{\Sigma}_2} \frac{\partial \varphi}{\partial \nu} \langle W, \nu \rangle \, d\sigma. \tag{5.31}$$

But $\langle W, \nu \rangle$ is zero and the integral (5.31) is also equal to zero: by definition of W

$$\begin{aligned} \langle W, \nu \rangle &= \langle V, \nu \rangle \langle \nu, \mathcal{A} \cdot \nabla y \rangle \\ &= \langle V, \nu \rangle \langle \mathcal{A} \cdot \nu, \nabla y \rangle = \langle V, \nu \rangle \frac{\partial y}{\partial \nu_A} = 0 \end{aligned}$$

on $\tilde{\Sigma}_2$. As for the first term on the right-hand side of (5.30), it can be evaluated with the help of identity (5.15):

$$\int_{\tilde{\Sigma}} \operatorname{div}_T W \varphi \, d\sigma = -\int_{\tilde{\Sigma}} \langle W, \nabla_T \varphi \rangle \, d\sigma - \int_{\tilde{\Sigma}} H \varphi \langle W, \nu \rangle \, d\sigma \tag{5.32}$$

with the vector field W chosen as in (5.29). On $\tilde{\Sigma}_1 \cup \tilde{\Sigma}_3$, W is zero since $\langle V, \nu \rangle$ is zero by our choice of the speed V. So we can substitute $\tilde{\Sigma}_2$ for $\tilde{\Sigma}$ in identity (5.32). Moreover, the second term in that identity is zero over $\tilde{\Sigma}_2$ since we have already shown that $\langle W, \nu \rangle$ is zero on $\tilde{\Sigma}_2$. Finally we obtain the following chain of identities:

$$\int_{\tilde{\Sigma}_2} \operatorname{div}_T W \varphi |_{\tilde{\Sigma}_2} \, d\sigma = -\int_{\tilde{\Sigma}_2} \langle W, \nabla_T \varphi \rangle \, d\sigma = -\int_{\tilde{\Sigma}_2} \langle W, \nabla \varphi \rangle \, d\sigma \tag{5.33}$$

for all φ in $H^{3/2+\sigma}(\tilde{\Omega})$. Notice that the above identities do not require that φ be zero on $\tilde{\Sigma}_3$. From identities (5.19) and (5.27)

$$\int_{\tilde{\Sigma}_2} \frac{\partial Y'}{\partial \nu_A} \varphi \, d\sigma = -\int_{\tilde{\Sigma}_2} \langle W, \nabla \varphi \rangle \, d\sigma, \forall \varphi \in H_0(\tilde{\Omega}) \cap H^{3/2+\sigma}(\tilde{\Omega}). \tag{5.34}$$

Combining identities (5.33) and (5.34)

$$\int_{\tilde{\Sigma}_2} \frac{\partial Y'}{\partial \nu_A} \varphi \bigg|_{\tilde{\Sigma}_2} d\sigma = \int_{\tilde{\Sigma}_2} \operatorname{div}_T W \varphi \bigg|_{\tilde{\Sigma}_2} d\sigma, \forall \varphi \in H_0(\tilde{\Omega}) \cap H^{3/2+\sigma}(\tilde{\Omega}). \tag{5.35}$$

For each h in $\mathcal{D}(\tilde{\Sigma}_2)$, there exists φ in $H_0(\tilde{\Omega}) \cap H^{3/2+\sigma}(\tilde{\Omega})$ such that

$$A\varphi = 0, \qquad \varphi \bigg|_{\tilde{\Sigma}_3} = 0, \frac{\partial \varphi}{\partial \nu_A}\bigg|_{\tilde{\Sigma}_2} = h, \qquad \frac{\partial \varphi}{\partial \nu_A}\bigg|_{\tilde{\Sigma}_1} = 0.$$

As a result

$$\int_{\tilde{\Sigma}_2} \frac{\partial Y'}{\partial \nu_A} h \, d\sigma = \int_{\tilde{\Sigma}_2} \operatorname{div}_T W h \, d\sigma, \qquad \forall h \in \mathcal{D}(\tilde{\Sigma}_2). \tag{5.36}$$

But for y in $H^{2+\varepsilon}(\tilde{\Omega})$, $\varepsilon \geqslant 0$, $W \in H^{1+\varepsilon}(\tilde{\Omega})$ and $\mathrm{div}_T W \in H^{-1/2+\varepsilon}(\tilde{\Sigma}_2)$ (use the fact that y is harmonic to define the trace $\mathrm{div}_T W$). So the right-hand side of (5.36) is continuous for the $H^{1/2-\varepsilon}(\tilde{\Sigma}_2)$-topology. By density of $\mathcal{D}(\tilde{\Sigma}_2)$ in $H^{1/2-\varepsilon}(\tilde{\Sigma}_2)$ for $\varepsilon \geqslant 0$, identity (5.36) holds for all h in $H^{1/2-\varepsilon}(\tilde{\Sigma}_2)$ and necessarily

$$\frac{\partial Y'}{\partial \nu_A} = \mathrm{div}_T W = \mathrm{div}_T(\mathcal{C}.\nabla y\langle V, \nu\rangle) \text{ in } H^{-1/2+\varepsilon}(\tilde{\Sigma}_2). \tag{5.37}$$

We summarize the following useful relations:

$$\left.\begin{array}{ll} \nu_A = \mathcal{C}.\nu & \text{(conormal associated with the operator } A \text{ of (3.11) and the normal vector } \nu) \\ \dfrac{\partial \varphi}{\partial \nu_A} = \langle \mathcal{C}.\nu, \nabla\varphi\rangle = \langle \nu_A, \nabla\varphi\rangle & \\ A\varphi = -\mathrm{div}(\mathcal{C}.\nabla\varphi). & \end{array}\right\} \tag{5.38}$$

If the *cotangential divergence* (with respect to the operator A) is defined as

$$\mathrm{div}_{T_A}(\vec{E}) = \mathrm{div}_T(\mathcal{C}.\vec{E}), \tag{5.39}$$

where $\vec{E}$ is a vector function from $\tilde{\Sigma}$ into $\mathbb{R}^3$, then (5.37) is equivalent to

$$\frac{\partial Y'}{\partial \nu_A} = \mathrm{div}_{T_A}(\nabla y\langle V, \nu\rangle). \tag{5.40}$$

6. Eulerian Derivative of the Cost and the Constraint

Recall from sec. 3.1 that the cost function to be minimized is of the form

$$J(\tilde{L}, \tilde{\rho}) = \tilde{L}J(\tilde{\Omega}), \tag{6.1}$$

where

$$J(\tilde{\Omega}) = \pi\int_0^1 \tilde{\rho}(\zeta)^2\, d\zeta \left(= \int_{\tilde{\Omega}} d\xi_1\, d\xi_2\, d\zeta, \text{ the volume of } \tilde{\Omega} \right). \tag{6.2}$$

Following J.-P. Zolésio [12, 14], define (whenever it exists) the *Eulerian derivative* $dJ(\tilde{\Omega}; V)$ (or *shape derivative*) at $\tilde{\Omega}$ in the direction of the field V as

$$dJ(\tilde{\Omega}; V) = \lim_{t \searrow 0} \left[J(\tilde{\Omega}_t) - J(\tilde{\Omega}) \right]/t. \tag{6.3}$$

6.1. *Shape Derivative of the Volume*

From J.-P. Zolésio [12] the shape derivative of $J(\tilde{\Omega})$ is

$$dJ(\tilde{\Omega}; V) = \int_{\tilde{\Sigma}} \langle V, \nu \rangle \, d\sigma, \tag{6.4}$$

where $\langle V, \nu \rangle$ is the normal component of the field V on $\tilde{\Sigma}$. It is equal to zero on $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_3$ and

$$\langle V, \nu \rangle = \omega \nu_r \text{ on } \tilde{\Sigma}_2. \tag{6.5}$$

Finally

$$\begin{aligned} dJ(\tilde{L}, \tilde{\rho}; V) &= \tilde{L}\, dJ(\tilde{\Omega}; V) = \tilde{L} \int_{\tilde{\Sigma}} \omega \nu_r \, d\sigma \\ &= \tilde{L} \int_0^1 2\pi \omega(\tilde{\rho}(\zeta), \zeta) \tilde{\rho}(\zeta) \, d\zeta. \end{aligned} \tag{6.6}$$

6.2. *Shape Derivative of the Constraint Functional f*

Recall that

$$f(\tilde{L}, \tilde{\rho}) = \int_{\tilde{\Sigma}_3} \left[\frac{\partial y}{\partial \zeta} + q \right]^- d\sigma = 2\pi \int_0^{\tilde{\rho}(1)} \left[\frac{\partial y}{\partial \zeta}(\rho, 1) + q \right]^- \rho \, d\rho \tag{6.7}$$

and that $Y(t, \rho, \zeta)$ is a smooth extension of $y = y(\tilde{\Omega})$. As a result for $t > 0$,

$$f(\tilde{L}, \tilde{\rho}_t) = 2\pi \int_0^{\tilde{\rho}_t(1)} \left[\frac{\partial y}{\partial \zeta}(t; \rho, 1) + q \right]^- \rho \, d\rho. \tag{6.8}$$

We consider two cases:

1) the constraint is saturated everywhere on the boundary $\tilde{\Sigma}_3$,

$$-\left[\frac{\partial y}{\partial \zeta}(\rho, 1) + q \right] > 0, \qquad \forall \rho, \quad 0 \leqslant \rho < \tilde{\rho}(1), \tag{6.9}$$

2) the constraint is not saturated on a subset of the boundary $\tilde{\Sigma}_3$ of non-zero measure.

In both cases we know that

$$\int_{\tilde{\Sigma}} \frac{\partial y}{\partial \nu_A} d\sigma = 0 \Rightarrow -\int_{\tilde{\Sigma}_3} \frac{\partial y}{\partial \nu_A} d\nu = \int_{\tilde{\Sigma}_1} \frac{\partial y}{\partial \nu_A} d\sigma = \pi. \tag{6.10}$$

As a result

$$-\frac{\partial y}{\partial \nu_A}(\rho,1) \leqslant q, \qquad \forall 0 \leqslant \rho \leqslant \tilde{\rho}(1) \Rightarrow \tilde{\rho}(1) \geqslant \sqrt{1/q} \tag{6.11}$$

$$-\frac{\partial y}{\partial \nu_A}(\rho,1) > q, \qquad \forall 0 \leqslant \rho < \tilde{\rho}(1) \Rightarrow \tilde{\rho}(1) < \sqrt{1/q}\,. \tag{6.12}$$

The condition

$$\tilde{\rho}(1) \geqslant \sqrt{1/q} \tag{6.13}$$

is *necessary* in order to satisfy the constraint on the boundary $\tilde{\Sigma}_3$. In case 1 condition (6.13) will be violated. In order to eliminate that situation we make the following hypothesis.

Hypothesis 6.1. The shape function $\tilde{\rho}$ will satisfy the following conditions

$$\tilde{\rho}(0) = 1 \quad \text{and} \quad \tilde{\rho}(1) \geqslant \sqrt{1/q}\,. \tag{6.14}$$

Remark 6.1. In case 1

$$\begin{aligned} f(\tilde{L},\tilde{\rho}) &= \int_{\tilde{\Sigma}_3}\left[\frac{\partial y}{\partial \nu_A} + q\right]^- d\sigma = -\int_{\tilde{\Sigma}_3}\frac{\partial y}{\partial \nu_A}\,d\sigma - q\pi\tilde{\rho}(1)^2 \\ &= \pi\left(1 - q\tilde{\rho}(1)^2\right) \end{aligned} \tag{6.15}$$

and

$$df(\tilde{L},\tilde{\rho};V) = -q2\pi\tilde{\rho}(1)\omega(\tilde{\rho}(1),1), \tag{6.16}$$

where $V=(\omega,0)$ is the speed of the points in the domain. In this situation the gradient says that the point $(1,\tilde{\rho}(1))$ has to be moved away from the ζ-axis, but *it does not say* what should be done with the remainder, $\{(\zeta,\tilde{\rho}(\zeta)) : 0 \leqslant \zeta < 1\}$, of the boundary $\tilde{\Sigma}_2$. Hypothesis 6.1 will ensure that case 1 and this situation do not occur.

In the second case, we introduce the following hypothesis.

Hypothesis 6.2. There exists a neighborhood N of the curve $C_3 = \tilde{\Sigma}_2 \cap \tilde{\Sigma}_3$ such that

$$\left[\frac{\partial y}{\partial \zeta}(\sigma) + q\right] = 0, \qquad \forall \sigma \in \tilde{\Sigma}_3 \cap N. \tag{6.17}$$

Under that hypothesis the integrand of the constraint (6.7) is zero in the neighborhood of $\tilde{\rho}(1)$. So there will be no boundary term on C_3 and the directional derivative of $f(\tilde{L}, \tilde{\rho}_t; V)$ at $t = 0$ will be given by

$$df(\tilde{L}, \tilde{\rho}; V) = \frac{d}{dt}\left\{2\pi \int_0^{\tilde{\rho}(1)} \left[\frac{\partial y}{\partial \zeta}(t; \rho, 1) + q\right]^- \rho \, d\rho \right\}_{t=0}, \tag{6.18}$$

where d/dt denotes the right-hand side derivative. By using general results in J.-P. Zolésio [12, 14], we know that expression (6.13) is equal to

$$-2\pi \int_0^{\tilde{\rho}(1)} \frac{\partial Y'}{\partial \zeta}(\rho, 1)\chi_+(\rho)\rho \, d\rho + 2\pi \int_0^{\tilde{\rho}(1)} \left[\frac{\partial Y'}{\partial \zeta}(\rho, 1)\right]^- \chi_0(\rho)\rho \, d\rho, \tag{6.19}$$

where

$$\chi_+(\rho)(\text{resp. } \chi_0(\rho)) = \begin{cases} 1, & \text{if } \dfrac{\partial y}{\partial \zeta}(\rho, 1) + q < 0 \text{ (resp. } = 0) \\ 0, & \text{otherwise.} \end{cases} \tag{6.20}$$

We introduce another hypothesis in order to obtain the linearity of the derivative $df(\tilde{\Omega}; V)$ with respect to V.

Hypothesis 6.3. For physical reasons, we shall assume that when $0 < q < 1$ the subset of all $\rho, 0 \leqslant \rho \leqslant \tilde{\rho}(1)$, such that

$$-\frac{\partial y}{\partial \nu_A}(\rho, 1) = q \tag{6.21}$$

has zero measure.

So the characteristic function χ_0 is zero almost everywhere in $[0, \tilde{\rho}(1)]$ and the second term in (6.19) is zero. Under Hypotheses 6.1, 6.2, and 6.3

$$df(\tilde{L}, \tilde{\rho}; V) = -2\pi \int_0^{\tilde{\rho}(1)} \frac{\partial Y'}{\partial \zeta}(\rho, 1)\chi_+(\rho)\rho \, d\rho. \tag{6.22}$$

Hypotheses 6.2 and 6.3 can be combined into a single stronger hypothesis.

Hypothesis 6.4. The function

$$\rho \to -\frac{\partial y}{\partial \zeta}(\rho, 1) : [0, \tilde{\rho}(1)] \to \mathbb{R} \tag{6.23}$$

is monotone strictly increasing.

6.3. *Derivative of the Penalized Cost*

Under hypotheses 6.1, 6.2, and 6.3 and in view of (4.3), (6.6) and (6.22), the Eulerian derivative of the penalized cost is given by the following expression:

$$\frac{1}{2\pi} dJ_\varepsilon(\tilde{L}, \tilde{\rho}; V) = \tilde{L} \int_0^1 \omega(\tilde{\rho}(\zeta), \zeta)\tilde{\rho}(\zeta) \, d\zeta - \frac{1}{\varepsilon} \int_0^{\tilde{\rho}(1)} \frac{\partial Y'}{\partial \zeta}(\rho, 1)\chi_+(\rho)\rho \, d\rho. \tag{6.24}$$

7. Derivative with Respect to the Parameter $\tilde{L}$

7.1. *Derivative of the Cost Function*

Trivially

$$\frac{\partial J}{\partial \tilde{L}}(\tilde{L},\tilde{\rho}) = J(\tilde{\Omega}) = \pi\int_0^1 \tilde{\rho}(\zeta)^2\,d\zeta \tag{7.1}$$

7.2. *Derivative of the Scaled Temperature*

Let y be the solution of problem (5.7) for $t=0$. By the implicit function theorem (cf. J. -P. Zolésio [14], G. Chavent [3]) we know that the maps $\tilde{L}\to y:\mathbb{R}_+\to H^2(\tilde{\Omega})$ is differentiable. So $y_L = \partial y/\partial\tilde{L}$ is the solution of the following variational problem: to find y_L in $H_0(\tilde{\Omega})$ such that

$$\begin{aligned}\int_{\tilde{\Omega}}\left[\tilde{L}^2\left(\frac{\partial y_L}{\partial \xi_1}\frac{\partial \psi}{\partial \xi_1}+\frac{\partial y_L}{\partial \xi_2}\frac{\partial \psi}{\partial \xi_2}\right)+\frac{\partial y_L}{\partial \zeta}\frac{\partial \psi}{\partial \zeta}\right]d\xi\,d\zeta\\ = -2\tilde{L}\int_{\tilde{\Omega}}\left(\frac{\partial y}{\partial \xi_1}\frac{\partial \psi}{\partial \xi_1}+\frac{\partial y}{\partial \xi_2}\frac{\partial \psi}{\partial \xi_2}\right)d\xi\,d\zeta\end{aligned} \tag{7.2}$$

for all ψ in $H_0(\tilde{\Omega})$.

We can further specify the boundary-value problem of which y_L is a solution:

$$Ay_L = -\left[\tilde{L}^2\left(\frac{\partial^2 y_L}{\partial \xi_1^2}+\frac{\partial^2 y_L}{\partial \xi_2^2}\right)+\frac{\partial^2 y_L}{\partial \zeta^2}\right] = 2\tilde{L}\left(\frac{\partial^2 y}{\partial \xi_1^2}+\frac{\partial^2 y}{\partial \xi_2^2}\right)\text{ in }\tilde{\Omega}, \tag{7.3}$$

$$\begin{aligned}\frac{\partial y_L}{\partial \nu_A} &= \tilde{L}^2\left(\nu_1\frac{\partial y_L}{\partial \xi_1}+\nu_2\frac{\partial y_L}{\partial \xi_2}\right)+\frac{\partial y_L}{\partial \zeta}\\ &= -2\tilde{L}\left(\nu_1\frac{\partial y}{\partial \xi_1}+\nu_2\frac{\partial y}{\partial \xi_2}\right)\text{ on }\tilde{\Sigma}_1\cup\tilde{\Sigma}_2\end{aligned} \tag{7.4}$$

Using equations (3.11)–(3.12) for y the two equations (7.3)–(7.4) are equivalent to

$$Ay_L = -\frac{2}{\tilde{L}}\frac{\partial^2 y}{\partial \zeta^2} \tag{7.3a}$$

$$\left.\frac{\partial y_L}{\partial \nu_A}\right|_{\tilde{\Sigma}_1} = 1,\ \left.\frac{\partial y_L}{\partial \nu_A}\right|_{\tilde{\Sigma}_2} = \frac{2}{\tilde{L}}\nu_\zeta\frac{\partial y}{\partial \zeta},\ \left.y_L\right|_{\tilde{\Sigma}_3} = 0. \tag{7.4a}$$

For the condition on $\tilde{\Sigma}_1$ we have used the fact that on $\tilde{\Sigma}_1$

$$\nu = (0,0,-1) \Rightarrow \nu_1 = \nu_2 = 0. \tag{7.5}$$

Remark 7.1. In view of the boundary conditions (7.4a), y_L has the same smoothness as y provided the boundary $\tilde{\Sigma}_2$ is sufficiently smooth.

7.3. Derivative of the Constraint Functional

As in section 6.2 we compute the right-hand side derivative:

$$\frac{df}{dL}(\tilde{L},\tilde{\rho}) = -2\pi\int_0^{\tilde{\rho}(1)}\frac{\partial y_L}{\partial \zeta}(\rho,1)\chi_+(\rho)\rho\,d\rho$$
$$+2\pi\int_0^{\tilde{\rho}(1)}\left[\frac{\partial y_L}{\partial \zeta}(\rho,1)+q\right]^-\chi_0(\rho)\rho\,d\rho, \tag{7.6}$$

where χ_0 and χ_+ are as defined in (6.11). For physical reasons we have already assumed that χ_0 is zero almost everywhere (cf. Hypothesis 6.3). As a result

$$\frac{d}{dL}f(\tilde{L},\tilde{\rho}) = -2\pi\int_0^{\tilde{\rho}(1)}\frac{\partial y_L}{\partial \zeta}(\rho,1)\chi_+(\rho)\rho\,d\rho. \tag{7.7}$$

7.4. Derivative of the Penalized Cost

In view of (4.3), (7.1) and (7.6)

$$\frac{1}{2\pi}\frac{\partial J_\varepsilon}{\partial \tilde{L}}(\tilde{L},\tilde{\rho}) = \frac{1}{2}\int_0^1\tilde{\rho}^2(\zeta)\,d\zeta - \frac{1}{\varepsilon}\int_0^{\tilde{\rho}(1)}\frac{\partial y_L}{\partial \zeta}(\rho,1)\chi_+(\rho)\rho\,d\rho. \tag{7.8}$$

8. Gradient Computations

Recall expressions (6.13) for the derivative of the penalized cost with respect to the shape of the volume $\tilde{\Omega}$ and expression (7.8) for the derivative of the penalized cost with respect to the parameter $\tilde{L}$. They both involve a term of the form

$$\lambda(\psi) = -\int_{\tilde{\Sigma}_3}\frac{\partial \psi}{\partial \nu_A}\chi_+\,d\sigma = -2\pi\int_0^{\tilde{\rho}(1)}\frac{\partial \psi}{\partial \zeta}(\rho,1)\chi_+(\rho)\rho\,d\rho \tag{8.1}$$

with $\psi = Y'$ and $\psi = y_L$, respectively.

Following a standard procedure we shall now introduce the solution p of an appropriate adjoint system and express $\lambda(Y')$ and $\lambda(y_L)$ as a function of y, p and the speed V. The following bilinear forms will be useful in the forthcoming discussion:

$$a(\psi,\varphi) = \int_{\tilde{\Omega}}\left[\tilde{L}^2\left(\frac{\partial \psi}{\partial \xi_1}\frac{\partial \varphi}{\partial \xi_1}+\frac{\partial \psi}{\partial \xi_2}\frac{\partial \varphi}{\partial \xi_2}\right)+\frac{\partial \psi}{\partial \zeta}\frac{\partial \varphi}{\partial \zeta}\right]d\xi\,d\zeta \tag{8.2}$$
$$(d\xi = d\xi_1\,d\xi_2),$$

$$b(\psi,\varphi) = -2\tilde{L}\int_{\tilde{\Omega}}\left(\frac{\partial \psi}{\partial \xi_1}\frac{\partial \varphi}{\partial \xi_1}+\frac{\partial \psi}{\partial \xi_2}\frac{\partial \varphi}{\partial \xi_2}\right)d\xi\,d\zeta \tag{8.3}$$

and

$$B(\psi,\varphi) = -\int_{\tilde{\Sigma}_2}\left[\tilde{L}^2\left(\frac{\partial\psi}{\partial\xi_1}\frac{\partial\varphi}{\partial\xi_1}+\frac{\partial\psi}{\partial\xi_2}\frac{\partial\varphi}{\partial\xi_2}\right)+\frac{\partial\psi}{\partial\zeta}\frac{\partial\varphi}{\partial\zeta}\right]\langle V,\nu\rangle\,d\sigma \tag{8.4}$$

8.1. Adjoint Equation

Let p be the solution of the system

$$\left.\begin{aligned}
&Ap = -\left[\tilde{L}^2\left(\frac{\partial^2 p}{\partial\xi_1^2}+\frac{\partial^2 p}{\partial\xi_2^2}\right)+\frac{\partial^2 p}{\partial\zeta^2}\right] = 0 \text{ in } \tilde{\Omega}\\
&p = \chi_+ \text{ on } \tilde{\Sigma}_3\\
&\frac{\partial p}{\partial\nu_A} = 0 \text{ on } \tilde{\Sigma}_1\cup\tilde{\Sigma}_2.
\end{aligned}\right\} \tag{8.5}$$

The function χ_+ is the characteristic function of a disk centered in $r=0$ on the surface $\tilde{\Sigma}_3$. So we only have

$$\chi_+ \in H^{1/2-\varepsilon}(\tilde{\Sigma}_3), \qquad \forall\varepsilon > 0. \tag{8.6}$$

As a result p belongs to $H^{1-\varepsilon}(\tilde{\Omega})$ and even to $D_A^{1-\varepsilon}(\tilde{\Omega})$, where for $0<s<2$,

$$\left.\begin{aligned}
&D_A^s(\tilde{\Omega}) = \{v\in H^s(\tilde{\Omega}): Av\in \Xi E^{s-2}(\tilde{\Omega})\}\\
&\Xi^{s-2}(\tilde{\Omega}) = (\Xi^{2-s}(\tilde{\Omega}))'(' = \text{topological dual})\\
&\Xi^{2-s}(\tilde{\Omega}) = \{v\in \mathcal{D}'(\tilde{\Omega})': d^{2-s}v\in H_0^{2-s}(\tilde{\Omega})\}
\end{aligned}\right\} \tag{8.7}$$

and $d(x,\tilde{\Sigma})$ is the distance from x to the boundary $\tilde{\Sigma}$ (cf. Lions and Magenes [11, Vol. 1, pp. 183, 199]).

We shall express $\lambda(\psi)$ as a function of p. For all φ and ψ in $H^2(\tilde{\Omega})$ the following identity holds

$$(A\psi,\varphi)_{L^2(\tilde{\Omega})}+\left(\frac{\partial\psi}{\partial\nu_A},\varphi\right)_{L^2(\tilde{\Sigma})} = (\psi,A\varphi)_{L^2(\tilde{\Omega})}+\left(\psi,\frac{\partial\varphi}{\partial\nu_A}\right)_{L^2(\tilde{\Sigma})}. \tag{8.8}$$

When φ and ψ are such that

$$A\varphi = 0, \frac{\partial\varphi}{\partial\nu_A} = 0 \text{ on } \tilde{\Sigma}_1\cup\tilde{\Sigma}_2 \text{ and } \psi|_{\tilde{\Sigma}_3} = 0, \tag{8.9}$$

the right hand side of identity (8.8) is zero and the following new identity holds

$$-\left(\frac{\partial\psi}{\partial\nu_A},\varphi\right)_{L^2(\tilde{\Sigma}_3)} = (A\psi,\varphi)_{L^2(\tilde{\Omega})}+\left(\frac{\partial\psi}{\partial\nu_A},\varphi\right)_{L^2(\tilde{\Sigma}_1\cup\tilde{\Sigma}_2)}. \tag{8.10}$$

By continuity the right-hand side of (8.10) makes sense for all φ in $H^{1-\varepsilon}(\tilde{\Omega})$, $\varepsilon < 1/2$, for which identities (8.9) are verified. In particular one can pick $\varphi = p$, the solution of the adjoint system (8.5). Finally

$$\lambda(\psi) = -\left(\frac{\partial\psi}{\partial\nu_A}, \chi_+\right)_{L^2(\tilde{\Sigma}_3)} = (A\psi, p)_{L^2(\tilde{\Omega})} + \left(\frac{\partial\psi}{\partial\nu_A}, p\right)_{L^2(\tilde{\Sigma}_1 \cup \tilde{\Sigma}_2)} \tag{8.11}$$

for all ψ in $H^2(\tilde{\Omega}) \cap H_0(\tilde{\Omega})$.

It is interesting that the right-hand side of expression (8.11) is continuous for ψ in $D_A^{1+\varepsilon}(\tilde{\Omega}) \cap H_0(\tilde{\Omega})$ since p belongs to $H^{1-\varepsilon}(\tilde{\Omega})$ and $p|_{\tilde{\Sigma}_1 \cup \tilde{\Sigma}_2}$ to $H^{1/2-\varepsilon}(\tilde{\Sigma}_1 \cup \tilde{\Sigma}_2)$. However in that case expression (8.11) must be written with duality products:

$$\lambda(\psi) = -\left\langle \frac{\partial\psi}{\partial\nu_A}, \chi_+\right\rangle_{H^{1/2-\varepsilon}(\tilde{\Sigma}_3)} = \langle A\psi, p\rangle_{\Xi^{1-\varepsilon}(\tilde{\Omega})} + \left\langle \frac{\partial\psi}{\partial\nu_A}, p\right\rangle_{H^{1/2-\varepsilon}(\tilde{\Sigma}_1 \cup \tilde{\Sigma}_2)} \tag{8.12}$$

for all ψ in $D_A^{1+\varepsilon}(\tilde{\Omega}) \cap H_0(\tilde{\Omega})$.

It is useful to summarize the above discussion in the following proposition.

Proposition 8.1. *For all* ε, $0 < \varepsilon < \frac{1}{2}$, *and all* ψ *in* $D_A^{1+\varepsilon}(\tilde{\Omega}) \cap H_0(\tilde{\Omega})$, *expression* (8.12) *holds. If, in addition,* ψ *belongs to* $H^2(\tilde{\Omega})$, *expression* (8.12) *reduces to expression* (8.11).

Remark 8.1. Equation (8.12) suggests that p could be obtained by transposition of the linear map

$$\psi \to A\psi, \left.\frac{\partial\psi}{\partial\nu_A}\right|_{\tilde{\Sigma}_1 \cup \tilde{\Sigma}_2} : D_A^{1+\varepsilon}(\tilde{\Omega}) \cap H_0(\tilde{\Omega}) \to \Xi^{-1+\varepsilon}(\tilde{\Omega}) \times H^{1/2+\varepsilon}(\tilde{\Sigma}_1 \cup \tilde{\Sigma}_2) \tag{8.13}$$

provided that it is an isomorphism and that the various spaces $D_A^{1+\varepsilon}$, $\Xi^{-1+\varepsilon}$ and $H^{1/2+\varepsilon}$ are well-defined. This would require an adaptation of a result in Lions and Magenes [11, Vol. 1, Thm. 7.4]. With such spaces, the linear map

$$\psi \to \lambda(\psi) = -\left\langle \frac{\partial\psi}{\partial\nu_A}, \chi_+\right\rangle_{H^{1/2-\varepsilon}(\tilde{\Sigma}_1 \cup \tilde{\Sigma}_2)} : D_A^{1+\varepsilon}(\tilde{\Omega}) \cap H_0(\tilde{\Omega}) \to \mathbb{R} \tag{8.14}$$

would be continuous. So by the method of transposition one could claim that there exists a unique pair

$$p_0, p_1 \in \Xi^{1-\varepsilon}(\tilde{\Omega}) \times H^{-1/2-\varepsilon}(\tilde{\Sigma}_1 \cup \tilde{\Sigma}_2) \tag{8.15}$$

such that

$$\langle A\psi, p_0\rangle_{\Xi^{1-\varepsilon}(\tilde{\Omega})} + \left\langle \frac{\partial\psi}{\partial\nu_A}, p_1\right\rangle_{H^{1/2-\varepsilon}(\tilde{\Sigma}_1 \cup \tilde{\Sigma}_2)} = \lambda(\psi) \tag{8.16}$$

for all ψ in $D_A^{1+\varepsilon}(\tilde{\Omega}) \cap H_0(\tilde{\Omega})$. Now by comparison of (8.12) and (8.16) and uniqueness of the solution to (8.16), we should formally obtain the identities

$$p_0 = p \text{ in } \tilde{\Omega} \quad \text{and} \quad p_1 = p|_{\tilde{\Sigma}_1 \cup \tilde{\Sigma}_2}. \tag{8.17}$$

Remark 8.2. When $\chi_+ = 1$ on $\tilde{\Sigma}_3$, the solution p of (8.5) is identically equal to 1 in $\bar{\tilde{\Omega}}$, the closure of $\tilde{\Omega}$.

8.2. Computation of $\lambda(Y')$.

Recall from section 5.5 that Y' is the solution of the boundary value problem

$$\begin{cases} AY' = 0 \text{ in } \tilde{\Omega} \\ \left.\dfrac{\partial Y'}{\partial \nu_A}\right|_{\tilde{\Sigma}_1} = 0, \quad \left.\dfrac{\partial Y'}{\partial \nu_A}\right|_{\tilde{\Sigma}_2} = \mathrm{div}_T(\mathcal{A} \cdot \nabla y\langle V, \nu\rangle), \qquad Y'|_{\tilde{\Sigma}_3 = 0}. \end{cases} \tag{8.18}$$

Also from Remark 5.1 the smoothness of Y' is related to the one of y as follows

$$y \in H^s(\tilde{\Omega}) \Rightarrow Y' \in H^{s-1}(\tilde{\Omega}), \qquad s \geqslant 2,$$

provided that the boundary $\tilde{\Sigma}_2$ is smooth enough and that the angles θ_1 and θ_3 are small enough.

So we assume for the moment that y belongs to $H^3(\tilde{\Omega})$ which means that Y' belongs to $H^2(\tilde{\Omega}) \cap H_0(\tilde{\Omega})$. Then set $\psi = Y'$ in expression (8.11):

$$\lambda(Y') = (AY', p)_{L^2(\tilde{\Omega})} + \left(\frac{\partial Y'}{\partial \nu_A}, p\right)_{L^2(\tilde{\Sigma}_1 \cup \tilde{\Sigma}_2)}. \tag{8.19}$$

In view of (8.18), identity (8.19) reduces to

$$\lambda(Y') = \left(\frac{\partial Y'}{\partial \nu_A}, p\right)_{L^2(\tilde{\Sigma}_2)}. \tag{8.20}$$

We have seen in section 5.5 that (cf. (5.31))

$$\frac{\partial Y'}{\partial \nu_A} = \mathrm{div}_T W, \qquad W = \mathcal{A}.\nabla y\langle V, \nu\rangle \tag{8.21}$$

and that for all φ in $H^{3/2+\sigma}(\tilde{\Omega})$ (cf. (5.27))

$$\int_{\tilde{\Sigma}_2} \mathrm{div}_T W \varphi|_{\tilde{\Sigma}_2}\, d\sigma = -\int_{\tilde{\Sigma}_2} \langle W, \nabla\varphi\rangle\, d\sigma. \tag{8.22}$$

We would like to set $\varphi = p$ in (8.22).

Consider two cases:

(i) $\chi_+ = 1$ a.e. on $\tilde{\Sigma}_3 \Rightarrow p = 1$ in $\tilde{\Omega} \Rightarrow p \in C^1(\tilde{\Omega})$.

(ii) $\chi_+ = 0$ on a subset of $\tilde{\Sigma}_3$ of nonzero measure.

Case (i) corresponds to complete saturation of the constraint on the boundary $\tilde{\Sigma}_3$. This cannot occur under Hypothesis 6.1. As for case (ii), we know that under Hypothesis 6.2, there exists a neighborhood N of the curve $C_3 = \tilde{\Sigma}_2 \cap \tilde{\Sigma}_3$ such that $p = \chi_+ = 0$ on $\bar{N} \cap \tilde{\Sigma}_3$ ($\bar{N}$, the closure of N). This makes it possible to show that p belongs to $H^{3/2+\sigma}(V_2)$ in some neighborhood V_2 of $\tilde{\Sigma}_2$. By construction, p globally belongs to $H^{1-\varepsilon}(\tilde{\Omega})$ for all $\varepsilon > 0$. However, in order to substitute p into (8.22) we only need to show that the restriction of p to V_2 belongs to $H^{3/2+\sigma}(V_2)$ for some neighborhood V_2 of $\tilde{\Sigma}_2$. This is precisely what Hypothesis 6.2 provides. We can always construct this neighborhood V_2 of $\tilde{\Sigma}_3$ such that $\tilde{\Sigma}_3 \cap V_2 \subset N$. This neighborhood does not contain the discontinuity of the characteristic function χ_+; so the solution p belongs to $H^{3/2+\sigma}(V_2)$ in that neighborhood of $\tilde{\Sigma}_2$.

Setting $\varphi = p$ in (8.22):

$$\int_{\tilde{\Sigma}_2} \frac{\partial Y'}{\partial \nu_A} p|_{\tilde{\Sigma}_2} d\sigma = -\int_{\tilde{\Sigma}_2} \left[\tilde{L}^2 \left(\frac{\partial y}{\partial \xi_1} \frac{\partial p}{\partial \xi_1} + \frac{\partial y}{\partial \xi_2} \frac{\partial p}{\partial \xi_2} \right) + \frac{\partial y}{\partial \zeta} \frac{\partial p}{\partial \zeta} \right] \langle V, \nu \rangle \, d\sigma. \tag{8.23}$$

By combining (8.20) and (8.23) we finally obtain

$$\lambda(Y') = B(y, p), \tag{8.24}$$

where B is as defined in (8.4).

Remark 8.2. In deriving expression (8.24) we have assumed that y at least belonged to $H^3(\tilde{\Omega})$. This meant that $\tilde{\rho}$ was sufficiently smooth and that the angles θ_1 and θ_3 were sufficiently small. However, in the end the bilinear form $B(y, p)$ is continuous and well-defined for y and p in $H^{3/2+\sigma}(V)$, $\sigma > 0$, for some neighborhood V of $\tilde{\Sigma}_2$. In view of the discussion at the end of section 3.4, this means that we can relax conditions on angles θ_1 and θ_3 and keep the following ones:

$$0 < \theta_1 < \pi, \qquad 0 < \theta_3 < \pi.$$

8.3. Computation of $\lambda(y_L)$

Recall that the smoothness of y_L (Remark 7.1) is the same as the smoothness of y provided that the boundary $\tilde{\Sigma}_2$ is sufficiently smooth and that the angles θ_1 and θ_3 are small enough. Assume that y belongs to $H^2(\tilde{\Omega})$. Thus y_L belongs to $H^2(\tilde{\Omega}) \cap H_0(\tilde{\Omega})$ and we can set $\psi = y_L$ in (8.11):

$$\lambda(y_L) = (Ay_L, p)_{L^2(\tilde{\Omega})} + \left(\frac{\partial y_L}{\partial \nu_A}, p \right)_{L^2(\tilde{\Sigma}_1)} + \left(\frac{\partial y_L}{\partial \nu_A}, p \right)_{L^2(\tilde{\Sigma}_2)}. \tag{8.25}$$

In view of identities (7.3)–(7.4) (resp. (7.3a)–(7.4a)) for y_L the above expression reduces to

$$\lambda(y_L) = 2\tilde{L}\int_{\tilde{\Omega}}\left(\frac{\partial^2 y}{\partial \xi_1^2} + \frac{\partial^2 y}{\partial \xi_2^2}\right) p\, d\xi\, d\zeta - 2\tilde{L}\int_{\tilde{\Sigma}_2}\left(\nu_1 \frac{\partial y}{\partial \xi_1} + \nu_2 \frac{\partial y}{\partial \xi_2}\right) p\, d\sigma \tag{8.26}$$

$$\lambda(y_L) = +\frac{2}{\tilde{L}}\left[-\int_{\tilde{\Omega}} \frac{\partial^2 y}{\partial \zeta^2} p\, d\xi\, d\zeta + \int_{\tilde{\Sigma}_2} \nu_\zeta \frac{\partial y}{\partial \zeta} p\, d\sigma\right]. \tag{8.26a}$$

Remark 8.3. If somehow we "regularize" the characteristic function χ_+ so that p belongs to $H^1(\tilde{\Omega})$, the right hand side of expressions (8.26) and (8.26a) can be integrated by parts to yield

$$\lambda(y_L) = -2\tilde{L}\int_{\tilde{\Omega}}\left(\frac{\partial y}{\partial \xi_1}\frac{\partial p}{\partial \xi_1} + \frac{\partial y}{\partial \xi_2}\frac{\partial p}{\partial \xi_2}\right) d\xi\, d\zeta = b(y, p) \tag{8.27}$$

(cf. notation (8.3)),

$$\lambda(y_L) = \frac{2}{\tilde{L}}\int_{\tilde{\Omega}} \frac{\partial y}{\partial \zeta}\frac{\partial p}{\partial \zeta}\, d\xi\, d\zeta. \tag{8.27a}$$

Remark 8.4. Expression (8.26) only requires that y belongs to $H^2(\tilde{\Omega}) \cap H_0(\tilde{\Omega})$, that the boundary $\tilde{\Sigma}_2$ be smooth (say $\tilde{\rho}$ belongs to $C^1(0,1)$) and that the angles θ_1 and θ_3 be such that

$$0 < \theta_1 < \pi, \qquad 0 < \theta_3 \leqslant \frac{\pi}{2}.$$

When p is "regularized" in such a way that it belongs to $H^1(\tilde{\Omega})$, the last requirement on θ_3 can be relaxed to the one

$$0 < \theta_3 < \pi.$$

8.4. *Final Expressions for Gradients in Cylindrical Coordinates*

Let Assumption 6.1 and 8.1 be verified. In cylindrical coordinates, expressions (8.2), (8.3) and (8.4) become

$$a(\psi,\varphi) = \int_0^1 d\zeta \int_0^{\tilde{\rho}(\zeta)} 2\pi\rho\, d\rho\left[\tilde{L}^2 \frac{\partial \psi}{\partial \rho}\frac{\partial \varphi}{\partial \rho} + \frac{\partial \psi}{\partial \zeta}\frac{\partial \varphi}{\partial \zeta}\right] \tag{8.28}$$

$$b(\psi,\varphi) = -2\tilde{L}\int_0^1 d\zeta \int_0^{\tilde{\rho}(\zeta)} 2\pi\rho\, d\rho \frac{\partial \psi}{\partial \rho}\frac{\partial \varphi}{\partial \rho} \tag{8.29}$$

$$B(\psi,\varphi) = -\int_0^1\left[\tilde{L}^2 \frac{\partial \psi}{\partial \rho}\frac{\partial \varphi}{\partial \rho} + \frac{\partial \psi}{\partial \zeta}\frac{\partial \varphi}{\partial \zeta}\right]\omega(\tilde{\rho}(\zeta),\zeta) 2\pi\tilde{\rho}(\zeta)\, d\zeta. \tag{8.30}$$

Recall that

$$\lambda(Y') = B(y, p), \tag{8.31}$$

where p is the solution of the boundary-value problem (8.5) and y is the solution of the variational problem: to find y in $H_0(\tilde{\Omega})$ such that

$$a(y,\varphi) = l(\varphi), \qquad \forall\varphi \in H_0(\tilde{\Omega}), \tag{8.32}$$

where

$$l(\varphi) = 2\pi\int_0^1 \varphi(0,\rho)\rho\, d\rho. \tag{8.33}$$

Expression (6.24) for the gradient of the penalized cost with respect to the shape of the volume $\tilde{\Omega}$ can now be rewritten in terms of p:

$$\frac{1}{2\pi}dJ_\varepsilon(\tilde{L},\tilde{\rho};V) = \int_0^1\left\{\tilde{L} - \frac{1}{\varepsilon}\left[\tilde{L}^2\frac{\partial y}{\partial\rho}\frac{\partial p}{\partial\rho} + \frac{\partial y}{\partial\zeta}\frac{\partial p}{\partial\zeta}\right]_{\tilde{\Sigma}_2}\right\}\tilde{\rho}(\zeta)\,\omega(\tilde{\rho}(\zeta),\zeta)\,d\zeta \tag{8.34}$$

Remark 8.5. Notice that the boundary conditions

$$\left.\frac{\partial p}{\partial\nu_A}\right|_{\tilde{\Sigma}_2} = \left.\frac{\partial y}{\partial\nu_A}\right|_{\tilde{\Sigma}_2} = 0$$

can be used to obtain

$$B(y,p) = -\int_0^1\left[1+\left(\frac{1}{\tilde{L}}\frac{d\tilde{\rho}(\zeta)}{d\zeta}\right)^2\right]\left.\frac{\partial y}{\partial\zeta}\right|_{\tilde{\Sigma}_2}\left.\frac{\partial p}{\partial\zeta}\right|_{\tilde{\Sigma}_2}\omega(\tilde{\rho}(\zeta),\zeta)2\pi\tilde{\rho}(\zeta)\,d\zeta.$$

This alternate expression can also be used in (8.34).
Similarly recall that

$$\lambda(y_L) = -\frac{4\pi}{\tilde{L}}\left[\int_0^1 d\zeta\int_0^{\tilde{\rho}(\zeta)}\rho\, d\rho\frac{\partial^2 y}{\partial\zeta^2}p + \int_0^1 d\zeta\tilde{\rho}(\zeta)\frac{d\tilde{\rho}(\zeta)}{d\zeta}\left.\frac{\partial y}{\partial\zeta}\right|_{\tilde{\Sigma}_2} p|_{\tilde{\Sigma}_2}\right] \tag{8.35}$$

(the latter expression for $\lambda(y_L)$ is preferred to the equivalent one

$$\lambda(y_L) = 4\pi\tilde{L}\left[\int_0^1 d\zeta\int_0^{\tilde{\rho}(\zeta)}d\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial y}{\partial\rho}\right)p - \int_0^1 d\zeta\tilde{\rho}(\zeta)\frac{\partial y}{\partial\rho}|_{\tilde{\Sigma}_2}p|_{\tilde{\Sigma}_2}\right]). \tag{8.35a}$$

Expression (7.8) for the derivative of the penalized cost with respect to the parameter $\tilde{L}$ can also be written in terms of p:

$$\frac{1}{2\pi}\frac{dJ}{d\tilde{L}}\varepsilon(\tilde{L},\tilde{\rho}) = \int_0^1\left\{\frac{1}{2}\tilde{\rho}(\zeta)^2 + \left[\frac{2}{\varepsilon\tilde{L}}\tilde{\rho}(\zeta)\frac{d\tilde{\rho}}{d\zeta}(\zeta)\frac{\partial y}{\partial\zeta}|_{\tilde{\Sigma}_2}p|_{\tilde{\Sigma}_2}\right.\right.$$
$$\left.\left. + \int_0^{\tilde{\rho}(\zeta)}\rho\frac{\partial^2 y}{\partial\zeta^2}p\, d\rho\right]\right\}d\zeta. \tag{8.36}$$

If p is "regularized" to have it in $H^1(\tilde{\Omega})$. Then p is the solution of the variational problem: to find p in $H_\chi(\tilde{\Omega})$ such that

$$a(\psi, p) = 0, \qquad \forall \psi \in H_0(\tilde{\Omega}) \tag{8.37}$$

where

$$H_\chi(\tilde{\Omega}) = \{\psi \in H^1(\tilde{\Omega}) : \psi|_{\tilde{\Sigma}_3} = \chi_+^R\} \tag{8.38}$$

and χ_+^R is the "regularized" χ_+. Moreover

$$\begin{aligned}\lambda(y_L) = b(y, p) &= \frac{2}{\tilde{L}} \int_0^1 d\zeta \int_0^{\tilde{\rho}(\zeta)} 2\pi\rho \, d\rho \frac{\partial y}{\partial \zeta} \frac{\partial p}{\partial \zeta} \\ &= -2\tilde{L} \int_0^1 d\zeta \int_0^{\tilde{\rho}(\zeta)} 2\pi\rho \, d\rho \frac{\partial y}{\partial \rho} \frac{\partial p}{\partial \rho}\end{aligned} \tag{8.39}$$

and

$$\frac{1}{2\pi} \frac{dJ_\varepsilon}{d\tilde{L}}(\tilde{L}, \tilde{\rho}) = \int_0^1 \left[\frac{1}{2} \tilde{\rho}(\zeta)^2 - \frac{2}{\varepsilon \tilde{L}} \int_0^{\tilde{\rho}(\zeta)} \rho \frac{\partial y}{\partial \zeta} \frac{\partial p}{\partial \zeta} d\rho \right] d\zeta. \tag{8.40}$$

9. Solution of the Penalized Problem

In this section we describe the method which will be used to compute and approximate the solution to the minimization problem

$$\operatorname{Inf}\{J_\varepsilon(\tilde{L}, \tilde{\rho}) : \tilde{L} \geqslant 0, \tilde{\rho} \text{ satisfying } (9.2)\}, \tag{9.1}$$

where

$$\tilde{\rho}(\zeta) \geqslant 0, \qquad 0 \leqslant \zeta \leqslant 1, \qquad \tilde{\rho}(0) = 1, \qquad \tilde{\rho}(1) \geqslant \sqrt{1/q}\,. \tag{9.2}$$

Problem (9.1) is first rewritten in the following equivalent form

$$\operatorname{Inf}\{\operatorname{Inf}\{J_\varepsilon(\tilde{L}, \tilde{\rho}) : \tilde{\rho} \text{ satisfying } (9.2)\} : \tilde{L} \geqslant 0\}. \tag{9.3}$$

Our original problem (9.1) is then divided into the following two subproblems:

I) given $\tilde{L}$, find $\tilde{\rho}_\varepsilon(\tilde{L})$ satisfying (9.2) such that

$$\forall \tilde{\rho} \text{ satisfying } (9.2), \qquad J_\varepsilon(\tilde{L}, \tilde{\rho}_\varepsilon(\tilde{L})) \leqslant J_\varepsilon(\tilde{L}, \tilde{\rho}), \tag{9.4}$$

II) Find $\tilde{L}_\varepsilon \geqslant 0$ such that

$$\forall \tilde{L} \geqslant 0, \qquad J_\varepsilon(\tilde{L}_\varepsilon, \tilde{\rho}_\varepsilon(\tilde{L}_\varepsilon)) \leqslant J_\varepsilon(\tilde{L}, \tilde{\rho}_\varepsilon(\tilde{L})). \tag{9.5}$$

Problem II will be solved by a one-dimensional search. So we only concentrate on problem I) for a fixed $\tilde{L}$. We have seen that expression (8.34) gives the "directional derivative" of $J_\varepsilon(\tilde{\Omega})$ with respect to the "speed" or "velocity field"

$$V(\rho,\zeta) = \begin{pmatrix} \omega(\rho,\zeta) \\ 0 \end{pmatrix}. \tag{9.6}$$

It is of the form (cf. (8.34))

$$j(v) = \int_0^1 f^0(\zeta)v(\zeta)\,d\zeta, \tag{9.7}$$

where

$$v(\zeta) = \omega(\tilde{\rho}(\zeta),\zeta) \tag{9.8}$$

and

$$f^0(\zeta) = \left\{\tilde{L} - \frac{1}{\varepsilon}\left[\tilde{L}^2\frac{\partial y}{\partial\rho}\frac{\partial p}{\partial\rho} + \frac{\partial y}{\partial\zeta}\frac{\partial p}{\partial\zeta}\right]\right\}2\pi\tilde{\rho}(\zeta). \tag{9.9}$$

The computation of f^0 involves the computation of the state y and the adjoint state p.

9.1. *Approximation of y and p*

We use a finite element method to approximate the solution y of problem (8.32) and the solution p of problem (8.5). Since both solutions are defined on the same domain $\tilde{\Omega}$, we shall use the same triangulation of the domain D which generates the domain $\tilde{\Omega}$ by revolution about the ζ-axis.

The triangulation of the domain D will be obtained by deformation of a triangulation of the unit square (cf. Fig. 5)

$$D_0 = \{(\rho,\zeta)|0<\rho<1, 0<\zeta<1\}. \tag{9.10}$$

Partition the ζ-axis into $M \geqslant 1$ intervals defined by a sequence $\{\zeta_j\colon j=1,2,\ldots,M$

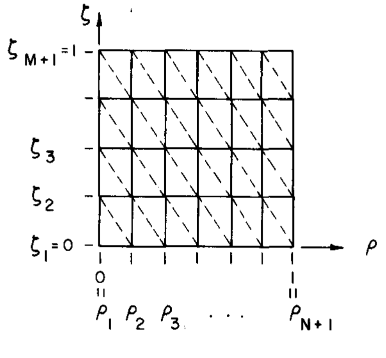

Fig. 5. Triangulation of the unit square D_0.

$+ 1\}$ of real numbers such that

$$\zeta_1 = 0, \qquad \zeta_{j+1} > \zeta_j, \qquad j = 1,2,\ldots,M, \qquad \zeta_{M+1} = 1. \tag{9.11}$$

Define the parameters

$$k = \operatorname{Max}\{\zeta_{j+1} - \zeta_j | j = 1,\ldots,M\}. \tag{9.12}$$

Partition the ρ-axis into $N \geqslant 1$ intervals defined by the sequence $\{\rho_i = (i-1)/N | i = 1,\ldots,N+1\}$ of real numbers. Consider the points

$$P_{ij} = (\rho_i, \zeta_j), \qquad i = 1,2,\ldots,N+1, \qquad j = 1,2,\ldots,M+1. \tag{9.13}$$

A triangulation of the domain D_0 is obtained by considering the set of all triangles with vertices

$$(P_{ij}, P_{i+1,j}, P_{i,j+1}), \qquad (P_{i,j+1}, P_{i+1,j}, P_{i+1,j+1}) \tag{9.14}$$

for $i = 1,2,\ldots,N+1$ and $j = 1,2,\ldots,M+1$.

The boundary $\tilde{\Sigma}_2$ of $\tilde{\Omega}$ and, a fortiori, the boundary S_2 of D, is completely specified by the shape function $\tilde{\rho}$. In the finite element approach the boundary S_2 has first to be approximated by a polygonal curve. This means that the function $\tilde{\rho}$ is approximated by a continuous shape function $\tilde{\rho}_k$ which is linear on each interval $[\zeta_j, \zeta_{j+1}]$, $j = 1,2,\ldots,M$. Equivalently, $\tilde{\rho}_k$ is completely characterized by a sequence $\{\tilde{\rho}_j | \tilde{\rho}_j > 0,\ j = 2,\ldots,M+1,\ \tilde{\rho}_1 = 1\}$ of real numbers in the following way

$$\tilde{\rho}_k(\zeta) = \frac{\zeta_{j+1} - \zeta}{\zeta_{j+1} - \zeta_j}\tilde{\rho}_j + \frac{\zeta - \zeta_j}{\zeta_{j+1} - \zeta_j}\tilde{\rho}_{j+1}, \qquad \zeta \in [\zeta_j, \zeta_{j+1}], \tag{9.15}$$

for $j = 1,\ldots,M$. This defines a polygonal domain

$$D_k = \{(\rho, \zeta) | 0 < \zeta < 1,\ 0 < \rho < \tilde{\rho}_k(\zeta)\} \tag{9.16}$$

which is an approximation of the original domain D.

The triangulation of the domain D_k is obtained from the triangulation of the unit square D_0 by moving each vertex P_{ij} in D_0 to the new position (cf. Fig. 6)

$$\tilde{P}_{ij} = (\tilde{\rho}_j \rho_i, \zeta_j) = ((i-1)\tilde{\rho}_j/N, \zeta_j) \tag{9.17}$$

in the (ρ, ζ)-plane for $j = 1,2,\ldots,M+1$ and $i = 1,2,\ldots,N+1$. Notice that the triangulation of D_0 has been chosen in such a way that the resulting triangles in the domain D_k are not too much distorted. Associate with the triangulation of D_k the parameter

$$h = \operatorname{Max}\{k, k'\}, k' = \operatorname{Max}\{\tilde{\rho}_j/N,\ j = 1,\ldots,M+1\}. \tag{9.18}$$

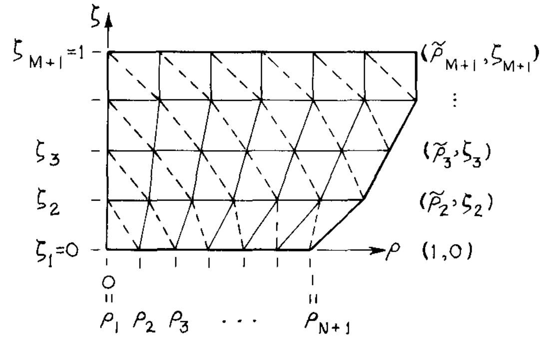

Fig. 6. Triangulation of the domain D_k.

Define the finite dimensional subspace $V_0^h(D_k)$ of $V_0(D_k)$ as

$$V_0^h(D_k) = \{v_h | v_h \in C^0(\bar{D}_k), v_h \text{ linear on each element}, \\ v_h(\rho,1)=0, 0 \leqslant \rho \leqslant \tilde{\rho}_k(1)\} \tag{9.19}$$

(the dimension of this space will be $M(N+1)$). The approximation y_h of y is defined as the solution of

$$\left.\begin{aligned} &y_h \in V_0^h(D_k) \text{ such that for all } v_h \text{ in } V_0^h(D_k) \\ &\int_{D_k}\left(\tilde{L}^2 \frac{\partial y_h}{\partial \rho}\frac{\partial v_h}{\partial \rho} + \frac{\partial y_h}{\partial \zeta}\frac{\partial v_h}{\partial \zeta}\right)\rho\, d\rho\, d\zeta = \int_0^1 v_h(\rho,0)\rho\, d\rho. \end{aligned}\right\} \tag{9.20}$$

By expressing y_h with respect to a basis of $V_0^h(D_k)$, problem (9.20) yields a linear system of algebraic equations and its (unique) solution provides the components of y_h.

The approximation p_h of p is obtained by first regularizing the characteristic function χ_+ in order to put the adjoint variable in the space $H^1(\tilde{\Omega})$. Then p_h is computed in a similar way to y_h by now using the variational problem (8.37)–(8.38). Choose as the regularized function χ_+^R the function χ_+^h which is defined by assigning to the points $\hat{\rho}_i = \tilde{\rho}_{M+1}\rho_i$, $i=1,\ldots,N+1$, the values

$$\chi_+^h(\hat{\rho}_i) = \begin{cases} 1, & \text{if } \left(\dfrac{\partial y_h}{\partial \zeta}(\hat{\rho}_i,1)+q\right) < 0 \\ 0, & \text{if } \left(\dfrac{\partial y_h}{\partial \zeta}(\hat{\rho}_i,1)+q\right) \geqslant 0 \end{cases} \tag{9.21}$$

and letting χ_+^h be continuous on the whole interval $[0, \hat{\rho}_{N+1}] = [0, \tilde{\rho}_{M+1}]$ and linear on each interval $[\hat{\rho}_i, \hat{\rho}_{i+1}]$, $i=1,\ldots,N$.

Associate with χ_+^h the function space

$$V_\chi^h(D_k) = \left\{ v_h \,\middle|\, \begin{aligned} &v_h \in C^0(\bar{D}_k)\, v_h \text{ linear on each element} \\ &v_h(\rho,1) = \chi_+^h(\rho), 0 \leqslant \rho \leqslant \tilde{\rho}_k(1) \end{aligned} \right\}.$$

Define p_h as the solution of the variational problem

$$\left.\begin{aligned} &p_h \in V_\chi^h(D_k) \text{ such that for all } v_h \text{ in } V_0^h(D_k) \\ &\int_{D_k}\left(\tilde{L}^2 \frac{\partial p_h}{\partial \rho}\frac{\partial v_h}{\partial \rho} + \frac{\partial p_h}{\partial \zeta}\frac{\partial v_h}{\partial \zeta}\right)\rho\, d\rho\, d\zeta = 0. \end{aligned}\right\} \tag{9.23}$$

Problem (9.23) also leads to an algebraic system which can be solved in an analogous way to problem (9.20).

9.2. *Gradient Computations for the Approximate Problem*

In the discretization process we have restricted our attention to domains $\tilde{\Omega}_h$ which are defined by revolution of the surface D_k around the ζ-axis. The surface D_k is itself completely specified by the function $\tilde{\rho}_k$ on $[0,1]$ which belongs to the subspace

$$P_1^1 = \left\{ \tilde{\rho} \in L^2(0,1) \,\middle|\, \begin{aligned} &\tilde{\rho} \text{ is continuous on } [0,1], \tilde{\rho}(0) = 1 \\ &\tilde{\rho} \text{ is linear on } [\zeta_j, \zeta_{j+1}],\ j = 1,\dots,M \end{aligned} \right\}. \tag{9.24}$$

We discretize expression (6.24) which gives the Eulerian derivative of the penalized cost for the speed $V = (\omega, 0)$:

$$dJ_\varepsilon(\tilde{L}, \tilde{\rho}; V) = 2\pi\left\{ \tilde{L}\int_0^1 \tilde{\rho}(\zeta)\omega(\tilde{\rho}(\zeta),\zeta)\, d\zeta - \frac{1}{\varepsilon}\int_0^{\tilde{\rho}(1)} \frac{\partial Y'}{\partial \zeta}(\rho,1)\chi_+(\rho)\rho\, d\rho \right\}. \tag{9.25}$$

The "discretized speeds" $V_h = (\omega_h, 0)$ are chosen in such a way that on $\tilde{\Sigma}_2^h$ (the lateral boundary of $\tilde{\Omega}_h$)

$$\omega_h(\tilde{\rho}(\zeta),\zeta) = v(\zeta), \tag{9.26}$$

where v belongs to the subspace

$$P_0^1 = \left\{ v \in L^2(0,1) \,\middle|\, \begin{aligned} &v \text{ is continuous on } [0,1], v(0) = 0 \\ &v \text{ is linear on } [\zeta_j, \zeta_{j+1}],\ j = 1,\dots,M \end{aligned} \right\} \tag{9.27}$$

The variable Y' (solution of (5.16)) is approximated by Y_h' the solution in $V_0^h(D_k)$ of the variational problem

$$\begin{aligned} &\int_{D_k}\left[\tilde{L}^2 \frac{\partial Y_h'}{\partial \rho}\frac{\partial \varphi}{\partial \rho} + \frac{\partial Y_h'}{\partial \zeta}\frac{\partial \varphi}{\partial \zeta}\right]\rho\, d\rho\, d\zeta \\ &\quad = -\int_0^1\left[\tilde{L}^2 \frac{\partial y}{\partial \rho}\frac{\partial \varphi}{\partial \rho} + \frac{\partial y}{\partial \zeta}\frac{\partial \varphi}{\partial \zeta}\right]2\pi\tilde{\rho}(\zeta)\omega(\tilde{\rho}(\zeta),\zeta)\, d\zeta \end{aligned} \tag{9.28}$$

for all φ in $V_0^h(D_k)$. The normal derivative $\partial Y'/\partial \nu_A$ on $\tilde{\Sigma}_3$ (which is equal to $(\partial Y'/\partial \zeta)(\rho, 1)$) is approximated in a weak sense by using Green's formula

$$\int_{\tilde{\Sigma}} \frac{\partial Y'}{\partial \nu_A} \varphi \, d\sigma = \int_{\tilde{\Omega}} \left[\tilde{L}^2 \left(\frac{\partial Y'}{\partial \xi_1} \frac{\partial \varphi}{\partial \xi_1} + \frac{\partial Y'}{\partial \xi_2} \frac{\partial \varphi}{\partial \xi_2} \right) + \frac{\partial Y'}{\partial \zeta} \frac{\partial \varphi}{\partial \zeta} \right] d\tilde{\Omega} + \int_{\tilde{\Omega}} - A Y' \varphi \, d\Omega \tag{9.29}$$

and the characterization (5.18)–(5.24) of Y'. In the end

$$\begin{aligned} \int_{\tilde{\Sigma}_3} \frac{\partial Y'}{\partial \nu_A} \varphi \, d\sigma = & \int_{\tilde{\Omega}} \left[\tilde{L}^2 \left(\frac{\partial Y'}{\partial \xi_1} \frac{\partial \varphi}{\partial \xi_1} + \frac{\partial Y'}{\partial \xi_2} \frac{\partial \varphi}{\partial \xi_2} \right) + \frac{\partial Y'}{\partial \zeta} \frac{\partial \varphi}{\partial \zeta} \right] d\tilde{\Omega} \\ & + \int_{\tilde{\Sigma}_2} \left[\tilde{L}^2 \left(\frac{\partial y}{\partial \xi_1} \frac{\partial \varphi}{\partial \xi_1} + \frac{\partial y}{\partial \xi_2} \frac{\partial \varphi}{\partial \xi_2} \right) + \frac{\partial y}{\partial \zeta} \frac{\partial \varphi}{\partial \zeta} \right] \langle V, \nu \rangle \, d\sigma \end{aligned} \tag{9.30}$$

for all φ in $H^{3/2+\sigma}(\tilde{\Omega})$ such that $\varphi = 0$ on $\tilde{\Sigma}_2 \cap \tilde{\Sigma}_3$. Using the approximation Y_h' of Y' and speeds V_h, the normal derivative $\dfrac{\partial Y'}{\partial \nu_A}$ is approximated by $\left(\dfrac{\partial Y'}{\partial \nu_A} \right)_h$ defined as

$$\begin{aligned} \int_{(\tilde{\Sigma}_3)_h} \left(\frac{\partial Y'}{\partial \nu_A} \right)_h \varphi_h \, d\sigma = & \int_{\tilde{\Omega}_h} \left[\tilde{L}^2 \left(\frac{\partial Y_h'}{\partial \xi_1} \frac{\partial \varphi_h}{\partial \xi_1} + \frac{\partial Y_h'}{\partial \xi_2} \frac{\partial \varphi_h}{\partial \xi_2} \right) + \frac{\partial Y_h'}{\partial \zeta} \frac{\partial \varphi_h}{\partial \zeta} \right] d\tilde{\Omega}_h \\ & + \int_{(\tilde{\Sigma}_2)_h} \left[\tilde{L}^2 \left(\frac{\partial y_h}{\partial \xi_1} \frac{\partial \varphi_h}{\partial \xi_1} + \frac{\partial y_h}{\partial \xi_2} \frac{\partial \varphi_h}{\partial \xi_2} \right) + \frac{\partial y_h}{\partial \zeta} \frac{\partial \varphi_h}{\partial \zeta} \right] \\ & \times \langle V_h, \nu \rangle \, d\sigma \end{aligned} \tag{9.31}$$

for all φ_h in $V^h(D_k)$,

$$V^h(D_k) = \{ v_h | v_h \in C^0(\overline{D}_k), v_h \text{ linear on each element} \} \tag{9.32}$$

such that $\varphi_h = 0$ on $(\tilde{\Sigma}_2)_h \cap (\tilde{\Sigma}_3)_h$. Finally the approximation of the Eulerian derivative of the penalized cost for speeds V_h is given by the expression

$$\begin{aligned} dJ_\varepsilon^h(\tilde{L}, \tilde{\rho}_k; V^h) = 2\pi \Big\{ & \tilde{L} \int_0^1 \tilde{\rho}_k(\zeta) \omega_h(\tilde{\rho}_k(\zeta), \zeta) \, d\zeta \\ & - \frac{1}{\varepsilon} \int_0^{\tilde{\rho}_k(1)} \left(\frac{\partial Y'}{\partial \zeta} \right)_h (\rho, 1) \chi_+(\rho) \rho \, d\rho. \end{aligned} \tag{9.33}$$

If we assume that χ_+ is zero on $\tilde{\Sigma}_3$ in a neighborhood of $\tilde{\Sigma}_2 \cap \tilde{\Sigma}_3$ and if we regularize χ_+ to χ_+^h, it is easy to verify that expression (9.32) becomes

$$dJ_\varepsilon^h(\tilde{L}, \tilde{\rho}_k; V^h) = \int_0^1 f_h^0(\zeta) v(\zeta) \, d\zeta \tag{9.34}$$

where

$$f_h^0(\zeta) = \left\{\tilde{L} - \frac{1}{\varepsilon}\left[\tilde{L}^2 \frac{\partial y_h}{\partial \rho}\frac{\partial p_h}{\partial \rho} + \frac{\partial y_h}{\partial \zeta}\frac{\partial p_h}{\partial \zeta}\right]_{(\tilde{\rho}_k(\zeta),\zeta)}\right\} 2\pi\tilde{\rho}_k(\zeta) \tag{9.35}$$

(substitute $\varphi_h = p_h$, the solution of (9.23), directly into (9.31)).

9.3. *Choice of the Speed*

Once the domain D_k has been fixed by specifying $\tilde{\rho}_k$, the state y_h and the adjoint state p_h can be computed. Then the gradient of the penalized cost $J_\varepsilon^h(\tilde{L}, \tilde{\rho}_k)$ can be obtained by computing $f_h^0(\zeta)$ from equation (9.35).

The approximate gradient in the direction of the speed v is of the form

$$j(v) = \int_0^1 f_h^0(\zeta) v(\zeta)\, d\zeta. \tag{9.36}$$

Ideally the speed which defines a direction of steepest descent is

$$v(\zeta) = -f_h^0(\zeta)/\|f_h^0\|_2. \tag{9.37}$$

Unfortunately the speeds v are required to belong to the subspace P_0^1 and the function f_h^0 is generally discontinuous at the points ζ_j (the partial derivative of y_h and p_h are at best piecewise constant on the boundary S_2) and $f_h^0(0) \neq 0$. So the best we can do is to project (in the L^2-sense) the function f_h^0 onto the subspace P_0^1. More precisely we shall choose the speed

$$v = -\mathcal{P} f_h^0/\|\mathcal{P} f_h^0\|_2, \tag{9.38}$$

where $\mathcal{P} f_h^0$ is the L^2-projection of f_h^0 onto P_0^1 (cf. Fig. 7).

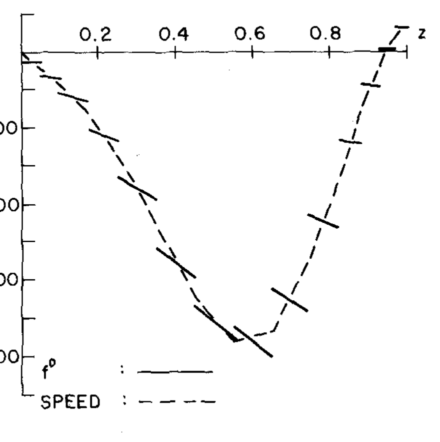

Fig. 7. Gradient f^0 and speed v. First iteration, $\varepsilon = 1/50$, $L = 2.75$.

9.4. *Method of Descent*

Start with an initial shape function $\tilde{\rho}_k^0$ such that

$$\tilde{\rho}_k^0(\zeta) \geqslant 0, \qquad 0 \leqslant \zeta \leqslant 1, \qquad \tilde{\rho}_k^0(0) = 1, \qquad \tilde{\rho}_k^0(1) \geqslant \sqrt{1/q}\,. \tag{9.39}$$

Assume that at step $n+1$ the shape function $\tilde{\rho}_k^n$ has been constructed in such a way that conditions (9.38) are verified with $\tilde{\rho}^n$ in place of $\tilde{\rho}^0$. To construct the new shape function $\tilde{\rho}^{n+1}$, we construct the new domain D_k^n from $\tilde{\rho}_k^n$ and compute the functions y_k^n and p_k^n on D_k^n. We compute the function $(f_h^0)^n$ and the speed v^n given by (9.37) with $(f_h^0)^n$ in place of f_h^0. Finally we construct the new shape function

$$\tilde{\rho}_k^{n+1}(\zeta) = \tilde{\rho}_k^n(\zeta) + t^{n+1}v^n(\zeta) \tag{9.40}$$

by choosing an appropriate $t^{n+1} > 0$ which minimizes $J_\varepsilon(\tilde{L}, \tilde{\rho}_k^{n+1}(\tilde{L}))$ and verifies condition (9.38) with $\tilde{\rho}_k^{n+1}$ in place of $\tilde{\rho}_k^0$.

9.5. *Numerical Example*

Three optimal shapes will be considered here, corresponding to the parameter $R_0 = 0.57$cm and the three values $1/5$, $1/10$ and $1/20$ of the flux ratio q.

We shall describe detailed computations for the intermediate case $q = 1/10$. The discretization parameters are the following:

$$\begin{aligned} N &= 10, \quad M = 14 \\ \{\zeta_j\} &= \{0.;0.05;0.1;0.175;0.25;0.35;0.45;0.55; \\ &\qquad 0.65;0.74;0.82;0.88;0.93;0.97;1.\}. \end{aligned}$$

A typical triangulation of the domain D_k is shown in Fig. 8 with an example of isotherms giving the temperature field inside the diffuser. For a fixed value of the

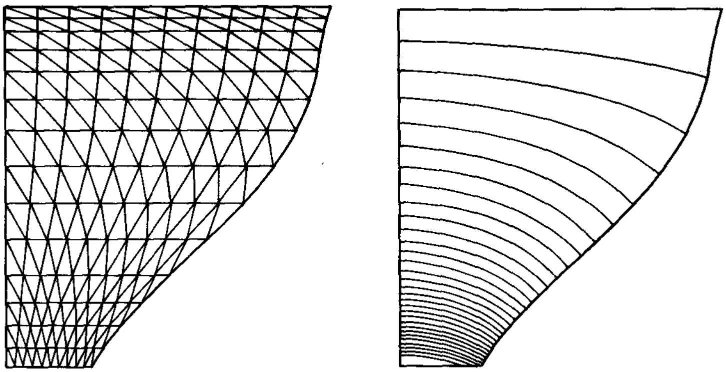

Fig. 8. Finite element grid and isotherms.

parameter $\tilde{L}$, a solution $\tilde{\rho}_\varepsilon(\tilde{L})$ of Problem I is found by the descent method of sec. 6.4. The initial shape $\tilde{\rho}_k^0$ is chosen as the linear function

$$\tilde{\rho}_k^0(\zeta) = 1 + 2\zeta/R_0, \qquad 0 \leqslant \zeta \leqslant 1, \qquad R_0 = 0.57\text{cm}. \tag{9.41}$$

In a first step, the parameter ε is set equal to $1/50$ in order to let the constraint be active during the iterations (cf. Fig. 9). The penalized cost can be plotted against the length $L = R_0\tilde{L}$ as shown in Fig. 10. From this graph it is readily seen that the length L leading to the lightest diffuser is around $L = 2.75$cm. Details of a typical computation are shown on Fig. 11. We obviously see a monotoneously decreasing penalized cost, a "dual behavior between the volume and the constraint (one increasing and the other decreasing). One of the striking features of the optimization search is that a small difference between two shapes can result in finding or not a "good" direction of descent. Fig. 11 shows the evolution of the shape function during the iteration for a fixed $L = \tilde{L}R_0 = 2.75$.

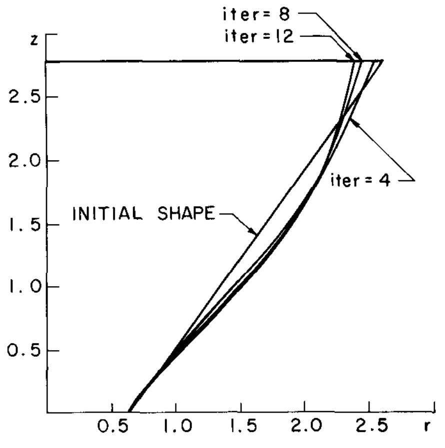

Fig. 9. Evolution of the shape; $\varepsilon = 1/50$.

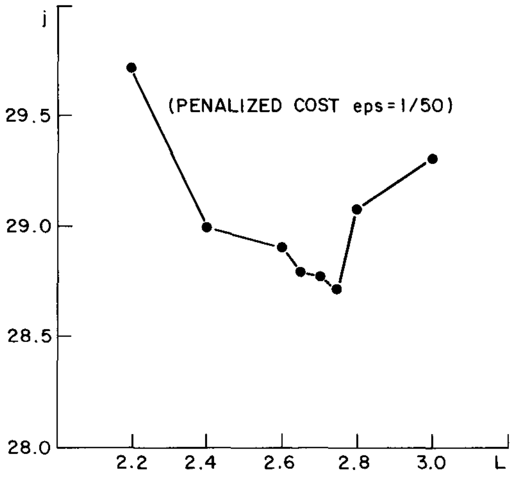

Fig. 10. Cost of optimal diffuser versus L.

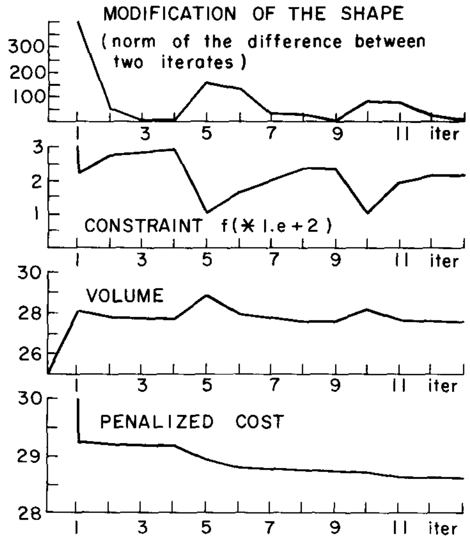

Fig. 11. Results for one optimization of the shape starting from a linear initial shape; $\varepsilon = 1/50$.

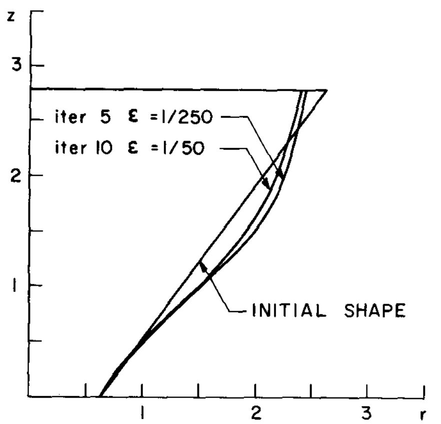

Fig. 12. Optimal diffusers for $L = 2.775$.

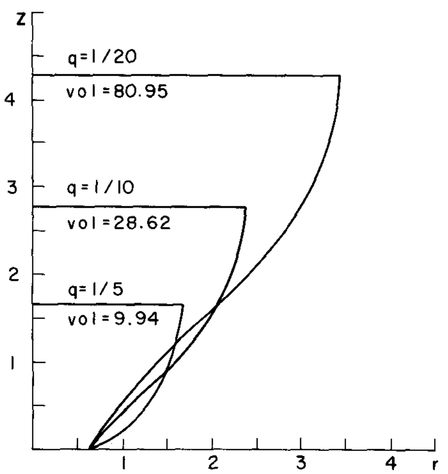

Fig. 13. Optimal diffusers for different ratios q.

In a second step, the parameter ε is reduced from the value $1/50$ to $1/250$. The initial shape function is chosen as the one found at the end of the iterations of the first step. In so doing we reduce the value of the constraint functional to practically zero by slightly enlarging the diffuser near its output surface $\tilde{\Sigma}_3$ (see Fig. 12). A one-dimensional search is also performed at this stage to determine the new value of the parameter $L = \tilde{L}R_0$ ($L = 2.775$) giving the smallest cost.

The optimal diffusers for the three values of the flux ratio q and $R_0 = 0.57$ have been drawn on Fig. 13.

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References

1. Céa J (1976) Une méthode numérique pour la recherche d'un domaine optimal. Publication IMAN, Université de Nice
2. Céa J (1978) Optimization: theory and algorithms—Tata Institute. Springer-Verlag, New York.
3. Chavent G (1971) Analyse fonctionnelle et identification des coefficients répartis dans les équations aux dérivées partielles. Thèse de doctorat d'état, Université de Paris VI
4. Delfour M, Payre G, Zolésio J-P (1981) Design of a mass-optimized diffuser. In: Céa J, Haug E J (eds) Optimization of distributed parameter structures: Vol II. Sijthoff and Nordhoff, Alphen aan den Rijn, pp 1250–1268.
5. Destuynder Ph. (1976) Etude théorique et numérique d'un algorithme d'optimisation de structures. Thèse de docteur ingénieur, Université de Paris VI
6. Do Carmo MP (1976) Differential geometry of curves and surfaces. Prentice-Hall, Englewood Cliffs
7. Grisvard P (1972) Alternative de Fredholm relative au problème de Dirichlet dans un polygone ou un polyèdre. Boll UMI (4) 5:132–164
8. Grisvard P (1975) Alternative de Fredholm relative au problème de Dirichlet dans un polyèdre. Ann Sc Norm Super Pisa, série IV, 2:359–388
9. Grisvard P (1975) Behavior of the solutions of an elliptic boundary value problem in a polygonal or polyhedral domain. In: Hubbard B (ed) SYNSPADE III, Symposium on Numerical Solution of Partial Differential Equations, Academic Press, New York, pp 207–274
10. Grisvard P (1980) Boundary value problems in non-smooth domains. Lecture Notes #19, University of Maryland, College Park
11. Lions JL, Magenes E (1968) Problèmes aux limites non homogènes et applications, vol 1, Dunod, Paris. Also in English translation in: Kenneth P (translator) Grundeehren der mathematischen wissenschaften, vol 181. Springer-Verlag Berlin Heidleberg New York
12. Zolésio J-P (1981) The material derivative. In: Céa J, Haug E J (eds) Optimization of distributed parameter structures: Vol II. Sijthoff and Nordhoff, Alphen aan den Rijn, pp 1089–1151
13. Zolésio J-P (1981a) Semi-derivative of repeated eigenvalues. In: Céa J, Haug E J (eds) Optimization of distributed parameter structures: Vol II. Sijthoff and Nordhoff, Alphen aan den Rijn, pp 1457–1473
14. Zolésio J-P (1979) Identification de domaines par déformations. Thèse de doctorat d'état, Université de Nice
15. Zolésio J-P (1977) An optimal design procedure for optimal control support. In: Auslender A (ed) Proceedings of a 1976 conference held at Murat le Quaire. Lecture Notes in Economical and Mathematical Systems, no. 14. Springer-Verlag New York Heidelberg Berlin

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